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Singular star-exponential functions

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Abstract. The $*$-exponential function $e_*^{tie^{av}*u}$ is defined in a transcendently extended Weyl algebra. In the Weyl ordering expression, this is given as the real analytic solution in t of

$$\begin{aligned}&\partial_t F_t(u,v) = ie^{av}\big\{\big(u+\frac{\hbar ia}{2}\big)F_t\big(u+\frac{\hbar ia}{2},v\big)-\frac{\hbar i}{2}\partial_v F_t\big(u+\frac{\hbar ia}{2},v\big)\big\},\\ &F_0(u,v)=1.\end{aligned}$$

For generic initial functions, this equation can be solved for all t, but the uniqueness holds only for one direction.

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§1. Introduction

In this paper, we treat the Weyl algebra $W_\hbar$ generated over $\mathbb{C}$ by two elements u, v. $W_\hbar$ is the associative algebra with the fundamental relation $u*v-v*u = -\hbar i$ where $\hbar$ is a positive constant.

In the Weyl ordering expression (cf. [9]), the Weyl algebra is understood as the space of polynomials with the *Moyal product* as follows:

$$\begin{aligned}f(u,v)*g(u,v) &= f\exp\Big\{\frac{\hbar i}{2}\overleftarrow{\partial_v}\dot\wedge\overrightarrow{\partial_u}\Big\}g\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{\hbar i}{2}\Big)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\big(\partial_v^{n-k}\partial_u^k f\big)\big(\partial_u^{n-k}\partial_v^k g\big),\end{aligned}\tag{1.1}$$

where $\overleftarrow{\partial_v}\dot\wedge\overrightarrow{\partial_u} = \overleftarrow{\partial_v}\cdot\overrightarrow{\partial_u}-\overleftarrow{\partial_u}\cdot\overrightarrow{\partial_v}$, and the arrow indicates to which side the operator acts. This product formula yields $u*v-v*u=-\hbar i$, and hence defines the Weyl algebra. The usual commutative product in the polynomial algebra plays only the supplementary role to write elements in a unique way.

Using this concrete product formula (1.1), we can extend the product $*$ as follows: Let $C^\infty(U)$ be the space of all C^∞-functions on an open subset U of the real 2-plane $\mathbb{R}^2$ with the C^∞-topology.

- $f*g$ is defined, if one of f,g is a polynomial.
- The associativity $f*(g*h)=(f*g)*h$ holds if two of f,g,h are polynomials.
- If p is a polynomial, then $p*$ and $*p$ are continuous linear mapping of $C^\infty(U)$ into itself.

Remark that such extension can be considered also for entire functions $Hol(\mathbb{C}^2)$ with compact open topology, instead of $C^\infty(U)$.

In such an extended system, which will be called a $\mathbb{C}[u,v]$-*module*, the first task we should do is to fix product formula of several transcendental functions, and to determine exponential functions of several elements with respect to the product $*$. Several results are already given in [10] for the $*$-exponential functions of quadratic forms.

If we change variables u,v by a transformation φ given as follows:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad ad-bc=1,$$

then we see $[u',v']=-\hbar i$ and $(\varphi^*f)*(\varphi^*g)=\varphi^*(f*g)$, if the product $f*g$ is defined. This is the most useful property of the Moyal product formula.

We call (f,g) a *quantum canonical conjugate pair*, if $[f,g]=-\hbar i$ holds. (u',v') is a quantum canonical conjugate pair which is linearly related to the original (u,v), but there are a lot of quantum canonical conjugate pair (f,g) which relates transcendentally to the original (u,v). For instance, $(u',v')=(\frac{u}{v},\frac{1}{2}v^2)$ is a quantum canonical conjugate pair, treated in [12] and [8].

In this paper, we treat a quantum canonical conjugate pair $(e^{av}*u,-\frac{1}{a}e^{-av})$ for $a>0$. We can easily check $[e^{av}*u,-\frac{1}{a}e^{-av}]=-\hbar i$ by a direct calculation using the Moyal product formula (1.1).

By the Moyal product formula, we see easily that $u*f(u,v)=uf(u,v)-\frac{\hbar i}{2}\partial_v f(u,v)$ and

$$e^{av}*f(u,v)=e^{av}\sum_k \frac{1}{k!}\Big(\frac{\hbar i a}{2}\Big)^k(\partial_u)^k f(u,v).$$

Since the above sum is the Taylor expansion of $f(u+\frac{\hbar ia}{2}, v)$, we extend the $*$-product by e^{av} as follows:

$$e^{av} * f(u,v) = e^{av} f\big(u + \frac{\hbar ia}{2}, v\big),$$

and

$$f(u,v) * e^{av} = e^{av} f\big(u - \frac{\hbar ia}{2}, v\big),$$

by a similar reasoning. Of course we assume that $e^{av} * f(u,v)$ is well defined if and only if $f(u,v)$ is a function such that $f(u+\frac{\hbar ia}{2}, v)$ is well defined.

To define the $*$-exponential function $e_*^{tie^{av}*u}$, we consider the linear equation

$$(1.2) \qquad \begin{cases} \dfrac{d}{dt}L_t = (ie^{av} * u) * L_t, \\ \\ L_0 = g(u,v), \end{cases} \qquad \text{resp.} \qquad \begin{cases} \dfrac{d}{dt}R_t = R_t * (ie^{av} * u), \\ \\ R_0 = g(u,v). \end{cases}$$

We call this the left (resp. right) equation.

Remark that if we set $\mu = e^{av} * u$ and $\nu = -\frac{1}{a}e^{-av}$, then μ, ν can play the same role as u, v after changing variables by such a transcendental transformation. These also generate the Weyl algebra which is isomorphic to $W_\hbar$. Thus, in a geometrical intuitive mind, $e_*^{tie^{av}*u}$ is expected to play as if e_*^{tu}. In the Moyal product formula, we see that e_*^{tu} is the ordinary exponential function e^{tu}. Hence left-(resp. right-) multiplication $e_*^{tu}*$ (resp. $*e_*^{tu}$) are invertible linear operator on $C^\infty(U)$.

It is remarkable that if we take the Fourier transform, the equation (1.2) turns out to be a simple differential equation.

In this paper, we show that the $*$-exponential function $e_*^{tie^{av}*u}$ is well-defined for all $t \in \mathbb{R}$ and it is real analytic. By the uniqueness of real analytic solutions, we see that $e_*^{tie^{av}*u}$ satisfies the exponential law:

$$e_*^{(s+t)ie^{av}*u} = e_*^{sie^{av}*u} * e_*^{tie^{av}*u}.$$

However, $e_*^{tie^{av}*u}$ behaves very strangely. We show that there are a lot of non-real analytic solutions and the uniqueness does not hold to the positive direction. In spite of this, the uniqueness holds for the negative direction. As a result, we can show the phenomenon of associativity breaking.

Throughout this paper, we concentrate to obtain the concrete formula of

$$e_*^{tie^{av}*u}, \quad e_*^{tie^{av}*u} * e^{\frac{2i}{\hbar}uv} \quad \text{and} \quad e_*^{tie^{av}*u} * e^{-\frac{2i}{\hbar}uv},$$

because it is known that functions $2e^{\frac{2i}{\hbar}uv}$ and $2e^{-\frac{2i}{\hbar}uv}$ play important roles in the construction of operator representation of our $\mathbb{C}[u,v]$-module. By the Moyal product formula, we see that

$$v * 2e^{\frac{2i}{\hbar}uv} = 0, \qquad u * 2e^{-\frac{2i}{\hbar}uv} = 0,$$

but

$$2e^{\frac{2i}{\hbar}uv} * 2e^{\frac{2i}{\hbar}uv} = 2e^{\frac{2i}{\hbar}uv}, \qquad 2e^{-\frac{2i}{\hbar}uv} * 2e^{-\frac{2i}{\hbar}uv} = 2e^{-\frac{2i}{\hbar}uv},$$

and

$$2e^{\frac{2i}{\hbar}uv} * 2e^{-\frac{2i}{\hbar}uv} = \text{diverge}.$$

By the formula $u * e^{-\frac{2i}{\hbar}uv} = 0$, we see that

$$(e^{av} * u) * e^{-\frac{2i}{\hbar}uv} = 0.$$

Hence, it is natural to expect that $e_*^{ite^{av}*u} * e^{-\frac{2i}{\hbar}uv} = e^{-\frac{2i}{\hbar}uv}$. Such an identity is obtained as a bi-product of our proof of the main theorem.

§2. $*$-exponential function of $e^{av} * u$

Since $\overline{e^{av} * u} = u * e^{av} = e^{av} * (u - \hbar ia)$, $\quad e^{av} * u$ is not an hermite element, though u and v are restricted in reals.

Set $L_t(g) = F_t(u,v)$. Since

$$ie^{av} * f(u,v) = ie^{av} f\big(u + \frac{\hbar ia}{2}, v\big),$$

we see that (1.2) turns out to be

$$\partial_t F_t(u,v) = ie^{av}\big\{\big(u + \frac{\hbar ia}{2}\big)F_t\big(u + \frac{\hbar ia}{2}, v\big) - \frac{\hbar i}{2}\partial_v F_t\big(u + \frac{\hbar ia}{2}, v\big)\big\},$$

with initial condition $F_0 = g(u,v)$. If we set $F_t(u,v) = G_t(u,v)e^{-\frac{2i}{\hbar}uv}$, then we have

$$\partial_t G_t(u,v) = \frac{\hbar}{2}e^{2av}\partial_v G_t\big(u + \frac{\hbar ia}{2}, v\big), \tag{2.1}$$

with the initial condition $G_0 = g(u,v)e^{\frac{2i}{\hbar}uv}$. This is not a differential equation, but an evolution equation of a differential-difference operator.

Real analytic solutions, if they exist, are unique with respect to initial functions. The solution, if exists, might be written as the $*$-exponential function

$e_*^{tie^{av}*u} * g$, where g is the initial function, and $e_*^{tie^{av}*u}$ is the real analytic solution with initial function 1.

For the sake of self-containedness, we repeat here the definition of real analyticity. Remark first that $C^\infty(U)$ is a Fréchet space whose topology is given by countable seminorms. Let $\mathbb{E}$ be a Fréchet space whose topology is given by countable seminorms $\| \|_k$. A smooth mapping $f : \mathbb{R} \to \mathbb{E}$ is *real analytic*, if for every $\| \|_k$, and for every $t_0 \in \mathbb{R}$, the Taylor series at every t_0 converges in $\| \|_k$ on some neighborhood of t_0 which may depend on k.

However, if G_t is restricted to periodic functions $G_t(u + \frac{\hbar i a}{2}, v) = G_t(u, v)$, then (2.1) turns out to be

$$\big(\partial_t - \frac{\hbar}{2} e^{2av} \partial_v\big) G_t(u, v) = 0,$$

and the solution is given by

$$G_t(u, v) = \varphi\big(u, \frac{\hbar t}{2} - \frac{1}{2a} e^{-2av}\big).$$

Thus, if the initial function G_0 is restricted furthermore to a periodic function $G_0(u, v + \frac{\pi i}{a}) = G_0(u, v)$, then the solution is written uniquely by the above shape.

Next, we want to restrict the variables u, v to the real line. To do this, denote by $\mathfrak{S}$ be the space of all rapidly decreasing functions of the variable ξ, and let $\mathfrak{S}'$ be its dual space, that is the space of all slowly increasing distributions.

First we assume that $G_t(u, v)$ is written as

$$G_t(u, v) = \int_{-\infty}^{\infty} a(t, \xi, v) e^{i\xi u}\, d\xi,$$

by using slowly increasing Schwartz distribution $a(t, \xi, v)$ with respect to ξ (i.e. $a(t, \xi, v)$ is a $\mathfrak{S}'$-valued C^∞ function). Then the above equation (2.1) is changed into the differential equation

$$\big(e^{a\hbar\xi/2} \partial_t - \frac{\hbar}{2} e^{2av} \partial_v\big) a(t, \xi, v) = 0. \tag{2.2}$$

It is remarkable that ξ plays only as a parameter. (2.2) shows that $a(t, \xi, v)$ is constant along the real analytic vector field $e^{a\hbar\xi/2} \partial_t - \frac{\hbar}{2} e^{2av} \partial_v$.

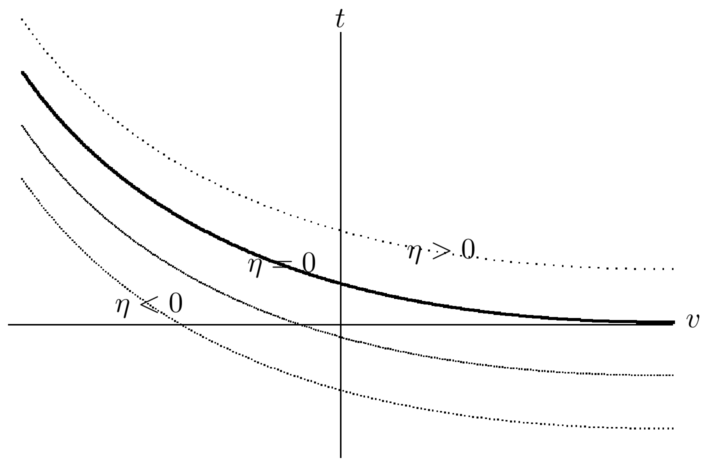

Figure 1: Level curves of $\frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av}$

Along the integral curves of this vector field, $\eta = \frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av}$ is constant.

Thus, by fixing ξ arbitrarily, and by replacing

$$t' = t - \frac{1}{a\hbar}e^{-2av}e^{a\hbar\xi/2}, \qquad v' = v,$$

the identities

$$\partial_{t'} = \partial_t, \quad \partial_{v'} = -\frac{2}{\hbar}e^{a\hbar\xi/2}e^{-2av}\partial_t + \partial_v$$

shows that if ξ is fixed in $\mathbb{R}$, the solutions $a(t,\xi,v)$ are given by arbitrary functions of t', not containing v'. That is, the solutions are given as arbitrary functions of $\frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av}$ by multiplying $\frac{\hbar}{2}e^{-a\hbar\xi/2}$ to both sides.

Our main theorem is as follows:

Theorem 1. *If $t \leq 0$ (resp. $t \geq 0$), then the equation*

$$\begin{cases} \dfrac{d}{dt}L_t = (ie^{av} * u) * L_t, \\ L_0 = g(u,v), \end{cases} \qquad \textit{resp.} \qquad \begin{cases} \dfrac{d}{dt}R_t = R_t * (ie^{av} * u), \\ R_0 = g(u,v), \end{cases} \tag{2.3}$$

has the unique solution $L_t(g)$, (resp. $R_{-t}(g)$) for almost all initial functions g with polynomial growth. However, if $t > 0$ (resp. $t < 0$), then the equation has a solution for almost all initial functions g with polynomial growth, but these are not unique, i.e. L_t, $t > 0$ is not defined as operators.

Precise definition of "almost all" will be clarified in the proof.

For every (t,v), we may view $a(t,\xi,v)$ as an $\mathfrak{S}'$-valued function $a(t,v)(\xi)$, that is for every test function $\psi(\xi)\in\mathfrak{S}$,

$$\int\Big(\partial_t a(t,v)(\xi)e^{a\hbar\xi/2}-\frac{\hbar}{2}e^{2av}\partial_v a(t,v)(\xi)\Big)\psi(\xi)\,d\xi=0.$$

The solution is written by using a $\mathfrak{S}'$-valued C^∞ function $\varphi(v)(\xi)=\varphi(\xi,v)$ as

$$a(t,\xi,v)=\varphi\big(\xi,\frac{\hbar t}{2}e^{-a\hbar\xi/2}-\frac{1}{2a}e^{-2av}\big). \tag{2.4}$$

The right hand side of (2.4) is the distribution defined for the test function $\psi(\xi)\in\mathfrak{S}$

$$\int\varphi\big(\xi,\frac{\hbar t}{2}e^{-a\hbar\xi/2}-\frac{1}{2a}e^{-2av}\big)\psi(\xi)\,d\xi.$$

The condition that has been imposed is as follows:

Condition 1. $\varphi(\xi,v)$ is a $\mathfrak{S}'$-valued C^∞ function such that for every (t,v),

$$\varphi(\xi,\frac{\hbar t}{2}e^{-a\hbar\xi/2}-\frac{1}{2a}e^{-2av})$$

is also a slowly increasing Schwartz distribution.

Let $\mathcal{S}_{u,v}$ be the set of all Fourier image of such functions. In particular, if $\operatorname{supp}\varphi$ is bounded below with respected to ξ, then φ satisfies the condition.

Setting $t=0$ in (2.4) gives the Fourier inverse image of the initial function. Remark

$$e^{\frac{ci}{\hbar}uv}=\int_{-\infty}^{\infty}\delta\big(\xi-\frac{cv}{\hbar}\big)e^{i\xi u}\,d\xi.$$

Then the solution φ with initial function g being $e^{\frac{2i}{\hbar}uv}$, 1 or $e^{-\frac{2i}{\hbar}uv}$, is given respectively as follows:

$$\varphi\big(\xi,-\frac{1}{2a}e^{-2av}\big)=\delta\big(\xi-\frac{4v}{\hbar}\big),\quad \delta\big(\xi-\frac{2v}{\hbar}\big),\quad \delta(\xi).$$

In general if $\hat{g}(\xi,v)=\int e^{-iu\xi}g(u,v)\,đu$ is the Fourier transform of $g(u,v)$, then the Fourier image of $G_0(u,v)=g(u,v)e^{\frac{2i}{\hbar}uv}$ with respect to the variable u is

$$\psi(\xi,v)=\int\delta\big(\xi'-\frac{2v}{\hbar}\big)\hat{g}(\xi-\xi',v)d\xi'=\hat{g}\big(\xi-\frac{2v}{\hbar},v\big),$$

and hence

$$\varphi\big(\xi, -\frac{1}{2a}e^{-2av}\big) = \hat{g}\big(\xi - \frac{2v}{\hbar}, v\big). \tag{2.5}$$

Thus the distribution $\varphi(\xi, \eta)$ which gives the initial function $G_0(u, v) = e^{\frac{2i}{\hbar}uv}$ or $G_0(u, v) = g(u, v)e^{\frac{2i}{\hbar}uv}$ is given respectively, by putting $-2a\eta = e^{-2av}$ at the place $\eta \leq 0$, as followings:

$$\varphi(\xi, \eta) = \delta\big(\xi + \frac{1}{a\hbar}\log(-2a\eta)\big),$$
$$\varphi(\xi, \eta) = \hat{g}\big(\xi + \frac{1}{a\hbar}\log(-2a\eta), -\frac{1}{2a}\log(-2a\eta)\big)$$

and these are arbitrary where $\eta > 0$.

By this, if the initial condition is $F_0(u, v) = 1$, then the solution is

$$\varphi\big(\xi, \frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av}\big) = \delta\Big(\xi + \frac{1}{a\hbar}\log\big(e^{-2av} - a\hbar te^{-a\hbar\xi/2}\big)\Big). \tag{2.6}$$

For the general initial function $F_0(u, v) = g(u, v)$, we see

$$= \hat{g}\Big(\xi + \frac{1}{a\hbar}\log\big(e^{-2av} - a\hbar te^{-a\hbar\xi/2}\big), -\frac{1}{2a}\log\big(e^{-2av} - a\hbar te^{-a\hbar\xi/2}\big)\Big). \tag{2.7}$$

In general, if $\eta \geq 0$, t is always positive > 0 on the curve defined by $\frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{2av} = \eta$ (see Figure 1). So this curve does not cross the initial surface $t = 0$.

Since we are trying to fix the solution by means of initial data, this is possible only for $\eta < 0$. For $\eta \geq 0$, φ is arbitrary. Hence the solution is not unique.

Remark 2. The family of curves $\frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av} = \eta$ is holomorphic. Thus, if (t, v) is complex, then every curve does cross the initial surface $t = 0$.

Recall that the solution of (2.2) is given by

$$a(t, \xi, v) = \varphi\big(\xi, \frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av}\big),$$

and the initial data is given as $a(\xi, v) = \varphi(\xi, -\frac{1}{2a}e^{-2av})$. Hence, the restriction for the initial data is only that $a(\xi, v)$ has the periodicity $a(\xi, v) = a(\xi, v + \frac{\pi i}{a})$.

§3. The solutions with initial data 1, $e^{\frac{2i}{\hbar}uv}$, $e^{-\frac{2i}{\hbar}uv}$

In particular, write the solution $L_t(1)$ with initial data 1 by $\tilde{F}_t(u,v)$. Then, $\tilde{F}_t(u,v)$ is given by

$$\tilde{F}_t(u,v) = e^{-\frac{2i}{\hbar}uv}\int_{-\infty}^{\infty}\delta\Big(\xi + \frac{1}{a\hbar}\log\big(e^{-2av} - a\hbar te^{-a\hbar\xi/2}\big)\Big)e^{i\xi u}\,d\xi.$$

For $t \le 0$, we see that

$$\xi' = \xi + \frac{1}{a\hbar}\log\big(e^{-2av} - a\hbar te^{-a\hbar\xi/2}\big)$$

gives a diffeomorphism of ξ-space. Thus it is better to change the variable by this diffeomorphism. Since the property of delta function gives $\tilde{G}_t(u,v) = e^{i\xi u}\frac{d\xi}{d\xi'}\Big|_{\xi'=0}$, we express this as a function of u, v. Since

$$\frac{d\xi'}{d\xi} = 1 + \frac{1}{2}\frac{a\hbar te^{-a\hbar\xi/2}}{e^{-2av} - a\hbar te^{-a\hbar\xi/2}},$$

the point $\xi' = 0$ is given as the solution of the equation $e^{-a\hbar\xi} + a\hbar te^{-\frac{a\hbar\xi}{2}} = e^{-2av}$. For both $t \le 0$ and $t > 0$, the solution is given by

$$2e^{-a\hbar\xi/2} = -a\hbar t + \sqrt{(a\hbar t)^2 + 4e^{-2av}}. \tag{3.1}$$

We write the right hand side of (3.1) as ψ, and

$$\frac{d\xi'}{d\xi}\Big|_{\xi'=0} = \frac{\psi}{\psi + a\hbar t}. \tag{3.2}$$

Thus, we have

$$\tilde{G}_t(u,v) = e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\psi}\frac{\psi}{\psi + a\hbar t},$$
$$\tilde{F}_t(u,v) = e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\psi}\frac{\psi}{\psi + a\hbar t}e^{-\frac{2i}{\hbar}uv}.$$

These are real analytic in (t,v). This solution can be extended naturally to the domain $t > 0$. It is also easy to check that this is a solution for all t by remarking the following identity:

$$\tilde{G}_t(u + \frac{a\hbar i}{2}, v) = e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\psi}\frac{\psi^2}{2(\psi + a\hbar t)},$$
$$\partial_t\psi = -a\hbar\frac{\psi}{\psi + a\hbar t}, \quad \partial_v\psi = -4a\frac{e^{-2av}}{\psi + a\hbar t}.$$

$\tilde{F}_t(u,v)$ is real analytic by its construction.

Lemma 3. *For every $s, t \in \mathbb{R}$, the exponential law*

$$\tilde{F}_s(u,v) * \tilde{F}_t(u,v) = \tilde{F}_{s+t}(u,v)$$

holds.

We can write $e_*^{ite^{av}*u} = \tilde{F}_t(u,v)$ precisely as

(3.3)
$$\tilde{F}_t(u,v) = e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}e^{av}\psi}\frac{e^{av}\psi}{e^{av}\psi + e^{av}a\hbar t}, \qquad e^{av}\psi = -a\hbar te^{av} + \sqrt{(a\hbar te^{av})^2+4}.$$

Moreover,

$$\sqrt{(\frac{1}{2}e^{av}a\hbar t)^2+1} + \frac{1}{2}e^{av}a\hbar t = \Big(\sqrt{(\frac{1}{2}e^{av}a\hbar t)^2+1} - \frac{1}{2}e^{av}a\hbar t\Big)^{-1}, \tag{3.4}$$

$$\left(1 - \frac{e^{av}a\hbar t}{\sqrt{(e^{av}a\hbar t)^2+4}}\right)\left(1 + \frac{e^{av}a\hbar t}{\sqrt{(e^{av}a\hbar t)^2+4}}\right) = \frac{4}{(e^{av}a\hbar t)^2+4}. \tag{3.5}$$

Remarking above, we see $e^{av}\psi(-t,v) = a\hbar te^{av} + \sqrt{(a\hbar te^{av})^2+4}$ and

$$\begin{aligned}\tilde{F}_{-t}(u,v) &= e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}e^{av}\psi(-t,v)}\frac{e^{av}\psi(-t,v)}{e^{av}\psi(-t,v) - e^{av}a\hbar t}\\ &= e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\left(e^{av}a\hbar t + \sqrt{(e^{av}a\hbar t)^2+4}\right)}\Big(1 + \frac{e^{av}a\hbar t}{\sqrt{(e^{av}a\hbar t)^2+4}}\Big).\end{aligned}$$

We have also

$$\overline{\tilde{F}_{-t}(u,v)} = \tilde{F}_t(u,v)\big(\sqrt{(e^{av}a\hbar t)^2+4} + e^{av}a\hbar t\big)^2.$$

The reason why $\tilde{F}_t(u,v)$ is not a unitary element is that $e^{av}*u$ is not hermite and $\overline{e^{av}*u} = u*e^{av}$.

Since $e^{av}*u = e^{av}(u+\frac{a\hbar i}{2})$ and $u*e^{av} = e^{av}(u-\frac{a\hbar i}{2})$, we have $e^{av}u = e^{av}*(u-\frac{a\hbar i}{2})$ is an hermite element. Thus, if we replace u by $u+\frac{a\hbar}{2}$ and construct

$$\hat{F}_t(u,v) = e_*^{ti(e^{av}u)},$$

then,

$$\hat{F}_t(u,v) = e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}(-e^{av}\hbar t + \sqrt{(e^{av}a\hbar t)^2+4})}\Big(\frac{2}{\sqrt{(e^{av}a\hbar t)^2+4}}\Big).$$

It is easy to see that

$$\overline{\hat{F}_{-t}(u,v)} = \hat{F}_t(u,v).$$

On the other hand, for the original $\tilde{F}_t(u,v)$ we can check that the bracket vanishes

$$\left[e^{av} * u,\ e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\left(\sqrt{(e^{av}a\hbar t)^2+4}+e^{av}a\hbar t\right)}\left(1+\frac{e^{av}a\hbar t}{\sqrt{(e^{av}a\hbar t)^2+4}}\right)\right] = 0,$$

by direct computations. By these, we see that

Proposition 4.

$$e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\left(\sqrt{(e^{-2v}a\hbar t)^2+4}+e^{av}a\hbar t\right)}\left(1+\frac{e^{av}a\hbar t}{\sqrt{(e^{av}a\hbar t)^2+4}}\right)$$

is the solution of the right equation

$$\frac{d}{dt}R_t = R_t * (ie^{av} * u), \qquad R_0 = 1.$$

3.1. The solution of the initial condition $e^{\frac{2i}{\hbar}uv}$. The distribution which gives the solution φ in (2.4) is given by

$$\varphi(\xi,\eta) = \delta\big(\xi + \frac{2}{a\hbar}\log(-2a\eta)\big).$$

Thus, the solution at the time t is

$$\varphi(\xi, \frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av}) = \delta\Big(\xi + \frac{2}{a\hbar}\log\big(e^{-2av} - a\hbar t e^{-a\hbar\xi/2}\big)\Big).$$

This is similar to (2.6), but since the coefficient is changed from $a\hbar$ to $a\hbar/2$, the behavior is changed as follows: If $2\hbar t < 1$, then we see

$$\xi + \frac{2}{a\hbar}\log\left(e^{-2av} - a\hbar t e^{-a\hbar\xi/2}\right) > 0,$$

and therefore the right hand side vanishes.

All together, we see the following:

Theorem 5. *For every $t \in \mathbb{R}$, $L_t(e^{\frac{2i}{\hbar}uv})$, $L_t(e^{-\frac{2i}{\hbar}uv})$ is a real analytic solution in t. Moreover, $L_t(e^{-\frac{2i}{\hbar}uv}) = e^{-\frac{2i}{\hbar}uv}$ and $L_t(e^{\frac{2i}{\hbar}uv}) = 0$ for $2\hbar t \leq 1$.*

Proof. For initial functions $e^{\frac{2i}{\hbar}uv}$ and $e^{-\frac{2i}{\hbar}uv}$, these Fourier images are $\delta(\xi-\frac{4v}{\hbar})$ and $\delta(\xi)$. They satisfy the Condition 1.

The first one is obtained by viewing the equation as the equation of vector field $(e^{av}*u)*e^{-\frac{2i}{\hbar}uv}=0$. Actually this is obtained by the fact that the initial function is $\varphi(\xi,\eta)=\delta(\xi)$ (cf. (3.6)). □

The relation $[\frac{1}{a}e^{-av}, e^{av}*u]=-\hbar i$ gives the following:

Proposition 6. *Let*

$$\tilde{F}_t(u,v)=e^{-\frac{2iu}{a\hbar}\log\frac{1}{2}\left(\sqrt{(e^{av}a\hbar t)^2+4}-e^{av}a\hbar t\right)}\Big(1-\frac{e^{av}a\hbar t}{\sqrt{(e^{av}a\hbar t)^2+4}}\Big),$$

then

$$[\frac{1}{a\hbar}e^{-av}, \tilde{F}_s(u,v)]=s\tilde{F}_s(u,v).$$

Proof. At $t=0$, we have $[e^{-av},\tilde{F}_0(u,v)]=0$. Taking the derivative

$$\begin{aligned}\frac{d}{dt}[\frac{1}{a}e^{-av},\tilde{F}_t(u,v)]&=[\frac{1}{a}e^{-av}, ie^{av}*u*\tilde{F}_t(u,v)]\\&=\hbar\tilde{F}_t(u,v)+ie^{av}*u*[\frac{1}{a}e^{-av},\ \tilde{F}_t(u,v)].\end{aligned}$$

Set $[\frac{1}{a}e^{-av},\tilde{F}_t(u,v)]=\tilde{F}_t(u,v)*g_t$ and looking for the solution, then $\tilde{F}_t(u,v)*\frac{d}{dt}g_t=\hbar\tilde{F}_t(u,v)$ gives $\hbar t\tilde{F}_t(u,v)$ is a real analytic solution. However, this does not give the uniqueness. This is given by the direct calculation: Since

$$\Big[e^{-av},\tilde{F}_t(u,v)\Big]=e^{-av}\big(\tilde{F}_t(u-\frac{\hbar ia}{2},v)-\tilde{F}_t(u+\frac{\hbar ia}{2},v)\big),$$

(3.4) gives that the result. □

Rewriting the above, for $(e^{-av}-a\hbar s)*\tilde{F}_s=\tilde{F}_s*e^{-av}$, then, we have the following.

Lemma 7. *If $s\geq 0$,*

$$\begin{aligned}\tilde{F}_{-s}(u,v)*(e^{-av}*\tilde{F}_s(u,v))-a\hbar s&=\tilde{F}_{-s}(u,v)*\big((e^{-av}-a\hbar s)*\tilde{F}_s(u,v)\big)\\&=\tilde{F}_{-s}(u,v)*(\tilde{F}_s(u,v)*e^{-av})=e^{-av}.\end{aligned}$$

Proof. $\tilde{F}_s(u,v) * e^{-av}$ is defined and the solution of left equation with the initial function e^{-av}. The first equality is given by the distributive law and the exponential law. On the other hand, for $t \geq 0$ the uniqueness for the left equation, the exponential law gives $\tilde{F}_{-t}(u,v) * (\tilde{F}_s(u,v) * g) = \tilde{F}_{-t+s}(u,v) * g$. Then, set $t = s$. □

This is indeed a dangerous equality. If the associativity holds then, applying F_s and F_{-s} to both side from left and right, we see that the above formula is valid for $s < 0$. It follows

$$1 = (F_{-s} * e^{av} * F_s) * (F_{-s} * e^{-av} * F_s) = (F_s * e^{av} * F_{-s}) * (e^{-av} + a\hbar s),$$

and hence $e^{-av} + a\hbar s$ is invertible for $s < 0$. This makes contradiction. Thus, we see that the associativity must break at some point.

We want to see how the associativity breaks down.

3.2. For general initial functions. For a general initial function g, the solution for $t \leq 0$ is given by (2.7) and this is

$$\begin{aligned} &G_t(u,v) \\ &= \int e^{i\xi u}\hat{g}\Big(\xi + \frac{1}{a\hbar}\log\big(e^{-2av} - a\hbar t e^{-a\hbar\xi/2}\big), -\frac{1}{2a}\log(e^{-2av} - a\hbar t e^{-a\hbar\xi/2})\Big)\,d\xi. \end{aligned}$$

This is determined uniquely by g. Denote this by $\tilde{L}_t(g)$ $(t \leq 0)$. The real analyticity in t does not hold unless g has some smooth property. But for $t \leq 0$, $\tilde{L}_t$ is a linear operator of $\mathcal{S}_{u,v}$ into $\mathcal{S}_{u,v}$.

For $t > 0$, there is a place such that $e^{-2av} - a\hbar t e^{-a\hbar\xi/2} < 0$. In such a place the solution is not determined by the initial function g, but φ may be chosen arbitrarily so that $G_t(u,v)$ is C^∞. This implies that for $t > 0$ the uniqueness does not hold for the left equation.

Avoiding this inconvenience, we fix the solution. One way to fix the solution is that we set $\varphi(\xi,\eta) = 0$ for $\eta > 0$.

This means we set to 0 on the domain $e^{-2av} - a\hbar t e^{-a\hbar\xi/2} < 0$. Here $G_t(u,v)$ must be C^∞. For this we must have that

$$\begin{aligned} &\int \varphi(\xi, \frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av})e^{i\xi u}\,d\xi \\ &= \int_{-\infty}^{\frac{2}{a\hbar}\log(a\hbar t) - \frac{1}{\hbar}4v} \varphi(\xi, \frac{\hbar t}{2}e^{-a\hbar\xi/2} - \frac{1}{2a}e^{-2av})e^{i\xi u}\,d\xi \end{aligned}$$

is C^∞ function and this looks like a new condition. But the imposed condition is in fact that this must be a slowly increasing distribution for every fixed t, v. Hence the C^∞-ness in t, v is satisfied automatically.

Thus, if we set

$$(3.6)\quad \varphi_0(\xi, \frac{\hbar t}{2}e^{-a\hbar\xi/2}-\frac{1}{2a}e^{-2av}) = \begin{cases} \hat{g}\Big(\xi+\frac{1}{a\hbar}\log\big(e^{-2av}-a\hbar te^{-a\hbar\xi/2}\big), -\frac{1}{2a}\log(e^{-2av}-a\hbar te^{-a\hbar\xi/2})\Big) & (e^{-2av}-a\hbar te^{-a\hbar\xi/2}>0) \\ 0, & (e^{-2av}-a\hbar te^{-a\hbar\xi/2}\leq 0), \end{cases}$$

then we have the solution for $t>0$. Write this by $L_t(g)$. The second line of (3.6) may not be used if $\lim_{v\to\infty} g(\xi,v)=0$. Thus, $L_t(g)$ is defined for all $g\in\mathcal{S}_{u,v}$. For $t\geq 0$, it is clear that $L_t(g)=\tilde{L}_t(g)$.

In any way, for every initial function $g(u,v)\in\mathcal{S}_{u,v}$, the solution of the equation (2.3) is defined for all $t\in\mathbb{R}$.

The direct calculation shows this is a solution. This proved Theorem 1.

L_t for $t>0$ is by the equality(3.6) has the property that $L_t(g)=0$ means $g=0$. That is to say $L_t:\mathcal{S}_{u,v}\to\mathcal{S}_{u,v}$ is a monomorphism for $t\geq 0$.

However there are other solutions, and the uniqueness does not hold. For $t>0$, $L_t(g)=\tilde{F}_t * g$ may not hold. Such identity holds only for $t\leq 0$.

The above observation tells us many: For $t\geq 0$, $\tilde{L}_{-t}(g)$ is determined by g. If we denote by g_t a solution with initial function g, then $\tilde{L}_{-t}(g_t)=g$ holds.

We have many g_t. Hence the linearity of the left equation gives for $t>0$ that $\tilde{L}_{-t}$ has the non trivial kernel ($\tilde{L}_{-t}(K)=0$). $\tilde{L}_{-t}(K)$ may be written as $\tilde{L}_{-t}(1)*K$, $\tilde{F}_t=\tilde{L}_t(1)$ and the exponential law holds for $\tilde{F}_t$. Since $\tilde{F}_t*\tilde{F}_{-t}=1$, $0=\tilde{F}_t*(\tilde{F}_{-t}*K)\neq K$ shows that the associativity breaks down at such place.

Lemma 8. *$\tilde{L}_{-t}:\mathcal{S}_{u,v}\to\mathcal{S}_{u,v}$ has non-trivial kernel for $t>0$, and $\tilde{L}_{-t}L_t=I$.*

3.3. $\mathrm{Ker}\,\tilde{L}_t$**.** For $t\leq 0$, $\tilde{L}_t(g)$ is uniquely determined for initial functions. Here we consider the case $t<0$. Recall (3.3) at first. In (3.6) the initial function $\hat{g}(\xi,v)$ satisfies for some $t<0$ that

$$(3.7)\quad \varphi(\xi,\ \frac{\hbar t}{2}e^{-a\hbar\xi/2}-\frac{1}{2a}e^{-2av}) = \hat{g}\Big(\xi+\frac{1}{a\hbar}\log\big(e^{-2av}-a\hbar te^{-a\hbar\xi/2}\big), -\frac{1}{2a}\log(e^{-2av}-a\hbar te^{-a\hbar\xi/2})\Big)=0.$$

Let $G(u,v)=\int \hat{g}(\xi,v)e^{i\xi u}\,d\xi$ be the initial function corresponding to $\hat{g}(\xi,v)$. Then setting $K(u,v)=G(u,v)e^{-\frac{2i}{\hbar}uv}$, we see that $L_t(K)=\tilde{F}_t(u,v)*K=0$.

We want to characterize $\hat{g}(\xi, v)$. Fix $t < 0$ and we first determine the domain

$$E_t = \{(\xi + \frac{1}{\hbar}\mu, -\frac{1}{2}\mu) \mid \mu = \frac{1}{a}\log(e^{-2av} - a\hbar t e^{-a\hbar\xi/2}), \xi, v \in \mathbb{R}\}. \tag{3.8}$$

If the supp $\hat{g}$ is in the complement of E_t, $\hat{g}$ satisfies this condition.

If $t < 0$ is fixed, the above μ in (3.8) moves in the range $\mu > \log(-a\hbar t e^{-a\hbar\xi/2})$ for every ξ, since e^{-2av} takes arbitrary positive number, By this, we see

$$E_t = \{(\xi, \eta) \mid \eta < \frac{\hbar}{2}\xi - \frac{1}{a}\log(-a\hbar t)\}.$$

A distribution, supported in $D_t = E_t^c$ and satisfying Condition 1, satisfies also (3.7). In particular for every η this must be a slowly increasing distribution with respect to ξ. By the property of the domain D_t, it is finite for the direction $\xi > 0$. If $0 \geq t > t'$, $D_t \subset D_{t'}$ is clear.

Initial function $\hat{g}(\xi, v)$ reflects $\tilde{L}_t(g)$ at $t < 0$ only for the part $(\xi, v) \in E_t$.

Proposition 9. *For $\hat{g}(\xi, v)$, if* $\operatorname{supp}\hat{g} \subset D_t$*, $t < 0$, Setting*

$$G(u, v) = \int \hat{g}(\xi, v)e^{i\xi u}\, d\xi, \qquad K(u, v) = G(u, v)e^{-\frac{2i}{\hbar}uv},$$

we have $L_t(K) = \tilde{F}_t(u, v) * K = 0$.

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