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Analyticity of Solutions to Nonlinear Schrödinger Equations

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Abstract. In this paper, we consider the Cauchy problem for the following nonlinear Schrödinger equations

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u = \mathcal{N}(u, \nabla u, \bar{u}, \nabla \bar{u}), & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \quad N \geq 3, \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}^N, \end{cases}$$

where

$$\mathcal{N}(u, v, \bar{u}, \bar{v}) = \sum_{\substack{2 \leq |\alpha|+|\beta| \leq l, \\ r \leq |\beta| \leq l}} \lambda_{\alpha\beta} u^{\alpha_1} \bar{u}^{\alpha_2} v^{\beta_1} \bar{v}^{\beta_2},$$

$\lambda_{\alpha\beta} \in \mathbb{C}$, $l \geq 2$, $r \geq 1$ if $N = 3, 4$, or $r \geq 0$ if $N \geq 5$. We study analyticity of global solutions for (NLS) with small initial data u_0. To be precise, we show that global solutions of (NLS) are analytic in space and time for $|t| \neq 0$ if the norm of the initial data is sufficiently small in analytical space with respect to $x \cdot \nabla$.

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§1. Introduction

In this paper, we consider the Cauchy problem for the following nonlinear Schrödinger equations

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u + \Delta u = \mathcal{N}(u, \nabla u, \bar{u}, \nabla \bar{u}), & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \quad N \geq 3, \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}^N, \end{cases}$$

where

$$\mathcal{N}(u, v, \bar{u}, \bar{v}) = \sum_{\substack{2 \leq |\alpha|+|\beta| \leq l, \\ r \leq |\beta| \leq l}} \lambda_{\alpha\beta} u^{\alpha_1} \bar{u}^{\alpha_2} v^{\beta_1} \bar{v}^{\beta_2},$$

with $\lambda_{\alpha\beta} \in \mathbb{C}$, $l \geq 2$, $r \geq 1$ if $N = 3, 4$, or $r \geq 0$ if $N \geq 5$.

Our purpose in this paper is to prove analyticity of solutions in space and time to nonlinear Schrödinger equations with nonlinearity of power greater than 2 including the derivative of unknown functions if the norm of the initial data is sufficiently small in analytical space with respect to $x \cdot \nabla$.

Global existence of solutions to (NLS) is started in [19, 20, 22]. They show the existence of global solutions under the condition $\mathbf{Re}\ \partial_v \mathcal{N} = 0$. N. Hayashi [8, 9] proves global existence of solutions to (NLS) with $l = 2$ without the condition $\mathbf{Re}\ \partial_v \mathcal{N} = 0$ by using the operators $P = x \cdot \nabla + 2t\partial_t$, $\Omega = (\Omega_{j,k})_{(1 \leq j < k \leq N)}$ with $\Omega_{j,k} = x_j \partial_k - x_k \partial_j$ and ∂ which commute with the linear Schrödinger equation. N. Hayashi and T. Ozawa [17] study global existence of solutions to (NLS) by using a gauge transformation and the above operators. N. Hayashi and H. Hirata [10] obtain global existence result to (NLS) by using the smoothing properties of solutions to linear Schrödinger equation. H. Chihara [1, 2, 3] proves global existence of solutions to (NLS) by applying the zeroth-order pseudo-differential operators in order to make use of the smoothing properties of solutions to linear Schrödinger equation (see also [6]). N. Hayashi, C. Miao and P.I. Naumkin [14] also apply the operators as H. Chihara [1, 2, 3] uses and show that global solution of (NLS) exists if the initial data u_0 is sufficiently small in $\mathbf{H}^{m,0} \cap \mathbf{H}^{m-2,2}$ with $m \geq [N/2] + 3$, where $\mathbf{H}^{m,s} = \{\phi \in \mathbf{L}^2\ ;\ \|(1 + |x|^2)^{s/2}(1 - \Delta)^{m/2}\phi\|_{\mathbf{L}^2} < \infty\}$, $m, s \in \mathbb{R}^+$.

N. Hayashi and K. Kato [13] prove analyticity of solutions to (NLS) with $l = 2$ in space if the initial data is analytical with respect to $x \cdot \nabla$ and ∂. N. Hayashi and K. Kato [12] study regularity in time for the nonlinear Schrödinger equations with nonlinear terms not including the unknown derivative function. They show that solution is in Gevrey class of order s (≥ 1) in time variable except for $t = 0$ if the initial data is in Gevrey class of order s with respect to $x \cdot \nabla$ and ∂. K. Kato and K. Taniguchi [18] treat Gevrey regularity for the nonlinear Schrödinger equations under the condition that the nonlinearity is in Gevrey class with respect to t, x and u. They prove analyticity of solutions for $t \neq 0$ if the initial data is in Gevrey class of order s (≥ 1) with respect to $x \cdot \nabla$.

H. Chihara [5] and N. Hayashi, P.I. Naumkin and P.N. Pipolo [16] study the analyticity for the cubic derivative nonlinear Schrödinger equations with nonlinear terms satisfying the gauge invariant condition that

$$\mathcal{N}\left(e^{i\theta}u, e^{i\theta}v, \overline{e^{i\theta}u}, \overline{e^{i\theta}v}\right) = e^{i\theta}\mathcal{N}(u, v, \overline{u}, \overline{v}) \qquad \text{for any } \theta \in \mathbb{R} \text{ and } u, v \in \mathbb{C}.$$

They make use of the smoothing properties of solutions to the linear Schrödinger equation to overcome the so-called loss of derivative. They do not use analyticity of the initial data. N. Hayashi, P.I. Naumkin and P.N. Pipolo [16] show that solution is analytic in one space dimension if the initial data is sufficiently small in Sobolev space $\mathbf{H}^{3,0}$ and decays exponentially. By a diagonalization technique which H. Chihara [4], N. Hayashi and E.I. Kaikina [11] and P.N. Pipolo [21] use, H. Chihara [5] announces gain of analyticity if the initial data without smallness is in Sobolev space $\mathbf{H}^{\theta,0}$ with $\theta > N/2+3$ and decays at infinity exponentially.

Applying the energy method to (NLS), so-called loss of derivative occurs because nonlinear terms of (NLS)do not satisfy the condition $\mathbf{Re}\partial_v \mathcal{N} = 0$. To overcome it, we use the following operator

$$\mathcal{S}(\varphi) = \prod_{j=1}^{N} \mathcal{S}_j(\varphi_j), \tag{1.1}$$

which is used to obtain a smoothing property of the linear Schrödinger equation in [14]. Here $\mathcal{S}_j(\varphi_j) = \cosh(\varphi_j) + \sinh(\varphi_j)\mathcal{H}_j$, $\mathcal{H}_j$ is Hilbert transformation with respect to the j-th variable. Here, we choose ϕ_j as follows:

$$\varphi_j(t, x_j) = \varepsilon^2 \langle t \rangle^{-\sigma} \int_{-\infty}^{x_j \langle t \rangle^{-\mu}} \langle y \rangle^{-2\mu} dy, \tag{1.2}$$

where $\mu = \frac{1}{2} + \sigma$ with $\sigma \in (0, \frac{1}{8})$. The original one of such operators is introduced by S. Doi [6] for Schrödinger type equations with derivatives (see also [2]).

$$\mathcal{S}^{(1)} = \exp\left(\sum_{j=1}^{N} \int_{-\infty}^{x_j} \left(1 + {x'_j}^2\right)^{-(1+\delta)} dx'_j \frac{D_j}{\langle D_j \rangle} \right),$$

where $\delta > 0$, $D_j = i\partial_j$ and $\langle D_j \rangle = (1 - \partial_j^2)^{1/2}$. H. Chihara [3, 4] uses the following operator based on $\mathcal{S}^{(1)}$:

$$\mathcal{S}^{(2)} = \exp\left(\sum_{j=1}^{N} \int_{-\infty}^{x_j} \|u(t, x'_j)\|^2_{\mathbf{L}^2 \backslash \mathbf{L}^2_{x_j}} dx'_j \frac{D_j}{\langle D_j \rangle} \right).$$

He applies the above operator to nonlinear Schrödinger equations with nonlinearity of power greater than 3. To use $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$, they need the knowledge concerning pseudo-differential operators and complicated calculation to obtain the generalized energy inequalities including these operators. However, for nonlinear Schrödinger equations obtained by transforming variables of Ishimori and Davey-Stewartson systems N. Hayashi and P.I. Naumkin [15] use

$$\mathcal{S}^{(3)}(\varphi) = \prod_{j=1}^{2} (\mathcal{S}_j(\varphi_j) = \cosh(\varphi_j) + \sinh(\varphi_j)\mathcal{H}_j)$$

where

$$\varphi_j(t,x_j) = \frac{1}{\varepsilon}\int_{-\infty}^{x_j} \|(1-\Delta)u(t,x_j')\|^2_{\mathbf{L}^2_{x_k}}\, dx_j' \qquad \text{for } j,k=1,2,\ j\neq k.$$

The calculation with communitators become explicit because of the operator $\mathcal{S}$. N. Hayashi, C. Miao and P.I. Naumkin [14] introduce the operator (1.1) based on the operator $\mathcal{S}^{(3)}$. We use the operator $\mathcal{S}$ to prove our result. In Lemma 2.4 of section 2, we can get an energy estimate in which we have the norm of half derivative of the unknown function u by using the operator $\mathcal{S}$. Owing to the above, we are able to overcome loss of derivative.

We easily see that the operator $\mathcal{S}$ acts continuously from $\mathbf{L}^2$ to $\mathbf{L}^2$ with the following estimate

$$\|\mathcal{S}(\varphi)\psi\| \le C\|\psi\|.$$

The inverse operator $\mathcal{S}_j^{-1} = (1+i\tanh(\varphi_j)\mathcal{H}_j)^{-1}\frac{1}{\cosh(\varphi_j)}$ exists and is continuous in $\mathbf{L}^2$:

$$\|\mathcal{S}^{-1}(\varphi)\psi\| \le (1-\tanh(\|\varphi\|_\infty))^{-1}\|\psi\| \le C\|\psi\|.$$

And to prove analyticity in space and time, we have the help of a operator $P = x\cdot\nabla + 2t\partial_t$. This operator have properties $[P,\mathcal{L}] = -2\mathcal{L}$,

$$(t\partial_t)^l = \frac{1}{2^l}\sum_{l=l_1+l_2}\frac{l!}{l_1!l_2!}(-x\cdot\nabla)^{l_1}P^{l_2}$$

and

$$t\Delta P^\nu u = -iP^{\nu+1}u + i(x\cdot\nabla)P^\nu u + 2t(P+2)^\nu\mathcal{N}.$$

where $\mathcal{L} = i\partial_t + \Delta$, $[A,B] = AB - BA$, $\mathcal{N}$ is a nonlinear term. From these properties, we show analyticity in space and time.

Notation and function spaces. We use Lebesgue space

$$\mathbf{L}^p = \{\phi : \phi \text{ is measurable on } \mathbb{R}^N, \|\phi\|_{\mathbf{L}^p} < \infty\},$$

where

$$\|\phi\|_{\mathbf{L}^p} = \begin{cases} \left(\int_{\mathbb{R}^N}|\phi(x)|^p dx\right)^{1/p}, & \text{if } 1\le p<\infty,\\ \text{ess.sup}\{|\phi(x)|; x\in\mathbb{R}^N\}, & \text{if } p=\infty.\end{cases}$$

Inner product on $\mathbf{L}^2$ is defined by $(f,g) = \int f\overline{g}dx$. We define weighted Sobolev space

$$\mathbf{H}^{m,s} = \{\phi\in\mathbf{L}^2; \|\phi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\phi\|_{\mathbf{L}^2} < \infty\}.$$

For convenience, $\mathbf{H}^m$ stands for $\mathbf{H}^{m,0}$, and we write $\|\cdot\|_m = \|\cdot\|_{m,0}$. We also use $\|\cdot\| = \|\cdot\|_{\mathbf{L}^2}$.

We let $\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We denote the operators $Q = x\cdot\nabla + 2it\Delta$, and $J = (J_j)_{(1\le j\le N)}$, with $J_j = x_j + 2it\partial_j$ and $x\cdot\nabla = x_1\partial_1+\cdots+x_N\partial_N$. We note these operators have the following relations:

$$\begin{aligned} Q &= P + 2it\mathcal{L} \\ &= J\cdot\nabla \\ &= \mathcal{U}(t)x\mathcal{U}(-t)\cdot\nabla, \end{aligned}$$

where $\mathcal{L} = i\partial_t + \Delta$ and

$$\mathcal{U}(t)\phi = (2\pi it)^{-N/2}\int \exp(i|x-x'|^2/4t)\phi(x')dx'.$$

We also have the following properties with the commutator:

$$\left\{\begin{aligned} &[Q,\nabla] = [P,\nabla] = -\nabla, \\ &[Q,J] = [P,J] = J, \\ &[P,Q] = [\Omega,P] = [\Omega,Q] = 0, \\ &[\partial_j, J_j] = \delta_{ij}, \\ &[\Omega_{j,k},\partial_l] = \delta_{kl}\partial_j - \delta_{jl}\partial_k, \\ &[\Omega_{j,k},\Delta] = 0, \end{aligned}\right. \tag{1.3}$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. We define operator vectors $\Gamma = (P,\Omega,\Delta,1)$ and $\Theta = (Q,\Omega,\Delta,1)$. We also use a operator vector $\Gamma^2 = (P,\Omega,\Delta,1)^2 = (P^2, P\Omega, P\Delta, P, \Omega^2, \Omega\Delta, \Omega, \Delta^2, \Delta, 1)$.

Let $\mathcal{F}_j\phi$ be the Fourier transform of $\phi \in C_0^\infty(\mathbb{R}^N)$ with respect to j-th variable, namely

$$\begin{aligned} &\mathcal{F}_j\phi(x_1,\ldots,x_{j-1},\xi_j,x_{j+1},\ldots,x_N) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_N)e^{-i\xi_j x_j}dx_j. \end{aligned}$$

We also denote by $\mathcal{F}_j^{-1}\psi$ the inverse Fourier transform of the function $\psi \in C_0^\infty(\mathbb{R}^N)$ with respect to j-th variable,

$$\begin{aligned} &\mathcal{F}_j^{-1}\psi(\xi_1,\ldots,\xi_{j-1},x_j,\xi_{j+1},\ldots,\xi_N) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \psi(\xi_1,\ldots,\xi_{j-1},\xi_j,\xi_{j+1},\ldots,\xi_N)e^{i\xi_j x_j}d\xi_j. \end{aligned}$$

We use the following notation $|\partial_j| = \mathcal{F}_j^{-1}|\xi_j|\mathcal{F}_j = -\mathcal{H}_j\partial_j$. The Hilbert transformation with respect to the variable x_1 is defined as follows

$$\begin{aligned}
&\mathcal{H}_j\phi(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_N)\\
&\equiv \mathcal{H}_{x_j}\phi(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_N)\\
&= \frac{1}{\pi}\mathrm{Pv}\int_{\mathbb{R}} \frac{\phi(x_1,\dots,x_{j-1},z,x_{j+1},\dots,x_N)}{x_j - z}dz\\
&= -i\mathcal{F}_j^{-1}\frac{\xi_j}{|\xi_j|}\mathcal{F}_j\phi,
\end{aligned}$$

where $j = 1,\dots,N$ and Pv means the principal value of the singular integral. The fractional derivative $|\partial_j|^\gamma$, $\gamma \in (0,1)$ is defined by

$$\begin{aligned}
|\partial_j|^\gamma\phi &= \mathcal{F}_j^{-1}|\xi_j|^\gamma\mathcal{F}_j\phi\\
&= C\int_{\mathbb{R}}(\phi(x_1,\dots,x_{j-1},x_j+z,x_{j+1},\dots,x_N)\\
&\qquad - \phi(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_N))\frac{dz}{|z|^{1+\gamma}}
\end{aligned}$$

and similarly we have

$$\begin{aligned}
|\partial_j|^\gamma\mathcal{H}_j\phi &= -i\mathcal{F}_j^{-1}\frac{\xi_j}{|\xi_j|}|\xi_j|^\gamma\mathcal{F}_j\phi\\
&= C\int_{\mathbb{R}}(\phi(x_1,\dots,x_{j-1},x_j+z,x_{j+1}\dots,x_N)\\
&\qquad - \phi(x_1,x_{j-1},\dots,x_j,x_{j+1},\dots,x_N))\frac{dz}{z|z|^\gamma},
\end{aligned}$$

with some constant C, see [23].

We note that factorial of zero and negative number is always considered as 1. In other word, ν and ν_j $(j = 1,2,3)$ are $\max(\nu,1)$ and $\max(\nu_j,1)$ respectively in the notation such as

$$\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu}{\nu_1\nu_2\nu_3!}\cdots \quad \text{or} \quad \sum_{\nu=\nu_1+\nu_2+\nu_3}\left(\frac{\nu}{\nu_1\nu_2\nu_3}\right)^2\cdots.$$

We define function spaces in order to show our main results.

$$\mathbf{Z}_{m,A} = \left\{\phi \in \mathbf{L}^2; \|\phi\|_{\mathbf{Z}_{m,A}} < \infty\right\},$$

where $m, A \in \mathbb{R}^+$,

$$\|\phi\|^2_{\mathbf{Z}_{m,A}} = \sum_{\nu=0}^{\infty}\frac{A^{2\nu}}{(\nu-1)!^2}\|(x\cdot\nabla)^\nu\phi\|^2_{\mathbf{B}_m}$$

and

$$\|\phi\|_{\mathbf{B}_m} = \sum_{|a|+b\le 2} \|\Omega^a (x\cdot\nabla)^b \phi\|_{(m-2|a|-2b)}.$$

And putting $\omega_j = \varepsilon\langle x_j \langle t\rangle^{-\mu}\rangle^{-\mu}$, the function space $\mathbf{Y}_{m,A}$ is defined by

$$\mathbf{Y}_{m,A} = \left\{\phi \in C(\mathbb{R}; \mathbf{L}^2(\mathbb{R}^n)); \|\phi\|_{\mathbf{Y}_{m,A}} < \infty\right\},$$

where $m, A \in \mathbb{R}^+$,

$$\|\phi\|^2_{\mathbf{Y}_{m,A}} = \sum_{\nu=0}^{\infty} \frac{A^{2\nu}}{(\nu-1)!^2} \|P^\nu \phi\|^2_{\mathbf{X}_m}$$

and

$$\begin{aligned}\|\phi\|^2_{\mathbf{X}_m} = &\sup_{t\in\mathbb{R}} \left\|\Gamma^2\phi\right\|^2_{m-4} + \sup_{t\in\mathbb{R}} \|\Gamma\Theta\phi\|^2_{m-4} \\ &+ \sup_{t\in\mathbb{R}} \|\Theta\phi\|^2_{m-2} + \sup_{t\in\mathbb{R}} \langle t\rangle^{-1} \left\|Q^2\phi\right\|^2_{m-4} \\ &+ \frac{1}{2}\sum_{k=1}^{N} \int_0^\infty \left\|\omega_k \mathcal{S}\sqrt{|\partial_k|}\Gamma^2 \partial_j^{m-4}\phi(\tau)\right\|^2 \frac{d\tau}{\langle\tau\rangle^{1/2+2\sigma}}.\end{aligned}$$

We state our main results.

Theorem 1.1. *Suppose that the initial data u_0 belong to $\mathbf{Z}_{m,A}$ with some positive constant A and $m \ge [\frac{N}{2}] + 6$ for $N \ge 3$ and satisfy $\|u_0\|_{\mathbf{Z}_{m,A}} \le \varepsilon$ with a sufficiently small positive number ε. Then (NLS) has a unique global solution $u \in C(\mathbb{R}; \mathbf{H}^m)$ such that*

$$P^\nu u \in C(\mathbb{R}; \mathbf{H}^m) \qquad (\nu = 0, 1, 2, \dots)$$

and satisfies the estimate

$$\sup_{t\in\mathbb{R}} \sum_{\nu=0}^{\infty} \left\| \frac{A^\nu P^\nu}{(\nu-1)!} u(t)\right\|^2_m \le C\varepsilon^2.$$

Remark 1.1. We give two function as example of the initial data.

(1) Let δ be sufficiently small. For a and b with $2b - N/2 - 1 > a > m - N/2$, $u_0(x) = \delta|x|^a(1+|x|^2)^{-b}$.

(2) Let δ sufficiently small. For $c \ge m + 2$, $u_0(x) = \delta|x|^c e^{-|x|}$.

These functions belong to $\mathbf{Z}_{m,A}$ for some $A > 0$.

Theorem 1.2. *Let u be the solution of (NLS) constructed in Theorem 1.1. Then there exist constants C_{17}, A_3, A_8 and A_9 such that*

$$\|a(x)^{|\mu|+2\kappa_1}\partial_t^{\kappa_1}\partial^\mu u\|_m \leq C_{17}|t|^{-\kappa_1}\max\{1,|t|^{-|\mu|-\kappa_1}\}A_3^{|\mu|}A_8^{|\mu|+\kappa_1}A_9^{\kappa_1}(|\mu|+\kappa_1)!,$$

for $t \neq 0$, any $\kappa_1 \in \mathbb{N}\cup\{0\}$ and any multi-index μ, where $a(x) = \langle x\rangle^{-N} = 1/(1+|x|^2)^{N/2}$.

Remark 1.2. Theorem 1.2 shows the analyticity in space and time of solutions to (NLS).

The rest of this paper is organized as follows. In section 2, we present the energy estimate including the operator $\mathcal{S}$, based on a smoothing property of the free linear Schrödinger equation and some estimates for nonlinearities. Section 3 is assigned as the proof of Theorem 1.1. In section 4, the result of analyticity is shown. Its proof has four steps.

§2. Preliminaries

In this section, we prepare some lemmas to prove the existence of solutions. We study the following linear Schrödinger equation to get the energy estimates of the solutions.

$$\begin{cases} \mathcal{L}u = f, & (t,x)\in\mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^N, \end{cases} \tag{2.1}$$

where $\mathcal{L} = i\partial_t + \Delta$.

In order to show two important lemmas, we prepare some lemmas.

Lemma 2.1 (The Gagliardo-Nirenberg inequality). *Let $1 \leq q, r \leq \infty$. Let integer number j, m satisfy the inequality $0 \leq j < m$. Let p be such that*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{(1-a)}{q},$$

where a satisfy $j/m \leq a < 1$ if $m - j - n/r \in \mathbb{N}\cup\{0\}$, and $j/m \leq a \leq 1$ otherwise. Then the following estimate is valid:

$$\sum_{|\alpha|=j}\|\partial^\alpha\phi\|_{\mathbf{L}^p} \leq C_{N,m,j,q,r}\sum_{|\beta|=m}\|\partial^\beta\phi\|_{\mathbf{L}^r}^a\|\phi\|_{\mathbf{L}^q}^{1-a},$$

provided that the right hand side is finite.

For the proof of Lemma 2.1, see, e.g., A. Friedman [7].

Lemma 2.2 (Hayashi-Miao-Naumkin [14], Hayashi-Naumkin [15]). *For* $0 < (1-\theta)/2 < \gamma < 1-\theta < 1$, *the following inequalities*

$$\|[|\partial_j|^\gamma, \phi]\psi\| \le C\|\phi_{x_j}\|_{\mathbf{L}^\infty}^{1-\gamma-\theta}(\|\phi\|_{\mathbf{L}^\infty} + \|\phi_{x_j}\|_{\mathbf{L}^\infty})^{\gamma+\theta}\|\psi\|$$

and

$$\|[|\partial_j|^\gamma \mathcal{H}_j, \phi]\psi\| \le C\|\phi_{x_j}\|_{\mathbf{L}^\infty}^{1-\gamma-\theta}(\|\phi\|_{\mathbf{L}^\infty} + \|\phi_{x_j}\|_{\mathbf{L}^\infty})^{\gamma+\theta}\|\psi\|$$

are valid, provided that the right hand sides are bounded.

In [14] and [15], they show the lemma in the case of $\theta = 1-\gamma$ and $\theta = 1/2$ respectively.

Lemma 2.3. *The following inequalities stand up for a sufficiently small constant* ε.

$$\|[|\partial_j|^{1/2}, \cosh(\varphi_j)\omega_j]\phi\| \le C\varepsilon\langle t\rangle^{-1/4}\|\phi\|,$$
$$\|\sinh(\varphi_j)\mathcal{H}_j[|\partial_j|^{1/2}, \omega_j]\phi\| \le C\varepsilon\langle t\rangle^{-1/4}\|\phi\|,$$
$$\|\sinh(\varphi_j)[|\partial_j|^{1/2}\mathcal{H}_j, \omega_j]\phi\| \le C\varepsilon\langle t\rangle^{-1/4}\|\phi\|,$$

and

$$\|[|\partial_j|^{1/2}, \sinh(\varphi_j)\omega_j]\mathcal{H}_j\phi\| \le C\varepsilon\langle t\rangle^{-1/4}\|\phi\|,$$

where $\omega_j = \varepsilon\langle x_j\langle t\rangle^{-\mu}\rangle^{-\mu}$.

Proof. We have

$$\|\cosh(\varphi_j)\phi\| \le \exp(\|\varphi_j\|_\infty)\|\phi\| \le C\|\phi\|$$

and

$$\|\sinh(\varphi_j)\phi\| \le \exp(\|\varphi_j\|_\infty)\|\phi\| \le C\|\phi\|.$$

where $\gamma = 1/2$ and $\theta = \sigma/(1+2\sigma)$. Hence, from Lemma 2.2, this lemma follows. □

In the next lemma, we can get an energy estimate in which we have the norm of half derivative of the unknown function u by using the operator $\mathcal{S}$ defined by (1.1).

Lemma 2.4 (Hayashi-Miao-Naumkin [14]). *The inequality*

$$\begin{aligned}\|u(\tau)\|^2 &+ \sum_{j=1}^n \int_0^t \|\omega_j \mathcal{S}|\partial_j|^{1/2}u(\tau)\|^2 \frac{d\tau}{\langle\tau\rangle^{1/2+2\sigma}} \\ &\le \|u_0\|^2 + C\int_0^t |\mathbf{Im}(\mathcal{S}u, \mathcal{S}f)|d\tau + C\varepsilon^2\int_0^t \|u(\tau)\|^2\frac{d\tau}{\langle\tau\rangle^{1+\sigma}}\end{aligned}$$

is valid for the solution u *of the Cauchy problem (2.1).*

We also need the next lemma in order to estimate the nonlinear terms of (NLS).

Lemma 2.5 (Hayashi-Miao-Naumkin [14]). *The estimate*

$$\begin{aligned}
&|\mathbf{Im}(\mathcal{S}w\partial_j v, \mathcal{S}u)| \\
\leq& C\varepsilon(\langle t\rangle^{-1/4}\|u\| + \|\omega_j\mathcal{S}|\partial_j|^{1/2}u\|) \\
&\times\left(\left\|\frac{w}{\omega_j^2}\right\|_{\mathbf{L}^\infty}\|\omega_j\mathcal{S}|\partial_j|^{1/2}v\| + C\left(\left\|\frac{\partial_j w}{\omega_j}\right\|_{\mathbf{L}^\infty} + \langle t\rangle^{-1/2}\left\|\frac{w}{\omega_j}\right\|_{\mathbf{L}^\infty}\right)^{1/2-\theta}\right. \\
&\times\left.\left(\left\|\frac{\partial_j w}{\omega_j}\right\|_{\mathbf{L}^\infty} + (1+\langle t\rangle^{-1/2})\left\|\frac{w}{\omega_j}\right\|_{\mathbf{L}^\infty}\right)^{1/2+\theta}\|v\| + C\langle t\rangle^{-1/4}\left\|\frac{w}{\omega_j}\right\|_{\mathbf{L}^\infty}\|v\|\right) \\
&+ C(\|w_{x_j}\|_{\mathbf{L}^\infty} + \langle t\rangle^{-1/2}\|w\|_{\mathbf{L}^\infty})\|u\|\|v\|.
\end{aligned}$$

is valid, provided that the right hand side is bounded.

We improve Lemma 2.2 in [14] to show our theorem and obtain the lemma. Hence, the lemma can be shown in the same way as in the proof of [14, Lemma 2.2] by using Lemma 2.3.

When we deal with both the norms of the half derivative term given by the linear part of (NLS) in Lemma 2.4 and ones given by the nonlinear part of (NLS) in Lemma 2.5, we can overcome the loss of derivative with the nonlinear terms of (NLS).

Next two lemmas are also shown by N. Hayashi, C. Miao and P.I. Naumkin in [14].

Lemma 2.6. *The following estimates are true:*

(1) For all $N \geq 3$,

$$\|\nabla\phi\|_p \leq C\langle t\rangle^{-a}\|\Theta\phi\|^a\|\nabla\phi\|^{1-a},$$

where $a = (N/2)(1-2/p) \geq 0$, $2 \leq p \leq 2N/(N-2)$.

(2) For all $N \geq 3$,

$$\|\phi\|_p \leq C\langle t\rangle^{-a}\|\Theta\phi\|^a\|\phi\|^{1-a},$$

where $a = (N/4)(1-2/p) \geq 0$, and p is such that $2 \leq p \leq \infty$ for $N = 3$, $2 \leq p < \infty$ for $N = 4$, and $2 \leq p \leq 2N/(N-4)$ for the case of the space dimension $N \geq 5$.

(3) For all $N \geq 5$,

$$\|\phi\|_p \leq C\langle t\rangle^{-1}\|\Theta\phi\|_m,$$

where $p \geq 2N/(N-4)$ and $m \geq N/2 - N/p - 2 \geq 0$.

(4) *For the case of $N = 3, 4$,*

$$\|\nabla\phi\|_p \le C\langle t\rangle^{-1-a}\|\Theta^2\phi\|^a\|\Theta\phi\|^{1-a},$$

where $a = (N/4)(1 - 2/N - 2/p) \ge 0$, $p \ge 2N/(N-2)$.

(5) *For all $N \ge 5$,*

$$\|\phi\|_p \le C\langle t\rangle^{-1-a}\|\Theta^2\phi\|^a\|\Theta\phi\|^{1-a},$$

where $a = (N/4)(1 - 2/p) - 1 \ge 0$, and p is such that $2 \le p \le \infty$ for $N = 5, 6, 7$, $2 \le p < \infty$ for $N = 8$, and $2 \le p \ge 2N/(N-8)$ for $N \le 9$.

(6) *For all $N \ge 9$,*

$$\|\phi\|_p \le C\langle t\rangle^{-2}\|\Theta^2\phi\|_m,$$

where $m \ge N/2 - N/p - 4$, $p \ge 2N/(N-8)$.

Lemma 2.7. *We have the estimate for all $n \ge 3$, $m \ge [\frac{N}{2}]+6$ and $\sigma \in (0, \frac{1}{8})$,*

$$\|\langle x\rangle^{1+2\sigma}\phi\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-2\sigma}\|\Theta^2\phi\|_{m-4}^{4\sigma}\|\Theta\phi\|_{m-4}^{1-4\sigma} + C\|\Theta\phi\|_{m-4}.$$

§3. The Proof of Theorem 1.1

We consider the linearized version of the Cauchy problem (NLS)

(LE)

$$\begin{cases} i\partial_t u + \Delta u = \mathcal{N}(v, \partial_1 v, \dots, \partial_N v, \overline{v}, \partial_1\overline{v}, \dots, \partial_N\overline{v}), & (t, x) \in \mathbb{R}\times\mathbb{R}^N, \\ u(0, x) = u_0(x), \qquad x \in \mathbb{R}^N. \end{cases}$$

We assume that $\|v\|_{\mathbf{Y}_{m,A}} \le \varepsilon$, $m \ge \left[\frac{N}{2}\right] + 6, m \in \mathbb{N}$ and $\varepsilon = \|u_0\|_{\mathbf{Z}_{m,A}}$. Then we define a mapping $u = \Psi v$ by the above problem and show that Ψ is a mapping from $(\mathbf{Y}_{m,A})^{N+1}$ into itself. We discuss the case $t > 0$ only since the case $t < 0$ can be treated similarly.

For simplicity, we prove this theorem only in the case that

$$\mathcal{N}(v, \overline{v}, \nabla v, \nabla\overline{v}) = \mathcal{N}_1 + \mathcal{N}_2$$

where

$$\mathcal{N}_1 = \lambda_1\sum_{k=1}^N \partial_k v^2 + \lambda_2 v\sum_{k=1}^N \partial_k\overline{v} + \lambda_3\overline{v}\sum_{k=1}^N \partial_k v + \lambda_4\sum_{k=1}^N \partial_k\overline{v}^2$$

and

$$\mathcal{N}_2 = \lambda_5 \left(\sum_{k=1}^N \partial_k v \right)^2 + \lambda_6 \left(\sum_{k=1}^N \partial_k v \right) \left(\sum_{k=1}^N \partial_k \overline{v} \right) + \lambda_7 \left(\sum_{k=1}^N \partial_k \overline{v} \right)^2 .$$

Since we have $(P+2)(uv) = (Pu)v + u(P+1)v + uv$ and $(P+2)(uv) = ((P+1)u)v + u(P+1)v$, we obtain identities

$$(P+2)^\nu(\phi\psi) = \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{\nu!}{\nu_1!\nu_2!\nu_3!} P^{\nu_1}\phi(P+1)^{\nu_2}\psi \tag{3.1}$$

and

$$(P+2)^\nu(\phi\psi) = \sum_{\nu=\nu_1+\nu_2} \frac{\nu!}{\nu_1!\nu_2!} (P+1)^{\nu_1}\phi(P+1)^{\nu_2}\psi. \tag{3.2}$$

Also, we have

$$(P+1)^\nu \partial_k \phi = \partial_k P^\nu \phi. \tag{3.3}$$

Multiplying (LE) by the operator $A^\nu P^\nu/(\nu-1)!$, using (3.1) and (3.2) in the case of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively and applying (3.3), we have

$$\begin{aligned}
&i\partial_t \frac{A^\nu P^\nu}{(\nu-1)!} u + \frac{1}{2}\Delta \frac{A^\nu P^\nu}{(\nu-1)!} u \\
&= \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{\nu A^{\nu_3}}{\nu_1!\nu_2!\nu_3!} \\
&\quad \times \Bigg(2\lambda_1 (A^{\nu_1}P^{\nu_1}v) \sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} v + \lambda_2 (A^{\nu_1}P^{\nu_1}v) \sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} \overline{v} \\
&\qquad + \lambda_3 (A^{\nu_1}P^{\nu_1}\overline{v}) \sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} v + 2\lambda_4 (A^{\nu_1}P^{\nu_1}\overline{v}) \sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} \overline{v} \Bigg) \\
&\quad + \sum_{\nu-1=\nu_1+\nu_2} \frac{\nu A^{\nu_3}}{\nu_1!\nu_2!} \\
&\quad \times \Bigg\{ \lambda_5 \left(\sum_{k=1}^N \partial_k A^{\nu_1}P^{\nu_1} v \right) \left(\sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} v \right) \\
&\qquad + \lambda_6 \left(\sum_{k=1}^N \partial_k A^{\nu_1}P^{\nu_1} v \right) \left(\sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} \overline{v} \right) \\
&\qquad + \lambda_7 \left(\sum_{k=1}^N \partial_k A^{\nu_1}P^{\nu_1} \overline{v} \right) \left(\sum_{k=1}^N \partial_k A^{\nu_2}P^{\nu_2} \overline{v} \right) \Bigg\}
\end{aligned}$$

Since we shall obtain the estimate of $\sum_{\nu=0}^{\infty}\left\|\Gamma^2\partial_l^{m-4}\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|^2$, we apply Lemmas 2.4 and 2.5 to the equation (LE). We note these lemmas are important to estimate the nonlinear terms which cause so-called loss of derivative, when we use the classical energy method. Here, we consider

$$\frac{A^\nu(P+2)^\nu}{(\nu-1)!}\mathcal{N}=\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_3!}\frac{A^{\nu_1}P^{\nu_1}}{\nu_1!}v\partial_k\frac{A^{\nu_2}P^{\nu_2}v}{\nu_2!}v,$$

as the nonlinearity. Differentiating this term $m-4$ times and multiplying the result by the operator $\mathcal{S}\Gamma^2$, the term becomes

$$\begin{aligned}(3.4)\qquad &\mathcal{S}\Gamma^2\partial_l^{m-4}\left(\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_3!}\frac{A^{\nu_1}P^{\nu_1}}{\nu_1!}v\partial_k\frac{A^{\nu_2}P^{\nu_2}}{\nu_2!}v\right)\\ &=\mathcal{S}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_3!}\frac{A^{\nu_1}\nu}{\nu_1!}v\partial_k\left(\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}\nu}{\nu_2!}v\right)+\mathcal{S}F.\end{aligned}$$

Using Lemma 2.6, the second term of the above is estimated by

$$(3.5)\qquad \|\mathcal{S}F\|\le C\langle t\rangle^{-1-\sigma}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}\right\|_{\mathbf{X}_m}\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}\right\|_{\mathbf{X}_m}.$$

Multiplying the first term of the right hand side of (3.4) by $\mathcal{S}\Gamma^2\partial_l^{m-4}(A^\nu P^\nu/(\nu-1)!)\overline{u}$, integrate it with respect to x in $\mathbb{R}^N$, taking the imaginary part of it and applying Lemma 2.5 to the result, we have the inequality

$$
\begin{aligned}
&\left|\left(\mathcal{S}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_3!}\frac{A^{\nu_1}P^{\nu_1}}{\nu_1!}v\partial_k\left(\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{\nu_2!}v\right),\mathcal{S}\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right)\right|\\
\le& C\langle t\rangle^{-3/4-2\sigma}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\\
&\quad\times\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|\left\|\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right\|\\
&+C\langle t\rangle^{-1/2-2\sigma}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\\
&\quad\times\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right\|\\
&+C\langle t\rangle^{-5/4+\sigma/2}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\\
&\quad\times\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}\left\|\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right\|\\
&+C\langle t\rangle^{-1+\sigma/2}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\\
&\quad\times\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right\|\\
&+C\langle t\rangle^{-5/4}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\\
&\quad\times\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}\left\|\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right\|\\
&+C\langle t\rangle^{-1}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\\
&\quad\times\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu}P^{\nu}}{(\nu-1)!}u\right\|,
\end{aligned}
$$

since we have the estimates $\left\| w/\omega_j^2 \right\|_{\mathbf{L}^\infty} \le C\langle t\rangle^{-1/2-2\sigma}\|w\|_{\mathbf{X}_m}$ and

$$\left(\left\| \frac{\partial_j w}{\omega_j} \right\|_{\mathbf{L}^\infty} + \langle t\rangle^{-1/2} \left\| \frac{w}{\omega_j} \right\|_{\mathbf{L}^\infty} \right)^{1/2-\sigma} \left(\left\| \frac{\partial_j w}{\omega_j} \right\|_{\mathbf{L}^\infty} + (1+\langle t\rangle^{-1/2}) \left\| \frac{w}{\omega_j} \right\|_{\mathbf{L}^\infty} \right)^{1/2+\sigma}$$
$$\le C\langle t\rangle^{-1+\sigma/2}\|w\|_{\mathbf{X}_m}$$

by Lemmas 2.6 and 2.7. Applying Lemma 2.4 to (LE), we obtain the inequality by the commutator's relations (1.3) , the estimate (3.5) and the above

$$\begin{aligned}
&\left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(t) \right\|^2 + \sum_{k=1}^N \int_0^t \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 d\tau \\
\le& \sum_{|a|+b\le 2} \left\| \frac{A^\nu \Omega^a (x\cdot\nabla)^{b+\nu}}{(\nu-1)!} u_0 \right\|^2_{(m-2|a|-2b)} \\
&+ C\int_0^t \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!} \langle\tau\rangle^{-3/4-(3/2)\sigma} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\| \\
&\quad \times \left\| \frac{A^{\nu_1} P^{\nu_1}}{(\nu_1-1)!} v \right\|_{\mathbf{X}_m} \sum_{k=1}^N \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^{\nu_2} P^{\nu_2}}{(\nu_2-1)!} v(\tau) \right\| d\tau \\
&+ C\int_0^t \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!} \sum_{k=1}^N \langle\tau\rangle^{-1/2-2\sigma} \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\| \\
&\quad \times \left\| \frac{A^{\nu_1} P^{\nu_1}}{(\nu_1-1)!} v \right\|_{\mathbf{X}_m} \sum_{k=1}^N \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^{\nu_2} P^{\nu_2}}{(\nu_2-1)!} v(\tau) \right\| d\tau \\
&+ C\int_0^t \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!} \langle\tau\rangle^{-1-\sigma} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\| \\
&\quad \times \left\| \frac{A^{\nu_1} P^{\nu_1}}{(\nu_1-1)!} v \right\|_{\mathbf{X}_m} \left\| \frac{A^{\nu_2} P^{\nu_2}}{(\nu_2-1)!} v(\tau) \right\|_{\mathbf{X}_m} d\tau \\
&+ C\int_0^t \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{A^{\nu_3}\nu}{\nu_1\nu_2\nu_3!} \langle\tau\rangle^{-3/4-(3/2)\sigma} \\
&\quad \times \sum_{k=1}^N \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\| \left\| \frac{A^{\nu_1} P^{\nu_1}}{(\nu_1-1)!} v \right\|_{\mathbf{X}_m} \left\| \frac{A^{\nu_2} P^{\nu_2}}{(\nu_2-1)!} v \right\|_{\mathbf{X}_m} d\tau \\
&+ C\varepsilon^2 \int_0^t \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 \frac{d\tau}{\langle\tau\rangle^{1+\sigma}} \\
\equiv& \left\| \frac{A^\nu (x\cdot\nabla)^\nu}{(\nu-1)!} u_0 \right\|^2_{\mathbf{B}_m} + \sum_{l=1}^4 I_l + C\int_0^t \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 \frac{d\tau}{\langle\tau\rangle^{1+\sigma}}.
\end{aligned}$$

We consider I_1. We have by the Schwarz's inequality

$$
\begin{aligned}
I_1 =& C\int_0^t \langle\tau\rangle^{-3/4-(-3/2)\sigma}\left\|\Gamma^2\partial_l^{m-4}\frac{A^\nu P^\nu}{(\nu-1)!}u(\tau)\right\| \\
&\times\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m} \\
&\qquad\sum_{k=1}^N\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(\tau)\right\|d\tau \\
\le& C\int_0^t \langle\tau\rangle^{-1/2-(1/2)\sigma}\left\|\Gamma^2\partial_l^{m-4}\frac{A^\nu P^\nu}{(\nu-1)!}u(\tau)\right\| \\
&\times\left(\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu^2}{\nu_1^2\nu_2^2\nu_3^2}\right)^{1/2}\left(\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{2\nu_3}}{(\nu_3-1)!^2}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}^2\right. \\
&\left.\times\langle\tau\rangle^{-1/2-2\sigma}\sum_{k=1}^N\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(\tau)\right\|^2\right)^{1/2}d\tau \\
\le& C\int_0^t \langle\tau\rangle^{-1/2-(1/2)\sigma}\left\|\Gamma^2\partial_l^{m-4}\frac{A^\nu P^\nu}{(\nu-1)!}u(\tau)\right\| \\
&\times\left(\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{2\nu_3}}{(\nu_3-1)!^2}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}^2\right. \\
&\left.\times\langle\tau\rangle^{-1/2-2\sigma}\sum_{k=1}^N\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(\tau)\right\|^2\right)^{1/2}d\tau \\
\le& C\left(\int_0^t\left\|\Gamma^2\partial_l^{m-4}\frac{A^\nu P^\nu}{(\nu-1)!}u(\tau)\right\|^2\frac{d\tau}{\langle\tau\rangle^{1+\sigma}}\right)^{1/2} \\
&\times\left(\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{2\nu_3}}{(\nu_3-1)!^2}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}^2\right. \\
&\left.\times\sum_{k=1}^N\int_0^t\left\|\omega_k\mathcal{S}|\partial_k|^{1/2}\Gamma^2\partial_l^{m-4}\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(\tau)\right\|^2\frac{d\tau}{\langle\tau\rangle^{1/2+2\sigma}}\right)^{1/2} \\
\le& C\left(\int_0^t\left\|\Gamma^2\partial_l^{m-4}\frac{A^\nu P^\nu}{(\nu-1)!}u(\tau)\right\|^2\frac{d\tau}{\langle\tau\rangle^{1+\sigma}}\right)^{1/2} \\
&\times\left(\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A^{2\nu_3}}{(\nu_3-1)!^2}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}^2\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}^2\right)^{1/2}
\end{aligned}
$$

since we have

$$\sup_{\nu} \sum_{\nu=\nu_1+\nu_2+\nu_3} \left(\frac{\nu}{\nu_1\nu_2\nu_3} \right)^2 \leq C.$$

Using the fact that the function $v(t,x)$ is in $\mathbf{Y}_{m,A}$, that is,

$$\|v\|_{\mathbf{Y}_{m,A}} = \sum_{\nu=0}^{\infty} \left\| \frac{A^\nu P^\nu}{(\nu-1)!} v \right\|_{\mathbf{X}_m}^2 \leq \varepsilon^2,$$

we have

$$\begin{aligned}
(3.6) \qquad & \sum_{\nu=0}^{\infty} \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{A^{2\nu_3}}{(\nu_3-1)!^2} \left\| \frac{A^{\nu_1} P^{\nu_1}}{(\nu_1-1)!} v \right\|_{\mathbf{X}_m}^2 \left\| \frac{A^{\nu_2} P^{\nu_2}}{(\nu_2-1)!} v \right\|_{\mathbf{X}_m}^2 \\
& \leq C \sum_{\nu_1=0}^{\infty} \left\| \frac{A^{\nu_1} P^{\nu_1}}{(\nu_1-1)!} v \right\|_{\mathbf{X}_m}^2 \sum_{\nu_2=0}^{\infty} \left\| \frac{A^{\nu_2} P^{\nu_2}}{(\nu_2-1)!} v \right\|_{\mathbf{X}_m}^2 \sum_{\nu_3=0}^{\infty} \frac{A^{2\nu_3}}{(\nu_3-1)!^2} \\
& \leq C\varepsilon^4
\end{aligned}$$

Hence we have the estimate

$$\sum_{\nu=0}^{\infty} I_1 \leq C\varepsilon^2 \int_0^t \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 \frac{d\tau}{\langle \tau \rangle^{1+\sigma}} + C\varepsilon^4.$$

In the same way as the above, we also obtain the estimates for the other terms

$$\begin{aligned}
& \sum_{\nu=0}^{\infty} \sum_{l=2}^{4} I_l \\
\leq & C\varepsilon^2 \sum_{\nu=0}^{\infty} \int_0^t \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\| \frac{d\tau}{\langle \tau \rangle^{1+\sigma}} \\
& + C\varepsilon^2 \sum_{\nu=0}^{\infty} \int_0^t \sum_{k=1}^{N} \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 \frac{d\tau}{\langle \tau \rangle^{1/2+2\sigma}} + C\varepsilon^4.
\end{aligned}$$

Hence the integral inequality becomes

$$
\begin{aligned}
(3.7)\quad & \sum_{\nu=0}^{\infty} \sup_{t\in[0,\infty)} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(t) \right\|^2 \\
& \quad + \sum_{\nu=0}^{\infty} \sum_{k=1}^{N} \int_0^\infty \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 \frac{d\tau}{\langle \tau \rangle^{1/2+2\sigma}} \\
& \le C \sum_{\nu=0}^{\infty} \left\| \frac{A^\nu (x\cdot\nabla)^\nu}{(\nu-1)!} u_0 \right\|_{\mathbf{B}_m}^2 \\
& \quad + C\varepsilon^2 \sum_{\nu=0}^{\infty} \sup_{t\in[0,\infty)} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(t) \right\|^2 \int_0^\infty \frac{d\tau}{\langle \tau \rangle^{1+\sigma}} \\
& \quad + C\varepsilon^2 \sum_{\nu=0}^{\infty} \int_0^\infty \sum_{k=1}^{N} \left\| \omega_k \mathcal{S} |\partial_k|^{1/2} \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 \frac{d\tau}{\langle \tau \rangle^{1/2+2\sigma}} + C\varepsilon^2
\end{aligned}
$$

If we choose a sufficiently small constant ε as $1-C\varepsilon^2>0$, we have from above estimate,

$$
\begin{aligned}
& \sum_{\nu=0}^{\infty} \sup_{t\in[0,\infty)} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(t) \right\|^2 \\
& \le C\varepsilon^2 + C\varepsilon^2 \sum_{\nu=0}^{\infty} \sup_{t\in[0,\infty)} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(\tau) \right\|^2 .
\end{aligned}
$$

Therefore we obtain one of the desired estimate

$$
\sum_{\nu=0}^{\infty} \sup_{t\in[0,\infty)} \left\| \Gamma^2 \partial_l^{m-4} \frac{A^\nu P^\nu}{(\nu-1)!} u(t) \right\|^2 \le C\varepsilon^2. \tag{3.8}
$$

Next, we consider

$$
\mathcal{N} = v \sum_{k=1}^{N} \partial_k v,
$$

as a nonlinearity again. By Lemma 2.6, we have

$$
\left\| \nabla \frac{A^\nu P^\nu}{(\nu-1)!} v(t) \right\|_{\mathbf{L}^3} \le C\langle t\rangle^{-1/2} \left\| \frac{A^\nu P^\nu}{(\nu-1)!} v(t) \right\|_{\mathbf{X}_m}
$$

and

$$
\left\| \nabla \frac{A^\nu P^\nu}{(\nu-1)!} v(t) \right\|_{\mathbf{L}^6} \le C\langle t\rangle^{-1} \left\| \frac{A^\nu P^\nu}{(\nu-1)!} v(t) \right\|_{\mathbf{X}_m} .
$$

It follows from identity $QP^\nu u = PP^\nu u + 2it\mathcal{L}P^\nu u = PP^\nu u + 2it(P+2)^\nu \mathcal{N}$, Hölder's inequality and the above estimates that

(3.9)
$$
\begin{aligned}
&\left\| Q\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-2}\\
\le&\left\|(1-\Delta)P\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}+2|t|\left\|\frac{A^\nu (P+2)^\nu}{(\nu-1)!}\mathcal{N}\right\|_{m-2}\\
\le&C\left\|\Gamma^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}\\
&+2|t|\left\|\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\sum_{k=1}^{N}\frac{A^{\nu_2}(P+1)^{\nu_2}}{(\nu_2-1)!}\partial_k v(t)\right\|_{m-2}\\
\le&C\left\|\Gamma^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}+C\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}
\end{aligned}
$$

and

(3.10)
$$
\begin{aligned}
&\left\| PQ\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}\\
\le&\left\|P^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}+2|t|\left\|P\frac{A^\nu (P+2)^\nu}{(\nu-1)!}\mathcal{N}\right\|_{m-4}+2|t|\left\|\frac{A^\nu (P+2)^\nu}{(\nu-1)!}\mathcal{N}\right\|_{m-4}\\
\le&C\left\|\Gamma^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}\\
&+2|t|\left\|P\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\sum_{k=1}^{N}\frac{A^{\nu_2}(P+1)^{\nu_2}}{(\nu_2-1)!}\partial_k v(t)\right\|_{m-4}\\
&+2|t|\left\|\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\sum_{k=1}^{N}\frac{A^{\nu_2}(P+1)^{\nu_2}}{(\nu_2-1)!}\partial_k v(t)\right\|_{m-4}\\
\le&C\left\|\Gamma^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}+C\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}
\end{aligned}
$$

By (3.6) and (3.8) - (3.10), we obtain

(3.11)
$$
\sum_{\nu=0}^{\infty}\sup_{t\in[0,\infty)}\left\|Q\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-2}^2+\sum_{\nu=0}^{\infty}\sup_{t\in[0,\infty)}\left\|PQ\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}^2\le C\varepsilon^2
$$

We denote $\mathbf{U} = (v, \nabla v, \overline{v}, \nabla \bar{v})$. Hence we obtain

$$
\begin{aligned}
Q\mathcal{N} = &\sum_{k=1}^{2N+2} ((x\cdot\nabla) + 2it\Delta)\mathbf{U}_k \partial_{\mathbf{U}_k}\mathcal{N} \\
&+ \sum_{k=1}^{2N+2}\sum_{l=1}^{2N+2} 2it\left(\sum_{j=1}^{N}\partial_j \mathbf{U}_k \partial_j \mathbf{U}_l\right)(\partial_{\mathbf{U}_k}\partial_{\mathbf{U}_l}\mathcal{N}).
\end{aligned}
$$

Therefore, by the Hölder inequality and Lemma 2.6, we get

$$
\begin{aligned}
&\left\| Q\frac{A^\nu (P+2)^\nu}{(\nu-1)!}\mathcal{N}\right\| \\
\le & C\sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\sum_{k=1}^{N}\Bigg\{\Bigg(\left\|Q\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\right\|\left\|\partial_k\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(t)\right\|_{\mathbf{L}^\infty} \\
&\qquad + \left\|Q\partial_k\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(t)\right\|\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\right\|_{\mathbf{L}^\infty}\Bigg) \\
&+ C|t|\sum_{l=1}^{N}\left(\left\|\partial_l\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\right\|_{\mathbf{L}^3} + \left\|\Delta\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\right\|_{\mathbf{L}^3}\right) \\
&\qquad \times\left(\left\|\partial_l\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(t)\right\|_{\mathbf{L}^6} + \left\|\Delta\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(t)\right\|_{\mathbf{L}^6}\right)\Bigg\} \\
\le & C\langle t\rangle^{-1/2}\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v(t)\right\|_{\mathbf{X}_m}\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v(t)\right\|_{\mathbf{X}_m}.
\end{aligned}
$$

We similarly have by the above inequality and (3.10),

$$
\begin{aligned}
&\langle t\rangle^{-1}\left\|Q^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}^2 \\
\le & C\langle t\rangle^{-1}\left\|PQ\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}^2 + C|t|\left\|Q\frac{A^\nu (P+2)^\nu}{(\nu-1)!}\mathcal{N}\right\|_{m-4}^2 \\
\le & C\left\|\Gamma^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4} \\
&+ C\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu A^{\nu_3}}{\nu_1\nu_2\nu_3!}\left\|\frac{A^{\nu_1}P^{\nu_1}}{(\nu_1-1)!}v\right\|_{\mathbf{X}_m}\left\|\frac{A^{\nu_2}P^{\nu_2}}{(\nu_2-1)!}v\right\|_{\mathbf{X}_m}.
\end{aligned}
$$

Hence by (3.6) and (3.8) we have

$$
\sum_{\nu=0}^{\infty}\sup_{t\in[0,\infty)}\langle t\rangle^{-1}\left\|Q^2\frac{A^\nu P^\nu}{(\nu-1)!}u(t)\right\|_{m-4}^2 \le C\varepsilon^2 \tag{3.12}
$$

By (3.7), (3.8), (3.11) and (3.12), we obtain the inequality

$$\|u\|^2_{\mathbf{Y}_{m,A}} \le C\varepsilon^2.$$

Hence it follows that $u = \Psi v$ defined by the linearized equation (LE) transforms a set $\{\phi \in \mathbf{Y}_{m,A} \ ; \ \|\phi\|_{\mathbf{Y}_{m,A}} \le C\varepsilon\}$ into itself. In the same way we are able to prove

$$\|\Psi v_1 - \Psi v_2\|_{\mathbf{Y}_{m,A}} \le \frac{1}{2}\|v_1 - v_2\|_{\mathbf{Y}_{m,A}}.$$

Therefore, the mapping Ψ is a contraction mapping. This completes the proof of Theorem 1.1.

§4. Analyticity

In this section, we prove the analyticity of solutions for (NLS) constructed in Theorem 1.1. To show Theorem 1.2, we use the following properties of the operator P as N. Hayashi and K. Kato [12] and K. Kato and K. Taniguchi [18]:

$$[P, \mathcal{L}] = -2\mathcal{L} \qquad \text{and} \quad (t\partial_t)^l = \frac{1}{2^l}\sum_{l=l_1+l_2}\frac{l!}{l_1!l_2!}(-x\cdot\nabla)^{l_1}P^{l_2}.$$

Moreover, K. Kato and K. Taniguchi also use a property

$$t\Delta P^\nu u = -iP^{\nu+1}u + i(x\cdot\nabla)P^\nu u + 2t(P+2)^\nu\mathcal{N}. \tag{4.1}$$

It is important to make use of the above properties in order to show Theorem 1.2. To estimate the norm of the term including the operator $x\cdot\nabla$, K. Kato and K. Taniguchi make use of C^∞-function $r(x)$ with the property $r(x) = 1$ if $|x| \le R$, or $r(x) = 0$ if $|x| > R$, where R is a positive constant. Instead of it, we use $a(x) = (1+|x|^2)^{-N/2}$.

We treat only case of $0 < t \le 1$. The case of $-1 \le t < 0$ can be proved similarly. And for $|t| > 1$, noting the inequalities (4.6), (4.13) and (4.16), the analyticity of solutions for (NLS) can be shown in the same way as the case of $t \in [-1,1]\backslash\{0\}$.

From Theorem 1.1, we have

$$\|P^\nu u\|_m \le C_1A_1^\nu\nu!, \qquad \text{for } m \ge [N/2]+6 \text{ and } \nu = 0,1,2,\dots. \tag{4.2}$$

Lemma 4.1. *Let u be the solution of (NLS) constructed in Theorem 1.1. Then we have positive constants C_2, A_2 and A_3 such that*

$$\|a(x)^{|\mu|}\partial^\mu P^\nu u\|_m \le \frac{C_2}{|t|^{|\mu|}}A_3^{|\mu|}A_2^{|\mu|+\nu}(|\mu|+\nu)!, \tag{4.3}$$

for any multi-index μ and $\nu = 0,1,2,\dots,$ where $a(x) = \langle x\rangle^{-N} = (1+|x|^2)^{-N/2}$.

We use two propositions to prove this lemma.

Proposition 4.1. *Let $m \geq [N/2]+1$. If $f, g \in \mathbf{H}^m$, then $fg \in \mathbf{H}^m$ with*

$$\|fg\|_m \leq C\|f\|_m \|g\|_m,$$

where C is a positive constant depending on N.

Proposition 4.2. *Let $\alpha_1, \dots, \alpha_k$ and α be multi-indices such as $\alpha_1 + \cdots + \alpha_k = \alpha$. For α and an integer l, let integers $\zeta_j \geq 1$ ($j = 1, \dots, k$) satisfy $\zeta_1 + \cdots + \zeta_k = |\alpha| + l$. Then, we have*

$$\sum_{\substack{\alpha_1+\cdots+\alpha_k=\alpha,\\ l_1+\cdots+l_k=l,\\ |\alpha_k|+l_k=\zeta_k}} \prod_{j=1}^{k} \frac{(|\alpha_j|+l_j)!}{\alpha_j! l!} = \frac{(|\alpha|+l)!}{\alpha! l!}.$$

Proof of Lemma 4.1. We prove the lemma only in the case that

$$\mathcal{N} = u \sum_{j=1}^{N} \partial_j u,$$

We get

$$\int_{\mathbb{R}^N} \left|(1-\Delta)^{m/2} a(x)\right|^2 dx \leq C_3^2. \tag{4.4}$$

We prove the lemma by induction with respect to $|\mu|$. The inequality (4.3) for $|\mu| = 0$ is nothing but the estimate (4.2). First, we prove (4.3) for $|\mu| = 1$. By using (4.4) and Proposition 4.1, we obtain

$$\begin{aligned} \|a(x)\partial P^\nu u\|_m &\leq \|\partial(a(x)P^\nu u)\|_m + \|(\partial a(x))P^\nu u\|_m \\ &\leq \|\partial(a(x)P^\nu u)\|_m + C_4 A_1^\nu \nu!, \end{aligned}$$

where $C_4 = C_1 C_3$. By Proposition 4.1, Leibniz's rule, the inequality (4.2) and

$$\|\partial^\chi f\|_m \leq \|\Delta f\|_m, \qquad \text{for } |\chi| = 2,$$

we have

$$\begin{aligned} &\|\partial(a(x)P^\nu u)\|_m \\ \leq& C\|\Delta(a(x)P^\nu u)\|_{m-1} + C\|a(x)P^\nu u\|_{m-1} \\ \leq& C\|(\Delta a(x))P^\nu u\|_{m-1} + 2C\left\|\nabla a(x) \cdot \nabla(P^\nu u)\right\|_{m-1} \\ &+ C\|a(x)\Delta P^\nu u\|_{m-1} + C_4 A_1^\nu \nu!. \end{aligned} \tag{4.5}$$

The first and second terms of the right hand side of (4.5) is estimated by $C_5 A_1^\nu \nu!$, where $C_5 = 3CC_1C_3$. We consider the third term. From (4.1), we have

(4.6)

$$\begin{aligned}&\|a(x)\Delta P^\nu u\|_{m-1}\\ &\le \frac{1}{|t|}\|a(x)P^{\nu+1}u\|_{m-1} + \frac{1}{|t|}\|a(x)(x\cdot\nabla)P^\nu u\|_{m-1} + 2\|a(x)(P+2)^\nu \mathcal{N}\|_{m-1}.\end{aligned}$$

We obtain by Proposition 4.1, inequalities (4.2) and (4.4)

$$\begin{aligned}\|a(x)P^{\nu+1}u\|_{m-1} &\le \|a(x)\|_{m-1}\|P^{\nu+1}u\|_{m-1}\\ &\le C_1C_3A_1^{\nu+1}(\nu+1)!.\end{aligned}$$

And the second term of the right hand side of (4.6) becomes

$$\begin{aligned}\|a(x)(x\cdot\nabla)P^\nu u\|_{m-1} &\le \sum_{j=1}^N \|(a(x)x_j)\partial_j P^\nu u\|_{m-1}\\ &\le C_3\sum_{j=1}^N \|\partial_j P^\nu u\|_{m-1}\\ &\le C_3\|P^\nu u\|_m\\ &\le C_1C_3A_1^\nu \nu!.\end{aligned}$$

By the identity

$$(P+2)^\nu(vw) = \sum_{\nu=\nu_1+\nu_2+\nu_3} \frac{\nu!}{\nu_1!\nu_2!\nu_3!} P^{\nu_1}v(P+1)^{\nu_2}w,$$

we obtain

$$\begin{aligned}(P+2)^\nu \mathcal{N} =&(P+2)^\nu\left(u\sum_{j=1}^N \partial_j u\right)\\ =&\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu!}{\nu_1!\nu_2!\nu_3!}(P^{\nu_1}u)\left(\sum_{j=1}^N \partial_j P^{\nu_2}u\right).\end{aligned}$$

Hence we have by Proposition 4.1 and the inequality (4.2)

$$\begin{aligned}\|(P+2)^\nu\mathcal{N}\|_{m-1} \le& C\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{\nu!}{\nu_1!\nu_2!\nu_3!}\|P^{\nu_1}u\|_{m-1}\|P^{\nu_2}u\|_m\\ \le& CC_1^2\sum_{\nu=\nu_1+\nu_2+\nu_3}\frac{A_1^{\nu_1+\nu_2}\nu!}{\nu_3!}\\ \le& C_6 e^{1/A_1}A_1^\nu(\nu+1)!,\end{aligned}$$

where $C_6 = CC_1^2$. Hence, we have

$$\|\partial(a(x)P^\nu u)\|_m \le \frac{C_7}{|t|} A_2^{\nu+1} A_3(\nu+1)!$$

where $C_7 A_2 = \max\{C_5, C_6, 2, A_1\}$, $A_3 = e^{1/A_1}$. Therefore we have the case of $|\mu|$=1 of the lemma.

Next, we prove that (4.3) is valid for $|\mu| = k+1$ with $k \ge 1$, assuming that (4.3) is valid for $|\mu| \le k$. Let $\mu = \beta + \chi$ with $|\beta| = k-1$ and $|\chi| = 2$. We have

$$\|a(x)^{|\mu|}\partial^\mu P^\nu u\|_m \le \|\partial^\chi(a(x)^{|\mu|}\partial^\beta P^\nu u)\|_m + \|[a(x)^{|\mu|}, \partial^\chi]\partial^\beta P^\nu u\|_m.$$

We estimate the second term of the right hand side of the above inequality. By calculations, we have

$$\partial_k a(x)^{|\mu|} = \frac{-N|\mu|}{1+|x|^2} x_k a(x)^{|\mu|}$$

and

$$\partial_k\partial_j a(x)^{|\mu|} = \begin{cases} \dfrac{N(k+1)\{N(k+1)+2\}}{(1+|x|^2)^2} x_k x_j a(x)^{|\mu|}, & (j \ne k), \\ \dfrac{-N(k+1)}{1+|x|^2} a(x)^{|\mu|} \\ \quad + \dfrac{N(k+1)\{N(k+1)+2\}}{(1+|x|^2)^2} x_k^2 a(x)^{|\mu|}, & (j = k). \end{cases}$$

Since

$$\begin{aligned} [a(x)^{|\mu|}, \partial^\chi] &= -\partial^\chi a(x)^{|\mu|} - (\partial_k a(x)^{|\mu|})\partial_j - (\partial_j a(x)^{|\mu|})\partial_k \\ &= -\partial^\chi a(x)^{|\mu|} - |\mu|(\partial_k a(x))a(x)^{|\mu|-1}\partial_j - |\mu|(\partial_j a(x))a(x)^{|\mu|-1}\partial_k \end{aligned}$$

and

$$\frac{-(k+1)}{(k+\nu)} + \frac{(k+1)\{N(k+1)+2\}}{N(k+\nu)(k+1+\nu)} \le C_{11},$$

we obtain

$$\begin{aligned} &\|[a(x)^{|\mu|}, \partial^\chi]\partial^\beta P^\nu u\|_m \\ \le& \frac{1}{|t|^{|\beta|+1}} C_{10}C_{11} A_3^{|\beta|+1} A_2^{|\beta|+1+\nu} (|\beta|+1+\nu)!(|\mu|+\nu) \\ &+ \frac{1}{|t|^{|\beta|}} C_{10}C_{11} A_3^{|\beta|} A_2^{|\beta|+\nu} (|\beta|+\nu)!(|\mu|-1+\nu)(|\mu|+\nu) \\ \le& \frac{1}{|t|^{|\mu|-1}} C_{10}C_{11} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu} (|\mu|+\nu)!. \end{aligned} \tag{4.7}$$

In the same way as the above inequality, we obtain

$$\|[a(x)^{|\mu|}, \Delta]\partial^\beta P^\nu u\|_m \leq \frac{N}{|t|^{|\mu|-1}} C_{10} C_{11} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu} (|\mu|+\nu)!. \tag{4.8}$$

Hence we have by the inequalities (4.7) and (4.8)

$$\begin{aligned}
&\|a(x)^{|\mu|}\partial^\mu P^\nu u\|_m \\
\leq &\|\partial^\chi a(x)^{|\mu|}\partial^\beta P^\nu u\|_m + \|[a(x)^{|\mu|}, \partial^\chi]\partial^\beta P^\nu u\|_m \\
\leq &C\|\Delta a(x)^{|\mu|}\partial^\beta P^\nu u\|_m + \frac{1}{|t|^{|\mu|-1}} C_{10} C_{11} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu}(|\mu|+\nu)! \\
\leq &\|a(x)^{|\mu|}\partial^\beta \Delta P^\nu u\|_m + \|[a(x)^{|\mu|}, \Delta]\partial^\beta P^\nu u\|_m \\
&+ \frac{1}{|t|^{|\mu|-1}} C_{10} C_{11} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu}(|\mu|+\nu)! \\
\leq &\|a(x)^{|\mu|}\partial^\beta \Delta P^\nu u\|_m + \frac{1}{|t|^{|\mu|-1}} C_{12} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu}(|\mu|+\nu)!.
\end{aligned} \tag{4.9}$$

By the assumption of induction, we have

$$\|a(x)^{|\mu|}\partial^\beta P^{\nu+1} u\|_m \leq \frac{1}{|t|^{|\beta|}} C_3 C_{10} A_3^{|\beta|} A_2^{|\beta|+1+\nu}(|\beta|+1+\nu)! \tag{4.10}$$

and

$$\begin{aligned}
&\|a(x)^{|\mu|}\partial^\beta (x\cdot\nabla) P^\nu u\|_m \\
\leq &|\beta|\|a(x)^{|\mu|}\partial^\beta P^\nu u\|_m + \sum_{j=1}^N \|a(x)^{|\mu|} x_j \partial_j \partial^\beta P^\nu u\|_m \\
\leq &\frac{1}{|t|^{|\beta|}} C_3 C_{10} A_3^{|\beta|} A_2^{|\beta|+\nu} |\beta|(|\beta|+\nu)! \\
&+ \frac{N}{|t|^{|\mu|-1}} C_3 C_{10} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu}(|\mu|-1+\nu)! \\
\leq &\frac{N+1}{|t|^{|\mu|-1}} C_3 C_{10} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu}(|\mu|-1+\nu)!.
\end{aligned} \tag{4.11}$$

Using Propositions 4.1 and 4.2, we obtain

(4.12)

$$\begin{aligned}
&\|a(x)^{|\mu|}\partial^\beta (P+2)^\nu \mathcal{N}\|_m \\
\le & C \sum_{\nu_1+\nu_2+\nu_3=\nu} \frac{\nu!}{\nu_1!\nu_2!\nu_3!} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \\
&\qquad \times \|a(x)^{|\beta_1|}\partial^{\beta_1} P^{\nu_1} u\|_m \sum_{j=0}^{N} \|a(x)^{|\beta_2|+1}\partial^{\beta_2}\partial_j P^{\nu_2} u\|_m \\
\le & C \sum_{\nu_1+\nu_2+\nu_3=\nu} \frac{\nu!}{\nu_1!\nu_2!\nu_3!} \sum_{\beta_1+\beta_2=\beta} \frac{\beta!}{\beta_1!\beta_2!} \\
&\qquad \times \frac{C_{10}}{|t|^{\beta_1}} A_3^{|\beta_1|} A_2^{|\beta_1|+\nu_1} (|\beta_1|+\nu_1)! \frac{C_{10}}{|t|^{|\beta|_2+1}} A_3^{|\beta_2|+1} A_2^{|\beta_2|+1+\nu_2} (|\beta_2|+1+\nu_2)! \\
\le & \frac{1}{|t|^{|\beta|+1}} C C_{10}^2 A_3^{|\beta|+1} A_2^{|\beta|+1+\nu} \sum_{\substack{\nu_1+\nu_2+\nu_3=\nu, \\ \beta_1+\beta_2=\beta}} \frac{1}{A_3^{\nu_3}\nu_3!} \frac{\nu!\beta!(|\beta_1|+\nu_1)!(|\beta_2|+1+\nu_2)!}{\nu_1!\nu_2!\beta_1!\beta_2!} \\
\le & \frac{1}{|t|^{|\beta|+1}} C C_{10}^2 e^{1/A_3} A_3^{|\beta|+1} A_2^{|\beta|+1+\nu} (|\beta|+1+\nu)! \\
\le & \frac{1}{|t|^{|\mu|-1}} C C_{10}^2 e^{1/A_3} A_3^{|\mu|} A_2^{|\mu|-1+\nu} (|\mu|-1+\nu)!,
\end{aligned}$$

where β_1 and β_2 are multi-indices. Considering the identity (4.1), we have by the inequalities (4.9)-(4.12)

$$\begin{aligned}
(4.13) \qquad & \|a(x)^{|\mu|}\partial^\beta \Delta P^\nu u\|_m \\
\le & \frac{1}{|t|}\|a(x)^{|\mu|}\partial^\beta P^{\nu+1} u\|_m + \frac{1}{|t|}\|a(x)^{|\mu|}\partial^\beta (x\cdot\nabla) P^\nu u\|_m \\
& + 2\|a(x)^{|\mu|}\partial^\beta (P+2)^\nu \mathcal{N}\|_m + \frac{1}{|t|^{|\mu|}} C_{12} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu} (|\mu|+\nu)! \\
\le & \frac{1}{|t|^{|\beta|}} C_3 C_{10} A_3^{|\beta|+1} A_2^{|\beta|+1+\nu} (|\beta|+1+\nu)! \\
& + \frac{N+1}{|t|^{|\mu|}} C_3 C_{10} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu} (|\mu|-1+\nu)! \\
& + \frac{2}{|t|^{|\mu|}} C C_{10}^2 e^{1/A_3} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu} (|\mu|-1+\nu)! \\
& + \frac{1}{|t|^{|\mu|}} C_{12} A_3^{|\mu|-1} A_2^{|\mu|-1+\nu} (|\mu|+\nu)! \\
\le & \frac{1}{|t|^{|\mu|}} C_{13} A_3^{|\mu|} A_2^{|\mu|+\nu} (|\mu|+\nu)!,
\end{aligned}$$

where $C_2 = \max\{C_{12}, (N+1)C_3C_{10}+1\}$, $A_2 = \max\{A_2, 1\}$ and $A_3 = \max\{e^{1/A_3}, A_3\}$. Therefore, we can have the desired result. □

Lemma 4.2. *Let u satisfy the inequality (4.3) in Lemma 4.1. Then there exist positive constants C_{14}, A_3, A_4 and A_5 such that*

$$\|a(x)^{|\mu|+2\sigma}\partial^\mu(x\cdot\nabla)^\sigma P^\nu u\|_m \le \frac{C_{14}}{|t|^{|\mu|+\sigma}}A_5^\sigma A_4^{|\mu|+\nu+\sigma}A_3^{|\mu|}(|\mu|+\nu+\sigma)!, \tag{4.14}$$

for any multi-index μ and $\sigma, \nu = 0, 1, 2, \dots$.

Proof. In the case of $\sigma = 0$, the inequality (4.14) is shown by Lemma 4.1. We assume that the inequality (4.14) holds for $\sigma = l$ and $|\mu|, \nu = 0, 1, 2, \dots$. We have the inequality

$$\begin{aligned}
&\|a(x)^{|\mu|}a(x)^{2(l+1)}\partial^\mu(x\cdot\nabla)^{l+1}P^\nu u\|_m \\
\le&\|a(x)^{|\mu|}a(x)^{2(l+1)}(x\cdot\nabla)\partial^\mu(x\cdot\nabla)^l P^\nu u\|_m \\
&+\|a(x)^{|\mu|}a(x)^{2(l+1)}[\partial^\mu,(x\cdot\nabla)](x\cdot\nabla)^l P^\nu u\|_m \\
\le&\sum_{j=1}^N\|x_j a(x)\|_m\|a(x)^{|\mu|+1}a(x)^{2l}\partial_j\partial^\mu(x\cdot\nabla)^l P^\nu u\|_m \\
&+\|a(x)^{|\mu|}a(x)^{2(l+1)}[\partial^\mu,(x\cdot\nabla)](x\cdot\nabla)^l P^\nu u\|_m \\
\le&C_3\sum_{j=0}^N\|a(x)^{|\mu|+1}a(x)^{2l}\partial_j\partial^\mu(x\cdot\nabla)^l P^\nu u\|_m \\
&+C_3|\mu|\|a(x)^{|\mu|}a(x)^{2l}\partial^\mu(x\cdot\nabla)^l P^\nu u\|_m \\
\le&C_3\frac{C_{14}}{|t|^{|\mu|+1+l}}A_5^l A_4^{|\mu|+1+\nu+l}A_3^{|\mu|}(|\mu|+1+\nu+l)! \\
&+C_3\frac{C_{14}}{|t|^{|\mu|+l}}A_5^l A_4^{|\mu|+\nu+l}A_3^{|\mu|+1}|\mu|(|\mu|+\nu+l)! \\
\le&\frac{C_{14}}{|t|^{|\mu|+l+1}}A_5^{l+1}A_4^{|\mu|+\nu+l+1}A_3^{|\mu|}(|\mu|+\nu+l+1)!,
\end{aligned}$$

where $A_4 = \max\{A_4, 1\}$ and $A_5 = \max\{C_3A_3, C_3\}$. Hence the lemma is completed by induction. □

Lemma 4.3. *Let u satisfy the inequality (4.14) in Lemma 4.2. Then we have positive constants C_{15}, A_3, A_5, A_6 and A_7 such that*

(4.15)

$$\|a(x)^{|\mu|+2\sigma+2\kappa}(t\partial_t)^\kappa\partial^\mu(x\cdot\nabla)^\sigma u\|_m \le \frac{C_{15}}{|t|^{|\mu|+\sigma+\kappa}}A_7^\kappa A_6^{|\mu|+\sigma+\kappa}A_5^\sigma A_3^{|\mu|}(|\mu|+\sigma+\kappa)!,$$

for any multi-index μ and $\sigma, \kappa = 0, 1, 2, \dots$.

Proof. Since $P = x \cdot \nabla + 2t\partial_t$, we have the identity

$$\begin{aligned}(t\partial_t)^l =& \frac{1}{2^l}(P - x \cdot \nabla)^l \\ =& \frac{1}{2^l} \sum_{l_1+l_2=l} \frac{l!}{l_1! l_2!} (-x \cdot \nabla)^{l_1} P^{l_2}.\end{aligned}$$

Let $B > 0$ satisfy $\|a(x)^{2k}\|_m \le B^k$. Hence we obtain by Lemma 4.2

$$\begin{aligned}(4.16) \qquad & \|a(x)^{|\mu|+2\sigma+2l}(t\partial_t)^l \partial^\mu (x \cdot \nabla)^\sigma u\|_m \\ \le& \frac{1}{2^l} \sum_{l_1+l_2=l} \frac{l!}{l_1! l_2!} \|a(x)^{|\mu|+2\sigma+2l} \partial^\mu (x \cdot \nabla)^{l_1+\sigma} P^{l_2} u\|_m \\ \le& \frac{1}{2^l} \sum_{l_1+l_2=l} \frac{l!}{l_1! l_2!} \frac{C_3 C_{14}}{|t|^{|\mu|+\sigma+l_1}} B^{l_2} A_5^{\sigma+l_1} A_4^{|\mu|+\sigma+l} A_3^{|\mu|} (|\mu| + \sigma + l)! \\ \le& \frac{C_{15}}{|t|^{|\mu|+\sigma}} \max\{1, |t|^{-l}\} \left(\frac{B + A_5}{2}\right)^l A_5^\sigma A_4^{|\mu|+\sigma+l} A_3^{|\mu|} (|\mu| + \sigma + l)!.\end{aligned}$$

Hence the lemma follows with $C_{15} = C_3 C_{14}$, $A_6 = A_4$ and $A_7 = (B + A_5)/2$. □

Lemma 4.4. *Let u satisfy the inequality (4.15) in Lemma 4.3. Then there exist positive constants C_{16}, A_3, A_7, A_8 and A_9 such that*

$$\begin{aligned}(4.17) \qquad & \|a(x)^{|\mu|+2(\kappa_1+\kappa_2)} \partial_t^{\kappa_1} (t\partial_t)^{\kappa_2} \partial^\mu u\|_m \\ (4.18) \qquad \le& \frac{C_{16}}{|t|^{|\mu|+2\kappa_1+\kappa_2}} A_9^{\kappa_1} A_8^{|\mu|+\kappa_1+\kappa_2} A_7^{\kappa_2} A_3^{|\mu|} (|\mu| + \kappa_1 + \kappa_2)!,\end{aligned}$$

for any multi-index μ and $\kappa_1, \kappa_2 = 0, 1, 2, \ldots$.

Proof. We have by Lemma 4.3 as $\sigma = 0$

$$\|a(x)^{|\mu|+2\kappa}(t\partial_t)^\kappa \partial^\mu u\|_m \le \frac{C_{15}}{|t|^{|\mu|+\kappa}} A_7^\kappa A_6^{|\mu|+\kappa} A_3^{|\mu|} (|\mu| + \kappa)!,$$

where $\kappa = 0, 1, 2, \ldots$. This is the case of $\kappa_1 = 0$ in this lemma. We shall prove the following estimate by the induction.

(4.19)

$$\|a(x)^{|\mu|+2(k+l)} t^k \partial_t^k (t\partial_t)^l \partial_x^\mu u\|_m \le \frac{C_{16}}{|t|^{|\mu|+k+l}} A_9^k A_8^{|\mu|+k+l} A_7^l A_3^{|\mu|} (\mu + k + l)!.$$

We consider the case of $k = j+1$ and $l = 0, 1, 2, \ldots$. From (4.19), we have the inequalities

$$
\begin{aligned}
&\|a(x)^{|\mu|+2(l+j+1)} t^{j+1} \partial_t^{j+1} (t\partial_t)^l \partial^\mu u\|_m \\
\leq &\|a(x)^{|\mu|+2(l+j+1)} t^j [t, \partial_t^j] \partial_t (t\partial_t)^l \partial^\mu u\|_m + \|a(x)^{|\mu|+2(l+j+1)} t^j \partial_t^j (t\partial_t)^{l+1} \partial^\mu u\|_m \\
\leq &|j| \|a(x)^{|\mu|+2(l+j+1)} t^j \partial_t^j (t\partial_t)^l \partial^\mu u\|_m + \|a(x)^{|\mu|+2(l+j+1)} t^j \partial_t^j (t\partial_t)^{l+1} \partial^\mu u\|_m \\
\leq &C_3 \frac{C_{16}}{|t|^{|\mu|+j+l}} A_9^j A_8^{|\mu|+j+l} A_7^l A_3^{|\mu|} |j| (|\mu| + j + l)! \\
&+ \frac{C_{16}}{|t|^{|\mu|+j+l+1}} A_9^j A_8^{|\mu|+j+l+1} A_7^{l+1} A_3^{|\mu|} (|\mu| + j + l + 1)! \\
\leq &\frac{C_{16}}{|t|^{|\mu|+j+1+l}} A_9^{j+1} A_8^{|\mu|+j+1+l} A_7^l A_3^{|\mu|} (|\mu| + j + 1 + l)!,
\end{aligned}
$$

where $A_8 = \max\{A_8, 1\}$ and $A_9 = \max\{C_3, A_7\}$. Hence we have the desired estimate. □

Corollary 4.1. *Let u satisfy the inequality (4.17) in Lemma 4.4. Then we have positive constants C_{17}, A_3, A_8, and A_9 such that*

$$\|a(x)^{|\mu|+2\kappa_1} \partial_t^{\kappa_1} \partial^{|\mu|} u\|_m \leq \frac{C_{17}}{|t|^{|\mu|+2\kappa_1}} A_3^{|\mu|} A_8^{\kappa_1+|\mu|} A_9^{\kappa_1} (|\mu| + \kappa_1)!,$$

for any multi-index μ and $\kappa_1 = 0, 1, 2, \ldots$.

Proof. When we consider $C_{17} = C_{16}$ and the inequality (4.17) as $\kappa_2 = 0$, this corollary holds. □

The Proof of Theorem 1.2. By Corollary 4.1, we can prove Theorem 1.2. □

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