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Application of Local Linking to Asymptotically Linear Elliptic Equations

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Abstract. Existence of a nontrivial solution to a semilinear elliptic equation is established by a variational method. We consider the weak solution to $-\triangle u = h(x,u)$, where $h(x,u)$ is asymptotically linear in u both as $u \to 0$ and $u \to \infty$. The proof is based on local linking theory, $(PS)^*$ condition and approximation by finite dimensional subspaces for the existence of a nontrivial critical point of a C^1-class functional.

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§1. Introduction

In this paper we consider the existence of a nontrivial solution to the following semilinear elliptic equation (P):

$$\text{(P)} \qquad \begin{cases} -\triangle u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. We treat the case where the non-linear term h is asymptotically linear at both 0 and ∞ in the following sense: There exist constants b_0 and b for which

$$\begin{aligned} g_0(x,\xi) &:= h(x,\xi) - b_0\xi = o(|\xi|) \text{ as } \xi \to 0 \text{ uniformly in } x \in \Omega, \\ g(x,\xi) &:= h(x,\xi) - b\xi = o(|\xi|) \quad \text{as } |\xi| \to \infty \text{ uniformly in } x \in \Omega. \end{aligned}$$

Many authors treated this problem by variational methods under some condition on b_0 and b. For example, there are papers studying nonresonant case ($b \notin \sigma(-\triangle)$) ([1], [7]), resonant case ($b_0 \neq b \in \sigma(-\triangle)$) ([3], [4], [6], [8], [10],

[13], [15]) and strong resonant case ($b_0 = b \in \sigma(-\triangle)$) ([5], [12]) (Note that some authors use the term "strong resonant case" in a slightly different sense). As for the resonant case with $b_0 \neq b$, Masiello and Pisani [8] and Mizoguchi [10] dealt with the case where g is bounded, while Bartsch and Li [3] considered the case where there exist some $\alpha > 0$, $C > 0$ such that

$$\begin{array}{l} G(x,\xi) - \frac{1}{2}g(x,\xi)\xi \geq C(|\xi|^{\alpha+1} - 1) \\ \text{or} \\ \frac{1}{2}g(x,\xi)\xi - G(x,\xi) \geq C(|\xi|^{\alpha+1} - 1) \end{array} \tag{1.1}$$

holds for $G(x,\xi) := \int_0^\xi g(x,s)\,ds$. On the other hand, Silva [13] considered g satisfying $\liminf_{|\xi|\to\infty} g_\xi(x,\xi) > b_0 - b$ or $\limsup_{|\xi|\to\infty} g_\xi(x,\xi) < b_0 - b$ where $g_\xi(x,\xi) := \partial_\xi g(x,\xi)$. Zou and Liu [15] dealt with the following condition

$$|g(x,\xi)| \leq c(1 + |\xi|^\beta) \text{ and } \liminf_{|\xi|\to\infty} \frac{\pm G(x,\xi)}{|\xi|^{\beta+1}} =: a^\pm(x) > 0 \text{ uniformly in } x \in \Omega, \tag{1.2}$$

where $0 < \beta < 1$.

In this paper, we introduce a new condition which guarantees the existence of a nontrivial solution to (P) even in the resonant case (see the condition (C2) in Section 3). For example, the following $g(x,\xi)$ satisfies our assumption (C2):

$$g(x,\xi) = a(x,\xi)|\xi|^\beta \operatorname{sgn}\xi + b(x,\xi)|\xi|^\alpha \operatorname{sgn}\xi,$$

where $a(x,\xi)$ and $b(x,\xi)$ are some suitable bounded functions and α, β are constants satisfying $0 < \alpha \leq \beta < 1$ and $2\beta < \alpha + 1$ (see the condition (C2) in Section 3). In general this case does not satisfy any of the conditions treated by the above-mentioned authors. (see Example 18 in Section 3)

The proof of this paper depends on the existence theory of a non-trivial critical point for a C^1-class functional proved in [11]. The proof of [11] is based on local linking, minimax theorem and $(WPS)^*$ condition which is a generalization of the $(PS)^*$ condition (see Definition 8 in [11]). Therefore in the following Section 2, we prepare some propositions and then sketch a proof of abstract theory in [11] by restricting to C^1-class functionals satisfying $(PS)^*$ condition. In Section 3, we prove the existence of a non-trivial weak solution to (P).

§2. Abstract theory

Throughout this section, we let E be a Hilbert space with inner $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $\Phi\colon E \to \mathbb{R}$ a C^1-class functional. We suppose $\{E_n\}_n$ is a sequence of finite dimensional subspace of E satisfying the following condition:

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots \subset E, \qquad E = \overline{\bigcup_{n=1}^{\infty} E_n}. \tag{2.1}$$

We define P_n as the orthogonal projection from E onto E_n.

Definition 1 (i) *A sequence $\{u_j\}_j$ in E is called a $(PS)_c^*$ sequence (w.r.t. Φ and $\{E_n\}_n$) provided $u_j \in E_{n_j}$, $n_j \to \infty$, $\Phi(u_j) \to c$ and $P_{n_j}(\nabla\Phi(u_j)) \to 0$ (as $j \to \infty$);*

(ii) *Φ is said to satisfy the $(PS)_c^*$ condition for $c \in \mathbb{R}$ if every $(PS)_c^*$ sequence has a norm convergent subsequence.*

(iii) *If there exists an orthogonal decomposition $E = V_0 \oplus W_0$ and an $r > 0$ satisfying the following condition, then Φ is said to have a local linking at 0 with respect to (V_0, W_0):*

$$\begin{cases} \Phi(u) \geq 0 \ (\forall u \in B_r V_0\,), \\ \Phi(u) \leq 0 \ (\forall u \in B_r W_0\,), \end{cases} \tag{2.2}$$

where $B_r V_0 := \{u \in V_0 \,:\, \|u\| \leq r\}$, $B_r W_0 := \{u \in W_0 \,:\, \|u\| \leq r\}$.

Definition 2 *A subset $\tilde{E}$ is defined by $\tilde{E} := \{\, u \in E \,:\, \nabla\Phi(u) \neq 0 \,\}$. A map $V : \tilde{E} \to E$ is called a pseudo-gradient vector field for Φ if V satisfies the following conditions on $\tilde{E}$:*

$$\begin{cases} \|V(u)\| \leq \frac{3}{2}\, \|\nabla\Phi(u)\|, \\ \langle\, \nabla\Phi(u)\,, V(u)\,\rangle \geq \frac{1}{2}\, \|\nabla\Phi(u)\|^2. \end{cases}$$

It is well known that there exists a locally Lipschitz continuous pseudo-gradient vector field V for every C^1 class functional Φ ([9, Lemma 6.1]).

For such a pseudo-gradient vector field V for Φ, the ordinary differential equation

$$\frac{du(t)}{dt} = -V(u(t)), \quad u(0) = u_0 \qquad (u_0 \in \tilde{E})$$

has a unique solution which is maximally defined in the positive direction of t. This maximal solution will be called the pseudo-gradient flow defined by V and (starting from) u_0.

We say that the sequence $\{E_n\}_n$ satisfying (2.1) is *compatible* with the orthogonal decomposition $V_0 \oplus W_0$ [*resp.* $V_\infty \oplus W_\infty$] if

$$E = V_0 \oplus W_0, \quad E_n = (E_n \cap V_0) \oplus (E_n \cap W_0) \quad \text{for every } n$$
$$[\textit{resp. } E = V_\infty \oplus W_\infty, \quad E_n = (E_n \cap V_\infty) \oplus (E_n \cap W_\infty) \quad \text{for every } n].$$

Now we prepare the conditions relevant to our abstract theory.

($\Phi1$) With respect to a sequence $\{E_n\}_n$ of finite dimensional subspaces satisfying (2.1), Φ satisfies $(PS)_c^*$ condition for every $c \in \mathbb{R}$.

($\Phi2$) Φ is bounded on every bounded set.

($\Phi3$) Φ has a local linking at 0 w.r.t. some orthogonal decomposition $E = V_0 \oplus W_0$.

($\Phi4$) There exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$ that satisfies the following (i) to (iii) for some number $\lambda \geq 0$, $\delta > 0$, $R_1 > 0$: where $u = w_\infty + v_\infty \quad (\, w_\infty \in W_\infty \, , \, v_\infty \in V_\infty \,)$

(i) $\left\langle \nabla\Phi(u) \, , \, v_\infty - \lambda\delta^2 \dfrac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0, \quad$ (if $\|v_\infty\| = \delta\|w_\infty\|^\lambda \, , \, \|v_\infty\| \geq R_1$),

(ii) $\langle \nabla\Phi(u) \, , \, v_\infty \rangle > 0, \quad$ (if $\|v_\infty\| \geq \delta\|w_\infty\|^\lambda \, , \, \|v_\infty\| = R_1$),

(iii) for every $c < 0$ there exists an $R > 0$ such that $\Phi(u) < c$ provided $\|v_\infty\| \leq \delta\|w_\infty\|^\lambda$ and $\|w_\infty\| \geq R$.

Remark. The conditions (i) and (ii) in ($\Phi4$) mean that the gradient vector $\nabla\Phi$ points outward to the shaded region on its boundary, as sketched in Figure 1.

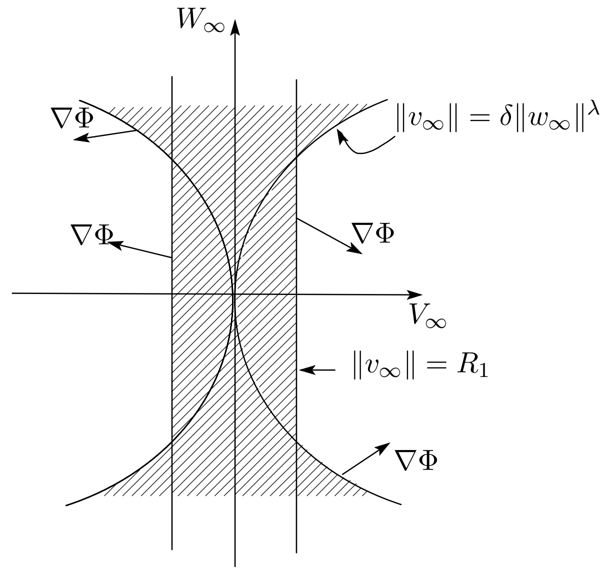

Figure 1: meaning of ($\Phi4$)

When we assume (Φ1), (Φ3) and that $\{E_n\}_n$ is compatible w.r.t $V_0 \oplus W_0$ and $V_\infty \oplus W_\infty$, we set the following notation

$$\begin{aligned}
&\Phi_n := \Phi|_{E_n}, \\
&\Phi_n^c := \big\{u \in E_n \,:\, \Phi(u) \le c\big\},\ (\Phi_n)_c := \big\{u \in E_n \,:\, \Phi(u) \ge c\big\}, \\
&E_n^1 := E_n \cap V_0, \qquad\qquad\qquad E_n^2 := E_n \cap W_0, \\
&B_n^j := B_r E \cap E_n^j \quad (j = 1, 2),\ S_n^j := \partial B_n^j \quad (j = 1, 2).
\end{aligned}$$

We need the following lemma, proposition and its corollary for our abstract theory.

Lemma 3 ([11, Lemma 11]) *If Φ satisfies* (Φ4) *with $\{E_n\}_n$ being compatible w.r.t. (V_∞, W_∞) , then $\Phi|_{E_n}$ satisfies* (Φ4) *with (V_∞, W_∞) replaced by $(V_\infty \cap E_n, W_\infty \cap E_n)$ for every $n \in \mathbb{N}$.*

Proposition 4 ([11, Proposition 4]) *Suppose that there exists an orthogonal decomposition $E = V_\infty \oplus W_\infty$, and Φ satisfies the following condition* (R). *Then there exists a locally Lipschitz continuous pseudo-gradient vector field V for Φ on $\tilde{E}$, for which the region*

$$U := \{\, (v_\infty, w_\infty) \mid \|v_\infty\| < \max\{R_1, \delta\|w_\infty\|^\lambda\}\,\} \tag{2.3}$$

encloses pseudo-gradient flows starting from its elements.

(R) *The following* (i), (ii) *hold for some $\lambda \ge 0$, $\delta > 0$, and $R_1 > 0$, where*

$$u = w_\infty + v_\infty \quad (\, w_\infty \in W_\infty\,,\, v_\infty \in V_\infty\,).$$

(i) $\left\langle \nabla\Phi(u)\,,\, v_\infty - \lambda\delta^2 \dfrac{w_\infty}{\|w_\infty\|^{2-2\lambda}} \right\rangle > 0 \quad (\text{if } \|v_\infty\| = \delta\|w_\infty\|^\lambda\,,\, \|v_\infty\| \ge R_1)$,

(ii) $\langle \nabla\Phi(u)\,,\, v_\infty \rangle > 0 \quad (\text{if } \|v_\infty\| \ge \delta\|w_\infty\|^\lambda\,,\, \|v_\infty\| = R_1)$.

Corollary 5 ([11, Corollary 6]) *Suppose that Φ satisfies the condition* (R) *in Proposition* 4 *and let U be as in* (2.3). *In addition, assume that the following conditions hold*:

(a) *Φ is bounded on every bounded sets.*

(b) *For every ε, $M > 0$ with $\varepsilon < M < \infty$,*

$$\inf\{\, \|\nabla\Phi(u)\| \mid u \in \Phi^{-1}([-M, -\varepsilon])\,\} > 0.$$

(c) *Under the notation that* Q_∞ *is the orthogonal projection onto* W_∞,

$$\lim_{R\to\infty} \sup\{\, \Phi(u) \mid u \in U,\ \|Q_\infty u\| \geq R \,\} = -\infty.$$

Then every continuous map $\varphi\colon S^n \to U \cap \{u \in \tilde{E} \mid \Phi(u) < 0\}$ *with* $n < \dim W_\infty - 1$ (S^n *: n-dimensional usual sphere*) *is homotopic to a constant map in* $U \cap \{u \in E \mid \Phi(u) < 0\}$.

The next lemma is stated as Lemma 6.5 in [9] and it can be proved by the standard deformation argument (*cf.* [14, Lemma 2.3]).

Lemma 6 (Deformation Lemma) *Suppose* (Φ1) *and* (Φ3) *hold and there exists no non-trivial critical point of* Φ. *Then there exist some* $\varepsilon > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every* $n \geq n_0$ *there exist continuous deformations* $\xi_n, \eta_n \in C([0,1] \times E_n\,,\, E_n)$ *satisfying the following conditions, where* $r > 0$ *satisfies* (2.2) *in* (Φ3).

(1) $\xi_n(0,\cdot) = \eta_n(0,\cdot) = id$,

(2) $\xi_n(t,\cdot), \eta_n(t,\cdot)$ *are homeomorphisms from* E_n *to* E_n *for every* $t \in [0,1]$,

(3) $\|\xi_n(t,u) - u\| \leq \frac{r}{2}\,,\ \|\eta_n(t,u) - u\| \leq \frac{r}{2}$ for every $(t,u) \in [0,1] \times E_n$,

(4) $\sup \Phi \circ \xi_n([0,1] \times B_n^2) = \inf \Phi \circ \eta_n([0,1] \times B_n^1) = 0$,

(5) $\Phi \circ \xi_n(t,\cdot)\,|_{S_n^2} < 0\,,\ \Phi \circ \eta_n(t,\cdot)\,|_{S_n^1} > 0$ for every $t \in (0,1]$,

(6) $\xi_n(1,u) \subset \Phi_n^{-\varepsilon}$ for every $u \in \big(B_{2r}E \cap \Phi_n^{\varepsilon}\big)\backslash B_{\frac{r}{3}}E$,

(7) $\eta_n(1,u) \subset (\Phi_n)_{\varepsilon}$ for every $u \in \big(B_{2r}E \cap (\Phi_n)_{-\varepsilon}\big)\backslash B_{\frac{r}{3}}E$.

Now we can prove our abstract result. We state a short proof of the following theorem for reader's convenience because it was proved under the general $(WPS)_c^*$ condition in [11, Theorem 12].

Theorem 7 ([11, Theorem 12]) *Suppose that* Φ *satisfies the conditions* (Φ1) *to* (Φ4), *and* $\{E_n\}_n$ *in* (Φ1) *is compatible with the decomposition* $V_0 \oplus W_0$ *in* (Φ3) *and* $V_\infty \oplus W_\infty$ *in* (Φ4). *Moreover, suppose that*

$$\limsup_{n\to\infty} \{\, \dim(W_\infty \cap E_n) - \dim(W_0 \cap E_n) \,\} > 0 \tag{2.4}$$

holds. Then Φ *has at least one non-trivial* (*i.e., non-zero*) *critical point.*

Remark 8 *We denote that if* $\dim W_\infty < \infty$ *and* $\dim W_0 < \infty$ *hold, then the condition* (2.4) *is satisfied if only if* $\dim W_\infty > \dim W_0$ *holds. And also if* $\mathrm{codim} W_\infty < \infty$ *and* $\mathrm{codim} W_0 < \infty$ *hold, then the condition* (2.4) *is satisfied if only if* $\mathrm{codim} W_\infty < \mathrm{codim} W_0$ *holds.*

Proof. We prove this theorem by contradiction. So suppose that there exists no critical points other than the origin.

Let U be the set defined by (2.3) and $r > 0$ satisfy (2.2) in (Φ3). We may assume $B_{2r}E \subset U$ by taking $r > 0$ small enough. We let n_0, $\varepsilon > 0$, $\xi_n, \eta_n \in C([0,1] \times E_n, E_n)$ satisfy the conditions (1) to (7) in Lemma 6, and set $A_n := \xi_n(1, S_n^2)$.

Since Φ satisfies the condition (Φ1) and Φ has no non-trivial critical points, for every $M > 0$ there exist $n_1 \in \mathbb{N}$ and $b > 0$ such that

$$\|\nabla \Phi_n(u)\| \geq b \quad \text{for every } u \in \Phi_n^{-1}(\,(-M, -\varepsilon]\,) \cup \Phi_n^{-1}(\,[\varepsilon, M)\,) \tag{2.5}$$

holds for every $n \geq n_1$.

Suppose $\dim W_0 > 0$. Then $\dim E_n \cap W_0 > 0$ holds for large n because of the compatibility of $\{E_n\}_n$ with the orthogonal decomposition $V_0 \oplus W_0$. By the assumption (2.4), there exists an increasing sequence $\{n_j\}_j$ of natural numbers satisfying $\dim E_{n_j} \cap W_\infty - \dim E_{n_j} \cap W_0 > 0$. We may also assume that $\dim E_{n_j} \cap W_0 > 0$.

We can identify the usual sphere S^m with $S_{n_j}^2$ where $m := \dim E_{n_j}^2$ and note that A_{n_j} is homeomorphic to $S_{n_j}^2$ by the condition (2) in Lemma 6. Since we can apply Corollary 5 with E replaced by E_{n_j} and Φ by Φ_{n_j} for sufficiently large j (see [11, Lemma 11] for detail), we obtain a continuous map $\tau_j \in C([0,1] \times A_{n_j}, E_{n_j})$ satisfying the following conditions:

$$\begin{array}{cl} \tau_j(0,u) = u & \text{for } u \in A_{n_j}, \\ \tau_j(1,u) = a_j & \text{for } u \in A_{n_j}, \\ \tau_j(t,u) \in U \cap \{u \in E_{n_j} | \Phi_{n_j}(u) < 0\} & \text{for } u \in A_{n_j},\ t \in [0,1], \end{array}$$

where $a_j \in U \cap \{u \in E_{n_j} | \Phi_{n_j}(u) < 0\}$. Moreover, because of the assumption (Φ4) and the construction of τ_j (see [11] for details), we may suppose that there exists a constant $C > 0$ independent of j such that $\|\tau_j(t,u)\| \leq C$ for every $u \in A_{n_j}$, $t \in [0,1]$. Therefore, $M := \sup\{\Phi(u) | \|u\| \leq C\} < \infty$ by the condition (Φ2).

Next we define $\gamma_j \in C(\partial([0,1] \times B_{n_j}), E_{n_j})$ by

$$\gamma_j(t,u) := \begin{cases} u & (u \in B_{n_j}^2,\ t = 0), \\ \xi_{n_j}(2t, u) & (u \in S_{n_j}^2,\ t \in (0, 1/2]), \\ \tau_j(2t-1, \xi_{n_j}(1,u)) & (u \in S_{n_j}^2,\ t \in (1/2, 1)), \\ a_j & (u \in B_{n_j}^2,\ t = 1). \end{cases}$$

Set $\Gamma_j := \{\, \rho \mid \rho \in C([0,1] \times B_{n_j}^2, E_{n_j}),\ \rho|_{\partial([0,1]\times B_{n_j}^2)} = \gamma_j \,\}$. Note that by the well known Dugundij extension theorem, there exists a $\rho \in \Gamma_j$ with values in the ball $\{\, u \in E_{n_j} \mid \|u\| \leq C \,\}$. Therefore,

$$c := \inf_{\rho \in \Gamma_j} \sup \{\, \Phi(u) \mid u \in \rho([0,1] \times B_{n_j}^2) \,\} \leq M$$

holds by the definition of M. By a standard argument using degree theory (*cf.* [2, Lemma 3.2]), it can be proved that $\rho([0,1]\times B^2_{n_j})\cap\eta_{n_j}(1,S^1_{n_j})\neq\emptyset$ for any $\rho\in\Gamma_j$. Hence $c\geq\varepsilon$. On the other hand, $c_0:=\sup\{\Phi(u)\mid u\in\gamma_j(\partial([0,1]\times B^2_{n_j}))\}\leq 0$ holds because of construction of γ_j. Therefore, by Ekeland's mini-max theorem ([9, Theorem 4.3]), there exists a point $u_j\in E_{n_j}$ such that

$$\varepsilon\leq\Phi_{n_j}(u_j)<M+1 \text{ and } \|\nabla\Phi_{n_j}(u_j)\|<1/j.$$

However we get a contradiction to (2.5) for j large enough.

In the case where $W_0=\{0\}$, then $\Phi(0)=0$ and $\Phi(\eta_{n_j}(1,u))\geq\varepsilon$ for $u\in S^1_{n_j}$. We note that $W_\infty\neq\{0\}$ by (2.4). By (iii) of (Φ4), there exists $e_{n_j}\in E_{n_j}$ such that $\Phi(e_{n_j})<0$ and $\|e_{n_j}\|>2r$. Set $\Gamma_j:=\{\rho\in C([0,1],E_{n_j})\mid\rho(0)=0,\rho(1)=e_{n_j}\}$. Then we similarly obtain $\rho([0,1])\cap\eta_{n_j}(1,S^1_{n_j})\neq\emptyset$ for any $\rho\in\Gamma_j$ by degree theory because of $E=V_0$. Therefore, applying Mountain pass lemma (cf. [9, Theorem 4.10]), we obtain a point $u_j\in E_{n_j}$ such that $\varepsilon\leq\Phi_{n_j}(u_j)$, $\sup_j\Phi_{n_j}(u_j)<\infty$ and $\|\nabla\Phi_{n_j}(u_j)\|<1/j$. Hence the same contradiction as for the previous case occurs. ∎

§3. Application

We consider the following semilinear elliptic problem:

$$\text{(P)}\qquad \begin{cases}-\triangle u=h(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ (C^2 class will suffice). The nonlinear term $h\in C\left(\overline{\Omega}\times\mathbb{R},\mathbb{R}\right)$ is assumed to satisfy the following conditions (h1) and (h2):

(h1) $h(x,0)=0$ for every $x\in\Omega$,

(h2) there exist constants b_0, $b\in\mathbb{R}$ that satisfy the following conditions:
$g_0(x,\xi):=h(x,\xi)-b_0\xi=o(|\xi|)$ as $\xi\to 0$ uniformly in $x\in\Omega$,
$g(x,\xi):=h(x,\xi)-b\xi=o(|\xi|)$ as $|\xi|\to\infty$ uniformly in $x\in\Omega$.

We set

$$b_0^+:=\min\{\,\lambda\mid\lambda\in\sigma(-\triangle),\,b_0<\lambda\,\},$$
$$b_0^-:=\max\{\,\lambda\mid\lambda\in\sigma(-\triangle)\cup\{-\infty\},\,b_0>\lambda\,\},$$

where $\triangle:=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_N^2$ denotes the usual Laplacian in $L^2(\Omega)$ with domain $H^2(\Omega)\cap H^1_0(\Omega)$.

Now we state the conditions (C1) to (C4) concerning the existence of a non-trivial weak solution to (P). A function u is said to be a weak solution to (P) if $u \in H_0^1(\Omega)$ and

$$\int_\Omega (-\triangle u)v\,dx = \int_\Omega h(x,u)v\,dx \quad \text{for every } v \in H_0^1(\Omega).$$

To state the assumptions, we set $G_0(x,\xi) := \int_0^\xi g_0(x,s)\,ds$ and $G(x,\xi) := \int_0^\xi g(x,s)\,ds$.

(C1) g is bounded and $G(x,\xi) \to +\infty$ as $|\xi| \to \infty$ uniformly in $x \in \Omega$.

(C2) The following condition (a1) or (a2) holds for some constants $0 < \alpha \leq \beta < 1$, $2\beta < \alpha + 1$, $c_1, c_2 > 0$ and $d_1, d_2 \geq 0$: for every $(x,\xi) \in \Omega \times \mathbb{R}$

(a1) $|g(x,\xi)| \leq c_1|\xi|^\beta + d_1$, $G(x,\xi) \geq c_2|\xi|^{\alpha+1} - d_2|\xi|$,
(a2) $|g(x,\xi)| \leq c_1|\xi|^\beta + d_1$, $G(x,\xi) \leq -c_2|\xi|^{\alpha+1} + d_2|\xi|$.

(C3) There exists a $\delta > 0$ such that $G_0(x,\xi) \geq 0$ if $|\xi| \leq \delta$.

(C4) There exists a $\delta > 0$ such that $G_0(x,\xi) \leq 0$ if $|\xi| \leq \delta$.

With these notations, our main theorem reads as follows, of which the cases referring to the condition (C2) are new (see the remark below the statement of the theorem).

Theorem 9 *Assume that the nonlinear term h satisfies* (h1) *and* (h2). *Moreover let b_0, g_0, b and g be as in* (h2). *Then the elliptic equation* (P) *has a non-trivial weak solution in each of the following cases*:

(A1) (*non-resonant case*) $b_0 \notin \sigma(-\triangle)$, $b \notin \sigma(-\triangle)$ *and* $b \notin [b_0^-, b_0^+)$.

(A2) (*case of resonance only at 0*) $b_0 \in \sigma(-\triangle)$, $b \notin \sigma(-\triangle)$ *and one of the following conditions holds*:

(1) $b \notin [b_0, b_0^+)$ *and* (C3),
(2) $b \notin [b_0^-, b_0)$ *and* (C4).

(A3) (*case of resonance only at* ∞) $b_0 \notin \sigma(-\triangle)$, $b \in \sigma(-\triangle)$ *and one of the following conditions holds*:

(1) $b_0 < b$ *and* (C1) *or* (a1) of (C2),
(2) $b_0 > b$ *and* (a2) of (C2).

(A4) (*case of resonance at 0 and* ∞) $b_0 \in \sigma(-\triangle)$, $b \in \sigma(-\triangle)$ *and one of the following conditions holds*:

(1) (C3), $b_0 < b$ *and* (C1) *or* (a1) of (C2),
(2) (C3), $b_0 \geq b$ *and* (a2) *of* (C2),
(3) (C4), $b_0 \leq b$ *and* (C1) *or* (a1) of (C2),
(4) (C4), $b_0 > b$ *and* (a2) *of* (C2).

Remark 10 *There exist many papers almost covering the cases* (A1), (A2), (A3) *with* (C1) *and* (A4) *with* (C1) *of Theorem* 9 (*cf.* [1], [3], [6], [7], [8], [10]). *However, the author considers that it is worthwhile to show that we can systematically prove the known results together with new ones.*

To prove theorem 9, we define a Hilbert space E and a C^1-class functional Φ on H. Namely, set $E := H_0^1(\Omega)$ with norm $\|u\|_E := \|\nabla u\|_2$, where $\|u\|_p$ is the usual L^p norm. **Throughout this section, we will write** $\|u\|_E = \|u\|$**,** $\langle \cdot,\cdot \rangle_E = \langle \cdot,\cdot \rangle$. The functional of our concern is defined as

$$\begin{aligned} \Phi(u) &:= \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u)\,dx && (3.1)\\ &= \frac{1}{2}\|u\|^2 - \frac{b}{2}\|u\|_2^2 - \int_\Omega G(x,u)\,dx && (3.2)\\ &= \frac{1}{2}\|u\|^2 - \frac{b_0}{2}\|u\|_2^2 - \int_\Omega G_0(x,u)\,dx && (3.3) \end{aligned}$$

where $H(x,\xi) := \int_0^\xi h(x,s)\,ds$. It is well known that Φ is a C^1-class functional on E and a critical point of Φ is a weak solution to (P). Moreover it is also well known that

$$\begin{aligned} \langle \nabla\Phi(u), v\rangle &= \langle u,v\rangle - \int_\Omega h(x,u(x))\,v(x)\,dx && (3.4)\\ &= \langle u,v\rangle - b\int_\Omega u(x)v(x)\,dx - \int_\Omega g(x,u(x))\,v(x)\,dx && (3.5)\\ &= \langle u,v\rangle - b_0\int_\Omega u(x)v(x)\,dx - \int_\Omega g_0(x,u(x))\,v(x)\,dx && (3.6) \end{aligned}$$

for every $u, v \in E$. Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ be the sequence of all eigenvalues of $-\triangle$ with Dirichlet boundary condition repeated as many times as their multiplicity, and let e_n be an eigenfunction of $-\triangle$ corresponding to λ_n. Note that each e_n belongs to $C(\bar{\Omega})$ by the regularity theorem and Sobolev embedding theorem. We define $X^+ :=$ lin.sp.$\{\, e_n : e_n$ corresponding to $\lambda_n > b\,\}$, $X^- :=$ lin.sp.$\{\, e_n : e_n$ corresponding to $\lambda_n < b\,\}$, $X_0^+ :=$ lin.sp.$\{\, e_n : e_n$ corresponding to $\lambda_n > b_0\,\}$, $X_0^- :=$ lin.sp.$\{\, e_n : e_n$ corresponding to $\lambda_n < b_0\,\}$, $X^0 := \ker(-\triangle - b)$, and $X_0^0 := \ker(-\triangle - b_0)$. $X^\pm$, X^0 are mutually orthogonal in E and also in L^2, and $X_0^\pm$, X_0^0 are orthogonal in E and in L^2. We can easily see that the following lemma holds by the definition of $X^\pm$ and $X_0^\pm$.

Lemma 11

$$\left\{\begin{aligned} a_0^+ &:= \inf_{u\in X_0^+,\|u\|=1}\int_\Omega |\nabla u|^2 - b_0\,|u|^2\,dx > 0\\ a^+ &:= \inf_{u\in X^+,\|u\|=1}\int_\Omega |\nabla u|^2 - b\,|u|^2\,dx > 0\\ -a_0^- &:= \sup_{u\in X_0^-,\|u\|=1}\int_\Omega |\nabla u|^2 - b_0\,|u|^2\,dx < 0\\ -a^- &:= \sup_{u\in X^-,\|u\|=1}\int_\Omega |\nabla u|^2 - b\,|u|^2\,dx < 0.\end{aligned}\right. \tag{3.7}$$

We also obtain the following result by $\dim(X^0\oplus X^-)<\infty$.

Lemma 12 (cf.[11, Lemma 22]) *If* (C1) *holds, then*

$$\int_\Omega G(x,u)\,dx \to \infty \quad \text{as } \|u\|\to\infty \text{ in } X^0\oplus X^-.$$

Lemma 13 *If h satisfies* (h1) *and* (h2)*, then every bounded $(PS)_c^*$ sequence has a convergent subsequence for every $c\in\mathbb{R}$.*

Proof. Let $\{u_j\}$ be a bounded $(PS)_c^*$ sequence for Φ. Then, by taking a subsequence if necessary, we may assume that there exists some $u\in E$ such that

$$u_j \rightharpoonup u \text{ in } E, \quad u_j\to u \text{ in } L^2, \tag{3.8}$$

$$h(x,u_j)\to h(x,u) \text{ in } L^2, \tag{3.9}$$

since $\{u_j\}$ is bounded and h satisfies $|h(x,u)|\le C\,|u|$ for some constant $C>0$. On the other hand, we have

$$\begin{aligned}\|u_j-u\|^2 =&\ \langle P_{n_j}\nabla\Phi(u_j)-\nabla\Phi(u),u_j-u\rangle\\ &+\int_\Omega (h(x,u_j)-h(x,u))\,(u_j-u)\,dx+\int_\Omega h(x,u_j)(P_{n_j}u-u)\,dx\end{aligned}$$

Therefore we obtain $u_j\to u$ in E by using (3.8) and (3.9). ∎

Lemma 14 *Assume that h satisfies* (h1) *and* (h2)*. In addition, suppose that one of $b\notin\sigma(-\triangle)$,* (C1) *or* (C2) *holds. Then Φ satisfies $(PS)_c^*$ condition for every $c\in\mathbb{R}$.*

Proof. Let $\{u_j\} \subset E$ be a $(PS)_c^*$ sequence w.r.t. Φ and $\{n_j\}$ be a sequence such that $u_j \in E_{n_j}$ and $n_j \to \infty$ as $j \to \infty$. By Lemma 13, it remains to show that $\{u_j\}$ is bounded. Throughout this proof, we let C and C_i $(i \in \mathbb{N})$ be positive constants independent of j, and we write $u_j = u_j^+ + u_j^0 + u_j^-$ where $u_j^\pm \in X^\pm$ and $u_j^0 \in X^0$. Because of the definition of $X^\pm$, $P_{n_j} u_j^\pm = u_j^\pm$ and (3.5), we have

$$\langle P_{n_j} \nabla \Phi(u_j), u_j^\pm \rangle = \int_\Omega |\nabla u_j^\pm|^2 - b|u_j^\pm|^2 \, dx - \int_\Omega g(x, u_j) u_j^\pm \, dx. \tag{3.10}$$

(i) The case of $b \notin \sigma(-\triangle)$.
In this case, $X^0 = \{0\}$ holds. By the condition (h2), for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that $|g(x,\xi)| \leq \varepsilon|\xi| + C_\varepsilon$ for every $\xi \in \mathbb{R}$ and $x \in \Omega$. Therefore by Hölder's inequality and Sobolev's embedding, we have

$$\left| \int_\Omega g(x, u_j) u_j^\pm \, dx \right| \leq \varepsilon \, C \|u_j^\pm\| \|u_j\| + C'_\varepsilon \|u_j^\pm\|$$

where C'_ε is a positive constant depending only on $\varepsilon > 0$ and Ω. Hence by recalling the definitions of a^+ and a^- in (3.7) and using (3.10), we obtain for j large enough

$$\|u_j^+\| \geq a^+ \|u_j^+\|^2 - \varepsilon C \|u_j^+\| \|u_j\| - C'_\varepsilon \|u_j^+\|,$$
$$\|u_j^-\| \geq a^- \|u_j^-\|^2 - \varepsilon C \|u_j^-\| \|u_j\| - C'_\varepsilon \|u_j^-\|.$$

Here, fixing an $\varepsilon > 0$ such that $0 < \varepsilon < \min\{a^+, a^-\}/2C$, we get

$$2(1 + C'_\varepsilon) \geq (\min\{a^+, a^-\} - 2\varepsilon C) \left(\|u_j^+\| + \|u_j^-\| \right)$$

and so $\{u_j\}$ is bounded.

(ii) The case of (C1).
Let $M := \sup_{x \in \Omega, \xi \in \mathbb{R}} |g(x,\xi)|$. Then by (3.10), we obtain for j large enough

$$\|u_j^+\| \geq a^+ \|u_j^+\|^2 - M \|u_j^+\|_1 \geq a^+ \|u_j^+\|^2 - MC \|u_j^+\|,$$

and so $MC + 1 \geq a^+ \|u_j^+\|$ holds, hence $\|u_j^+\|$ is bounded. Similarly we obtain $MC + 1 \geq a^- \|u_j^-\|$, and so $\|u_j^-\|$ is bounded. Next if $\|u_j^0\|$ is not bounded, we may assume $\|u_j^0\| \to \infty$ (as $j \to \infty$), going if necessary to a subsequence. Since G satisfies the following equation

$$\begin{aligned}
\int_\Omega G(x, u_j) \, dx &= \int_0^1 \frac{d}{ds} G(x, u_j^0 + s(u_j^+ + u_j^-)) \, ds \\
&= \int_\Omega G(x, u_j^0) \, dx \\
&\quad + \int_\Omega \int_0^1 g(x, u_j^0 + s(u_j^+ + u_j^-))(u_j^+ + u_j^-) \, ds \, dx,
\end{aligned}$$

the boundedness of $\|u_j^\pm\|$ and g yield

$$\begin{aligned}\Phi(u_j) &= \frac{1}{2}\int_\Omega |\nabla u_j|^2 - b|u_j|^2\,dx - \int_\Omega G(x,u_j^0)\,dx \\ &\quad - \int_\Omega\int_0^1 g(x,u_j^0 + s(u_j^+ + u_j^-))(u_j^+ + u_j^-)\,ds\,dx \\ &\leq C - \int_\Omega G(x,u_j^0)\,dx.\end{aligned}$$

Now by Lemma 12, $\int_\Omega G(x,u_j^0)\,dx \to \infty$ $(j \to \infty)$ holds and so we obtain $\Phi(u_j) \to -\infty$ $(j \to \infty)$. This contradicts the assumption that $\{u_j\}$ is a $(PS)_c^*$ sequence.

(iii) The case of (C2).
We treat the case where (a1) of the condition (C2) holds because the case (a2) can be handled similarly. We let p, q be positive constants such that $\max\{2, 1/\beta\} \leq p \leq 2/\beta$, $1/p + 1/q = 1$, then the inclusions $E \hookrightarrow L^{p\beta}$ and $E \hookrightarrow L^q$ are continuous since $1 \leq p\beta \leq 2$ and $1 \leq q \leq 2$. Therefore by the assumption (a1), H lder's inequality and Sobolev embedding theorem, we have

$$\begin{aligned}\left|\int_\Omega g(x,u_j)u_j^\pm\,dx\right| &\leq c_1\|u_j\|_{p\beta}^\beta\|u_j^\pm\|_q + d_1\|u_j^\pm\|_1 \\ &\leq C_1\|u_j\|^\beta\|u_j^\pm\| + C_2\|u_j^\pm\|.\end{aligned}$$

Then because of (3.10), we obtain if j is sufficiently large

$$\begin{aligned}\|u_j^+\| &\geq a^+\|u_j^+\|^2 - C_1\|u_j\|^\beta\|u_j^+\| - C_2\|u_j^+\|, \\ \|u_j^-\| &\geq a^-\|u_j^-\|^2 - C_1\|u_j\|^\beta\|u_j^-\| - C_2\|u_j^-\|,\end{aligned}$$

and so these yield

$$\begin{aligned}2(1+C_2) &\geq \min\{a^+,a^-\}(\|u_j^+\| + \|u_j^-\|) - 2C_1\|u_j\|^\beta \\ &\geq \min\{a^+,a^-\}(\|u_j^+\| + \|u_j^-\|) - C_3(\|u_j^+\| + \|u_j^-\|)^\beta - C_3\|u_j^0\|^\beta.\end{aligned}$$

Hence we have

$$C_3\|u_j^0\|^\beta \geq \min\{a^+,a^-\}(\|u_j^+\| + \|u_j^-\|) - C_3(\|u_j^+\| + \|u_j^-\|)^\beta - 2(1+C_2).$$

Let $y_j := \|u_j^+\| + \|u_j^-\|$, if $\{y_j\}$ is not bounded, then we may assume, going if necessary to a subsequence, $y_j \to \infty$ (as $j \to \infty$). Since

$$C_3\|u_j^0\|^\beta \geq \min\{a^+,a^-\}y_j - C_3 y_j^\beta - 2(1+C_2)$$

holds for j large enough, there exists some $C_4 > 0$ such that

$$\|u_j^0\|^\beta \geq C_4\,y_j \tag{3.11}$$

for sufficiently large j. On the other hand, because of the assumption (a1) and $\dim X^0 < \infty$, we have

$$\begin{aligned}\int_\Omega G(x,u^0)\,dx &\geq c_2\|u^0\|_{\alpha+1}^{\alpha+1} - d_2\|u^0\|_1 \\ &\geq C_5\|u^0\|^{\alpha+1} - C_6\|u^0\|.\end{aligned}$$

Similarly, we can also show that

$$\begin{aligned}&\left|\int_\Omega\int_0^1 g(x,u_j^0+s(u_j^+ + u_j^-))(u_j^+ + u_j^-)\,ds\,dx\right| \\ &\leq C\int_\Omega |u_j^0|^\beta |u_j^+ + u_j^-|\,dx + C\|u_j^+ + u_j^-\|^{\beta+1} + C\|u_j^+ + u_j^-\| \\ &\leq C\|u_j^0\|^\beta\|u_j^+ + u_j^-\| + C\|u_j^+ + u_j^-\|^{\beta+1} + C\|u_j^+ + u_j^-\|.\end{aligned}$$

So by using (3.11), we see for sufficiently large j that

$$\begin{aligned}\Phi(u_j) &= \frac{1}{2}\int_\Omega |\nabla u_j|^2 - b|u_j|^2\,dx - \int_\Omega G(x,u_j^0)\,dx \\ &\quad - \int_\Omega\int_0^1 g(x,u_j^0+s(u_j^+ + u_j^-))(u_j^+ + u_j^-)\,ds\,dx \\ &\leq Cy_j^2 - C_5\|u_j^0\|^{\alpha+1} + C_6\|u_j^0\| + C\|u_j^0\|^\beta y_j + Cy_j^{\beta+1} + Cy_j \qquad (3.12) \\ &\leq C\|u_j^0\|^{2\beta} - C_5\|u_j^0\|^{\alpha+1} + C_6\|u_j^0\| + C\|u_j^0\|^{\beta(\beta+1)} + C\|u_j^0\|^\beta.\end{aligned}$$

Hence we obtain $\Phi(u_j) \to -\infty$ (as $j \to \infty$) since $\beta(\beta+1) < 2\beta < \alpha+1$ and $\|u_j^0\| \to \infty$ by (3.11). This is a contradiction. Thus $\{y_j\}$ is bounded. Next if $\|u_j^0\|$ is not bounded, we may similarly assume that, going if necessary to a subsequence, $\|u_j^0\| \to \infty$ (as $j \to \infty$). Then by using (3.12), we have

$$\Phi(u_j) \leq C - C_5\|u_j^0\|^{\alpha+1} + C_6\|u_j^0\| + C\|u_j^0\|^\beta,$$

and we similarly obtain $\Phi(u_j) \to -\infty$ (as $j \to \infty$), which is a contradiction. Thus $\|u_j^0\|$ is bounded, and so $\{u_j\}$ is bounded. ∎

Lemma 15 *Suppose that h satisfies* (h1) *and* (h2)*, and that $b \notin \sigma(-\triangle)$ holds. Then Φ satisfies* (Φ4) *with $\lambda = 1$, $V_\infty = X^+$ and $W_\infty = X^-$. Moreover $-\Phi$ satisfies* (Φ4) *with $\lambda = 1$, $V_\infty = X^-$ and $W_\infty = X^+$.*

Proof. We treat only the case of Φ because we can similarly prove the case of $-\Phi$. In correspondence with the decomposition $E = X^+ \oplus X^-$, we write $u = u^+ + u^-$ where $u^\pm \in X^\pm$. Note that $X^0 = \{0\}$ since $b \notin \sigma(-\triangle)$. Set

$C_0 := \sup_{u\in X^+,\|u\|=1} \int_\Omega |\nabla u|^2 - b|u|^2\,dx$. We fix a $\delta > 0$ such that $\delta^2 < \min\{a^+, a^-\}/C_0 \le 1$ and let $\lambda = 1$. Then by the condition (h2), for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that $|g(x,\xi)| \le \varepsilon|\xi| + C_\varepsilon$. Hence we have

$$\left|\int_\Omega g(x,u)u^\pm\,dx\right| \le \varepsilon C_1\|u\|\|u^\pm\| + C_\varepsilon C_2\|u^\pm\|,$$

where C_1, C_2 are positive constants independent of $\varepsilon > 0$ and

$$\pm\langle\nabla\Phi(u), u^\pm\rangle \ge a^\pm\|u^\pm\|^2 \mp \int_\Omega g(x,u)u^\pm\,dx$$

Here we fix an $\varepsilon_1 > 0$ with $0 < \varepsilon_1 < \delta\min\{a^+, a^-\}/2C_1$, then it holds that for u with $\|u^+\| = \delta\|u^-\|$

$$\begin{aligned}\langle\nabla\Phi(u), u^+ - \delta^2 u^-\rangle &\ge a^+\|u^+\|^2 + a^-\delta^2\|u^-\|^2 \\ &\qquad - \left(\varepsilon_1 C_1\|u\| + C'_{\varepsilon_1}\right)\left(\|u^+\| + \delta^2\|u^-\|\right) \\ &\ge 2\left(\min\{a^+, a^-\} - \varepsilon_1 C_1(1+1/\delta)\right)\|u^+\|^2 - 2C'_{\varepsilon_1}\|u^+\|.\end{aligned}$$

Therefore there exists an $R_1 > 0$ such that $\langle\nabla\Phi(u), u^+ - \delta^2 u^-\rangle > 0$ provided $\|u^+\| = \delta\|u^-\|$, $\|u^+\| \ge R_1$.

Similarly for u with $\|u^+\| \ge \delta\|u^-\|$ we have

$$\begin{aligned}\langle\nabla\Phi(u), u^+\rangle &\ge a^+\|u^+\|^2 - \varepsilon_1 C_1\|u\|\|u^+\| - C'_{\varepsilon_1}\|u^+\| \\ &\ge \left(a^+ - \varepsilon_1 C_1(1+1/\delta)\right)\|u^+\|^2 - C'_{\varepsilon_1}\|u^+\|.\end{aligned}$$

Hence $\langle\nabla\Phi(u), u^+\rangle > 0$ holds for u with $\|u^+\| \ge \delta\|u^-\|$ and $\|u^+\| \ge R_1$, and so the conditions (i) and (ii) of (Φ4) are satisfied.

Next note that for every ε_2 with $0 < \varepsilon_2 < (a^- - C_0\delta^2)/4C_1$ there exists some constant $C_{\varepsilon_2} > 0$ such that

$$\left|\int_\Omega G(x,u)\,dx\right| = \left|\int_\Omega\int_0^1 g(x,su)u\,ds\,dx\right| \le \varepsilon_2 C_1\|u\|^2 + C_{\varepsilon_2}C_2\|u\|$$

for all $u \in E$, because of the assumptions (h2) and $g(x,0) = 0$.

Thus for u with $\|u^+\| \le \delta\|u^-\|$ we have

$$\begin{aligned}\Phi(u) &\le \frac{1}{2}C_0\|u^+\|^2 - \frac{a^-}{2}\|u^-\|^2 + \varepsilon_2 C_1\|u\|^2 + C_{\varepsilon_2}C_2\|u\| \\ &\le \frac{1}{2}C_0\delta^2\|u^-\|^2 - \frac{a^-}{2}\|u^-\|^2 + \varepsilon_2 C_1(1+\delta^2)\|u^-\|^2 + C_{\varepsilon_2}C_2(1+\delta)\|u^-\| \\ &\le -\frac{1}{2}\left(a^- - C_0\delta^2 - 4\varepsilon_2 C_1\right)\|u^-\|^2 + 2C_2 C_{\varepsilon_2}\|u^-\|.\end{aligned}$$

Hence for every $c < 0$ there exists an $R > 0$ such that $\Phi(u) < c$ provided $\|u^+\| \le \delta\|u^-\|$, $\|u^-\| \ge R$.

∎

Lemma 16 *Suppose that* h *satisfies* (h1) *and* (h2), *and assume that* (C1) *holds. Then* Φ *satisfies* (Φ4) *with* $\lambda = 0$, $V_\infty = X^+$ *and* $W_\infty = X^- \oplus X^0$.

Proof. Let $M := \sup_{x\in\Omega,\xi\in\mathbb{R}} |g(x,\xi)|$, and we write $u = u^+ + u^0 + u^-$ where $u^\pm \in X^\pm$, $u^0 \in X^0$. Then we have

$$\begin{aligned}\langle \nabla\Phi(u), u^+\rangle &= \int_\Omega |\nabla u^+|^2 - b|u^+|^2\,dx - \int_\Omega g(x,u)u^+\,dx \\ &\geq a^+\|u^+\|^2 - M\|u^+\|_1 \\ &\geq a^+\|u^+\|^2 - MC\|u^+\|.\end{aligned}$$

Hence there exists an $R_1 > 0$ such that $\langle \nabla\Phi(u), u^+\rangle > 0$ provided $\|u^+\| \geq R_1$. Next since G satisfies the equality

$$\int_\Omega G(x,u)\,dx = \int_\Omega G(x,u^0+u^-)\,dx + \int_\Omega\int_0^1 g(x,u^0+u^-+su^+)u^+\,ds\,dx,$$

we obtain for u with $\|u^+\| \leq R_1$

$$\begin{aligned}\Phi(u) &\leq C_0\|u^+\|^2 - \frac{a^-}{2}\|u^-\|^2 - \int_\Omega G(x,u^0+u^-)\,dx + MC\|u^+\| \\ &\leq C_0R_1^2 + MCR_1 - \frac{a^-}{2}\|u^-\|^2 - \int_\Omega G(x,u^0+u^-)\,dx\end{aligned}$$

where $C_0 := \sup_{u\in X^+,\|u\|=1}\int_\Omega |\nabla u|^2 - b|u|^2\,dx$. Therefore, by Lemma 12, Φ satisfies that $\Phi(u) \to -\infty$ as $\|u^0 + u^-\| \to \infty$. And so (iii) of (Φ4) holds. ∎

Lemma 17 *If* h *satisfies* (h1) *and* (h2), *then the following assertions* (1) *and* (2) *hold.*

(1) *if* (a1) *of* (C2) *holds, then* Φ *satisfies* (Φ4) *with* $V_\infty = X^+$ *and* $W_\infty = X^- \oplus X^0$;

(2) *if* (a2) *of* (C2) *holds, then* $-\Phi$ *satisfies* (Φ4) *with* $V_\infty = X^-$ *and* $W_\infty = X^+ \oplus X^0$.

Proof. We treat only the case of (1), since we can similarly show that $-\Phi$ satisfies (Φ4) in the case of (2) using the finite dimension condition of X^0. So we assume that the condition (a1) of (C2) holds and shall show that Φ satisfies (Φ4) with $\delta = 1$, $V_\infty = X^+$ and $W_\infty = X^- \oplus X^0$. By the assumption on α and β, we can choose λ, p and q satisfying $\max\{1/2,\beta\} < \lambda < (\alpha+1)/2$, $\max\{2, 1/\beta\} < p < 2\lambda/\beta$ and $1/p + 1/q = 1$.

We write $u = u^+ + u^0 + u^-$ where $u^\pm \in X^\pm$, $u^0 \in X^0$ and let C, C_i $(i \in \mathbb{N})$ be suitable positive constants independent of $u \in E$ and $x \in \Omega$.

With the aid of Hölder's inequality, Young's inequality and the Sobolev's embedding theorem, we have

$$\int_\Omega |u|^\beta |v|\,dx \le C_1\|u\|^{p\beta} + C_2\|v\|^q \quad \text{for any } u, v \in E. \tag{3.13}$$

Combining (3.13) with the equality

$$\int_\Omega G(x,u)\,dx = \int_\Omega G(x,u^0+u^-)\,dx + \int_\Omega \int_0^1 g(x,u^0+u^-+su^+)u^+\,ds\,dx,$$

Sobolev inequality and (a1) of (C2), we obtain

$$\begin{aligned}
&\left|\int_\Omega \int_0^1 g(x,u^0+u^-+su^+)u^+\,ds\,dx\right| \\
&\le C\|u^+\|^{\beta+1} + C\|u^+\| + C\int_\Omega |u^0+u^-|^\beta |u^+|\,dx \\
&\le C\|u^+\|^{\beta+1} + C\|u^+\| + C_1'\|u^0+u^-\|^{p\beta} + C_2'\|u^+\|^q.
\end{aligned}$$

Since all norms on $X^0 \oplus X^-$ are mutually equivalent, the condition (a1) of (C2) yields

$$\int_\Omega G(x,u^0+u^-)\,dx \ge C_3\|u^0+u^-\|^{\alpha+1} - C_4\|u^0+u^-\|.$$

Therefor for u with $\|u^+\| \le \|u^0+u^-\|^\lambda$ we have

$$\begin{aligned}
\Phi(u) \le\ & \frac{C_0}{2}\|u^+\|^2 - \frac{a^-}{2}\|u^-\|^2 - \int_\Omega G(x,u^0+u^-)\,dx + C\|u^+\|^{\beta+1} + C\|u^+\| \\
& + C_1'\|u^0+u^-\|^{p\beta} + C_2'\|u^+\|^q \\
\le\ & \frac{C_0}{2}\|u^+\|^2 - \frac{a^-}{2}\|u^-\|^2 - C_3\|u^0+u^-\|^{\alpha+1} + C_4\|u^0+u^-\| \\
& + C\|u^+\|^{\beta+1} + C\|u^+\| + C_1'\|u^0+u^-\|^{p\beta} + C_2'\|u^+\|^q \\
\le\ & \frac{C_0}{2}\|u^0+u^-\|^{2\lambda} - \frac{a^-}{2}\|u^-\|^2 - C_3\|u^0+u^-\|^{\alpha+1} + C_4\|u^0+u^-\| \\
& + C\|u^0+u^-\|^{\lambda(\beta+1)} + C\|u^0+u^-\|^\lambda + C_1'\|u^0+u^-\|^{p\beta} \\
& \quad + C_2'\|u^0+u^-\|^{\lambda q}
\end{aligned}$$

where $C_0 := \sup_{u\in X^+,\|u\|=1}\int_\Omega |\nabla u|^2 - b|u|^2\,dx$. Now because of $\lambda(\beta+1) < 2\lambda$, $p\beta < 2\lambda < \lambda q < 2\lambda$ and $2\lambda < \alpha+1$, it implies that $\Phi(u) \to -\infty$ as

$\|u^0+u^-\| \to \infty$ with $\|u^+\| \le \|u^0+u^-\|^\lambda$. Therefore the condition (iii) of (Φ4) holds.

Next by the inequality (3.13), we have

$$\left|\int_\Omega g(x,u)u^+\,dx\right| \le C\|u^+\|^{\beta+1} + C\|u^+\| + C\int_\Omega |u^0+u^-|^\beta |u^+|\,dx$$
$$\le C\|u^+\|^{\beta+1} + C\|u^+\| + C_1'\|u^0+u^-\|^{p\beta} + C_2'\|u^+\|^q. \tag{3.14}$$

Therefore we obtain for u with $\|u^0+u^-\|^\lambda \le \|u^+\|$

$$\langle \nabla\Phi(u), u^+\rangle$$
$$\ge a^+\|u^+\|^2 - C\|u^+\|^{\beta+1} - C_1'\|u^0+u^-\|^{p\beta} - C\|u^+\| - C_2'\|u^+\|^q \tag{3.15}$$
$$\ge a^+\|u^+\|^2 - C\|u^+\|^{\beta+1} - C\|u^+\| - C_1'\|u^+\|^{p\beta/\lambda} - C_2'\|u^+\|^q.$$

Because of $p\beta < 2\lambda$, there exists an $R_2 > 0$ such that $\langle\nabla\Phi(u), u^+\rangle > 0$ provided $\|u^+\| \ge R_2$, $\|u^0+u^-\|^\lambda \le \|u^+\|$.

Similarly using (3.13) we have

$$\left|\int_\Omega g(x,u)(u^0+u^-)\,dx\right|$$
$$\le C\|u^0+u^-\|^{\beta+1} + C\|u^0+u^-\| + C\int_\Omega |u^+|^\beta|u^0+u^-|\,dx$$
$$\le C\|u^0+u^-\|^{\beta+1} + C\|u^0+u^-\| + C_1'\|u^+\|^{p\beta} + C_2'\|u^0+u^-\|^q,$$

and combining with (3.14) and (3.15), we obtain for u with $\|u^0+u^-\|^\lambda = \|u^+\|$

$$\left\langle \nabla\Phi(u)\,,\, u^+ - \lambda\frac{u^0+u^-}{\|u^0+u^-\|^{2-2\lambda}}\right\rangle_E$$
$$\ge a^+\|u^+\|^2 - \left|\int_\Omega g(x,u)\,u^+\,dx\right| - \frac{\lambda}{\|u^0+u^-\|^{2-2\lambda}}\left|\int_\Omega g(x,u)(u^0+u^-)\,dx\right|$$
$$\ge a^+\|u^+\|^2 - C\|u^+\|^{\beta+1} - C\|u^+\| - C_1'\|u^0+u^-\|^{p\beta} - C_2'\|u^+\|^q$$
$$-\frac{\lambda}{\|u^0+u^-\|^{2-2\lambda}}\left\{C\|u^0+u^-\|^{\beta+1} + C\|u^0+u^-\| + C_1'\|u^+\|^{p\beta}\right.$$
$$\left.+C_2'\|u^0+u^-\|^q\right\}$$
$$\ge a^+\|u^0+u^-\|^{2\lambda} - C\|u^0+u^-\|^{\lambda(\beta+1)} + C\|u^0+u^-\|^\lambda - C_1'\|u^0+u^-\|^{p\beta}$$
$$-C_2'\|u^0+u^-\|^{\lambda q} - \lambda\left\{C\|u^0+u^-\|^{\beta+2\lambda-1} + C\|u^0+u^-\|^{2\lambda-1}\right\}$$
$$-\lambda\left\{C_1'\|u^0+u^-\|^{p\beta-(2-2\lambda)} + C_2'\|u^0+u^-\|^{q-(2-2\lambda)}\right\}.$$

Therefore there exists an $R_3 > 0$ such that

$$\left\langle \nabla\Phi(u)\,,\, u^+ - \lambda \frac{u^0 + u^-}{\|u^0 + u^-\|^{2-2\lambda}} \right\rangle > 0$$

provided $\|u^0 + u^-\|^\lambda = \|u^+\|$, $\|u^+\| \geq R_3$ since $2\lambda > \max\{\lambda(\beta+1), p\beta, \lambda q, \beta + 2\lambda - 1, p\beta - (2-2\lambda), q - (2-2\lambda)\}$. Therefore, if we set $R_1 := \max\{R_2, R_3\}$, then the conditions (i) and (ii) of ($\Phi 4$) are satisfied. ∎

Lemma 18 *Let h satisfy* (h1) *and* (h2) *and suppose that one of the following conditions holds*:

(1) $b_0 \in \sigma(-\triangle)$ *and* (C3) *holds*;

(2) $b_0 \in \sigma(-\triangle)$ *and* (C4) *holds*;

(3) $b_0 \notin \sigma(-\triangle)$.

Then Φ has a local linking at 0 *w.r.t. the following decomposition $E = V_0 \oplus W_0$ in each of the cases*:

$$\begin{cases} V_0 = X_0^+,\ W_0 = X_0^0 \oplus X_0^- & \text{in the case of (1)}, \\ V_0 = X_0^0 \oplus X_0^+,\ W_0 = X_0^- & \text{in the case of (2)}, \\ V_0 = X_0^+,\ W_0 = X_0^- & \text{in the case of (3)}. \end{cases} \tag{3.16}$$

Proof. We fix $2 < p \leq 2N/(N-2)$. By the assumption (h2), for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$|G_0(x,\xi)| \leq \varepsilon\,|\xi|^2 + C_\varepsilon |\xi|^p, \tag{3.17}$$

Note that E is continuously embedding in $L^p(\Omega)$. This readily yields the proof in cases of (1) and (3). Indeed, we can obtain for $u^\pm \in X_0^\pm$

$$\begin{aligned} \pm\Phi(u^\pm) &\geq \frac{a_0^\pm}{2}\|u^\pm\|^2 \mp \int_\Omega G_0(x,u^\pm)\,dx \\ &\geq \frac{a_0^\pm}{2}\|u^\pm\|^2 - \varepsilon C\|u^\pm\|^2 - C_\varepsilon C\|u^\pm\|^p \end{aligned} \tag{3.18}$$

for some constant $C > 0$ independent of $u^\pm$. Therefore $\pm\Phi(u^\pm) \geq 0$ for $\|u^\pm\|$ small enough. Furthermore in the case of (1), by $\dim(X_0^0 \oplus X_0^-) < \infty$, there exists some constant $M > 0$ such that $\|u^0 + u^-\|_\infty \leq M\|u^0 + u^-\|$. Hence if $\|u^0 + u^-\| \leq \delta/M$ where δ is a constant satisfying the condition $(C3)$, then $\|u^0 + u^-\|_\infty \leq \delta$ and $\int_\Omega G_0(x, u^0 + u^-)\,dx \geq 0$. Therefore it remains to prove the case of (2).

Using the inequality (3.18), if we choose sufficiently small $\varepsilon > 0$, there exists an $r > 0$ such that

$$\Phi(u) \leq 0 \quad \text{if } u \in W_0,\ \|u\| \leq r.$$

Next for every $u \in V_0$, we write $u = u^+ + u^0$ where $u^+ \in X_0^+$ and $u^0 \in X_0^0$. Since X_0^0 is a finite-dimensional space, there exists some constant $M > 0$ such that $\|u^0\|_\infty \leq M\|u^0\|$. Let $u \in V_0$ be such that $\|u\| \leq \delta/2M$ where δ is a constant satisfying (C4) and set $\Omega_1 := \{x \in \Omega\,;\, |u^+(x)| \leq \delta/2\}$, $\Omega_2 := \Omega \setminus \Omega_1$. On Ω_1, we have $|u(x)| \leq |u^+| + |u^0| \leq \delta$ since $\|u^0\|_\infty \leq M\|u^0\| \leq M\|u\| < 2/\delta$. Hence the assumption (C4) yields

$$\int_{\Omega_1} G_0(x, u)\, dx \leq 0.$$

On the other hand, on Ω_2, we have $|u(x)| \leq 2|u^+(x)|$ and

$$\int_{\Omega_2} G_0(x, u)\, dx \leq 4\varepsilon\|u^+\|_2^2 + 2^p C_\varepsilon \|u^+\|_p^p$$

by the inequality (3.17). Therefore for every u with $u \in V_0, \|u\| \leq \delta/2M$ we have

$$\begin{aligned}\Phi(u) &\geq \frac{a_0^+}{2}\|u^+\|^2 - 4\varepsilon\|u^+\|_2^2 - 2^p C_\varepsilon\|u^+\|_p^p - \int_{\Omega_1} G_0(x, u)\, dx \\ &\geq \frac{a_0^+}{2}\|u^+\|^2 - \varepsilon C_3\|u^+\|^2 - C'_\varepsilon\|u^+\|^p.\end{aligned}$$

Therefore, if we fix $\varepsilon > 0$ sufficiently small, then there exists some $0 < r' \leq \delta/2M$ such that

$$\Phi(u) \geq 0 \quad \text{if } u \in V_0,\ \|u\| \leq r'.$$

∎

Proof of Theorem 9. We show that we can apply Theorem 7 to either Φ defined by (3.1) or $-\Phi$ and obtain a non-trivial critical point of Φ, which yields a non-trivial weak solution to (P). Therefore we shall show that Φ or $-\Phi$ satisfy the assumptions of Theorem 7 in each of the cases stated Theorem 9. We define $E_n := \text{lin.sp.}\{e_1, \cdots, e_n\}$, and we note that E_n satisfies compatibility condition w.r.t. $V_0 \oplus W_0$ and $V_\infty \oplus W_\infty$ which are stated below.

(i) Condition (Φ1) ($(PS)_c^*$ condition):
If one of the conditions (A1) to (A4) holds, then the one of the assumptions $b \notin \sigma(-\triangle)$, (C1) and (C2) is satisfied. Therefore by Lemma 14, Φ satisfies $(PS)_c^*$ condition for every $c \in \mathbb{R}$. This yields that $-\Phi$ also satisfies $(PS)_c^*$ for every $c \in \mathbb{R}$.

(ii) Condition ($\Phi 2$):
The nonlinear term h satisfies $|h(x,\xi)| \le C|\xi|$ for every $\xi \in \mathbb{R}$. This yields that $|\int_\Omega H(x,u)\,dx| \le C\|u\|^2$ for all $u \in E$. Hence Φ and $-\Phi$ satisfy the condition ($\Phi 2$).

(iii) Condition ($\Phi 3$):
If one of the conditions (A1) to (A4) holds, then one of the assumptions (i) to (iii) in Lemma 18 is satisfied, hence Φ satisfies ($\Phi 3$) w.r.t. (V_0, W_0) as in (3.16). $-\Phi$ satisfies w.r.t. (V_0, W_0) replaced V_0 by W_0 in (3.16).

(iv) Condition ($\Phi 4$):
We note that if one of the conditions (A1) to (A4) holds, then one of the assumptions $b \notin \sigma(-\triangle)$, (C1), (a1), (a2) of (C2) is satisfied. If $b \notin \sigma(-\triangle)$ holds, then Φ satisfies ($\Phi 4$) with $V_\infty = X^+$ and $W_\infty = X^-$, and $-\Phi$ satisfies ($\Phi 4$) with $V_\infty = X^-$ and $W_\infty = X^+$ by Lemma 15. If (C1) holds, then ($\Phi 4$) is satisfied with $V_\infty = X^+$ and $W_\infty = X^- \oplus X^0$ by Lemma 16. If (a1) of (C2) holds, then Φ satisfies ($\Phi 4$) with $V_\infty = X^+$ and $W_\infty = X^- \oplus X^0$ by Lemma 17. If (a2) of (C2) holds, then $-\Phi$ satisfies ($\Phi 4$) with $V_\infty = X^-$ and $W_\infty = X^+ \oplus X^0$ by Lemma 17.

(v) dimension condition (2.4):
The following Claim is checked easily, where $E(\lambda) := \ker(-\triangle - \lambda)$ for $\lambda \in \sigma(-\triangle)$.

Claim *The following inclusions hold.*

(1) if $b_0 \notin \sigma(-\triangle)$ and $b \notin [b_0^-, b_0^+)$, then

$$X^- \oplus X^0 \supset X_0^- \oplus E(b_0^+) \quad (\text{if } b \ge b_0^+),$$
$$X^- \oplus X^0 \oplus E(b_0^-) \subset X_0^- \quad (\text{if } b < b_0^-).$$

(2) if $b_0 \in \sigma(-\triangle)$ and $b \notin [b_0^-, b_0)$, then

$$X^- \oplus X^0 \supset X_0^- \oplus E(b_0) \quad (\text{if } b \ge b_0),$$
$$X^- \oplus X^0 \oplus E(b_0^-) \subset X_0^- \quad (\text{if } b < b_0^-).$$

(3) if $b_0 \in \sigma(-\triangle)$ and $b \notin [b_0, b_0^+)$, then

$$X^- \oplus X^0 \supset X_0^- \oplus X_0^0 \oplus E(b_0^+) \quad (\text{if } b \ge b_0^+),$$
$$X^- \oplus X^0 \oplus E(b_0) \subset X_0^- \oplus X_0^0 \quad (\text{if } b < b_0).$$

(4) We note that if $b = b_0 \in \sigma(-\triangle)$, then we have

$$X^- \oplus E(b_0) = X_0^- \oplus X_0^0,$$

and if $b = b_0^-$, then we have

$$X^- \oplus E(b_0^-) = X_0^-.$$

Using this Claim, we shall deal with only the case (1) and (2) of (A4) since the other cases would be similarly handled. First we treat the case (1) of (A4). Then Φ has a local linking at 0 w.r.t. $(V_0, W_0) = (X_0^+, X_0^0 \oplus X_0^-)$ and satisfies $(\Phi 4)$ w.r.t. $(V_\infty, W_\infty) = (X^+, X^0 \oplus X^-)$. Since we are assuming $b, b_0 \in \sigma(-\triangle)$ and $b_0 < b$, we have $b_0 < b_0^+ \leq b$. Therefore by the case (3) in the Claim, we obtain for large n

$$E_n \cap \left(X_0^0 \oplus X_0^-\right) \subsetneq E_n \cap \left(X_0^0 \oplus X_0^- \oplus E(b_0^+)\right) \subset E_n \cap \left(X^0 \oplus X^-\right).$$

Hence

$$\liminf_n \{\dim(W_\infty \cap E_n) - \dim(W_0 \cap E_n)\} \geq \dim E(b_0^+) > 0.$$

Finally we show the case (2) of (A4) with $-\Phi$. Then $-\Phi$ has a local linking at 0 w.r.t. $(V_0, W_0) = (X_0^0 \oplus X_0^-, X_0^+)$ and satisfies $(\Phi 4)$ w.r.t. $(V_\infty, W_\infty) = (X^-, X^0 \oplus X^+)$. Using the Claim, we similarly obtain for large n

$$E_n \cap X^- \subsetneq E_n \cap \left(X^- \oplus E(b_0)\right) \subset E_n \cap \left(X_0^0 \oplus X_0^-\right).$$

Hence

$$\liminf_n \{\dim(V_0 \cap E_n) - \dim(V_\infty \cap E_n)\} \geq \dim E(b_0) > 0.$$

Therefore

$$\limsup_n \{\dim(W_\infty \cap E_n) - \dim(W_0 \cap E_n)\} > 0.$$

∎

Example 19 *The following $g(x,\xi)$ satisfies our assumption* (C2):

$$g(x,\xi) = a(x,\xi)|\xi|^\beta \operatorname{sgn}\xi + b(x,\xi)|\xi|^\alpha \operatorname{sgn}\xi,$$

where $a(x,\xi)$ and $b(x,\xi)$ are some suitable bounded functions and α, β are constants satisfying $0 < \alpha \leq \beta < 1$ and $2\beta < \alpha + 1$.

(i) *If $a(x,\xi) = c + \sin\xi$ with $1 < c < (1+\beta)/(1-\beta)$ and $b(x,\xi) = 0$, then g does not satisfy either of the condition* (1.1) *or Silva's because of $\liminf_{|\xi|\to\infty} \pm\{G(x,\xi) - g(x,\xi)\xi/2\} = -\infty$ and $\liminf_{|\xi|\to\infty} \pm g_\xi(x,\xi) = -\infty$, but our assumption* (a1) *of* (C2) *is satisfied. Indeed, we obtain the following inequality for $\xi > 0$*

$$G(x,\xi) - \frac{1}{2}g(x,\xi)\xi \leq \xi^{\beta+1}\left\{\frac{c}{\beta+1} - \frac{c}{2} - \frac{1}{2}\sin\xi - \frac{1}{\xi}\cos\xi + \frac{1}{\xi}\right\}.$$

Therefore if we put $\xi_n := 2n\pi + \pi/2$, then we have $\lim_{n\to\infty} G(x,\xi_n) - g(x,\xi_n)\xi_n/2 = -\infty$. Similarly we can check the other assumptions .

(ii) *If* $a(x,\xi) = \sin\xi$ *and* $b(x,\xi)$ *is a constant, then the condition* (1.2) *of Zou and Liu cannot be satisfied since* $\liminf_{|\xi|\to\infty} \pm G(x,\xi)/|\xi|^{\beta+1} \le 0$, *but our assumption* (C2) *is satisfied.*

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