SUT Journal of Mathematics
Vol. 38, No. 1 (2002), 75–82

Totally contact-umbilical semi-invariant submanifolds of a Sasakian manifold

S. H. Kon and Tee-How Loo

(Received November 13, 2001)

Abstract. This paper gives a characterization of totally contact-umbilical semi-invariant submanifolds of a Sasakian manifold.

AMS 1991 *Mathematics Subject Classification.* Primary 53B20; Secondary 53B25, 53C25.

Key words and phrases. Sasakian manifold, generalized Hopf manifold, totally contact-umbilical semi-invariant submanifold.

§1. Introduction

Bejancu [1] introduced the notion of CR-submanifolds and begin the study of CR-submanifolds of a Kaehler manifold. In particular the geometry of totally umbilical CR-submanifolds of a Kaehler manifold has been studied by many differential geometers. Bejancu [3] and Chen [6] independently classified a totally umbilical CR-submanifold M of a Kaehler manifold and showed that either (i) M is totally geodesic; or (ii) M is anti-invariant; or (iii) the anti-invariant distribution $D^{\perp}$ is of dimension 1. Further, Toyonari and Nemoto [8] characterized totally umbilical CR-submanifolds of a Kaehler manifold, which occurs in the third case ($\dim D^{\perp} = 1$), i.e., they proved the following

Theorem 1.1. *Let M be a connected non-totally geodesic, totally umbilical proper m-dimensional CR-submanifold in a Kaehler manifold,* $(m > 4)$. *Then it is homothetic to a Sasakian manifold.*

Motivated by this, we obtain a characterization of totally contact-umbilical semi-invariant submanifolds of a Sasakian manifold (cf. Theorem 4.2).

§2. Preliminaries

Let N be a $(2n+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g). Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.2}$$

for any vector fields X and Y tangent to N. We denote by $\overline{\nabla}$ the Levi-Civita connection on N and $\overline{R}$ the curvature tensor corresponding to $\overline{\nabla}$. Then we have [11]

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad \overline{\nabla}_X \xi = -\phi X, \tag{2.3}$$

$$\begin{aligned} \overline{R}(X, Y)\phi Z \;&=\; \phi\overline{R}(X, Y)Z + g(\phi X, Z)Y - g(Y, Z)\phi X \\ &\quad + g(X, Z)\phi Y - g(\phi Y, Z)X, \end{aligned} \tag{2.4}$$

$$\begin{aligned} g(\overline{R}(\phi X, \phi Y)\phi Z, \phi W) &= g(\overline{R}(X, Y)Z, W) - \eta(Y)\eta(Z)g(X, W) \\ &\quad -\eta(X)\eta(W)g(Y, Z) + \eta(Y)\eta(W)g(X, Z) + \eta(X)\eta(Z)g(Y, W), \end{aligned} \tag{2.5}$$

$$\overline{R}(X, \xi)Y = -(\overline{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{2.6}$$

for any vector fields X, Y, Z and W tangent to N.

An m-dimensional submanifold M of N is said to be a *semi-invariant submanifold* if there exists a pair of orthogonal distributions $(D, D^\perp)$ satisfying the conditions [5]

(i) $TM = D \bigoplus D^\perp \bigoplus \{\xi\}$;

(ii) the distribution D is invariant by ϕ, i.e., $\phi(D_x) = D_x, x \in M$;

(iii) the distribution $D^\perp$ is anti-invariant, i.e., $\phi(D_x^\perp) \subset T_x M^\perp, x \in M$

where TM and $TM^\perp$ denote the tangent bundle and normal bundle to M respectively. It follows that the normal bundle splits as $TM^\perp = \phi D^\perp \bigoplus \nu$, where ν is an invariant sub-bundle of $TM^\perp$ by ϕ. If $D = \{0\}$ (resp. $D^\perp = \{0\}$) then M is said to be an *anti-invariant* (resp. *invariant*) submanifold. We say that M is *proper* if it is neither invariant nor anti-invariant.

For any vector bundle S over M we denote by $\Gamma(S)$ the module of all differentiable sections on S. Let ∇ be the induced Levi-Civita connection on M and $\nabla^\perp$ the induced normal connection on $TM^\perp$. Then the Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\overline{\nabla}_X\zeta = -A_\zeta X + \nabla^\perp_X\zeta$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where h is the second fundamental form of M and the shape opertor A_ζ is related to h by

$$g(A_\zeta X, Y) = g(h(X,Y),\zeta).$$

The projection morphism of TM on D and $D^\perp$ are denoted by P and Q respectively. For $\zeta \in \Gamma(TM^\perp)$ we denote by $t\zeta$ the tangential part and $f\zeta$ the normal part of $\phi\zeta$ respectively. Also, we put $\psi = \phi \circ P$ and $\omega = \phi \circ Q$. Then we have [2]

$$(\nabla_X\psi)Y = th(X,Y) + A_{\omega Y}X + g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$(\nabla_X\omega)Y = fh(X,Y) - h(X,\psi Y), \tag{2.8}$$

$$(\nabla_X f)\zeta = -h(X,t\zeta) - \omega A_\zeta X, \tag{2.9}$$

$$h(X,\xi) = -\omega X, \quad \nabla_X\xi = -\psi X \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$.

Now we recall the definition of a locally conformal Kaehler manifold. Let M be a Hermitian manifold with complex structure J. Then M is called a *locally conformal Kaehler manifold* if there exists a closed 1-form τ, called the *Lee form*, on M such that

$$d\Omega = \tau \wedge \Omega$$

or equivalently,

$$(\nabla_X J)Y = \frac{1}{2}\{\theta(Y)X - \tau(Y)JX - \Omega(X,Y)B - g(X,Y)A)\} \tag{2.11}$$

for $X, Y \in \Gamma(TM)$, where $\Omega(X,Y) = g(X,JY)$, B is the *Lee vector field* such that $g(B,X) = \tau(X), \theta = \tau \circ J$ is the *anti-Lee 1-form* and $A = -JB$ is the *anti-Lee vector field*. Moreover, a *generalized Hopf manifold* is a locally conformal Kaehler manifold whose Lee form is parallel, i.e., $\nabla\tau = 0$ (cf. [9]).

§3. Geometry of Totally Contact-umbilical Semi-invariant Submanifolds

A submanifold M is said to be *totally umbilical* if $h(X,Y) = g(X,Y)\overline{H}$, for all $X, Y \in \Gamma(TM)$, where $\overline{H} = \frac{1}{m}$(trace of h), is the mean curvature vector of M. If the mean curvature vector $\overline{H} = 0$ then M is called a *totally geodesic submanifold*.

Now, it follows from (2.10) that a Sasakian manifold N does not admit any non-totally geodesic, totally umbilical semi-invariant submanifold (cf. [10, p.47, Proposition 1.2]). From this point of view, Bejancu [4] considered the concept of totally contact-umbilical semi-invariant submanifolds. The notion of totally contact-umbilical submanifold was first defined by Kon [7].

A semi-invariant submanifold M is said to be *totally contact-umbilical* if

$$
\begin{aligned}
h(X,Y) &= g(\phi X,\phi Y)H+\eta(Y)h(X,\xi)+\eta(X)h(Y,\xi) \\
&= \{g(X,Y)-\eta(X)\eta(Y)\}H-\eta(Y)\omega X-\eta(X)\omega Y
\end{aligned} \tag{3.1}
$$

or equivalently,

$$A_\zeta X=g(H,\zeta)X-\{\eta(X)g(H,\zeta)+g(\omega X,\zeta)\}\xi+\eta(X)t\zeta \tag{3.2}$$

for any $X,Y\in\Gamma(TM)$ and $\zeta\in\Gamma(TM^\perp)$, where H is a normal vector field on M. If $H\equiv 0$ then M is called a *totally contact-geodesic submanifold*. Bejancu [4] has shown the following

Theorem 3.1. *Any totally contact-umbilical proper semi-invariant submanifold of a Sasakian manifold N with $\dim D^\perp>1$ is a totally contact-geodesic submanifold.*

In the rest of this section, suppose M, ($\dim M>4$), is a connected non-totally contact-geodesic, totally contact-umbilical proper semi-invariant submanifold of a Sasakian manifold N. It follows from Theorem 3.1 that $\dim D^\perp=1$. We first state

Lemma 3.2. $H\in\Gamma(\phi D^\perp)$.

Proof. By putting $Y=X\in\Gamma(D)$ in (2.8) and taking account of (3.1) we obtain

$$-\omega\nabla_X X=g(X,X)fH.$$

Note that the left side and the right side of the above equation is respectively in $\Gamma(\phi D^\perp)$ and $\Gamma(\nu)$, hence $fH=0$ or $H\in\Gamma(\phi D^\perp)$.

Lemma 3.3. $\nabla^\perp_X H\in\Gamma(\phi D^\perp)$, *for any* $X\in\Gamma(TM)$.

Proof. By putting $\zeta=H$ in (2.9) and taking account of the fact that $fH=0$, we obtain

$$-f\nabla^\perp_X H=-h(X,tH)-\omega A_H X.$$

Note that the left side of this equation is in $\Gamma(\nu)$ while the right side is in $\Gamma(\phi D^\perp)$ by virtue of (3.1) and Lemma 3.2. It follows that $f\nabla^\perp_X H=0$ and so $\nabla^\perp_X H\in\Gamma(\phi D^\perp)$.

Lemma 3.4.

$$\begin{aligned}[\overline{R}(X,Y)W]^{\perp} = & \{g(Y,W)-\eta(Y)\eta(W)\}\nabla^{\perp}_X H \\ & -\{g(X,W)-\eta(X)\eta(W)\}\nabla^{\perp}_Y H \\ & -g(\psi Y,W)\omega X + g(\psi X,W)\omega Y + 2g(\psi X,Y)\omega W,\end{aligned}$$

for any $X, Y, W \in \Gamma(TM)$.

Proof. For any $X, Y, W \in \Gamma(TM)$, by using (2.8), (2.10) and (3.1) we obtain

$$\begin{aligned}(\nabla_X h)(Y,W) &= \{g(Y,W)-\eta(Y)\eta(W)\}\nabla^{\perp}_X H - \{(\nabla_X\eta)Y\cdot\eta(W) \\ &\quad +\eta(Y)(\nabla_X\eta)W\}H - (\nabla_X\eta)Y\cdot\omega W - \eta(Y)(\nabla_X\omega)W \\ &\quad -(\nabla_X\eta)W\cdot\omega Y - \eta(W)(\nabla_X\omega)Y \\ &= \{g(Y,W)-\eta(Y)\eta(W)\}\nabla^{\perp}_X H + \{g(Y,\psi X)\eta(W) \\ &\quad +\eta(Y)g(W,\psi X)\}H + g(Y,\psi X)\omega W - \eta(Y)\{fh(X,W) \\ &\quad -h(X,\psi W)\} + g(W,\psi X)\omega Y \\ &\quad -\eta(W)\{fh(X,Y)-h(X,\psi Y)\}.\end{aligned}$$

It follows from (3.1) and Lemma 3.2 that this equation reduces to

$$(\nabla_X h)(Y,W) = \{g(Y,W)-\eta(Y)\eta(W)\}\nabla^{\perp}_X H + g(Y,\psi X)\omega W + g(W,\psi X)\omega Y.$$

Exchanging X and Y in the above equation, we have

$$(\nabla_Y h)(X,W) = \{g(X,W)-\eta(X)\eta(W)\}\nabla^{\perp}_Y H + g(X,\psi Y)\omega W + g(W,\psi Y)\omega X.$$

From these equations and the Codazzi equation we obtain the Lemma.

Since M is non-totally contact-geodesic, we may choose a connected open set G on M such that H is nowhere zero on G. For the moment, we restrict our arguments on such an open set G. Define a unit vector field Z in $D^{\perp}$ by $Z = -\frac{1}{\mu}\phi H$, where $\mu = \| H \|$. Then we have the following

Lemma 3.5. $\nabla_X Z = \mu\psi X$, *for any* $X \in \Gamma(TM)$.

Proof. For any $X \in \Gamma(TM)$, we have

$$g(\nabla_X Z, Z) = 0 \quad \text{and} \quad g(\nabla_X Z, \xi) = -g(Z, \nabla_X \xi) = g(Z, \psi X) = 0.$$

Next, by using (2.7) we obtain

$$-\psi\nabla_X Z = th(X,Z) + A_{\omega Z}X + g(X,Z)\xi.$$

By applying ψ to this equation and taking account of (3.2) we get

$$\nabla_X Z = \psi A_{\omega Z}X = g(H,\omega Z)\psi X = \mu\psi X.$$

Remark. Lemma 3.2 to Lemma 3.5 also hold when $\dim M = 4$.

Lemma 3.6. *The normal vector field H is parallel.*

Proof. Let $Y \in \Gamma(D)$ be a unit vector field. Then from (2.6) and Lemma 3.4

$$\nabla^{\perp}_{\xi} H = [\overline{R}(\xi, Y)Y]^{\perp} = 0.$$

Now, consider a unit vector field $X \in \Gamma(D)$ with $g(X, Y) = g(X, \psi Y) = 0$. Then by (2.4) we have

$$\overline{R}(\phi Z, X)\phi^2 X = \phi \overline{R}(\phi Z, X)\phi X - \phi Z.$$

By taking inner product with Y we get

$$g(\overline{R}(\phi Z, X)X, Y) = g(\overline{R}(\phi Z, X)\phi X, \phi Y)$$

or

$$g(\overline{R}(Y, X)X, \phi Z) = g(\overline{R}(\phi Y, \phi X)X, \phi Z).$$

Together with Lemma 3.4, we obtain

$$g(\nabla^{\perp}_{Y} H, \phi Z) = 0.$$

Next, by making use of (2.5) we obtain

$$g(\overline{R}(Z, Y)Y, \phi Z) = g(\overline{R}(\phi Z, \phi Y)\phi Y, \phi^2 Z) = -g(\overline{R}(\phi Z, \phi Y)\phi Y, Z).$$

On the other hand, it follows from Lemma 3.4 that we obtain

$$g(\overline{R}(Z, Y)Y, \phi Z) = g(\overline{R}(\phi Z, \phi Y)\phi Y, Z) = g(\nabla^{\perp}_{Z} H, \phi Z).$$

These two equations imply that $g(\nabla^{\perp}_{Z} H, \phi Z) = 0$. All this amount to say that $\nabla^{\perp}_{X} H \in \Gamma(\nu)$, for all $X \in \Gamma(TM)$. Together with Lemma 3.3, we obtain that H is parallel.

It follows from Lemma 3.6 that μ is a constant on G. Since M is connected, μ is a nonzero constant on M. Hence we have

Lemma 3.7. *Z is a unit vector field defined on the whole of M.*

§4. Characterization of Totally Contact-umbilical Semi-invariant Submanifolds

We first prove

Theorem 4.1. *Let M be a connected proper, non-totally contact-geodesic, totally contact-umbilical m-dimensional semi-invariant submanifold of a Sasakian manifold N, $(m > 4)$. Then it is a generalized Hopf manifold.*

Proof. From our assumption and Theorem 3.1, we can see that $\dim D^{\perp} = 1$. Hence, for any $X \in \Gamma(TM)$, we may put

$$X = PX + \alpha(X)Z + \eta(X)\xi = -\psi^2 X + \alpha(X)Z + \eta(X)\xi$$

where $\alpha(X) = g(X, Z)$. Now we define a tensor field J of type (1,1) on M by

$$JX = \psi X + \alpha(X)\xi - \eta(X)Z. \tag{4.1}$$

It is clear that J is an almost complex structure on M. Furthermore, we define a vector field B and a 1-form τ on M by

$$B = 2(\mu\xi + Z), \quad \tau(X) = g(B, X) = 2(\alpha(X) + \mu\eta(X)) \tag{4.2}$$

for any $X \in \Gamma(TM)$.

It follows from (2.10), (4.2) and Lemma 3.5 that, we have $(\nabla_X \tau)Y = 0$, for any $X, Y \in \Gamma(TM)$. Hence, τ is parallel (and so is closed).

Finally, we shall show that (2.11) holds. For any $X, Y \in \Gamma(TM)$, it follows from (2.7), (2.10), (4.1) and Lemma 3.5 that

$$\begin{aligned}(\nabla_X J)Y &= (\nabla_X \psi)Y + (\nabla_X \alpha)Y \cdot \xi + \alpha(Y)\nabla_X \xi - (\nabla_X \eta)Y \cdot Z - \eta(Y)\nabla_X Z \\ &= th(X, Y) + \alpha(Y)A_{\omega Z}X + g(X, Y)\xi - \eta(Y)X + \mu g(\psi X, Y)\xi \\ &\quad -\alpha(Y)\psi X + g(\psi X, Y)Z - \mu\eta(Y)\psi X.\end{aligned}$$

Now, from (3.1) and (3.2) the above equation becomes

$$\begin{aligned}(\nabla_X J)Y &= -\{g(X, Y) - \eta(X)\eta(Y)\}\mu Z + \eta(X)\alpha(Y)Z + \eta(Y)\alpha(X)Z \\ &\quad \alpha(Y)\{\mu X - \mu\eta(X)\xi - \eta(X)Z - \alpha(X)\xi\} + g(X, Y)\xi - \eta(Y)X \\ &\quad \mu g(\psi X, Y)\xi - \alpha(Y)\psi X + g(\psi X, Y)Z - \mu\eta(Y)\psi X.\end{aligned}$$

This, together with (4.1) and (4.2) give

$$\begin{aligned}(\nabla_X J)Y &= \frac{1}{2}\{g(X, Y)JB - g(JB, Y)X + g(JX, Y)B - g(B, Y)JX\} \\ &= \frac{1}{2}\{g(X, Y)JB - g(X, JY)B + \tau(JY)X - \tau(Y)JX\}.\end{aligned}$$

This completes the proof of the Theorem.

As an immediate consequence of Theorem 3.1 and Theorem 4.1, we obtain the following

Theorem 4.2. *Let M be a connected totally contact-umbilical m-dimensional semi-invariant submanifold of a Sasakian manifold N, $(m > 4)$. Then either*

(i) *M is totally contact-geodesic; or*

(ii) *M is anti-invariant; or*

(iii) *M is a generalized Hopf manifold.*

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S. H. Kon
Institute of Mathematical Science, University of Malaya
50603 Kuala Lumpur, Malaysia

Tee-How Loo
School of Arts and Science, Tunku Abdul Rahman College
P.O. Box 10979, 50932 Kuala Lumpur, Malaysia