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Weighted Three Point Identities and their Bounds

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Abstract. The weighted three point rule is investigated in the current article. It involves $f^{(n)}(t)$ being of bounded variation for $t \in [a, b]$. The rule consists of evaluations at the ends of the interval and at an interior point x. Weighted Ostrowski and Trapezoidal rules and their related bounds are recaptured as particular instances of the current development. The unweighted results of Ostrowski, Trapezoidal and three point rules are also procured if we take the weight to be unity.

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§1. Introduction

Cerone and Dragomir [5] obtained the following identity involving $n-$time differentiable functions with evaluation at an interior point and at the end points.

For $f : [a, b] \to \mathbb{R}$ a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ with $\alpha : [a, b] \to [a, b]$ and $\beta : [a, b] \to [a, b]$, $\alpha \le x \le \beta$, then for all $x \in [a, b]$ the following identity holds

$$(1.1)\quad (-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\, dt = \int_a^b f(t)\, dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x)\, f^{(k-1)}(x) + S_k(x)\right],$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) := \begin{cases} \frac{(t-\alpha(x))^n}{n!}, & t \in [a,x] \\ \frac{(t-\beta(x))^n}{n!}, & t \in (x,b], \end{cases} \tag{1.2}$$

(1.3)
$$\begin{cases} & R_k(x) = (\beta(x) - x)^k + (-1)^{k-1}(x - \alpha(x))^k \\ \text{and} & \\ & S_k(x) = (\alpha(x) - a)^k f^{(k-1)}(a) + (-1)^{k-1}(b - \beta(x))^k f^{(k-1)}(b). \end{cases}$$

They obtained inequalities for $f^{(n)} \in L_p[a,b]$, $p \geq 1$. In an earlier paper [2] the same authors treated the case $n = 1$ but also examined the results emanating from the Riemann-Stieltjes integral $\int_a^b K_1(x,t)\,df(t)$ and obtained bounds for f being of bounded variation, Lipschitzian or monotonic. Applications to numerical quadrature were investigated covering rules of Newton-Cotes type containing the evaluation of the function at three possible points: the interior and extremities. The development included the midpoint, trapezoidal and Simpson type rules. However, unlike the classical rules (see Atkinson [1]), the results were not as restrictive in that the bounds were derived in terms of the behaviour of at most the first derivative and the Peano kernel $K_1(x,t)$. Perturbed rules were also obtained using Grüss type inequalities. (For other particular instances of the work [5], see also [2] – [7]).

In 1938, Ostrowski (see for example [15, p. 468]) proved the following integral inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathring{I}$ ($\mathring{I}$ is the interior of I), and let $a, b \in \mathring{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.4}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

Fink [10] used the integral remainder from a Taylor series expansion to show that for $f^{(n-1)}$ absolutely continuous on $[a,b]$, then the identity

$$\int_a^b f(t)\,dt = \frac{1}{n}\left((b-a)f(x) + \sum_{k=1}^{n-1} F_k(x) \right) + \int_a^b K_F(x,t) f^{(n)}(t)\,dt \tag{1.5}$$

is shown to hold where

$$K_F(x,t) = \frac{(x-t)^{n-1}}{(n-1)!} \cdot \frac{p(x,t)}{n} \tag{1.6}$$

with $p(x,t)$ being given by

$$p(x,t) = \begin{cases} t-a, & t \in [a,x] \\ t-b, & t \in (x,b] \end{cases}$$

and

$$F_k(x) = \frac{n-k}{k!}\left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right].$$

Fink then proceeds to obtain a variety of bounds from (1.5), (1.6) for $f^{(n)} \in L_p[a,b]$. It may be noticed that (1.5) is again an identity that involves function evaluations at three points to approximate the integral from the resulting inequalities. See Mitrinović, Pečarić and Fink [15, Chapter XV] for further related results.

The following theorem was obtained in Cerone and Dragomir [5]

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and, let $\alpha : [a,b] \to [a,b]$ and $\beta : [a,b] \to [a,b]$, $\alpha \le x \le \beta$. Then the following inequalities hold for all $x \in [a,b]$*

$$\begin{aligned} &|P_n(x)| \\ :&= \left|\int_a^b f(t)\,dt - \sum_{k=1}^{n} \frac{1}{k!}\left[R_k(x) f^{(k-1)}(x) + S_k(x)\right]\right| \end{aligned} \tag{1.7}$$

$$\le \begin{cases} \frac{\left\|f^{(n)}\right\|_\infty}{n!} Q_n(1,x) & \text{if } f^{(n)} \in L_\infty[a,b], \\ \frac{\left\|f^{(n)}\right\|_p}{n!} \left[Q_n(q,x)\right]^{\frac{1}{q}} & \text{if } f^{(n)} \in L_p[a,b] \\ & \text{with } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{\left\|f^{(n)}\right\|_1}{n!} M^n(x), & \text{if } f^{(n)} \in L_1[a,b], \end{cases}$$

where

$$\begin{aligned} Q_n(q,x) &= \frac{1}{nq+1}\Big[(\alpha(x)-a)^{nq+1} + (x-\alpha(x))^{nq+1} \\ &\quad + (\beta(x)-x)^{nq+1} + (b-\beta(x))^{nq+1}\Big], \end{aligned} \tag{1.8}$$

$$
\begin{aligned}
(1.9) \qquad & M(x) \\
= \; & \frac{1}{2}\left\{\frac{b-a}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+b}{2}\right|\right. \\
& \left.+\left|x-\frac{a+b}{2}+\left|\alpha(x)-\frac{a+x}{2}\right|+\left|\beta(x)-\frac{x+b}{2}\right|\right|\right\},
\end{aligned}
$$

$R_k(x)$, $S_k(x)$ are given by (1.3), and

$$
\begin{aligned}
(1.10) \qquad \left\|f^{(n)}\right\|_\infty &:= ess \sup_{t\in[a,b]}\left|f^{(n)}(t)\right| < \infty \\
\text{and } \left\|f^{(n)}\right\|_p &:= \left(\int_a^b \left|f^{(n)}(t)\right|^p\right)^{\frac{1}{p}},\ 1 \le p < \infty.
\end{aligned}
$$

Specialisations of the above results were also considered such as taking

$$
(1.11) \qquad \alpha(x) = (1-\gamma)a + \gamma x \quad \text{and} \quad \beta(x) = \gamma x + (1-\gamma)b.
$$

They obtained results involving Taylor series and procured explicit expressions for composite rules including a priori estimates of the error.

It is the express aim of the current article to obtain weighted generalisations of the identity (1.1) and its corresponding bounds (1.7). Bounds insisting on weaker conditions of bounded variation rather than absolute continuity will be obtained since the identity will involve a Riemann-Stieltjes integral of $f^{(n)}(t)$ rather than a Riemann integral of $f^{(n+1)}(t)$.

The analysis will be based on some results obtained for the weighted trapezoidal rules by Cerone and Roumeliotis [8]. Earlier, Matić et al. [14] considered the weighted Ostrowski problem in which expressions involve evaluation at one point $x \in [a,b]$ rather than trapezoidal type results that involve the end points a and b. The current development contains these two as special cases and recaptures earlier results involving unweighted Newton-Cotes rules. The weighted rules to be investigated here are related to rules known as product integration rules (see Atkinson [1]).

§2. Some Notation and an Identity Involving Three Points

Before proceeding to develop identities, it is worthwhile to introduce some notation. The notation of Cerone and Roumeliotis [8] will be utilised.

Let $w(\cdot)$ be a weight function and suppose that $w : [a,b] \to [0,\infty)$ is integrable on the interval $[a,b]$ and such that

$$
\int_a^b w(t)\,dt > 0.
$$

Also, let

$$m_k(c,d;w)=\int_c^d u^k w(u)\,du \tag{2.1}$$

represent the k^{th} moment about the origin of the weight function $w(\cdot)$ over the interval $[c,d]\subseteq[a,b]$. Further, let

$$L_n(c,d;w)=\frac{1}{n!}\int_c^d (u-c)^n w(u)\,du, \tag{2.2}$$

$$U_n(c,d;w)=\frac{1}{n!}\int_c^d (d-u)^n w(u)\,du, \tag{2.3}$$

and

$$\nu_n(c,\gamma,d;w)=\frac{1}{n!}\int_c^d |u-\gamma|^n w(u)\,du. \tag{2.4}$$

Then

$$L_n(c,d;w)=\nu_n(c,c,d;w)$$

and

$$U_n(c,d;w)=\nu_n(c,d,d;w),$$

which are incidentally all nonnegative.

We may notice from (2.2) and (2.3)

$$\begin{aligned} 0 \leq\; & L_n(a,x;w)=\frac{1}{n!}\int_a^x (u-a)^n w(u)\,du \\ =\; & \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-a)^{n-k} m_k(a,x;w) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} 0 \leq\; & U_n(x,b;w)=\frac{1}{n!}\int_x^b (b-u)^n w(u)\,du \\ =\; & \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} b^{n-k}(-1)^k m_k(x,b;w). \end{aligned} \tag{2.6}$$

It may be observed that for $x\in[a,b]$

$$L_0(a,x;w)+U_0(x,b;w)=\int_a^b w(t)\,dt=m_0(a,b;w)$$

and

$$L_n(a,x;1) = \frac{(x-a)^{n+1}}{(n+1)!}, \quad U_n(x,b;1) = \frac{(b-x)^{n+1}}{(n+1)!}. \tag{2.7}$$

We introduce the kernel (here we explicitly show the dependence on a and b of Q_n)

(2.8)

$$Q_n(a,x,b;t;w) := \begin{cases} \dfrac{1}{(n-1)!}\displaystyle\int_x^t (t-u)^{n-1} w(u)\,du, & n \in \mathbb{N}, \\ & \\ w(t), & n = 0, \end{cases} \quad x,t \in [a,b]$$

which satisfies

$$\frac{\partial Q_n}{\partial t} = Q_{n-1}, \quad n \in \mathbb{N}. \tag{2.9}$$

The kernel may further be written, using (2.5) and (2.6), as

$$Q_n(a,x,b;t;w) := \begin{cases} (-1)^n L_{n-1}(t,x;w), & a \le t \le x, \\ & \\ U_{n-1}(x,t;w), & x < t \le b, \end{cases} \quad n \in \mathbb{N} \tag{2.10}$$

and $Q_0(a,x,b;t;w) = w(t)$, $x,t \in [a,b]$.

Further, define the functional

(2.11) $T_n(a,x,b;f;w)$

$$:= \int_a^b w(t) f(t)\,dt - \sum_{k=0}^{n} \left[L_k(a,x;w) f^{(k)}(a) + (-1)^k U_k(x,b;w) f^{(k)}(b) \right]$$

for $f : [a,b] \to \mathbb{R}$, $x \in [a,b]$ and $w(\cdot)$ is a weight function with $L_k(c,d;w)$, $U_k(c,d;w)$ as defined by (2.5) and (2.6). The following theorem was obtained by Cerone and Roumeliotis [8].

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ with $a < b$. For $n = 0,1,2,\ldots$ let $Q_{n+1}(a,x,b;t;w)$ be as given by (2.8) and $Q_0(a,x,b;t;w) = w(t)$. Further, suppose that for some $n \in \mathbb{N} \cup \{0\}$, $f^{(n)}(t)$ exists for $t \in [a,b]$, where $f^{(0)}(t) \equiv f(t)$ then for $f^{(n)}(\cdot)$ of bounded variation the identity*

$$T_n(a,x,b;f;w) = (-1)^{n+1} \int_a^b Q_{n+1}(a,x,b;t;w)\,df^{(n)}(t) \tag{2.12}$$

holds where T_n and Q_{n+1} are as defined by (2.9) and (2.8) respectively.

The following identity involves function evaluation at three points.

Theorem 3. *Let the conditions of Theorem 2 hold, then for* $a < \alpha \leq x \leq \beta < b$:

$$\mathcal{T}_n(a,\alpha,x,\beta,b;f;w) = (-1)^{n+1}\int_a^b \kappa_{n+1}(a,\alpha,x,\beta,b;t;w)\,df^{(n)}(t), \tag{2.13}$$

where

$$\begin{aligned} &\mathcal{T}_n(a,\alpha,x,\beta,b;f;w) \\ :&= \int_a^b w(t)f(t)\,dt - \sum_{k=0}^{n}\Big\{L_k(a,\alpha;w)f^{(k)}(a) \\ &+ \Big[(-1)^k U_k(\alpha,x;w) + L_k(x,\beta;w)\Big]f^{(k)}(x) \\ &+ (-1)^k U_k(\beta,b;w)f^{(k)}(b)\Big\} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\kappa_{n+1}(a,\alpha,x,\beta,b;t;w) \\ =& \begin{cases} \dfrac{1}{n!}\displaystyle\int_\alpha^t (t-u)^n w(u)\,du, & t\in[a,x],\ n\in\mathbb{N}, \\ \dfrac{1}{n!}\displaystyle\int_\beta^t (t-u)^n w(u)\,du, & t\in(x,b], \\ w(t), & t\in[a,b],\ n=0. \end{cases} \end{aligned} \tag{2.15}$$

Proof. The proof follows directly from (2.12) of Theorem 2. An application of the theorem on the interval $[a,x]$ gives

$$T_n(a,\alpha,x;f;w) = (-1)^{n+1}\int_a^x Q_{n+1}(a,\alpha,x,t;w)\,df^{(n)}(t) \tag{2.16}$$

and similarly on $(x,b]$

$$T_n(x,\beta,b,f;w) = (-1)^{n+1}\int_x^b Q_{n+1}(x,\beta,b,f;w)\,df^{(n)}(t). \tag{2.17}$$

Adding (2.16) and (2.17) produces, on utilising (2.11) and (2.8), (2.13) with its elements being as presented in (2.14) and (2.15). □

Remark 1. We note that if we take $\alpha = \beta = x$, then identity (2.12) for the generalised weighted trapezoidal rule is recaptured. If α and β are chosen so that $\alpha = a$ and $\beta = b$, then the identity obtained by Matić et al. [14] is recaptured as a special case.

With the definitions (2.2) and (2.3), we may write, from (2.15),

(2.18)

$$\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)=\begin{cases}(-1)^{n+1}L_n(t,\alpha;w), & t\in[a,\alpha]\\[2ex] U_n(\alpha,t;w), & t\in(\alpha,x]\\[2ex] (-1)^{n+1}L_n(t,\beta;w), & t\in(x,\beta]\\[2ex] U_n(\beta,t;w), & t\in(\beta,b]\\[2ex] w(t), & t\in[a,b],\ n=0.\end{cases}\quad,\ n\in\mathbb{N},$$

§3. Inequalities for the Weighted Three Point Rule

The following well known lemmas (see [2] for proofs) will prove useful for procuring bounds for a Riemann-Stieltjes integral. They will be stated here for lucidity.

Lemma 1. *Let $g,v:[a,b]\to\mathbb{R}$ be such that g is continuous and v is of bounded variation on $[a,b]$. Then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and is such that*

$$\left|\int_a^b g(t)\,dv(t)\right|\le\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(v),\tag{3.1}$$

where $\bigvee_a^b(v)$ is the total variation of v on $[a,b]$.

Lemma 2. *Let $g,v:[a,b]\to\mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is $\mathcal{L}-$Lipschitzian on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right|\le\mathcal{L}\int_a^b|g(t)|\,dt\tag{3.2}$$

with v is $\mathcal{L}-$Lipschitzian if it satisfies

$$|v(x)-v(y)|\le\mathcal{L}|x-y|$$

for all $x,y\in[a,b]$.

Lemma 3. *Let $g,v:[a,b]\to\mathbb{R}$ be such that g is Riemann integrable on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$. Then*

$$\left|\int_a^b g(t)\,dv(t)\right|\le\int_a^b|g(t)|\,dv(t).\tag{3.3}$$

Note that if v is nonincreasing, then $-v$ is nondecreasing.

Theorem 4. *Let the conditions of Theorems 2 and 3 continue to hold such that $f^{(n)}(t)$ is of bounded variation for $t \in [a,b]$. We then have for $\alpha, x, \beta \in [a,b]$, $\alpha < x < \beta$ and $n \in \mathbb{N} \cup \{0\}$*

$$(3.4)\quad |\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)| \le \begin{cases} \max\{A_n(a,\alpha,x;w), B_n(x,\beta,b;w)\} \frac{1}{2} \bigvee_a^b \left(f^{(n)}\right), \\ \mathcal{L}\left[L_{n+1}(a,\alpha;w) + U_{n+1}(\alpha,x;w) + L_{n+1}(x,\beta;w) + U_{n+1}(\beta,b;w)\right], \\ \qquad\qquad f^{(n)} \text{ is } \mathcal{L}-\text{Lipschitzian}, \\ L_n(a,\alpha;w)\left[f^{(n)}(\alpha) - f^{(n)}(a)\right] + U_n(\alpha,x;w)\left[f^{(n)}(x) - f^{(n)}(\alpha)\right] \\ +L_n(x,\beta;w)\left[f^{(n)}(\beta) - f^{(n)}(x)\right] + U_n(\beta,b;w)\left[f^{(n)}(b) - f^{(n)}(\beta)\right], \\ \qquad\qquad f^{(n)} \text{ is monotonic nondecreasing} \end{cases}$$

where $\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)$ is given by (2.14),

$$(3.5)\quad A_n(a,\alpha,x;w) = L_n(a,\alpha;w) + U_n(\alpha,x;w) + |U_n(\alpha,x;w) - L_n(a,\alpha;w)|,$$

$$(3.6)\quad B_n(x,\beta,b;w) = L_n(x,\beta;w) + U_n(\beta,b;w) + |U_n(\beta,b;w) - L_n(x,\beta;w)|$$

and $L_n(\cdot,\cdot;w)$, $U_n(\cdot,\cdot;w)$ are given by (2.2) and (2.3) respectively.

Here, by $\bigvee_a^b(h)$ is meant to represent the total variation of $h(t)$ for $t \in [a,b]$. That is, $\bigvee_a^b(h) = \int_a^b |h(t)|\,dt$.

Proof. Taking the modulus of (2.13) and utilising Lemma 1, we have

$$(3.7)\quad \begin{aligned} |\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)| &= \left|\int_a^b \kappa_{n+1}(a,\alpha,x,\beta,b;t;w)\,df^{(n)}(t)\right| \\ &\le \sup_{t\in[a,b]} |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)| \bigvee_a^b \left(f^{(n)}\right). \end{aligned}$$

Now, from (2.15) or the more explicit form (2.18), we have,

$$(3.8)\quad \begin{aligned} &\sup_{t\in[a,b]} |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)| \\ &= \frac{1}{n!}\max\left\{\int_a^\alpha (u-a)^n w(u)\,du, \int_\alpha^x (x-u)^n w(u)\,du,\right. \\ &\qquad\qquad \left.\int_x^\beta (u-x)^n w(u)\,du, \int_\beta^b (b-u)^n w(u)\,du\right\} \\ &= \max\{L_n(a,\alpha;w), U_n(\alpha,x;w), L_n(x,\beta;w), U_n(\beta,b;w)\} \\ &= \frac{1}{2}\max\{A_n(a,\alpha,x;w), B_n(x,\beta,b;w)\} \end{aligned}$$

where

$$A_n(a,\alpha,x;w)=2\max\{L_n(a,\alpha;w),U_n(\alpha,x;w)\}$$

and

$$B_n(x,\beta,b;w)=2\max\{L_n(x,\beta;w),U_n(\beta,b;w)\}$$

and, using the fact that $\max\{u,v\}=\frac{1}{2}[u+v+|u-v|]$, we have from (3.7) and (3.8) the first inequality in (3.4).

If $f^{(n)}(\cdot)$ is $\mathcal{L}-$Lipschitzian on $[a,b]$, then from Lemma 2 and (2.13) we have

$$\begin{aligned}|\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)| &= \left|\int_a^b \kappa_{n+1}(a,\alpha,x,\beta,b;t;w)\,df^{(n)}(t)\right| \\ &\le \mathcal{L}\int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)|\,dt.\end{aligned} \tag{3.9}$$

Making use of (2.18) we have

$$\begin{aligned}&\int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)|\,dt \\ &=\int_a^\alpha L_n(t,\alpha;w)\,dt+\int_\alpha^x U_n(\alpha,t;w)\,dt+\int_x^\beta L_n(t,\beta;w)\,dt+\int_\beta^b U_n(\beta,t;w)\,dt.\end{aligned} \tag{3.10}$$

We may simplify the expression on the right by an interchange of the order of integration to give

$$\begin{aligned}\int_a^\alpha L_n(t,\alpha;w)\,dt &= \frac{1}{n!}\int_a^\alpha\int_t^\alpha (u-t)^n w(u)\,dudt \\ &= \frac{1}{n!}\int_a^\alpha w(u)\int_a^u (u-t)^n\,dtdu \\ &= \frac{1}{(n+1)!}\int_a^\alpha (u-a)^{n+1} w(u)\,du = L_{n+1}(a,\alpha;w)\end{aligned}$$

and

$$\begin{aligned}\int_\alpha^x U_n(\alpha,t;w)\,dt &= \frac{1}{n!}\int_\alpha^x\int_\alpha^t (t-u)^n w(u)\,dudt \\ &= \frac{1}{n!}\int_\alpha^x w(u)\int_u^x (t-u)^n\,dtdu \\ &= \frac{1}{(n+1)!}\int_\alpha^x (x-u)^{n+1} w(u)\,du = U_{n+1}(\alpha,x;w).\end{aligned}$$

In a similar fashion, or alternatively making the appropriate associations, we have

$$\int_x^\beta L_n(t,\beta;w)\,dt = L_{n+1}(x,\beta;w)$$

and

$$\int_\beta^b U_n(\beta,t;w)\,dt = U_{n+1}(\beta,b;w).$$

Thus, from (3.9) and (3.10),

$$\begin{aligned} (3.11) \qquad & \int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)|\,dt \\ = \ & L_{n+1}(a,\alpha;w) + U_{n+1}(\alpha,x;w) + L_{n+1}(x,\beta;w) + U_{n+1}(\beta,b;w), \end{aligned}$$

giving the second inequality in (3.4).

For the final inequality in (3.4) when $f^{(n)}(t)$ is monotonic nondecreasing on $[a,b]$, we have from the identity (2.13) and utilising Lemma 3

$$|\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)| \le \int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)|\,df^{(n)}(t). \tag{3.12}$$

Now, from (2.18),

$$\begin{aligned} (3.13) \qquad & \int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)|\,df^{(n)}(t) \\ = \ & \int_a^\alpha L_n(t,\alpha;w)\,df^{(n)}(t) + \int_\alpha^x U_n(\alpha,t;w)\,df^{(n)}(t) \\ & + \int_x^\beta L_n(t,\beta;w)\,df^{(n)}(t) + \int_\beta^b U_n(\beta,t;w)\,df^{(n)}(t). \end{aligned}$$

We have for $t \le \gamma$

$$L_n(t,\gamma;w) = \frac{1}{n!}\int_t^\gamma (u-t)^n w(u)\,du \tag{3.14}$$

and so

$$L_n'(t,\gamma;w) = \begin{cases} -L_{n-1}(t,\gamma;w), & n \in \mathbb{N} \\ -w(t), & n = 0, \end{cases} \tag{3.15}$$

where the dash represents differentiation with respect to t.

Also for $t \geq \gamma$

$$U_n(\gamma, t; w) = \frac{1}{n!} \int_{\gamma}^{t} (t-u)^n w(u)\, du \tag{3.16}$$

differentiation with respect to t gives

$$U_n'(\gamma, t; w) = \begin{cases} U_{n-1}(\gamma, t; w), & n \in \mathbb{N} \\ w(t), & n = 0. \end{cases} \tag{3.17}$$

Thus integration by parts of each of the integrals on the right hand side of (3.13) and using (3.14) – (3.17) gives

$$\begin{aligned} &\int_a^b |\kappa(a, \alpha, x, \beta, b; t; w)|\, df(t) \\ = {} & L_0(a, \alpha; w) f(a) + \int_a^{\alpha} w(t) f(t)\, dt + U_0(\alpha, x; w) f(x) \\ & - \int_{\alpha}^{x} w(t) f(t)\, dt - L_0(x, \beta; w) f(x) \\ & + \int_x^{\beta} w(t) f(t)\, dt + U_0(\beta, b; w) f(b) - \int_{\beta}^{b} w(t) f(t)\, dt \\ \leq {} & L_0(a, \alpha; w)[f(\alpha) - f(a)] + U_0(\alpha, x; w)[f(x) - f(\alpha)] \\ & + L_0(x, \beta; w)[f(\beta) - f(x)] + U_0(\beta, b; w)[f(b) - f(\beta)]. \end{aligned} \tag{3.18}$$

Here we have used the facts that if $g(t) > 0$ and $f(t)$ is monotonic nondecreasing for $t \in [a, b]$, then

$$\begin{cases} \int_a^b g(t) f(t)\, dt \leq f(b) \int_a^b g(t)\, dt & \text{and,} \\ -\int_a^b g(t) f(t)\, dt \leq -f(a) \int_a^b g(t)\, dt. \end{cases} \tag{3.19}$$

Further, for $n \in \mathbb{N}$, from (3.13) and using (3.14) – (3.17) gives on integration by parts

$$
\begin{aligned}
(3.20)\quad & \int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)|\, df^{(n)}(t) \\
= \ & -L_n(a,\alpha;w) f^{(n)}(a) + \int_a^\alpha L_{n-1}(t,\alpha;w) f^{(n)}(t)\,dt \\
& +U_n(\alpha,x;w) f^{(n)}(x) - \int_\alpha^x U_{n-1}(\alpha,t;w) f^{(n)}(t)\,dt \\
& -L_n(x,\beta;w) f^{(n)}(x) + \int_x^\beta L_{n-1}(t,x;w) f^{(n)}(t)\,dt \\
& +U_n(\beta,b;w) f^{(n)}(b) - \int_\beta^b U_{n-1}(x,t;w) f^{(n)}(t)\,dt \\
\le \ & L_n(a,\alpha;w)\left[f^{(n)}(\alpha) - f^{(n)}(a)\right] + U_n(\alpha,x;w)\left[f^{(n)}(x) - f^{(n)}(\alpha)\right] \\
& +L_n(x,\beta;w)\left[f^{(n)}(\beta) - f^{(n)}(x)\right] + U_n(\beta,b;w)\left[f^{(n)}(b) - f^{(n)}(\beta)\right],
\end{aligned}
$$

where we have utilised (3.19). We notice that the inequality in (3.20) includes that in (3.18) on taking $n = 0$. Thus substitution of (3.20) into (3.12) gives the third inequality in (3.4). $\square$

The following theorem gives bounds on $|\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)|$ in terms of $\left\|f^{(n+1)}\right\|_p$, $p \ge 1$, the Lebesgue norms as defined by (1.10).

Theorem 5. *Let the conditions of Theorem 4 hold and further let $f^{(n)}(t)$ be absolutely continuous for $t \in [a,b]$ then*

$$
(3.21)\quad |\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)|
$$
$$
\le \begin{cases}
[L_{n+1}(a,\alpha;w) + U_{n+1}(\alpha,x;w) + L_{n+1}(x,\beta;w) \\
\qquad +U_{n+1}(\beta,b;w)] \left\|f^{(n+1)}\right\|_\infty, & f^{(n+1)} \in L_\infty[a,b]; \\
\\
\|\kappa_{n+1}(a,\alpha,x,\beta,b;\cdot;w)\|_q \left\|f^{(n+1)}\right\|_p, & f^{(n+1)} \in L_p[a,b], \\
& p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\
\max\{A_n(a,\alpha,x;w), B_n(x,\beta,b;w)\} \frac{\left\|f^{(n+1)}\right\|_1}{2}, & f^{(n+1)} \in L_1[a,b],
\end{cases}
$$

where $\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)$ is as given by (2.14) and $L_n(\cdot,\cdot;w)$, $U_n(\cdot,\cdot;w)$ by (2.2), (2.3),

$$
A_n(a,\alpha,x;w) = 2\max\{L_n(a,\alpha;w), U_n(\alpha,x;w)\}
$$

and

$$
B_n(x,\beta,b;w) = 2\max\{L_n(x,\beta;w), U_n(\beta,b;w)\}.
$$

Proof. From identity (2.13) we have for $f^{(n)}(t)$ absolutely continuous on $[a,b]$ that $df^{(n)}(t) = f^{(n+1)}(t)\,dt$ giving the identity

$$\mathcal{T}_n(a,\alpha,x,\beta,b;f;w) = (-1)^{n+1}\int_a^b \kappa_{n+1}(a,\alpha,x,\beta,b;w)\, f^{(n+1)}(t)\,dt. \tag{3.22}$$

Thus using the well known properties of the modulus and integral, we have from (3.22)

$$|\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)| \le \int_a^b \left|\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)\, f^{(n+1)}(t)\right| dt. \tag{3.23}$$

Now, for $f^{(n+1)} \in L_\infty[a,b]$

$$\int_a^b \left|\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)\, f^{(n+1)}(t)\right| dt \le \left\|f^{(n+1)}\right\|_\infty \int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)|\,dt$$

and so from (3.11) produces the first inequality.

For the second inequality we use Hölder's integral inequality in (3.23) to give

$$\begin{aligned}
&\int_a^b \left|\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)\, f^{(n+1)}(t)\right| dt \\
\le\ & \left(\int_a^b |\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)|^q\,dt\right)^{\frac{1}{q}} \times \left(\int_a^b \left|f^{(n+1)}(t)\right|^p dt\right)^{\frac{1}{p}} \\
=\ & \|\kappa_{n+1}(a,\alpha,x,\beta,b;\cdot;w)\|_q \left\|f^{(n+1)}\right\|_p,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1.
\end{aligned}$$

The final inequality is obtained for $f^{(n+1)} \in L_1[a,b]$ from (3.23) to give

$$\int_a^b \left|\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)\, f^{(n+1)}(t)\right| dt \le \sup_{t\in[a,b]} |\kappa_{n+1}(a,\alpha,x,\beta,b,t;w)| \left\|f^{(n+1)}\right\|_1,$$

which from (3.8) and (3.5), (3.6) give the required result. ∎

Remark 2. If we take $w(t) \equiv 1$ in Theorem 5 and reduce n by one, we obtain the results of Theorem 1 the unweighted three point rule for $n-$time differentiable function $f(t)$ of Cerone and Dragomir [5]. Taking $\alpha = \beta = x$ gives the corresponding trapezoidal type result of Cerone et al. [7] and $\alpha = a$, $\beta = b$ reproduces the Ostrowski type results of Cerone et al. [6].

Remark 3. Taking $\alpha = \beta = x$ gives the generalised weighted trapezoidal rule of Cerone and Roumeliotis [8] while if $\alpha = a$ and $\beta = b$, the weighted Ostrowski type results of Matić et al. [10]. The results of Cerone et al. [9] are also recaptured for $n = 1$ consisting of bounds involving $f''(\cdot)$.

§4. Some Coarser Bounds and Other Results

The results given by (3.21) are valid for any $\alpha, x, \beta \in [a,b]$ with $\alpha \le x \le \beta$.With regards to the first and third inequality in (3.21), let

$$I(\alpha,x,\beta) \;:\; = L_{n+1}(a,\alpha;w) + U_{n+1}(\alpha,x;w) + L_{n+1}(x,\beta;w) + U_{n+1}(\beta,b;w) \tag{4.1}$$

and

$$J(\alpha,x,\beta) := \max\left\{L_n(a,\alpha;w), U_n(\alpha,x;w), L_n(x,\beta;w), U_n(\beta,b;w)\right\}. \tag{4.2}$$

The following lemma investigates obtaining coarser bounds which may prove useful in practice. It involves obtaining bounds on

$$\|\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)\|_1 = I(\alpha,x,\beta),$$

where $I(\alpha,x,\beta)$ is given by (4.1).

Lemma 4. *Let $w(t)$ be a weight function defined on $[a,b]$ and $\alpha, x, \beta \in [a,b]$ with $\alpha \le x \le \beta$. Then,*

$$\begin{aligned} |I(\alpha,x,\beta)| &= \|\kappa_{n+1}(a,\alpha,x,\beta,b;t;w)\|_1 \\ &\le \begin{cases} D(n+2)\|w\|_\infty, & w \in L_\infty[a,b]; \\ D^{\frac{1}{q}}(q(n+1)+1)\|w\|_p, & w \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\theta^{n+1}}{(n+1)!}\|w\|_1, & w \in L_1[a,b], \end{cases} \end{aligned} \tag{4.3}$$

where

$$\eta D(\eta) = (\alpha-a)^\eta + (x-\alpha)^\eta + (\beta-x)^\eta + (b-\beta)^\eta \tag{4.4}$$

and

$$\begin{aligned} \theta &= \frac{1}{2}\left\{\frac{b-a}{2} + \left|\alpha - \frac{a+x}{2}\right| + \left|\beta - \frac{x+b}{2}\right| \right. \\ &\quad \left. + \left| x - \frac{a+b}{2} + \left|\alpha - \frac{a+x}{2}\right| - \left|\beta - \frac{x+b}{2}\right|\right|\right\}. \end{aligned} \tag{4.5}$$

Proof. From (4.1) with (2.2) and (2.3), it may be noticed that

$$I(\alpha,x,\beta) = \frac{1}{(n+1)!}\int_a^b \phi^{n+1}(a,\alpha,x,\beta,b;u)\,w(u)\,du, \tag{4.6}$$

where

$$\phi(a,\alpha,x,\beta,b;u)=\begin{cases} u-a, & u\in[a,\alpha], \\ x-u, & u\in(\alpha,x], \\ u-x, & u\in(x,\beta], \\ b-u, & u\in(\beta,b]. \end{cases} \tag{4.7}$$

Now, for $w \in L_p[a,b]$, $1<p<\infty$, then

$$(n+1)!I(\alpha,x,\beta)\le\left(\int_a^b \phi^{q(n+1)}(a,\alpha,x,\beta,b;u)\,du\right)^{\frac{1}{q}}\left(\int_a^b w^p(u)\,du\right)^{\frac{1}{p}}. \tag{4.8}$$

Explicitly, from (4.7),

$$\begin{aligned}
&\left(\int_a^b \phi^{q(n+1)}(a,\alpha,x,\beta,b;u)\,du\right)^{\frac{1}{q}} \\
=\ &\left\{\int_a^\alpha (u-a)^{q(n+1)}\,du+\int_\alpha^x (x-u)^{q(n+1)}\,du\right. \\
&\left.+\int_x^\beta (u-x)^{q(n+1)}\,du+\int_\beta^b (b-u)^{q(n+1)}\,du\right\}^{\frac{1}{q}} \\
=\ &\left\{\frac{(\alpha-a)^{q(n+1)+1}+(x-\alpha)^{q(n+1)+1}+(\beta-x)^{q(n+1)+1}+(b-\beta)^{q(n+1)+1}}{q(n+1)+1}\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence, from (4.8) the second inequality in (4.3) results. The first inequality is also procured on noting that it corresponds to the case $q=1$.

To obtain the final inequality, we note from (4.6) that for $w\in L_1[a,b]$

$$(n+1)!I(\alpha,x,\beta)\le\sup_{u\in[a,b]}\phi^{n+1}(a,\alpha,x,\beta,b;u)\,\|w\|_1. \tag{4.9}$$

Now,

$$\sup_{u\in[a,b]}\phi^{n+1}(a,\alpha,x,\beta,b;u)=\max{}^{n+1}\{\alpha-a,x-\alpha,\beta-x,b-\beta\}. \tag{4.10}$$

Further, using the fact that $\max\{X,Y\}=\frac{X+Y}{2}+\frac{|X-Y|}{2}$, then

$$m_1=\max\{\alpha-a,x-\alpha\}=\frac{x-a}{2}+\left|\alpha-\frac{a+x}{2}\right|$$

and

$$m_2 = \max\{\beta - x, b - \beta\} = \frac{b-x}{2} + \left|\beta - \frac{x+b}{2}\right|$$

giving

$$\begin{aligned} & \max\{\alpha - a, x - \alpha, \beta - x, b - \beta\} = \max\{m_1, m_2\} \\ = \; & \frac{1}{2}\left\{\frac{b-a}{2} + \left|\alpha - \frac{a+x}{2}\right| + \left|\beta - \frac{x+b}{2}\right| \right. \\ & \left. + \left|x - \frac{a+b}{2} + \left|\alpha - \frac{a+x}{2}\right| - \left|\beta - \frac{x+b}{2}\right|\right|\right\} \\ = \; & \theta, \quad \text{as given above by (4.5).} \end{aligned}$$

Thus, from (4.9) and (4.10) we readily obtain the third inequality in (4.3). □

Karamata [10] proved the following theorem.

Theorem 6. *Let $g, w : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and suppose $m \leq g(t) \leq M$ and $0 < c \leq w(t) \leq \lambda c$ for $t \in [a,b]$ and some constants m, M, c and λ. If G and $A(g,w)$ are defined as*

$$G := \frac{1}{b-a}\int_a^b g(t)\,dt \quad \text{and} \quad A(g,w) := \frac{\int_a^b g(t)\,w(t)\,dt}{\int_a^b w(t)\,dt} \tag{4.11}$$

then

$$\frac{\lambda m(M-G) + M(G-m)}{\lambda(M-G) + (G-m)} \leq A(g,w) \leq \frac{m(M-G) + \lambda M(G-m)}{(M-G) + \lambda(G-m)}. \tag{4.12}$$

Using the above theorem of Karamata, the third inequality in (4.3) may be improved.

If we associate $\phi^{n+1}(a, \alpha, x, \beta, b; w)$, as defined by (4.7), with $g(t)$ above, then

$$0 \leq \phi(a, \alpha, x, \beta, b; u) \leq \theta = \max\{\alpha - a, x - \alpha, \beta - x, b - \beta\},$$

where θ may be represented by (4.5), and

$$G = \frac{1}{b-a}\int_a^b \phi^{n+1}(a, \alpha, x, \beta, b; u)\,du = \frac{1}{b-a}D(n+2),$$

where $D(\eta)$ is as defined by (4.4).

Hence from (4.11) and (4.12) we have

$$\begin{aligned} I(\alpha, x, \beta) &= \|\kappa_{n+1}(a, \alpha, x, \beta, b; t; w)\|_1 \\ &\leq \frac{\lambda \theta^{n+1} D(n+2)\,\|w\|_1}{(b-a)\,\theta^{n+1} - D(n+2) + \lambda D(n+2)}. \end{aligned}$$

Remark 4. The above results obtain coarser bounds for $I(\alpha, x, \beta)$ which are perhaps easier to implement. The parameters α, x and β may be chosen in many ways such that $a \leq \alpha \leq x \leq \beta \leq b$. If for example $\alpha = \beta = x$ then from (2.14) $\mathcal{T}_n(a, x, x, x, b; f; w)$ produces a product trapezoidal rule for the weighted integral with bounds provided by (3.4) or (3.21).

If we choose α, x and β to be at their respective midpoints, then $\mathcal{T}_n\left(a, \frac{2a+b}{2}, \frac{a+b}{2}, \frac{a+2b}{2}, b; f; w\right)$ may be bounded by either (3.4) or (3.21) with $\alpha = \frac{a+x}{2}$, $x = \frac{a+b}{2}$ and $\beta = \frac{x+b}{2}$.

If we choose α, x and β satisfying

$$L_n(a, \tilde{\alpha}; w) = U_n(\tilde{\alpha}, \tilde{x}; w), \quad L_n\left(\tilde{x}, \tilde{\beta}; w\right) = U_n\left(\tilde{\beta}, b; w\right) \tag{4.13}$$

and

$$L_n\left(\tilde{x}, \tilde{\beta}; w\right) = U_n(\tilde{\alpha}, \tilde{x}; w)$$

then from (2.14)

$$\begin{aligned} \mathcal{T}_n\left(a, \tilde{\alpha}, \tilde{x}, \tilde{\beta}, b; f; w\right) = {} & \int_a^b w(t) f(t)\, dt - \sum_{k=0}^{n} \tilde{W}_k\left(\tilde{\alpha}, \tilde{x}, \tilde{\beta}\right) \\ & \times \left\{ f^{(k)}(a) + \left[(-1)^k + 1\right] f^{(k)}(x) + f^{(k)}(b) \right\} \end{aligned} \tag{4.14}$$

where

$$\tilde{W}_k\left(\tilde{\alpha}, \tilde{x}, \tilde{\beta}\right) = L_k(a, \tilde{\alpha}; w) = U_k(\tilde{\alpha}, \tilde{x}; w) = L_k\left(\tilde{x}, \tilde{\beta}; w\right) = U_k\left(\tilde{\beta}, b; w\right). \tag{4.15}$$

The bounds for (4.14) may be obtained from (3.4) or (3.11) with $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$ and $x = \tilde{x}$. In particular, for $f^{(n+1)} \in L_1(a, b)$

$$\begin{aligned} \left|\mathcal{T}_n\left(a, \tilde{\alpha}, \tilde{x}, \tilde{\beta}, b; f; w\right)\right| & \leq J\left(\tilde{\alpha}, \tilde{x}, \tilde{\beta}\right) \frac{\left\|f^{(n+1)}\right\|_1}{2} \\ & = \tilde{W}_n\left(\tilde{\alpha}, \tilde{x}, \tilde{\beta}\right) \frac{\left\|f^{(n+1)}\right\|_1}{2}, \end{aligned} \tag{4.16}$$

where $\tilde{W}_n\left(\tilde{\alpha}, \tilde{x}, \tilde{\beta}\right)$ is as defined in (4.15). If $w(t) \equiv 1$, then $\tilde{\alpha}, \tilde{x}$ and $\tilde{\beta}$, as may be shown from solving (4.13), would be at their respective midpoints, recovering the first choice given above.

If only α and β are taken at their respective midpoints so that $\alpha = \alpha^* = \frac{a+x}{2}$ and $\beta = \beta^* = \frac{x+b}{2}$, then we may choose $x = x^*$ to satisfy

$$U_n(\alpha^*, x^*; w) = L_n(x^*, \beta^*; w). \tag{4.17}$$

From (2.14) we would then have

$$\mathcal{T}_n(a,\alpha^*,x^*,\beta^*,b;f;w)=\int_a^b w(t)f(t)\,dt \\ -\sum_{k=0}^{n}\Big\{L_k(a,\alpha^*;w)f^{(k)}(a)+\Big[(-1)^k+1\Big]W_k^*(\alpha^*,x^*,\beta^*)f^{(k)}(x^*) \\ +(-1)^k U_k(\beta^*,b;w)f^{(k)}(b)\Big\}, \tag{4.18}$$

where

$$W_k^*(\alpha^*,x^*,\beta^*)=U_n(\alpha^*,x^*,\beta^*)=L_n(\alpha^*,x^*,\beta^*) \tag{4.19}$$

with

$$\alpha^*=\frac{a+x^*}{2},\quad \beta^*=\frac{x^*+b}{2}. \tag{4.20}$$

Again the bounds may be obtained from either (3.4) or (3.21) depending on the assumption regarding $f^{(n+1)}(\cdot)$.

In particular, for $f^{(n+1)}\in L_\infty[a,b]$ then from (3.21), (4.1) and (4.18)

$$|\mathcal{T}_n(a,\alpha^*,x^*,\beta^*,b;f;w)|\le I(\alpha^*,x^*,\beta^*)\left\|f^{(n+1)}\right\|_\infty .$$

The question of the rule $\mathcal{T}_n(a,\alpha,x,\beta,b;f;w)$ providing the tightest bounds for various assumptions on the behaviour of $f^{(n+1)}(t)$, $t\in[a,b]$ remains an open issue.

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