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On a complex equiaffine immersion of general codimension

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Abstract. We study a complex equiaffine immersion of general codimension by regarding a frame of the complex determinant bundle of a complex vector bundle as a complex volume form. Some results on a complex affine hypersurface with volume form, especially a complex equiaffine hypersurface, are generalized to the case of general codimension. Especially, we obtain the fundamental theorems for a complex equiaffine immersion to a complex affine space of general codimension.

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§1. Introduction.

For a complex affine hypersurface, if the transversal vector field is parallel with respect to the transversal connection, then the immersion is called a complex equiaffine hypersurface and the transversal vector field is often called equiaffine. For a complex affine hypersurface in a complex affine space, the existence of an equiaffine transversal vector field is studied in [3] and [4]. Such a hypersurface is studied in [11], where the author use a complex Ricci tensor. On the other hand, an equiaffine immersion of general codimension is studied in [7] and [8], where they take a certain frame of its transversal bundle to define an induced volume form.

The main purpose of this paper is to study a complex affine immersion with volume form of general codimension, especially a complex equiaffine immersion, and generalize some of the results in [3], [4], [9] and [11]. In Section 2, to apply them to the transversal bundle of a complex affine immersion, we prepare the notions in terms of a complex vector bundle. In detail, for a complex vector bundle with connection, we regard a frame of its complex determinant bundle as a complex volume form and derive some fundamental results for the complex determinant bundle in terms of connections, which contain generalizations of those for the tangent bundle of a complex manifold in [3], [4] and [11]. In Section 3, for a decomposition of a complex vector bundle with connection and volume form, we study the relation between the induced connections, the second fundamental forms determined by the decomposition and the connections on the complex determinant bundles. In Section 4, we apply these results in the previous sections to a complex affine immersion and derive some results which include generalizations of those in [9] and [11]. We also define and study a complex equiaffine immersion of general codimension, where we obtain some results which can be considered as generalizations of those in [4] and [11]. In Section 5, we state and prove the fundamental theorems for a complex equiaffine immersion to a complex affine space of general codimension, which are complex versions of those given in [5].

§2. Preliminaries.

Throughout this paper, all manifolds are assumed to be connected and all objects and morphisms are assumed to be smooth. Let M be a manifold, TM its tangent bundle and T^*M its cotangent bundle. We use letters E, $\widetilde{E}$ to denote vector bundles over M. The fibre of a vector bundle E at $x \in M$ is denoted by E_x, the space of all cross sections of E by $\Gamma(E)$ and the set of all connections on E by $\mathcal{C}(E)$. We denote by $A^p(E) = \Gamma(\wedge^p T^*M \otimes E)$ the space of all E-valued p-forms on M. Let $\mathrm{Hom}(\widetilde{E}, E)$ be the vector bundle of which fibre $\mathrm{Hom}(\widetilde{E}, E)_x$ at $x \in M$ is the vector space $\mathrm{Hom}_{\mathbb{R}}(\widetilde{E}_x, E_x)$ of linear maps from $\widetilde{E}_x$ to E_x. Let $\mathrm{HOM}(\widetilde{E}, E)$ be the space of all vector bundle homomorphisms from $\widetilde{E}$ to E and $\mathrm{END}(E) := \mathrm{HOM}(E, E)$. We note that $\mathrm{HOM}(\widetilde{E}, E)$ can be identified with $\Gamma(\mathrm{Hom}(\widetilde{E}, E))$. For $\Phi \in \mathrm{HOM}(\widetilde{E}, E)$ and $x \in M$, put $\Phi_x := \Phi|_{E_x}$. The space of all vector bundle isomorphisms from $\widetilde{E}$ to E is denoted by $\mathrm{ISO}(\widetilde{E}, E)$.

In order to make our paper self-contained, we begin by preparing definitions and fundamental properties about a complex vector bundle, some of the results are given in [1].

Let (V, J^V) be a $2r$-dimensional real vector space with complex structure J^V. We can regard (V, J^V) as a complex vector space by defining scalar multiplication by

$$(a + b\sqrt{-1})\xi := a\xi + bJ^V\xi$$

for $a, b \in \mathbb{R}$ and $\xi \in V$. We denote the complex vector space defined above by $(V, J^V)_{\mathbb{C}}$. We call $(s_1, \ldots, s_r, J^V s_1, \ldots, J^V s_r)$ a *complex basis* of (V, J^V) with respect to J^V if $(s_1, \dots, s_r)$ is a basis of $(V, J^V)_{\mathbb{C}}$. For simplicity, we denote $(s_1, \dots, s_r, J^V s_1, \dots, J^V s_r)$ by $(s_1, \dots, J^V s_r)$. For $\xi \in V$ and a complex basis $(s, \dots, J^V s_r)$, we denote by $\hat{\xi} := {}^t(\xi^1, \dots, \xi^{2r})$ the column vector which is a component of ξ with respect to $(s_1, \dots, J^V s_r)$, that is, $\xi = (s_1, \dots, J^V s_r)\hat{\xi} = \xi^l s_l + \xi^{l+r} J^V s_l = (s_1, \dots, J^V s_r)\, {}^t(\xi^1, \dots, \xi^{2r})$, where ${}^t(\,\cdot\,)$ denote the transpose of $(\,\cdot\,)$. Then, we put $\mathrm{re}\hat{\xi} := {}^t(\xi^1, \dots, \xi^r)$ and $\mathrm{im}\hat{\xi} := {}^t(\xi^{r+1}, \dots, \xi^{2r})$. For $(\widetilde{V}, J^{\widetilde{V}})$, let $\mathrm{Hom}_{\mathbb{R}}((\widetilde{V}, J^{\widetilde{V}}),\ (V, J^V)) := \{\psi \in \mathrm{Hom}_{\mathbb{R}}(\widetilde{V}, V) \mid \psi J^{\widetilde{V}} = J^V \psi\}$, which can be identified with $\mathrm{Hom}_{\mathbb{C}}((\widetilde{V}, J^{\widetilde{V}})_{\mathbb{C}},\ (V, J^V)_{\mathbb{C}})$. Especially, we denote the standard basis of $\mathbb{R}^{2r}$ by $(e_1, \dots, e_{2r})$ and define the standard complex structure J_0 on $\mathbb{R}^{2r}$ by

$$J_0 e_k = e_{k+r}, \qquad J_0 e_{k+r} = -e_k$$

for $k = 1, \dots, r$. We will generally use the same symbol to denote an endomorphism of $\mathbb{R}^{2r}$ and its matrix representation with respect to the standard basis. $M(2r, 2p; J_0)$ denotes all real $(2r, 2p)$-matrices which commute with J_0 of $\mathbb{R}^{2r}$ and J_0 of $\mathbb{R}^{2p}$. For a $(2r, 2p)$-matrix A, we see that $A \in M(2r, 2p; J_0)$ if and only if A is the following type:

$$A = \begin{pmatrix} B & -C \\ C & B \end{pmatrix},$$

where B, C are (r, p)-matrices. Then we put $\mathrm{re}A := B$ and $\mathrm{im}A := C$.

Let (E, J^E) be a real vector bundle E of rank $2r$ over M with complex structure $J^E \in \mathrm{END}(E)$ such that $(J^E)^2 = -id_E$ and we call (E, J^E) a complex vector bundle. A complex vector bundle (E, J^E) can be turned into a complex vector bundle in the usual sense when we regard (E_x, J^E_x) as a complex vector space $(E_x, J^E_x)_{\mathbb{C}}$ for each $x \in M$. For (E, J^E), an open set $U \subset M$ and $u_k \in \Gamma(E|_U)$, if $(u_1(x), \dots, u_r(x))$ is a basis of $(E_x, J^E_x)_{\mathbb{C}}$ for each $x \in U$, we call $(u_1, \dots, J^E u_r)$ a complex local frame field of (E, J^E) with respect to J^E and put $U := \mathrm{Dom}(u_1, \dots, J^E u_r)$. Let $\mathfrak{LF}(E, J^E)$ be the set of all complex local frame fields of (E, J^E). Hereafter in this paper, $(\widetilde{E}, J^{\widetilde{E}})$ and (E, J^E) denote complex vector bundles over M. Let $\mathrm{Hom}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E))$ be the vector bundle of which fibre at $x \in M$ is $\mathrm{Hom}_{\mathbb{R}}((\widetilde{E}_x, J^{\widetilde{E}}_x), (E_x, J^E_x))$. Moreover, we put

$$\begin{aligned} \mathrm{HOM}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E)) &:= \{\Phi \in \mathrm{HOM}(\widetilde{E}, E) \mid \Phi J^{\widetilde{E}} = J^E \Phi\}, \\ \mathrm{ISO}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E)) &:= \mathrm{ISO}(\widetilde{E}, E) \cap \mathrm{HOM}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E)). \end{aligned}$$

Hereafter in this paper, we denote by (M, J) a complex manifold with complex structure $J \in \mathrm{END}(TM)$ which is integrable.

Definition 2.1 If a complex vector bundle $(E, J^E)_{\mathbb{C}}$ has a holomorphic vector bundle structure in the usual sense, we call $(u_1, \dots, J^E u_r)$ a *real holomorphic* *local frame field* of (E, J^E) for a holomorphic local frame field $(u_1, \ldots, u_r)$ of $(E, J^E)_{\mathbb{C}}$ with respect to the structure. A *holomorphic vector bundle structure* $\mathcal{H}$ of (E, J^E) is the set of all real holomorphic local frame fields of (E, J^E) and we denote a holomorphic vector bundle (E, J^E) with $\mathcal{H}$ by $(E; \mathcal{H})$.

For an $M(2r, 2p; J_0)$-valued function A on an open set $V \subset M$, we say A is *real holomorphic* if the equation

$$X\mathrm{re}A = (JX)\mathrm{im}A$$

holds for any $X \in T_xM$, $x \in V$. When we regard $\mathrm{re}A + \sqrt{-1}\mathrm{im}A$ as a complex (r, p)-matrix-valued function, A is real holomorphic if and only if $\mathrm{re}A + \sqrt{-1}\mathrm{im}A$ is holomorphic.

Definition 2.2 For a holomorphic vector bundle $(E; \mathcal{H})$ and an open set V of M, a section $\xi \in \Gamma(E|_V)$ is said to be *holomorphic with respect to* $\mathcal{H}$ if for any $(u_1 \ldots, J^E u_r) \in \mathcal{H}$ such that $V \cap U \neq \phi$ and the local representation $\xi|_{V \cap U} = (u_1, \ldots, J^E u_r)\hat{\xi}$ on $V \cap U$, $\hat{\xi}$ is a real holomorphic function, that is,

$$X(\mathrm{re}\hat{\xi}) = (JX)(\mathrm{im}\hat{\xi})$$

for any $X \in T_xM$, $x \in U \cap V$, where $U = \mathrm{Dom}(u_1, \ldots, J^E u_r)$. The space of all holomorphic sections of $E|_V$ with respect to $\mathcal{H}$ is denoted by $\Gamma^h(E|_V; \mathcal{H})$.

Note that if $(E, J^E) = (TM, J)$, a holomorphic section is often called a real holomorphic vector field.

For a complex vector bundle (E, J^E), a connection ∇ on E is said to be *complex* if $\nabla_X J^E = J^E \nabla_X$ for any $X \in T_xM$, $x \in M$. We denote by $\mathcal{C}(E, J^E)$ the set of all complex connections on E.

Definition 2.3 For a holomorphic vector bundle $(E; \mathcal{H})$, we say that $\nabla \in \mathcal{C}(E, J^E)$ is *adapted to* $\mathcal{H}$ if for any open set $V \subset M$, $\xi \in \Gamma^h(E|_V; \mathcal{H})$ and $X \in T_xM$, $x \in V$,

$$\nabla_{JX}\xi = J^E \nabla_X \xi.$$

The set of all connections on (E, J^E) adapted to $\mathcal{H}$ is denoted by $\mathcal{C}^a(E; \mathcal{H})$.

Definition 2.4 For complex vector bundles $(\widetilde{E}, J^{\widetilde{E}})$ and (E, J^E), a 1-form $K \in A^1(\mathrm{Hom}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E)))$ is said to be *complex* if

$$K_{JX} = J^E K_X$$

for any $X \in T_xM$, $x \in M$. We denote by $A^{1,0}(\mathrm{Hom}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E)))$ the set of all complex $\mathrm{Hom}((\widetilde{E}, J^{\widetilde{E}}), (E, J^E))$-valued 1-forms.

For $\nabla \in \mathcal{C}(E, J^E)$, we define the curvature form $R \in A^2(\mathrm{End}(E, J^E))$ of ∇ by

$$R_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for $X, Y \in \Gamma(TM)$.

Lemma 2.5 *For $\nabla \in \mathcal{C}^a(E; \mathcal{H})$, we have*

$$R_{X,Y} + J^E R_{JX,Y} + J^E R_{X,JY} - R_{JX,JY} = 0 \tag{2.1}$$

for any $X, Y \in T_xM$, $x \in M$.

The following result is given by paraphrasing the result in [6].

Lemma 2.6 *For $\nabla \in \mathcal{C}(E, J^E)$, if the curvature form R of ∇ satisfies* (2.1)*, then (E, J^E) has a unique holomorphic vector bundle structure to which ∇ is adapted.*

Definition 2.7 We denote by $\mathcal{C}^a(E, J^E)(\subset \mathcal{C}(E, J^E))$ the set of all complex connections on E whose curvature form satisfies the equation (2.1) and call an element of $\mathcal{C}^a(E, J^E)$ an *adapted connection*. Especially we denote by $\mathcal{H}^\nabla$ the holomorphic vector bundle structure of (E, J^E) determined by $\nabla \in \mathcal{C}^a(E, J^E)$.

We prepare three classes of connections in terms of their curvature forms.

Definition 2.8 For $\nabla \in \mathcal{C}(E, J^E)$, we say ∇ is *holomorphic* if $R_{JX,Y} = J^E R_{X,Y}$, *anti-holomorphic* if $R_{JX,Y} = -J^E R_{X,Y}$ and *of type* $(1,1)$ if $R_{JX,JY} = R_{X,Y}$ for any $X, Y \in T_xM$, $x \in M$.

Note that a connection of type $(1,1)$ is defined in [2]. The Hermitian connection on a Hermitian holomorphic vector bundle in [6] is of type $(1,1)$. When $(E, J^E) = (TM, J)$, torsion free affine connections of type $(1,1)$ are called *affine Kähler connections* in [9]. We see that a flat connection $\nabla \in \mathcal{C}(E, J^E)$ satisfies all of the three conditions above and a connection $\nabla \in \mathcal{C}(E, J^E)$ which satisfies at least two of the conditions above is flat.

For a p-dimensional vector space V, we define $\mathrm{Det}V$ by

$$\mathrm{Det}V := \{\omega : V^p \to \mathbb{R} \mid \omega \text{ is skew-symmetric, } \mathbb{R}\text{-multilinear}\}.$$

A non-zero element of $\mathrm{Det}V$ is called a *volume form* on V. For a real vector bundle E of rank p, we can define a smooth vector bundle $\mathrm{Det}E$ by requiring

$$(\mathrm{Det}E)_x := \mathrm{Det}E_x$$

for each $x \in M$, which is called the *determinant bundle* of E. We call a global frame field of $\mathrm{Det}E$ a *volume form* on E. For $\nabla \in \mathcal{C}(E)$, we denote the induced connection by $\nabla^{\mathrm{Det}E} \in \mathcal{C}(\mathrm{Det}E)$, that is,

$$(\nabla_X^{\mathrm{Det}E}\omega)(\xi_1, \ldots, \xi_p) = X(\omega(\xi_1, \ldots, \xi_p)) - \sum_{i=1}^{p} \omega(\xi_1, \ldots, \nabla_X \xi_i, \ldots, \xi_p)$$

for any $\omega \in \Gamma(\mathrm{Det}E)$, $X \in T_xM$, $x \in M$ and $\xi_i \in \Gamma(E)$. Then we can prove

$$\mathrm{tr}_{\mathbb{R}}(R_{X,Y})\,\omega \quad = \quad -R_{X,Y}^{\mathrm{Det}E}\omega \tag{2.2}$$

for any $X, Y \in T_xM$, $x \in M$, where $R^{\mathrm{Det}E}$ is the curvature form of $\nabla^{\mathrm{Det}E}$. Note that $\nabla^{\mathrm{Det}E}$ is usually denoted by ∇. For a volume form $\omega \in \Gamma(\mathrm{Det}E)$ and $\nabla \in \mathcal{C}(E)$, if $\nabla_X^{\mathrm{Det}E}\omega = 0$ for any $X \in T_xM$, $x \in M$, then we say that ω is *parallel with respect to* ∇. For the sake of simplicity, we write $\mathrm{tr}_{\mathbb{R}}R = 0$ when $\mathrm{tr}_{\mathbb{R}}R_{X,Y} = 0$ for any $X, Y \in T_xM$, $x \in M$. Since there is a local parallel frame field on a neighbourhood of each point for a flat connection on a vector bundle, (2.2) implies

Lemma 2.9 *For* $\nabla \in \mathcal{C}(E)$, $\mathrm{tr}_{\mathbb{R}}R = 0$ *if and only if there exists a local volume form on* E *which is parallel with respect to* ∇ *defined on a neighbourhood of each point.*

Let $(\mathbb{R}^2, J_0)$ be a real vector space $\mathbb{R}^2$ with the standard complex structure J_0. Hereafter we always identify $(\mathbb{R}^2, J_0)$ with $\mathbb{C}$ under the natural correspondence. Let (V, J^V) be a complex vector space. For an $\mathbb{R}$-linear map $T : V \to V$, we define the complex trace $\mathrm{tr}_{\mathbb{C}}T$ by

$$\mathrm{tr}_{\mathbb{C}}T := \frac{1}{2}\left(\mathrm{tr}_{\mathbb{R}}T - \sqrt{-1}\,\mathrm{tr}_{\mathbb{R}}J^V T\right).$$

We note that $\mathrm{tr}_{\mathbb{C}}T = 0$ if and only if $\mathrm{tr}_{\mathbb{R}}T = 0$ and $\mathrm{tr}_{\mathbb{R}}J^V T = 0$. From the definition of the complex trace, we obtain

Lemma 2.10 *If* $\nabla \in \mathcal{C}(E, J^E)$ *is holomorphic or anti-holomorphic, then* $\mathrm{tr}_{\mathbb{C}}R = 0$ *if and only if* $\mathrm{tr}_{\mathbb{R}}R = 0$.

For a $2r$-dimensional complex vector space (V, J^V), we define $\mathrm{Det}_{\mathbb{C}}V$ by

$$\begin{aligned}\mathrm{Det}_{\mathbb{C}}V := \{\Omega : V^r \to \mathbb{C} \mid\ & \Omega \text{ is skew-symmetric, } \mathbb{R}\text{-multilinear,} \\ & \sqrt{-1}\Omega(v_1, \ldots, v_r) = \Omega(J^V v_1, v_2, \ldots, v_r) \text{ for any } v_1, \ldots, v_r \in V\}.\end{aligned}$$

A non-zero element of $\mathrm{Det}_{\mathbb{C}}V$ is called a *complex volume form* on (V, J^V). For (E, J^E), we can define a smooth vector bundle $\mathrm{Det}_{\mathbb{C}}E$ by requiring

$$(\mathrm{Det}_{\mathbb{C}}E)_x = \mathrm{Det}_{\mathbb{C}}E_x$$

for each $x \in M$ and call it the *complex determinant bundle* of E. We call a global frame field of $\Gamma(\mathrm{Det}_{\mathbb{C}}E)$ a *complex volume form* on (E, J^E). Let $J^{\mathrm{Det}_{\mathbb{C}}E}$ be the induced complex structure on $\mathrm{Det}_{\mathbb{C}}E$ from J^E, that is,

$$(J^{\mathrm{Det}_{\mathbb{C}}E}\Theta)(\xi_1, \dots, \xi_r) = \Theta(J^E\xi_1, \dots, \xi_r) = \sqrt{-1}(\Theta(\xi_1, \dots, \xi_r))$$

for any $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}E)$, $\xi_i \in E(x)$, $x \in M$, $i = 1, \dots, r$. Let $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ be the connection on $\mathrm{Det}_{\mathbb{C}}E$ induced from $\nabla \in \mathcal{C}(E, J^E)$, that is,

$$\nabla_X^{\mathrm{Det}_{\mathbb{C}}E}\Theta(\xi_1, \dots, \xi_r) = X(\Theta(\xi_1, \dots, \xi_r)) - \sum_{i=1}^{r} \Theta(\xi_1, \dots, \nabla_X\xi_i, \dots \xi_r)$$

for any $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}E)$, $X \in T_xM$, $x \in M$ and $\xi_i \in \Gamma(E)$. Then $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ is a complex connection with respect to $J^{\mathrm{Det}_{\mathbb{C}}E}$. As an analogue of (2.2), for any $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}E)$ and $X, Y \in T_xM$, $x \in M$ we get

$$\mathrm{tr}_{\mathbb{C}}(R_{X,Y})\Theta \;=\; -R_{X,Y}^{\mathrm{Det}_{\mathbb{C}}E}\,\Theta, \tag{2.3}$$

where $R^{\mathrm{Det}_{\mathbb{C}}E}$ is the curvature form of $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$. As an analogue of Lemma 2.9, we have the following from (2.3).

Lemma 2.11 *For $\nabla \in \mathcal{C}(E, J^E)$, $\mathrm{tr}_{\mathbb{C}}R = 0$ if and only if there exists a local complex volume form on (E, J^E) which is parallel with respect to ∇ defined on a neighbourhood of each point.*

Note that Lemma 2.11 has been already shown in the case where $(E, J^E) = (TM, J)$ in [11].

Lemma 2.12 *For $\nabla \in \mathcal{C}(E, J^E)$ and a complex volume form $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}E)$,*

$$\theta \;:=\; \left(\sqrt{-1}\right)^r \Theta\wedge\bar{\Theta} \in \Gamma(\mathrm{Det}E)$$

is a real volume form, where $\bar{\Theta}$ is defined by

$$\bar{\Theta}(\xi_1, \dots, \xi_r) \;:=\; \overline{\Theta(\xi_1, \dots, \xi_r)}$$

for any $\xi_i \in \Gamma(E)$, $i = 1, \dots, r$. Moreover if Θ is parallel with respect to ∇, then θ is parallel with respect to ∇.

Proof. Since Θ is a complex volume form, we get

$$\Theta(\dots, \eta, \dots, J^E\eta, \dots) = 0$$

for any $\eta \in E_x$, $x \in M$. Thus, for a complex basis $\xi_1, \ldots, J^E\xi_r$ of E_x, $x \in M$, we have

$$\begin{aligned}
&(\Theta\wedge\bar{\Theta})(\xi_1, \ldots, J^E\xi_r) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_r=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_r}\Theta((J^E)^{\varepsilon_1}\xi_1, \ldots, (J^E)^{\varepsilon_r}\xi_r) \\
&\qquad\qquad \overline{\Theta}((J^E)^{1-\varepsilon_1}\xi_1, \ldots, (J^E)^{1-\varepsilon_r}\xi_r) \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_r=0}^{1} (-1)^{\varepsilon_1+\cdots+\varepsilon_r}(\sqrt{-1})^{\varepsilon_1+\cdots+\varepsilon_r}\overline{(\sqrt{-1})}^{r-(\varepsilon_1+\cdots+\varepsilon_r)} \\
&\qquad\qquad \Theta(\xi_1, \ldots, \xi_r)\overline{\Theta(\xi_1, \ldots, \xi_r)} \\
&= \sum_{\varepsilon_1,\ldots,\varepsilon_r=0}^{1} (\overline{\sqrt{-1}})^r\Theta(\xi_1, \ldots, \xi_r)\overline{\Theta(\xi_1, \ldots, \xi_r)} \\
&= (\overline{\sqrt{-1}})^r 2^r\Theta(\xi_1, \ldots, \xi_r)\overline{\Theta(\xi_1, \ldots, \xi_r)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\theta(\xi_1, \ldots, J^E\xi_r) &= \left(\sqrt{-1}\right)^r \Theta\wedge\overline{\Theta}(\xi_1, \ldots, J^E\xi_r) \\
&= \left(\sqrt{-1}\right)^r (\overline{\sqrt{-1}})^r 2^r\Theta(\xi_1, \ldots, \xi_r)\overline{\Theta(\xi_1, \ldots, \xi_r)} \\
&= 2^r\Theta(\xi_1, \ldots, \xi_r)\overline{\Theta(\xi_1, \ldots, \xi_r)}.
\end{aligned}$$

Hence $\theta \in \Gamma(\mathrm{Det}E)$. Since Θ is a volume form, so is θ. From the definition of θ, if Θ is parallel with respect to ∇, then θ is parallel with respect to ∇. □

Lemma 2.13 *If $\nabla \in \mathcal{C}(E, J^E)$ is holomorphic or anti-holomorphic and a volume form $\theta \in \Gamma(\mathrm{Det}E)$ is parallel with respect to ∇ , then there exists a local complex volume form on E which is parallel with respect to ∇ defined on a neighbourhood of each point.*

Proof. From Lemma 2.9, we see that $\mathrm{tr}_{\mathbb{R}}R = 0$. Since ∇ is holomorphic or anti-holomorphic, we get $\mathrm{tr}_{\mathbb{C}}R = 0$ by Lemma 2.10. Hence from Lemma 2.11, we get the result. □

We mention that Lemmas 2.12 and 2.13 are given in [4] for $(E, J^E) = (TM, J)$. By a straightforward calculation, we have

Lemma 2.14 *For $\nabla \in \mathcal{C}(E, J^E)$, $\nabla^{\mathrm{Det}_{\mathbb{C}}E} \in \mathcal{C}(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$ is holomorphic if and only if $\mathrm{tr}_{\mathbb{C}}(R_{JX,Y}) = \sqrt{-1}\mathrm{tr}_{\mathbb{C}}(R_{X,Y})$ and is of type $(1,1)$ if and only if $\mathrm{tr}_{\mathbb{C}}(R_{JX,JY}) = \mathrm{tr}_{\mathbb{C}}(R_{X,Y})$ for any $X, Y \in T_xM$, $x \in M$.*

Especially, we consider the case where $(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$ is a holomorphic vector bundle. For (E, J^E) and $K \in A^2(\mathrm{End}(E, J^E))$, we define $K^{(0,2)}{}_{X,Y}$ by

$$K^{(0,2)}{}_{X,Y} := \frac{1}{4}(K_{X,Y} + J^E K_{JX,Y} + J^E K_{X,JY} - K_{JX,JY})$$

for $X, Y \in T_xM$, $x \in M$. Note that for $\nabla \in \mathcal{C}(E, J^E)$, if $R^{(0,2)}{}_{X,Y} = 0$, then there is a unique holomorphic vector bundle structure of (E, J^E) to which ∇ is adapted from Lemma 2.6. From the equation (2.3) and Lemma 2.5, we get

Lemma 2.15 *For $\nabla \in \mathcal{C}(E, J^E)$, if $\nabla^{\mathrm{Det}_{\mathbb{C}}E} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$, then it holds that $\mathrm{tr}_{\mathbb{C}}((R^{(0,2)})_{X,Y}) = 0$ for any $X, Y \in T_xM$, $x \in M$.*

Note that if $\nabla \in \mathcal{C}^a(E, J^E)$, then it holds that $\nabla^{\mathrm{Det}_{\mathbb{C}}E} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$ and there is a holomorphic vector bundle structure of $(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$. More generally, we get the following by using (2.3) and Lemma 2.6.

Lemma 2.16 *For $\nabla \in \mathcal{C}(E, J^E)$, if it holds that $\mathrm{tr}_{\mathbb{C}}((R^{(0,2)})_{X,Y}) = 0$ for any $X, Y \in T_xM$, $x \in M$, then $(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$ has a unique holomorphic vector bundle structure to which $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ is adapted.*

Proof. Since (2.3) holds, we get

$$((R^{\mathrm{Det}_{\mathbb{C}}E})^{(0,2)})_{X,Y} = \frac{1}{4}\mathrm{tr}_{\mathbb{C}}((R^{(0,2)})_{X,Y}) = 0$$

for any $X, Y \in T_xM$, $x \in M$. Then from Lemma 2.6, we obtain the result. □

Hereafter in this section, we consider the case where $(E, J^E) = (TM, J)$. For a manifold M, we denote by $\mathcal{C}_0(TM)$ the set of all torsion free affine connections on M and we call a volume form on TM a *volume form* on M.

For a complex manifold (M, J), we denote by $\mathcal{C}_0(TM, J)$ the set of all torsion free affine connections $\nabla \in \mathcal{C}_0(TM)$ such that $\nabla_X J = J\nabla_X$ for and any $X \in T_xM$, $x \in M$. Such connections are called *complex affine connections.* Denote by $\nabla^{\mathrm{Det}_{\mathbb{C}}TM}$ the connection on $(\mathrm{Det}_{\mathbb{C}}TM, J^{\mathrm{Det}_{\mathbb{C}}TM})$ induced from $\nabla \in \mathcal{C}_0(TM, J)$. We note that $\nabla^{\mathrm{Det}_{\mathbb{C}}TM} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}TM, J^{\mathrm{Det}_{\mathbb{C}}TM})$.

We define a complex Ricci tensor ric of $\nabla \in \mathcal{C}_0(TM, J)$ which is $\mathbb{C}$-valued by

$$ric_{X,Y} \;\; := \;\; \frac{1}{2}(Ric_{X,Y} - \sqrt{-1}Ric_{X,JY})$$

for $X, Y \in T_xM$, $x \in M$. If ric is symmetric, then Ric is symmetric. The converse holds if ∇ is holomorphic or anti-holomorphic. Since $ric_{X,JY} = \sqrt{-1}ric_{X,Y}$ for any $X, Y \in T_xM$, $x \in M$, by using Lemma 2.14 and Bianchi's identity, we get

Lemma 2.17 *For* $\nabla^{\mathrm{Det}_{\mathbb{C}}TM} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}TM, J^{\mathrm{Det}_{\mathbb{C}}TM})$*, we have the following.*
(1) $\nabla^{\mathrm{Det}_{\mathbb{C}}TM}$ *is holomorphic if and only if* $ric_{JX,Y} = ric_{X,JY}$ *for any* $X, Y \in T_xM$*,* $x \in M$*.*
(2) $\nabla^{\mathrm{Det}_{\mathbb{C}}TM}$ *is of type* $(1,1)$ *if and only if* $ric_{X,Y} - ric_{Y,X} = ric_{JX,JY} - ric_{JY,JX}$ *for any* $X, Y \in T_xM$*,* $x \in M$*.*

Next we prepare the definition of a complex equiaffine structure.

Definition 2.18 For a complex volume form $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}TM)$ on a complex manifold (M, J) and the induced connection $\nabla^{\mathrm{Det}_{\mathbb{C}}TM} \in \mathcal{C}(\mathrm{Det}_{\mathbb{C}}TM, J^{\mathrm{Det}_{\mathbb{C}}TM})$ from $\nabla \in \mathcal{C}_0(TM, J)$, the pair (∇, Θ) which satisfies $\nabla_X^{\mathrm{Det}_{\mathbb{C}}TM}\Theta = 0$ for any $X \in T_xM$, $x \in M$ is called a *complex equiaffine structure*.

From the equation (2.3), Lemma 2.11 and Bianchi's identity, we have

Proposition 2.19 *For* $\nabla^{\mathrm{Det}_{\mathbb{C}}TM} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}TM, J^{\mathrm{Det}_{\mathbb{C}}TM})$ *and a complex volume form* $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}TM)$*, we have*

$$R_{X,Y}^{\mathrm{Det}_{\mathbb{C}}TM}\Theta = -\mathrm{tr}_{\mathbb{C}}(R_{X,Y})\Theta = ric_{X,Y}\Theta - ric_{Y,X}\Theta$$

for any $X, Y \in T_xM$*,* $x \in M$*. Moreover, the followings are equivalent.*
(1) *there is a local complex equiaffine structure on* (M, J) *defined on a neighbourhood of each point.*
(2) $\mathrm{tr}_{\mathbb{C}}R = 0$*,*
(3) ric *is symmetric.*

We remark that Proposition 2.19 is shown in [4] and [11].

§3. A decomposition of a vector bundle with connection and volume forms.

In this section, we study a decomposition of a vector bundle with connection and volume forms to apply results obtained here to the theory of complex affine immersions in Sections 4 and 5. We prepare fundamental results on a decomposition of a vector bundle with connection, some of which are already given in [1]. Let E be a real vector bundle over M, E_1 and E_2 are subbundles of E such that

$$E = E_1 \oplus E_2.$$

Throughout this section, we assume that $i, j = 1, 2$ and $i \neq j$. Let $\pi_i : E \to E_i$ and $\iota_i : E_i \to E$ be the projection homomorphisms and the inclusions, respectively. Then the following equations hold:

$$\pi_i\iota_i = id_{E_i}, \quad \pi_j\iota_i = 0, \quad \iota_1\pi_1 + \iota_2\pi_2 = id_E. \tag{3.1}$$

Definition 3.1 For $\nabla \in \mathcal{C}(E)$, we define

$$\nabla^i_X := (\pi_i \nabla \iota_i)_X := \pi_i \nabla_X \iota_i, \qquad B^i_X := (\pi_j \nabla \iota_i)_X := \pi_j \nabla_X \iota_i$$

for $X \in T_xM$, $x \in M$. We call $\nabla^i \in \mathcal{C}(E_i)$ the *induced connection* on E_i for ∇ and $B^i \in A^1(\mathrm{Hom}(E_i, E_j))$ the *second fundamental form* of E_i for ∇.

From the definitions, we have

$$\nabla = \iota_1 \nabla^1 \pi_1 + \iota_2 B^1 \pi_1 + \iota_2 \nabla^2 \pi_2 + \iota_1 B^2 \pi_2, \tag{3.2}$$
$$\nabla_X \iota_i = \iota_i \nabla^i_X + \iota_j B^i_X \tag{3.3}$$

for and any $X \in T_xM$, $x \in M$. Note that the formula (3.3) corresponds to Gauss and Weingarten formulas in submanifold theory. From (3.1) and (3.3), by straightforward calculations, we have

Lemma 3.2 *Let R and R^i be the curvature forms of $\nabla \in \mathcal{C}(E)$ and $\nabla^i \in \mathcal{C}(E_i)$, respectively. Then we get*

$$\pi_i R_{X,Y} \iota_i = R^i_{X,Y} + B^j_X B^i_Y - B^j_Y B^i_X, \tag{3.4}$$
$$\pi_j R_{X,Y} \iota_i = B^i_X \nabla^i_Y + \nabla^j_X B^i_Y - B^i_Y \nabla^i_X - \nabla^j_Y B^i_X - B^i_{[X,Y]} \tag{3.5}$$

for any $X, Y \in \Gamma(TM)$. Moreover, when we fix a torsion free affine connection $\nabla^M \in \mathcal{C}_0(TM)$, the equation (3.5) is rewritten as

$$\pi_j R_{X,Y} \iota_i = (\hat{\nabla}^j_X B^i)_Y - (\hat{\nabla}^j_Y B^i)_X,$$

where $(\hat{\nabla}^j_X B^i)_Y$ is defined by

$$(\hat{\nabla}^j_X B^i)_Y \xi_i = \nabla^j_X (B^i_Y \xi_i) - B^i_Y \nabla^i_X \xi_i - B^i_{\nabla^M_X Y} \xi_i$$

for any $X, Y \in \Gamma(TM)$ and $\xi_i \in \Gamma(E_i)$.

The equations above correspond to the structure equations in submanifold theory. To be more precisely, the equation (3.4) corresponds to Gauss and Ricci equations and the equation (3.5) corresponds to Codazzi equation. Conversely, we have the following, which we will use to prove the existence theorem in Section 5.

Lemma 3.3 *Let E_i be a vector bundle, $\nabla^i \in \mathcal{C}(E_i)$ and $B^i \in A^1(\mathrm{Hom}(E_i, E_j))$, $i, j = 1, 2$, $i \neq j$. Then ∇ given by*

$$\nabla = \iota_1 \nabla^1 \pi_1 + \iota_2 B^1 \pi_1 + \iota_2 \nabla^2 \pi_2 + \iota_1 B^2 \pi_2 \tag{3.6}$$

is a connection on $E_1 \oplus E_2$ *such that the induced connection on* E_i *for* ∇ *is* ∇^i, B^i *is the second fundamental form of* E_i *for* ∇ *and the curvature form* R *of* ∇ *satisfies* (3.4) *and* (3.5) *in Lemma* 3.2*. Moreover, if* ∇^i *and* B^i *satisfy*

$$R^i_{X,Y} + B^j_X B^i_Y - B^j_Y B^i_X = 0,$$
$$B^i_X \nabla^i_Y + \nabla^j_X B^i_Y - B^i_Y \nabla^i_X - \nabla^j_Y B^i_X \ - B^i_{[X,Y]} = 0$$

for any $X, Y \in \Gamma(TM)$*, then* ∇ *is flat.*

For a vector bundle homomorphism, we have

Lemma 3.4 *Let* E (*resp.* $\widetilde{E}$) *be a vector bundle such that* $E = E_1 \oplus E_2$ (*resp.* $\widetilde{E} = \widetilde{E}_1 \oplus \widetilde{E}_2$), $\nabla \in \mathcal{C}(E)$ (*resp.* $\widetilde{\nabla} \in \mathcal{C}(\widetilde{E})$), $\nabla^i \in \mathcal{C}(E_i)$ (*resp.* $\widetilde{\nabla}^i \in \mathcal{C}(\widetilde{E}_i)$) *and* $B^i \in A^1(\mathrm{Hom}(E_i, E_j))$ (*resp.* $\widetilde{B}^i \in A^1(\mathrm{Hom}(\widetilde{E}_i, \widetilde{E}_j))$) *be a connection on* E (*resp.* $\widetilde{E}$)*, the induced connection on* E_i (*resp.* $\widetilde{E}_i$) *for* ∇ (*resp.* $\widetilde{\nabla}$) *the second fundamental form of* E_i (*resp.* $\widetilde{E}_i$) *for* ∇ (*resp.* $\widetilde{\nabla}$), $i = 1, 2$ *and* $\Phi \in \mathrm{HOM}(\widetilde{E}, E)$*. Assume that it holds that* $\Phi(\widetilde{E}_i) = E_i$, $i = 1, 2$*. Then*

$$\Phi \widetilde{\nabla}_X = \nabla_X \Phi$$

if and only if

$$\nabla^i_X \Phi_i = \Phi_i \widetilde{\nabla}^i_X, \qquad B^j_X \Phi_j = \Phi_i \widetilde{B}^j_X$$

for any $X \in T_xM$, $x \in M$*, where* $\Phi_i \in \mathrm{ISO}(\widetilde{E}_i, E_i)$ *is defined by* $\Phi_i := \widetilde{\pi}_i \Phi \iota_i$*.*

From (3.4) in Lemma 3.2, we get

Lemma 3.5 *For* $\nabla \in \mathcal{C}(E)$*, we have*

$$\mathrm{tr}_{\mathbb{R}}(R_{X,Y}) \ = \ \mathrm{tr}_{\mathbb{R}}(R^1_{X,Y}) + \mathrm{tr}_{\mathbb{R}}(R^2_{X,Y}) \tag{3.7}$$

for any $X, Y \in T_xM$, $x \in M$*.*

As a corollary , we have

Corollary 3.6 *If* $\mathrm{tr}_{\mathbb{R}} R = 0$*, then* $\mathrm{tr}_{\mathbb{R}} R^1 = 0$ *if and only if* $\mathrm{tr}_{\mathbb{R}} R^2 = 0$*.*

For a real vector bundle E, $\nabla \in \mathcal{C}(E)$, a subbundle E_i, $i = 1, 2$ such that $E = E_1 \oplus E_2$, we denote by $\nabla^{\mathrm{Det}E_i} \in \mathcal{C}(\mathrm{Det}E_i)$ the connection induced from $\nabla^i \in \mathcal{C}(E_i)$, $i = 1, 2$. For volume forms on E, E_1 and E_2, we have

Lemma 3.7 *For* $\nabla \in \mathcal{C}(E)$ *and a volume form* θ_i *on* E_i, $\pi_1^*\theta_1 \wedge \pi_2^*\theta_2$ *is a volume form of* E*. If* θ_i *is parallel with respect to* ∇^i, $i = 1, 2$*, then* $\pi_1^*\theta_1 \wedge \pi_2^*\theta_2$ *is parallel with respect to* ∇*. If* $\pi_1^*\theta_1 \wedge \pi_2^*\theta_2$ *is parallel with respect to* ∇*, then* θ_1 *is parallel with respect to* ∇^1 *if and only if* θ_2 *is parallel with respect to* ∇^2*.*

For a complex vector bundle (E, J^E), if E_i is J^E-invariant, then we call $J_i := \pi_i J^E \iota_i$ the induced complex structure of E_i. Hereafter in this section, we assume that both (E_1, J_1) and (E_2, J_2) are J^E-invariant subbundles of a complex vector bundle (E, J^E) such that $E = E_1 \oplus E_2$. If $E = \widetilde{E}$ and $\Phi = J^E$ in Lemma 3.4, we have

Corollary 3.8 *For $\nabla \in \mathcal{C}(E)$, we see that $\nabla \in \mathcal{C}(E, J^E)$ if and only if*

$$\nabla^i \in \mathcal{C}(E_i, J_i), \qquad B^i \in A^1(\mathrm{Hom}((E_i, J_i), (E_j, J_j))).$$

We will use Corollary 3.8 to prove the existence theorem for a complex equiaffine immersion in Section 5. From Lemma 3.5, we have

Lemma 3.9 *For $\nabla \in \mathcal{C}(E, J^E)$, we have*

$$\mathrm{tr}_{\mathbb{C}}(R_{X,Y}) = \mathrm{tr}_{\mathbb{C}}(R^1_{X,Y}) + \mathrm{tr}_{\mathbb{C}}(R^2_{X,Y}) \tag{3.8}$$

for any $X, Y \in T_xM$, $x \in M$.

By Lemma 3.9, we obtain

Corollary 3.10 *For $\nabla \in \mathcal{C}(E, J^E)$, if $\mathrm{tr}_{\mathbb{C}} R = 0$, then the following conditions are equivalent.*
(1) $\mathrm{tr}_{\mathbb{C}} R^1 = 0$,
(2) $\mathrm{tr}_{\mathbb{C}} R^2 = 0$.

Lemma 2.10 and Corollary 3.10 yield

Corollary 3.11 *For $\nabla \in \mathcal{C}(E, J^E)$, if $\mathrm{tr}_{\mathbb{C}} R = 0$ and both ∇^i, $i = 1, 2$, are holomorphic or anti-holomorphic, then the following conditions are equivalent:*
(1) *and* (2) *of Corollary* 3.10,
(3) $\mathrm{tr}_{\mathbb{R}} R^1 = 0$,
(4) $\mathrm{tr}_{\mathbb{R}} R^2 = 0$.

For $\nabla \in \mathcal{C}(E, J^E)$, let $\nabla^{\mathrm{Det}_{\mathbb{C}} E_i}$ be the connection induced from $\nabla^i \in \mathcal{C}(E_i, J_i)$, $i = 1, 2$. Lemmas 2.11 and 3.9 imply

Proposition 3.12 *For $\nabla \in \mathcal{C}(E, J^E)$, if $\mathrm{tr}_{\mathbb{C}} R^i = 0$, $i = 1, 2$, then there are local complex volume forms $\Theta_i \in \Gamma(\mathrm{Det}_{\mathbb{C}} E_i, J^{\mathrm{Det}_{\mathbb{C}} E_i})$ and $\pi_1^* \Theta_1 \wedge \pi_2^* \Theta_2 \in \Gamma(\mathrm{Det}_{\mathbb{C}} E, J^{\mathrm{Det}_{\mathbb{C}} E})$ which are parallel with respect to ∇^i and ∇ defined on a neighbourhood of each point.*

From Lemmas 2.12 and 3.7, we have

Proposition 3.13 *For a complex volume form Θ_i of E_i, $i=1,2$, θ_i given by*

$$\theta_i := \left(\sqrt{-1}\right)^{r_i} \Theta_i \wedge \overline{\Theta}_i$$

is a real volume form on E_i, $i=1,2$ and $\widetilde{\theta}$ given by

$$\widetilde{\theta} := \left(\sqrt{-1}\right)^{r} (\pi_1^*(\Theta_1 \wedge \overline{\Theta}_1)) \wedge (\pi_2^*(\Theta_2 \wedge \overline{\Theta}_2))$$

is a real volume form on E, where r_i is the rank of E_i. Moreover, for $\nabla \in \mathcal{C}(E, J^E)$, if both Θ_1 and Θ_2 are parallel with respect to ∇^1 and ∇^2 respectively, then θ_i is parallel with respect to ∇^i, $i=1,2$ and $\widetilde{\theta}$ is parallel with respect to ∇.

By virtue of Lemmas 2.13 and 3.7, we obtain

Proposition 3.14 *If both ∇^i are holomorphic or anti-holomorphic and there are parallel volume forms on E_i with respect to ∇^i, $i=1,2$, then there are local complex volume forms on E and E_i which are parallel with respect to ∇ and ∇^i defined on a neighbourhood of each point, $i=1,2$.*

For $\nabla \in \mathcal{C}(E, J^E)$, it follows from Lemma 3.9 that

$$\mathrm{tr}_{\mathbb{C}}((R^{(0,2)})_{X,Y}) = \mathrm{tr}_{\mathbb{C}}(((R^1)^{(0,2)})_{X,Y}) + \mathrm{tr}_{\mathbb{C}}(((R^2)^{(0,2)})_{X,Y}) \tag{3.9}$$

for any $X, Y \in T_xM$, $x \in M$. From (3.8), (3.9) and Lemma 2.14, we get

Lemma 3.15 *For $\nabla \in \mathcal{C}(E, J^E)$, we have the following.*
(1) If $\nabla^{\mathrm{Det}_{\mathbb{C}}E} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$, then $\nabla^{\mathrm{Det}_{\mathbb{C}}E_1} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E_1, J^{\mathrm{Det}_{\mathbb{C}}E_1})$ if and only if $\nabla^{\mathrm{Det}_{\mathbb{C}}E_2} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E_2, J^{\mathrm{Det}_{\mathbb{C}}E_2})$.

(2) If both $\nabla^{\mathrm{Det}_{\mathbb{C}}E_i}$ satisfies $\nabla^{\mathrm{Det}_{\mathbb{C}}E_i} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E_i, J^{\mathrm{Det}_{\mathbb{C}}E_i})$, $i=1,2$, then $\nabla^{\mathrm{Det}_{\mathbb{C}}E} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$.

(3) If $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ is holomorphic, then $\nabla^{\mathrm{Det}_{\mathbb{C}}E_1}$ is holomorphic if and only if $\nabla^{\mathrm{Det}_{\mathbb{C}}E_2}$ is.

(4) If both $\nabla^{\mathrm{Det}_{\mathbb{C}}E_i}$ are holomorphic, $i=1,2$, then $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ is holomorphic.

(5) If $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ is of type $(1,1)$, then $\nabla^{\mathrm{Det}_{\mathbb{C}}E_1}$ is of type $(1,1)$ if and only if $\nabla^{\mathrm{Det}_{\mathbb{C}}E_2}$ is.

(6) If both $\nabla^{\mathrm{Det}_{\mathbb{C}}E_i}$ are of type $(1,1)$, $i=1,2$, then $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ is of type $(1,1)$.

Lemma 2.16 and (3.9) yield

Proposition 3.16 *For $\nabla \in \mathcal{C}(E, J^E)$, if $\mathrm{tr}_{\mathbb{C}}(((R^i)^{(0,2)})_{X,Y}) = 0$ for any $X, Y \in T_xM$, $x \in M$, then there are holomorphic vector bundle structures of $(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$ and $(\mathrm{Det}_{\mathbb{C}}E_i, J^{\mathrm{Det}_{\mathbb{C}}E_i})$ $i=1,2$, to which $\nabla^{\mathrm{Det}_{\mathbb{C}}E}$ and $\nabla^{\mathrm{Det}_{\mathbb{C}}E_i}$ are adapted respectively.*

For $\nabla \in \mathcal{C}(E, J^E)$, if $\nabla \in \mathcal{C}^a(E, J^E)$ and B^1 is complex, then $\nabla^i \in \mathcal{C}^a(E_i, J_i)$, $i = 1, 2$, holds. From this fact, Lemmas 2.15 and 2.16, we get

Proposition 3.17 *If $\nabla \in \mathcal{C}^a(E, J^E)$ and B^1 is complex, then it holds that $\nabla^{\mathrm{Det}_{\mathbb{C}}E} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E, J^{\mathrm{Det}_{\mathbb{C}}E})$ and $\nabla^{\mathrm{Det}_{\mathbb{C}}E_i} \in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}E_i, J^{\mathrm{Det}_{\mathbb{C}}E_i})$, $i = 1, 2$.*

§4. Complex affine immersions.

In this section, we prepare notations of an affine immersion and a complex affine immersion with transversal bundle. We apply results given in Section 3 to the decomposition determined by an affine immersion.

Let M and $\widetilde{M}$ be manifolds, $f : M \to \widetilde{M}$ a smooth map, $f^{\#}T\widetilde{M}$ and $f_{\#} : f^{\#}T\widetilde{M} \to T\widetilde{M}$ the induced bundle and its bundle map. We define $i^f : TM \to f^{\#}T\widetilde{M}$ by $i^f_x := (f_{\#x})^{-1} f_{*x}$ for each $x \in M$. Hereafter we consider the case where f is an immersion in this section. For a subbundle N of $f^{\#}T\widetilde{M}$, if

$$f^{\#}T\widetilde{M} \;=\; i^f(TM) \oplus N, \tag{4.1}$$

then we call such an immersion an *immersion with transversal bundle N*. Let $\iota_f : i^f(TM) \to f^{\#}T\widetilde{M}$ and $\iota_N : N \to f^{\#}T\widetilde{M}$ be inclusions and $\pi_f : f^{\#}T\widetilde{M} \to i^f(TM)$ and $\pi_N : f^{\#}T\widetilde{M} \to N$ projection homomorphisms. We put $\hat{i}^f := \pi_f i^f \in \mathrm{ISO}(TM, i^f(TM))$. Let (M, ∇) and $(\widetilde{M}, \widetilde{\nabla})$ be manifolds with torsion free affine connections ∇ and $\widetilde{\nabla}$. We denote by $f^{\#}\widetilde{\nabla}$ the pull-back of $\widetilde{\nabla}$. For an immersion $f : M \to \widetilde{M}$ with transversal bundle N, if the induced connection $\pi_f(f^{\#}\widetilde{\nabla})\iota_f$ on $i^f(TM)$ for $f^{\#}\widetilde{\nabla}$ coincides with $\hat{i}^f\nabla(\hat{i}^f)^{-1}$, we call such a morphism $(f, N) : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ an *affine immersion with transversal bundle* N and denote it by $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ for simplicity if the transversal bundle is stated. In this case, we define the affine fundamental form $B \in A^1(\mathrm{Hom}(TM, N))$, the shape tensor $A \in A^1(\mathrm{Hom}(N, TM))$ and the transversal connection $\nabla^N \in \mathcal{C}(N)$ by

$$B := \pi_N(f^{\#}\widetilde{\nabla})\iota_f \hat{i}^f,\; A := -(\hat{i}^f)^{-1}\pi_f(f^{\#}\widetilde{\nabla})\iota_N,\; \nabla^N := \pi_N(f^{\#}\widetilde{\nabla})\iota_N.$$

Since $\widetilde{\nabla}$ is torsion free, B is symmetric, that is, $B_X Y = B_Y X$ for any $X, Y \in T_x M$, $x \in M$. Note that $B_X Y$ (resp. $A_X \xi$) is usually denoted by $\alpha(X, Y)$ (resp. $A_\xi X$) for any $X, Y \in T_x M$ and $\xi \in N_x$, $x \in M$. Then we can write Gauss and Weingarten formulas as

$$\begin{aligned}(f^{\#}\widetilde{\nabla})_X i^f Y &= i^f \nabla_X Y + B_X Y,\\ (f^{\#}\widetilde{\nabla})_X \xi &= -i^f A_X \xi + \nabla^N_X \xi\end{aligned}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$. When we apply Lemma 3.2 to the decomposition (4.1), Gauss, Codazzi and Ricci equations are given by

$$\begin{aligned}
(\hat{i}^f)^{-1}\pi_f \widetilde{R}_{X,Y} i^f Z &= R_{X,Y}Z - A_X B_Y Z + A_Y B_X Z, \\
\pi_N \widetilde{R}_{X,Y} i^f Z &= (\hat{\nabla}_X B)_Y Z - (\hat{\nabla}_Y B)_X Z, \\
(\hat{i}^f)^{-1}\pi_f \widetilde{R}_{X,Y}\xi &= -(\hat{\nabla}_X A)_Y \xi + (\hat{\nabla}_Y A)_X \xi, \\
\pi_N \widetilde{R}_{X,Y}\xi &= R^N_{X,Y}\xi - B_X A_Y \xi + B_Y A_X \xi,
\end{aligned}$$

where $\widetilde{R}$, R, R^N are the curvature forms of $(f^{\#}\widetilde{\nabla})$, ∇, ∇^N, respectively, $(\hat{\nabla}_X A)$ and $(\hat{\nabla}_X B)$ are given by

$$\begin{aligned}
(\hat{\nabla}_X B)_Y Z &= \nabla^N_X (B_Y Z) - B_{\nabla_X Y} Z - B_Y \nabla_X Z, \\
(\hat{\nabla}_X A)_Y \xi &= \nabla_X (A_Y \xi) - A_{\nabla_X Y}\xi - A_Y \nabla^N_X \xi
\end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$ and $\xi \in \Gamma(N)$.

For complex manifolds (M, J), $(\widetilde{M}, \widetilde{J})$ and an immersion $f : M \to \widetilde{M}$, if f is a holomorphic map, that is, $f_* J = \widetilde{J} f_*$, then we call f a *holomorphic immersion*.

Definition 4.1 For complex manifolds (M, J), $(\widetilde{M}, \widetilde{J})$, $\nabla \in \mathcal{C}_0(TM, J)$, $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M}, \widetilde{J})$ and an affine immersion $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, if f is holomorphic and N is an $(f^{\#}\widetilde{J})$-invariant subbundle of $f^{\#}T\widetilde{M}$, that is, $(f^{\#}\widetilde{J})(N) = N$, then we call such an immersion a *complex affine immersion* and denote it by $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ and the induced complex structure of N by $J^N := \pi_N (f^{\#}\widetilde{J})\iota_N$.

For a complex affine immersion $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ with transversal bundle N, we have

$$\begin{aligned}
&\nabla^N \in \mathcal{C}^a(N, J^N), \\
&B \in A^{1,0}(\mathrm{Hom}((TM, J), (N, J^N))), \\
&A \in A^1(\mathrm{Hom}((N, J^N), (TM, J))).
\end{aligned}$$

Hereafter in this paper, we denote by (M, J) and $(\widetilde{M}, \widetilde{J})$ complex manifolds and always assume that $\nabla \in \mathcal{C}_0(TM, J)$, $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M}, \widetilde{J})$. For a complex affine immersion $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ with transversal bundle N, we denote by $\nabla^{\mathrm{Det}_{\mathbb{C}}TM}$ and $\nabla^{\mathrm{Det}_{\mathbb{C}}N}$ the induced connections on $\mathrm{Det}_{\mathbb{C}}TM$ and $\mathrm{Det}_{\mathbb{C}}N$ respectively. Proposition 3.17 yields

Lemma 4.2 *Let $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ be a complex affine immersion with transversal bundle N. Then we have*

$$\begin{aligned}
\nabla^{\mathrm{Det}_{\mathbb{C}}TM} &\in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}TM, J^{\mathrm{Det}_{\mathbb{C}}TM}), \\
\nabla^{\mathrm{Det}_{\mathbb{C}}N} &\in \mathcal{C}^a(\mathrm{Det}_{\mathbb{C}}N, J^{\mathrm{Det}_{\mathbb{C}}N}).
\end{aligned}$$

Next we study a relation between the Ricci tensor and the transversal connection of a complex affine immersion with transversal bundle. The complex trace $\mathrm{tr}_{\mathbb{C}} R^N$ of R^N is defined by

$$\mathrm{tr}_{\mathbb{C}} R^N_{X,Y} \quad := \quad \frac{1}{2}\left(\mathrm{tr}_{\mathbb{R}}(R^N_{X,Y}) - \sqrt{-1}\,\mathrm{tr}_{\mathbb{R}}(J^N R^N_{X,Y})\right)$$

for $X, Y \in T_xM$, $x \in M$. From Lemmas 2.9, 2.10, 2.11, 2.14 and Corollaries 3.10, 3.11 we have

Proposition 4.3 *For a complex affine immersion $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ with transversal bundle N, assume that $\mathrm{tr}_{\mathbb{C}} \widetilde{R} = 0$. Then the following are equivalent.*
(1) *ric is symmetric,*
(2) $\mathrm{tr}_{\mathbb{C}} R^N = 0$,
(3) $\mathrm{tr}_{\mathbb{C}} R = 0$,
(4) *there exists a local complex equiaffine structure on (M, J) defined on a neighbourhood of each point,*
(5) *there exists a local complex volume form on N which is parallel with respect to ∇^N defined on a neighbourhood of each point.*
Moreover, if both $\nabla^{\mathrm{Det}_{\mathbb{C}} TM}$ and $\nabla^{\mathrm{Det}_{\mathbb{C}} N}$ are holomorphic or anti-holomorphic, the conditions (1), (2), (3), (4), (5) *and the followings are equivalent.*
(6) *Ric is symmetric,*
(7) $\mathrm{tr}_{\mathbb{R}} R^N = 0$,
(8) $\mathrm{tr}_{\mathbb{R}} R = 0$.

We mention that if $p = 1$ in Proposition 4.3, then (3) is equivalent to $R^N = 0$. We note that Proposition 4.3 generalizes some of results in [11] for $(\widetilde{M}, \widetilde{J}, \widetilde{\nabla}) = (\mathbb{R}^{2(m+1)}, \widetilde{J}, D)$, where we denote by $(\mathbb{R}^{2(m+1)}, \widetilde{J}, D)$ a $2(m+1)$-dimensional real affine space with the standard affine connection D and the standard complex structure $\widetilde{J}$ which is induced from J_0 on $T\mathbb{R}^{2(m+1)}$. Proposition 4.3 yields

Proposition 4.4 *For a complex affine immersion $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ with transversal bundle N, assume that*

$$\begin{aligned} \mathrm{tr}_{\mathbb{C}} \widetilde{R} = 0, &\quad \mathrm{tr}_{\mathbb{R}}((\hat{\imath}^f)^{-1} \pi_{i^f(TM)} \widetilde{R}_{\cdot X} i^f Y) = 0, \\ \mathrm{tr}_{\mathbb{R}}(A_{\cdot}\xi) = 0, &\quad \mathrm{tr}_{\mathbb{R}} A_{JX} B_Y = -\mathrm{tr}_{\mathbb{R}} J A_X B_Y \end{aligned}$$

for any $\xi \in N_x$ and $X, Y \in T_xM$, $x \in M$. Then the following conditions are equivalent: (1), (2), (3), (4) *and* (5) *from Proposition* 4.3 *and*
(9) $Ric = 0$.

Proof. From the assumption that $\mathrm{tr}_{\mathbb{R}}(A.\xi) = 0$, we see that

$$Ric_{X,Y} \quad = \quad -\mathrm{tr}_{\mathbb{R}} A_X B_Y \tag{4.2}$$

for any $X, Y \in T_xM$ and $\xi \in N_x$, $x \in M$. First we assume (1), that is, ric is symmetric. Then we get

$$Ric_{JY,X} = Ric_{X,JY} = Ric_{Y,JX} \tag{4.3}$$

for any $X, Y \in T_xM$, $x \in M$. From (4.2), it holds that

$$\begin{aligned} Ric_{Y,JX} &= -\mathrm{tr}_{\mathbb{R}} A_Y B_{JX} = -\mathrm{tr}_{\mathbb{R}} J A_Y B_X, \\ Ric_{JY,X} &= -\mathrm{tr}_{\mathbb{R}} A_{JY} B_X = \mathrm{tr}_{\mathbb{R}} J A_Y B_X \end{aligned}$$

for any $X, Y \in T_xM$, $x \in M$. Combining these equations and (4.3), we get $Ric = 0$. The converse is obvious. □

Note that under the assumption in Proposition 4.4, $\nabla^{\mathrm{Det}_{\mathbb{C}} TM}$ is of type $(1,1)$. This Proposition can be considered as a generalization of the corresponding result in [9] and [11] for $(\widetilde{M}, \widetilde{J}, \widetilde{\nabla}) = (\mathbb{R}^{2(m+1)}, \widetilde{J}, D)$.

To state the next corollary, we prepare the notion of H-projectively flatness. For a complex manifold (M, J), the H-projective curvature P of $\nabla \in \mathcal{C}_0(TM, J)$ is defined by

$$\begin{aligned} P_{X,Y}Z \quad := \quad & R_{X,Y}Z - N_{X,Z}Y + N_{Y,Z}X - N_{X,Y}Z + N_{Y,X}Z \\ & + N_{X,JZ}JY - N_{Y,JZ}JX + N_{X,JY}JZ - N_{Y,JX}JZ, \end{aligned}$$

where $N_{X,Y}$ is defined by

$$\begin{aligned} N_{X,Y} \quad := \quad -\frac{1}{2m+2}\Big\{ Ric_{X,Y} + \frac{1}{2m-2}\,(& Ric_{X,Y} + Ric_{Y,X} \\ & - Ric_{JX,JY} - Ric_{JY,JX}) \Big\} \end{aligned}$$

for $X, Y \in T_xM$, $x \in M$. An affine connection is said to be *H-projectively flat* if around each point there is a H-projective change of the connection to a flat affine connection. We recall the following result in [12].

Theorem ([12]) *For a complex manifold (M, J) and $\nabla \in \mathcal{C}_0(TM, J)$, we have the following.*
(1) *If $\dim M \geqq 6$, then ∇ is H-projectively flat if and only if $P = 0$. Moreover, in this case it holds that*

$$(\hat{\nabla}_X N)_{Y,Z} \quad = \quad (\hat{\nabla}_Y N)_{X,Z}, \tag{4.4}$$

where $(\hat{\nabla}_X N)_{Y,Z}$ *is defined by*

$$(\hat{\nabla}_X N)_{Y,Z} \quad := \quad X(N_{Y,Z}) - N_{\nabla_X Y,Z} - N_{Y,\nabla_X Z}$$

for any $X, Y, Z \in \Gamma(TM)$.
(2) *If* $\dim M = 4$*, then* ∇ *is* H*-projectively flat if and only if* $P = 0$ *and the equation* (4.4) *holds.*

Note that the space of constant holomorphic sectional curvature is H-projectively flat. By using this result, we have the following corollary for Proposition 4.4.

Corollary 4.5 *Let* $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ *be a complex affine immersion with transversal bundle* N*. Assume that* $\dim M \geqq 4$*,* $\widetilde{\nabla}$ *is* H*-projectively flat,* $\mathrm{tr}_{\mathbb{C}} \widetilde{R} = 0$, $\mathrm{tr}_{\mathbb{R}}(A.\xi) = 0$ *and* $\mathrm{tr}_{\mathbb{R}} A_{JX} B_Y = -\mathrm{tr}_{\mathbb{R}} J A_X B_Y$ *for any* $X, Y \in T_x M$ *and* $\xi \in N_x$*,* $x \in M$*, then* (1), (2), (3), (4) *and* (5) *from Proposition* 4.3 *and* (9) *from Proposition* 4.4 *are equivalent.*

Proof. Since $\widetilde{\nabla}$ is H-projectively flat, we see that

$$\mathrm{tr}_{\mathbb{R}}((\hat{i}^f)^{-1} \pi_{i^f(TM)} \widetilde{R}._X i^f Y) = 0$$

for any $X, Y \in T_x M$, $x \in M$ by a direct calculation. □

From Lemmas 2.17 and 3.15, we have the following propositions.

Proposition 4.6 *For a complex affine immersion* $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ *with transversal bundle* N*, assume that* $\nabla^{\mathrm{Det}_{\mathbb{C}} T\widetilde{M}}$ *is holomorphic. Then the following conditions are equivalent.*
(1) $\nabla^{\mathrm{Det}_{\mathbb{C}} TM}$ *is holomorphic,*
(2) $\nabla^{\mathrm{Det}_{\mathbb{C}} N}$ *is holomorphic,*
(3) $ric_{JX,Y} = ric_{X,JY}$ *for any* $X, Y \in T_x M$*,* $x \in M$*,*
(4) $Ric_{JX,Y} = Ric_{X,JY}$ *for any* $X, Y \in T_x M$*,* $x \in M$*.*

Proposition 4.7 *For a complex affine immersion* $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ *with transversal bundle* N*, assume that* $\nabla^{\mathrm{Det}_{\mathbb{C}} T\widetilde{M}}$ *is of type* $(1,1)$*. Then the following conditions are equivalent.*
(1) $\nabla^{\mathrm{Det}_{\mathbb{C}} TM}$ *is of type* $(1,1)$*,*
(2) $\nabla^{\mathrm{Det}_{\mathbb{C}} N}$ *is of type* $(1,1)$*,*
(3) $ric_{X,Y} - ric_{Y,X} = ric_{JX,JY} - ric_{JY,JX}$ *for any* $X, Y \in T_x M$*,* $x \in M$*,*
(4) $Ric_{X,Y} - Ric_{Y,X} = Ric_{JX,JY} - Ric_{JY,JX}$ *for any* $X, Y \in T_x M$*,* $x \in M$*.*

We prepare the notion of an equiaffine immersion of general codimension, which is defined in [7] and is also studied in [8]. For real manifolds M, $\nabla \in \mathcal{C}_0(TM)$ and a volume form $\theta \in \mathrm{Det}TM$, if $\nabla_X^{\mathrm{Det}TM}\theta = 0$ for any $X \in T_xM$, $x \in M$, then we call (∇, θ) an *equiaffine structure*. For a real manifold M, $\widetilde{M}$, $\nabla \in \mathcal{C}_0(TM)$, $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$, an affine immersion $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, let $\widetilde{\theta} \in \Gamma(\mathrm{Det}T\widetilde{M})$ be a volume form on $\widetilde{M}$ and $\theta^N \in \Gamma(\mathrm{Det}N)$ be a volume form on N. Then a volume form $\theta \in \Gamma(\mathrm{Det}TM)$ on M defined by

$$(((\hat{i}^f)^{-1} \circ \pi_{i^f(TM)})^*\theta) \wedge (\pi_N^*\theta^N) = f^\#\widetilde{\theta}$$

is called the *induced volume form* for (N, θ^N), where $f^\#\widetilde{\theta}$ is the pull-back of $\widetilde{\theta}$. Under the assumption that $(\widetilde{\nabla}, \widetilde{\theta})$ is an equiaffine structure, if the induced volume form (∇, θ) is an equiaffine structure, we say that (N, θ^N) is *equiaffine*. We note that if $(\widetilde{\nabla}, \widetilde{\theta})$ is an equiaffine structure, then (∇, θ) is an equiaffine structure if and only if θ^N is parallel with respect to ∇^N.

Definition 4.8 For real manifolds M, $\widetilde{M}$, $\nabla \in \mathcal{C}_0(TM)$, $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$, equiaffine structures (∇, θ) on M, $(\widetilde{\nabla}, \widetilde{\theta})$ on $\widetilde{M}$ and an immersion $f : M \to \widetilde{M}$, we call f an *equiaffine immersion* from (M, θ, ∇) to $(\widetilde{M}, \widetilde{\theta}, \widetilde{\nabla})$ with transversal bundle (N, θ^N) if $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ is an affine immersion with transversal bundle N and θ is the induced volume form for (N, θ^N). We denote such an affine immersion by $f : (M, \theta, \nabla) \to (\widetilde{M}, \widetilde{\theta}, \widetilde{\nabla})$.

Next we prepare the definition of a complex equiaffine immersion of general codimension which is an analogue of that of an equiaffine immersion of general codimension. For complex manifolds (M, J), $(\widetilde{M}, \widetilde{J})$, $\nabla \in \mathcal{C}_0(TM, J)$, $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M}, \widetilde{J})$, a complex affine immersion $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ with transversal bundle N, let $\widetilde{\Theta} \in \Gamma(\mathrm{Det}_{\mathbb{C}}T\widetilde{M})$ be a complex volume form on $(\widetilde{M}, \widetilde{J})$ and $\Theta^N \in \Gamma(\mathrm{Det}_{\mathbb{C}}N)$ a complex volume form on N. Then a complex volume form $\Theta \in \Gamma(\mathrm{Det}_{\mathbb{C}}TM)$ on (M, J) defined by

$$(((\hat{i}^f)^{-1} \circ \pi_{i^f(TM)})^*\Theta) \wedge (\pi_N^*\Theta^N) = f^\#\widetilde{\Theta}$$

is called the *induced volume form* for (N, Θ^N), where $f^\#\widetilde{\Theta}$ is the pull-back of $\widetilde{\Theta}$. Under the assumption that $(\widetilde{\nabla}, \widetilde{\Theta})$ is a complex equiaffine structure, if the induced volume form (∇, Θ) is a complex equiaffine structure, we say that (N, Θ^N) is *complex equiaffine*. We note that if $(\widetilde{\nabla}, \widetilde{\Theta})$ is a complex equiaffine structure, then (∇, Θ) is a complex equiaffine structure if and only if Θ^N is parallel with respect to ∇^N.

Definition 4.9 For a complex equiaffine structure (∇, Θ) (resp.$(\widetilde{\nabla}, \widetilde{\Theta})$) on a complex manifold (M, J) (resp. $(\widetilde{M}, \widetilde{J})$) with $\nabla \in \mathcal{C}_0(TM, J)$ (resp. $\widetilde{\nabla} \in$

$\mathcal{C}_0(T\widetilde{M}, \widetilde{J}))$ and a holomorphic immersion $f : (M, J) \to (\widetilde{M}, \widetilde{J})$, we call f a *complex equiaffine immersion* from (M, J, Θ, ∇) to $(\widetilde{M}, \widetilde{J}, \widetilde{\Theta}, \widetilde{\nabla})$ with transversal bundle (N, Θ^N) if $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ is a complex affine immersion with transversal bundle N and the induced volume form for (N, Θ^N) is Θ. We denote such an affine immersion by $f : (M, J, \Theta, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\Theta}, \widetilde{\nabla})$.

By Lemma 2.12 and Proposition 3.13, we obtain

Proposition 4.10 *For a complex equiaffine immersion $f : (M, J, \Theta, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\Theta}, \widetilde{\nabla})$ with transversal bundle (N, Θ^N), θ and $\widetilde{\theta}$ given by*

$$\begin{aligned} \theta &:= \left(\sqrt{-1}\right)^m \Theta \wedge \overline{\Theta}, \\ \widetilde{\theta} &:= \left(\sqrt{-1}\right)^{m+p} \widetilde{\Theta} \wedge \overline{\widetilde{\Theta}} \end{aligned}$$

are volume forms on M and $\widetilde{M}$ such that (∇, θ) and $(\widetilde{\nabla}, \widetilde{\theta})$ are equiaffine structures on M and $\widetilde{M}$ and θ^N defined by

$$\theta^N := \left(\sqrt{-1}\right)^p \Theta^N \wedge \overline{\Theta^N}$$

is a volume form on N which is parallel with respect to ∇^N and the affine immersion f is an equiaffine immersion from (M, θ, ∇) to $(\widetilde{M}, \widetilde{\theta}, \widetilde{\nabla})$ with transversal bundle (N, θ^N).

Note that a similar result as Proposition 4.10 is already given in [4] in the case where $(\widetilde{M}, \widetilde{J}, \widetilde{\Theta}, \widetilde{\nabla}) = (\mathbb{R}^{2(m+1)}, \widetilde{J}, \hat{\Theta}, D)$, where $\hat{\Theta}$ denotes the standard complex volume form of $\mathbb{R}^{2(m+1)}$. The converse is not always true. From Lemma 2.13 and Proposition 3.14, we have

Proposition 4.11 *Let $f : (M, J, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\nabla})$ be an complex affine immersion with transversal bundle N and (∇, θ), $(\widetilde{\nabla}, \widetilde{\theta})$ equiaffine structures on M, $\widetilde{M}$. If both ∇ and ∇^N are holomorphic or anti-holomorphic and $f : (M, \theta, \nabla) \to (\widetilde{M}, \widetilde{\theta}, \widetilde{\nabla})$ is an equiaffine immersion with transversal bundle (N, θ^N) in the real sense, then there are local complex equiaffine structures (∇, Θ), $(\widetilde{\nabla}, \widetilde{\Theta})$ of M and $\widetilde{M}$ defined on a neighbourhood of each point and a local complex volume form $\Theta^N \in \Gamma(\mathrm{Det}_{\mathbb{C}} N)$ which is parallel with respect to ∇^N defined on a neighbourhood of each point such that the affine immersion $f : (M, J, \Theta, \nabla) \to (\widetilde{M}, \widetilde{J}, \widetilde{\Theta}, \widetilde{\nabla})$ is a complex equiaffine immersion with transversal bundle (N, Θ^N) locally.*

§5. The fundamental theorems for a complex equiaffine immersion.

In this section, we state and prove the fundamental theorems for a complex equiaffine immersion to a complex affine space of general codimension. Note that the fundamental theorems, that is, the existence theorem and the equivalence theorem for a complex affine immersion to a complex affine space of general codimension is given in [1]. To prove our theorems, we use a similar method as in [1], [5] and [10], which is self-contained and rather elementary. We note that the fundamental theorems for an equiaffine immersion of general codimension is already given in [5].

For an n-dimensional real affine space $\mathbb{R}^n$, we denote by $(e_1,\ldots,e_n)$ the standard basis of $\mathbb{R}^n$ and $\overline{e}_\alpha$ the global parallel tangent vector field obtained from e_α, $\alpha=1,\ldots,n$. Let M be a manifold and $f: M\to\mathbb{R}^n$ be a smooth map. For the standard basis, we write $f=f^\alpha e_\alpha$, where f^α is a smooth function on M for $\alpha=1,\ldots,n$. For any $X\in T_xM$, $x\in M$ we have

$$f_*X = (df^\alpha)(X)(\overline{e}_\alpha)_{f(x)}. \tag{5.1}$$

We define $i^f: TM\to f^\# T\mathbb{R}^n$ by $i^f_x=(f_{\# x})^{-1}f_{*x}$ and $(f^\#\overline{e}_\alpha)\in\Gamma(f^\# T\mathbb{R}^n)$ by $(f^\#\overline{e}_\alpha)_x := (f_{\# x})^{-1}(\overline{e}_\alpha)_{f(x)}$ for each $x\in M$. In this case, $(f^\#\overline{e}_\alpha)$ is $(f^\# D)$-parallel and it follows that

$$i^f_x = (f_{\# x})^{-1}(df^\alpha\overline{e}_\alpha) = df^\alpha(f^\#\overline{e}_\alpha)_x \tag{5.2}$$

for each $x\in M$. Hereafter in this paper, we denote by $(\mathbb{R}^{2(m+p)},\widetilde{J},D)$ a $2(m+p)$-dimensional real affine space with the standard affine connection D and the standard complex structure $\widetilde{J}$ on $T\mathbb{R}^{2(m+p)}$ which is induced from J_0. Although the proofs of fundamental theorems are similar to those in [1], we write the proof here to make them self-contained.

Theorem 5.1 *Let (M,J) be a $2m$-dimensional complex manifold with complex structure J, $\nabla\in\mathcal{C}_0(TM,J)$, (∇,Θ) a complex equiaffine structure on M, (F,J^F) a complex vector bundle over M of rank $2p$ with complex structure J^F, $\nabla^F\in\mathcal{C}(F,J^F)$, Θ_F a complex volume form on F which is parallel with respect to ∇^F,*

$$\widetilde{B}\in A^{1,0}(\mathrm{Hom}((TM,J),(F,J^F)))$$

a symmetric section, that is, $\widetilde{B}_XY=\widetilde{B}_YX$ for $X,Y\in\Gamma(TM)$ and

$$\widetilde{A}\in A^1(\mathrm{Hom}((F,J^F),(TM,J)))$$

a section such that for $X,Y\in\Gamma(TM)$,

$$R_{X,Y}-\widetilde{A}_X\widetilde{B}_Y+\widetilde{A}_Y\widetilde{B}_X = 0, \tag{5.3}$$

$$\widetilde{B}_X\nabla_Y+\nabla^F_X\widetilde{B}_Y-\widetilde{B}_Y\nabla_X-\nabla^F_Y\widetilde{B}_X-\widetilde{B}_{[X,Y]} = 0, \tag{5.4}$$

$$\widetilde{A}_X\nabla^F_Y+\nabla_X\widetilde{A}_Y-\widetilde{A}_Y\nabla^F_X-\nabla_Y\widetilde{A}_X-\widetilde{A}_{[X,Y]} = 0, \tag{5.5}$$

$$R^F_{X,Y}-\widetilde{B}_X\widetilde{A}_Y+\widetilde{B}_Y\widetilde{A}_X = 0, \tag{5.6}$$

where R, R^F are curvature forms of ∇, ∇^F, respectively. If M is simply connected, then there exist a complex equiaffine affine immersion $f : (M, J, \Theta, \nabla) \to (\mathbb{R}^{2(m+p)}, \widetilde{J}, \widetilde{\Theta}, D)$ with transversal bundle (N, Θ^N) with complex structure J^N and $\Psi \in \mathrm{ISO}((F, J^F), (N, J^N))$ such that

$$B_X = \Psi \widetilde{B}_X,\ A_X \Psi = \widetilde{A}_X,\ \nabla^N_X \Psi = \Psi \nabla^F_X,\ \Psi^* \Theta_N = \Theta_F \tag{5.7}$$

hold for any $X \in T_xM$, $x \in M$, where B, A and ∇^N are the affine fundamental form, the shape tensor and the transversal connection of f, respectively, $(D, \widetilde{\Theta})$ is a complex equiaffine structure on $(\mathbb{R}^{2(m+p)}, \widetilde{J})$ and Θ^N is a complex volume form on N which is parallel with respect to ∇^N.

Proof. We set $\widetilde{E} := TM \oplus F$ and

$$\nabla^{\widetilde{E}} := \widetilde{\iota}_1 \nabla \widetilde{\pi}_1 + \widetilde{\iota}_2 \widetilde{B} \widetilde{\pi}_1 - \widetilde{\iota}_1 \widetilde{A} \widetilde{\pi}_2 + \widetilde{\iota}_2 \nabla^F \widetilde{\pi}_2, \tag{5.8}$$

where $\widetilde{\iota}_1 : TM \to \widetilde{E}$ and $\widetilde{\iota}_2 : F \to \widetilde{E}$ are the inclusions, $\widetilde{\pi}_1 : \widetilde{E} \to TM$ and $\widetilde{\pi}_2 : \widetilde{E} \to F$ are the projection homomorphisms. By virtue of Lemma 3.3, we see that $\nabla^{\widetilde{E}}$ is flat. Next we put $J^{\widetilde{E}}$ by

$$J^{\widetilde{E}} := \widetilde{\iota}_1 J \widetilde{\pi}_1 + \widetilde{\iota}_2 J^F \widetilde{\pi}_2,$$

then $J^{\widetilde{E}}$ is a complex structure on $\widetilde{E}$ and it holds that $\nabla^{\widetilde{E}} \in \mathcal{C}(\widetilde{E}, J^{\widetilde{E}})$ from (5.8) and Corollary 3.8.

Fix a point $x_0 \in M$. Let $(\zeta_1, \ldots, J^{\widetilde{E}}\zeta_{m+p})$ be a complex frame of $\widetilde{E}_{x_0}$ with respect to $J^{\widetilde{E}}_{x_0}$. Since M is simply connected and $\nabla^{\widetilde{E}} \in \mathcal{C}(\widetilde{E}, J^{\widetilde{E}})$ is flat, there are unique global parallel extensions $\widetilde{\zeta}_1, \ldots, J^{\widetilde{E}}\widetilde{\zeta}_{m+p}$. Let $(\widetilde{\zeta}^1, \ldots, \widetilde{\zeta}^{2(m+p)})$ be the dual frame field of $(\widetilde{\zeta}_1, \ldots, J^{\widetilde{E}}\widetilde{\zeta}_{m+p})$. Then $\omega^\alpha := \widetilde{\zeta}^\alpha \circ \widetilde{\iota}_1 = \widetilde{\zeta}^\alpha|_{TM}$ are closed form for $\alpha = 1, \ldots, 2(m+p)$ since ∇ is torsion free and $\widetilde{B}$ is symmetric. Then there are smooth functions f^α on M such that $df^\alpha = \omega^\alpha$ for $\alpha = 1, \ldots, 2(m+p)$ because M is simply connected. Define a map $f : M \to \mathbb{R}^{2(m+p)}$ by $f = f^\alpha e_\alpha$. By virtue of (5.1), we have, for any $X \in T_xM$, $x \in M$

$$f_{*x}(X) = (df^\alpha)(X)(\overline{e}_\alpha)_{f(x)} = \omega^\alpha(X)(\overline{e}_\alpha)_{f(x)}. \tag{5.9}$$

We define $\Phi \in \mathrm{ISO}(\widetilde{E}, f^\# T\mathbb{R}^{2(m+p)})$ by

$$\Phi_x(\widetilde{\zeta}_\alpha)_x := (f^\# \overline{e}_\alpha)_x, \quad \alpha = 1, \ldots, 2(m+p)$$

for each $x \in M$. Therefore we get $f_* = f_\# \Phi \widetilde{\iota}_1$ from (5.9) and the definition of Φ. Thus we see that f is an immersion since $f_\#$ is a linear isomorphism at each point, Φ is an isomorphism and $\widetilde{\iota}_1$ is the inclusion. It follows from a direct calculation that

$$\Phi J^{\widetilde{E}} = (f^\# \widetilde{J}) \Phi. \tag{5.10}$$

Since Φ sends $\nabla^{\widetilde{E}}$-parallel frame field $\widetilde{\zeta}_1, \dots, J^{\widetilde{E}}\widetilde{\zeta}_{m+p}$ to $(f^{\#}D)$-parallel frame field $f^{\#}\overline{e}_1, \dots, f^{\#}\overline{e}_{2(m+p)}$, Φ preserves the connection, that is,

$$\Phi\nabla^{\widetilde{E}}_X \;=\; (f^{\#}D)_X\Phi \tag{5.11}$$

for any $X \in \Gamma(TM)$. If we put $N := \Phi(F)$, we get the following decomposition

$$f^{\#}T\mathbb{R}^{2(m+p)} = \Phi(\widetilde{E}) = i^f(TM) \oplus N.$$

We put $J^N := \pi_N(f^{\#}\widetilde{J})\iota_N$ and $\Psi := \pi_N\Phi\widetilde{\iota}_2$. Since (5.10) holds, we have

$$\begin{aligned} \hat{i}^f &\in \mathrm{HOM}((TM, J), (i^f(TM), \pi_f(f^{\#}\widetilde{J})\iota_f)), \\ \Psi &\in \mathrm{ISO}((F, J^F), (N, J^N)). \end{aligned} \tag{5.12}$$

From (5.12), we can prove that f is holomorphic. Since the equation (5.11) holds, by using Lemma 3.4, we see that the induced connections

$$\pi_f(f^{\#}D)\iota_f \in \mathcal{C}(i^f(TM), \pi_f(f^{\#}\widetilde{J})\iota_f), \quad \nabla^N \in \mathcal{C}(N, J^N)$$

for $(f^{\#}D)$ and the second fundamental forms

$$\begin{aligned} B(\hat{i}^f)^{-1} &\in A^1(\mathrm{Hom}((i^f(TM), \pi_f(f^{\#}\widetilde{J})\iota_f), (N, J^N))), \\ -\hat{i}^f A &\in A^1(\mathrm{Hom}((N, J^N), (i^f(TM), \pi_f(f^{\#}\widetilde{J})\iota_f))) \end{aligned}$$

for $(f^{\#}D)$ satisfy

$$\begin{aligned} \hat{i}^f\nabla_X = \pi_f(f^{\#}D)_X\iota_f\hat{i}^f, &\quad \Psi\nabla^F_X = \nabla^N_X\Psi, \\ B_X(\hat{i}^f)^{-1}\hat{i}^f = \Psi\widetilde{B}_X, &\quad -\hat{i}^f A_X\Psi = -\hat{i}^f\widetilde{A}_X \end{aligned} \tag{5.13}$$

for any $X \in \Gamma(TM)$. From (5.13), we can conclude that f is a complex affine immersion with transversal bundle N and (5.7) is satisfied.

Define $\Theta^{\widetilde{E}} \in \Gamma(\mathrm{Det}_{\mathbb{C}}\widetilde{E})$ by

$$\Theta^{\widetilde{E}} := (\widetilde{\pi}_1^*\Theta) \wedge (\widetilde{\pi}_2^*\Theta_F).$$

Then $\Theta^{\widetilde{E}}$ is a complex volume form which is parallel with respect to $\nabla^{\widetilde{E}}$ since both Θ and Θ_F are complex volume forms that are parallel with respect to ∇ and ∇^F respectively. Then, $\widetilde{\Theta} := ((f_{\#})^{-1}\Phi^{-1})^*\Theta^{\widetilde{E}}$ is a complex equiaffine structure on $(\mathbb{R}^{2(m+p)}, \widetilde{J})$. We define Θ_N by $\Psi^*\Theta_N := \Theta_F$. Then $\Theta_N \in \Gamma(\mathrm{Det}_{\mathbb{C}}N)$ since Θ_F is a complex volume form on F. Since Θ_F is parallel with respect to ∇^F, $\Psi \in \mathrm{ISO}((F, J^F), (N, J^N))$ and (5.14), we get

$$\Psi^*(\nabla^{\mathrm{Det}_{\mathbb{C}}N}_X\Theta_N) \;=\; (\nabla^{\mathrm{Det}_{\mathbb{C}}F}_X\Theta_F)$$

for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(F)$. Hence Θ_N is a complex volume form which is parallel with respect to ∇^N. Since we get

$$\begin{aligned}(f^{\#}\widetilde{\Theta}) &= (\Phi^{-1})^*\widetilde{\Theta}^{\widetilde{E}} \\ &= (\Phi^{-1})^*((\widetilde{\pi}_1^*\Theta) \wedge (\widetilde{\pi}_2^*\Theta_F)) \\ &= (((\hat{i}^f)^{-1} \circ \pi_{i^f(TM)})^*\Theta) \wedge (\pi_N^*\Theta_N),\end{aligned}$$

f is a complex equiaffine immersion. □

Next, we prove the equivalence theorem.

Theorem 5.2 *Let (M, J, ∇) be a $2m$-dimensional complex manifold with complex structure J with $\nabla \in \mathcal{C}_0(TM, J)$ and f (resp. g) : $(M, J, \Theta, \nabla) \to (\mathbb{R}^{2(m+p)}, \widetilde{J}, \widetilde{\Theta}, D)$ be a complex equiaffine immersion with transversal bundle (N^f, Θ^{N^f}) (resp. (N^g, Θ^{N^g})) with the induced complex structure J^f (resp. J^g). The affine fundamental form, the shape tensor and the transversal connection of f (resp. g) are denoted by B^f, A^f and ∇^{N^f} (resp. B^g, A^g and ∇^{N^g}). If M is connected and there exists $\Psi \in \mathrm{ISO}((N^f, J^f), (N^g, J^g))$ such that*

$$B^g_X = \Psi B^f_X,\ A^g_X\Psi = A^f_X,\ \nabla^{N^g}_X\Psi = \Psi\nabla^{N^f}_X,\ \Psi^*\Theta^{N^g} = \Theta^{N^f} \tag{5.14}$$

for any $X \in \Gamma(TM)$, then there exists a complex affine transformation ϕ : $(\mathbb{R}^{2(m+p)}, \widetilde{J}, D) \to (\mathbb{R}^{2(m+p)}, \widetilde{J}, D)$ such that $g = \phi f$ and $(\phi_*)^*\widetilde{\Theta} = \widetilde{\Theta}$ and the bundle isomorphism induced by ϕ_* coincides with Ψ on N^f.

Proof. We define $\Phi_1 \in \mathrm{ISO}(i^f(TM), i^g(TM))$ by

$$\Phi_1 := \hat{i}^g(\hat{i}^f)^{-1}.$$

Then we have

$$\Phi_1\hat{i}^f J(\hat{i}^f)^{-1} = \hat{i}^g J(\hat{i}^g)^{-1}\Phi_1, \tag{5.15}$$

$$\Phi_1\hat{i}^f \nabla(\hat{i}^f)^{-1} = \hat{i}^g \nabla(\hat{i}^g)^{-1}\Phi_1. \tag{5.16}$$

By (5.14), it holds that

$$\Phi_1\hat{i}^f A^f_X = \hat{i}^g A^g_X\Psi, \tag{5.17}$$

$$\Psi B^f_X(\hat{i}^f)^{-1} = B^g_X(\hat{i}^g)^{-1}\Phi_1 \tag{5.18}$$

for any $X \in \Gamma(TM)$. We define $\Phi : f^{\#}T\mathbb{R}^{2(m+p)} \to g^{\#}T\mathbb{R}^{2(m+p)}$ by

$$\Phi := \iota_g\Phi_1\pi_f + \iota_{N^g}\Psi\pi_{N^f}.$$

From (5.15) and the assumption that $\Psi \in \mathrm{ISO}((N^f, J^f), (N^g, J^g))$, we see that

$$\Phi(f^{\#}\widetilde{J}) = (g^{\#}\widetilde{J})\Phi. \tag{5.19}$$

By virtue of (5.14), (5.16), (5.17) and (5.18), Lemma 3.4 yields

$$(g^{\#}D)_X\Phi = \Phi(f^{\#}D)_X \tag{5.20}$$

for any $X \in \Gamma(TM)$.

Fix a point $x_0 \in M$, we can write

$$\Phi_{x_0}((f^{\#}\overline{e}_\alpha)_{x_0}) = a_\alpha^\beta(g^{\#}\overline{e}_\beta)_{x_0}$$

for $\alpha, \beta = 1, \ldots, 2(m+p)$. Since (5.20) holds and M is connected, we have

$$\Phi_x((f^{\#}\overline{e}_\alpha)_x) = a_\alpha^\beta(g^{\#}\overline{e}_\beta)_x \tag{5.21}$$

for each $x \in M$. We define an affine transformation $\phi : \mathbb{R}^{2(m+p)} \to \mathbb{R}^{2(m+p)}$ by

$$\phi(f(x_0) + e_\alpha) := g(x_0) + a_\alpha^\beta e_\beta$$

and define a map $\widetilde{\phi} : M \to \mathbb{R}^{2(m+p)}$ by

$$\widetilde{\phi}(x) := \phi(f(x)) - g(x) = (a_\alpha^\beta f^\alpha(x) + g^\beta(x_0) - a_\gamma^\beta f^\gamma(x_0) - g^\beta(x))e_\beta$$

for $x \in M$. From (5.2) and (5.21), we obtain

$$\Phi_x(df^\alpha(f^{\#}\overline{e}_\alpha)_x) = df^\alpha a_\alpha^\beta(g^{\#}\overline{e}_\beta)_x \tag{5.22}$$

for each $x \in M$. By (5.22), it holds that

$$d(a_\beta^\alpha f^\beta + g^\alpha(x_0) - a_\gamma^\alpha f^\gamma(x_0) - g^\alpha) a_\beta^\alpha df^\beta - dg^\alpha = 0$$

and $\widetilde{\phi}(x_0) = 0$, we obtain $\widetilde{\phi} = 0$, that is,

$$g = \phi \circ f \tag{5.23}$$

on M. By virtue of (5.21) and (5.23), we get

$$\Phi_x = (g_{\# x})^{-1}\phi_{*x}(f_{\# x}) \tag{5.24}$$

for each $x \in M$. Then (5.19) and (5.24) imply $\phi_*\widetilde{J} = \widetilde{J}\phi_*$, that is, ϕ is a complex affine transformation. Moreover form (5.24), we see that Ψ coincides with the bundle isomorphism induced by ϕ_* on N^f.

Since both f and g are complex equiaffine immersions, we have

$$\begin{aligned} f^{\#}\widetilde{\Theta} &= ((\hat{i}^f)^{-1} \circ \pi_f)^*\Theta \wedge \pi_{N^f}^*\Theta^{N^f}, \\ g^{\#}\widetilde{\Theta} &= ((\hat{i}^g)^{-1} \circ \pi_g)^*\Theta \wedge \pi_{N^g}^*\Theta^{N^g}. \end{aligned}$$

From the definition of Φ, we get

$$\Phi^*(g^{\#}\widetilde{\Theta}) = \Phi^*(((\hat{i}^g)^{-1} \circ \pi_g)^*\Theta \wedge \pi_{N^g}^*\Theta^{N^g}) = f^{\#}\widetilde{\Theta}.$$

Combining this and (5.24), we see that $(\phi_*)^*\widetilde{\Theta} = \widetilde{\Theta}$. □

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