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# DEHN FILLING AND THE THURSTON NORM

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**Abstract**

For a compact, orientable, irreducible 3–manifold with toroidal boundary that is not the product of a torus and an interval or a cable space, each boundary torus has a finite set of slopes such that, if avoided, the Thurston norm of a Dehn filling behaves predictably. More precisely, for all but finitely many slopes, the Thurston norm of a class in the second homology of the filled manifold plus the so-called winding norm of the class will be equal to the Thurston norm of the corresponding class in the second homology of the unfilled manifold. This generalizes a result of Sela and is used to answer a question of Baker-Motegi concerning the Seifert genus of knots obtained by twisting a given initial knot along an unknot which links it.

## 1. Introduction

How does the Thurston norm behave under Dehn filling?

Let $N$ be a compact, orientable 3–manifold with toroidal boundary and let $T \subset \partial N$ be a particular component. Consider the Dehn fillings $N_T(b)$ along slopes $b$ in $T$. For each slope $b$ in $T$, the Dehn filling induces a natural inclusion of $N$ into $N_T(b)$ that induces the monomorphism

$$\iota_b \colon H_2(N, \partial N - T) \to H_2(N_T(b), \partial N_T(b))$$

defined as follows. If $z \in H_2(N, \partial N - T)$ is represented by a properly embedded surface $S$ in $N$ with $\partial S \cap T = \varnothing$, then $\iota_b(z) = \widehat{z}$ is also represented by $S$ under the inclusion. Consequently,

$$x(z) \geq x(\widehat{z}) \qquad (*)$$

on the Thurston norms of classes $z \in H_2(N, \partial N - T)$ and $\iota_b(z) = \widehat{z} \in H_2(N_T(b), \partial N_T(b))$.

Gabai and Sela both address when Inequality $(*)$ is an equality. Gabai shows that for a fixed class $z \in H_2(N, \partial N - T)$, $x(z) = x(\widehat{z})$ for all except at most one slope $b$ in $T$ [Gab87a, Corollary 2.4]. Sela extends this

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result showing that the equality $x(z) = x(\widehat{z})$ holds for every class $z \in H_2(N, \partial N - T)$ and induced class $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$ for all Dehn fillings except along a finite number of slopes $b$ in $T$ [Sel90, Theorem 3].[1]

In this article we extend consideration to all classes in $H_2(N, \partial N)$. To do so, for each slope $b$ in $T$ we consider the restriction of the Dehn filling $N_T(b)$ to $N$ rather than the inclusion of $N$ into $N_T(b)$. Restriction gives a monomorphism

$$\rho_b \colon H_2(N_T(b), \partial N_T(b)) \to H_2(N, \partial N)$$

defined as follows. If $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$ is represented by a properly embedded surface $\widehat{S}$ that is transverse to $K_b$, then $\rho_b(\widehat{z}) = z$ is represented by $S = \widehat{S} \cap N$. Here, and throughout, we take $K_b \subset N_T(b)$ to be the core of the filling with tubular neighborhood $\mathcal{N}(K_b)$ so that $N = N_T(b) - \mathcal{N}(K_b)$, and we orient $K_b$ and its meridian $b$ so that $b$ links $K_b$ positively. The algebraic intersection number with the core $K_b$ is a linear form on homology, so its absolute value is a pseudo-norm. That is, the pseudo-norm **winding number** of $K_b$ about a homology class $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$ is defined to be

$$\mathrm{wind}_{K_b}(\widehat{z}) = |[K_b] \cdot \widehat{z}|.$$

The winding number enables the following extension of Inequality $(*)$, whose proof is given in Section 2.2.

**Lemma 1.1.** *Let $N$ be a compact, orientable, irreducible $3$–manifold whose boundary is a union of tori. Let $T$ be a component of $\partial N$ and let $b$ be a slope in $T$. If $N_T(b)$ has no $S^1 \times D^2$ or $S^1 \times S^2$ summands, then for all classes $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$,*

$$x(z) \geq x(\widehat{z}) + \mathrm{wind}_{K_b}(\widehat{z}) \tag{$\dagger$}$$

*where $\rho_b(\widehat{z}) = z$.*

Our main goal in this paper is to address when Inequality $(\dagger)$ is an equality, i.e. when

$$x(z) = x(\widehat{z}) + \mathrm{wind}_{K_b}(\widehat{z}). \tag{$\ddagger$}$$

For convenience, if there exists a class $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$ for which Equality $(\ddagger)$ fails, then we say the slope $b$ is a **norm-reducing** slope, the class $z = \rho_b(\widehat{z}) \in H_2(N, \partial N)$ is a **norm-reducing** class with respect to the norm-reducing slope $b$, and the class $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$ is a **norm-reducing** class with respect to the knot $K_b$.

**Theorem 4.6.** *Let $N$ be a compact, connected, orientable, irreducible $3$–manifold whose boundary is a union of tori. Then either*

1) *$N$ is a product of a torus and an interval,*

[1] Sela uses [Gab87a, Theorem 1.8] which required an atoroidality hypothesis. However, [Gab87a, Corollary 2.4] can be used instead to avoid such an additional hypothesis. Lackenby discusses such atoroidality hypotheses in the Appendix to [Lac97a].

2) *$N$ is a cable space, or*
3) *for each torus component $T \subset \partial N$ there is a finite set of slopes $\mathcal{R} = \mathcal{R}(N,T)$ in $T$ such that if $b \notin \mathcal{R}$ then $b$ is not norm-reducing.*

In Corollary 4.4 we obtain a bound on the size of $\mathcal{R}(N,T)$ in terms of the Thurston norms of two integral classes of two different fillings and the distance between the two filling slopes. Since $\mathrm{wind}_{K_b}(\widehat{z}) = 0$ when $\rho_b(\widehat{z}) \in H_2(N, \partial N - T)$, Theorem 4.6 generalizes Sela's result (with the additional assumption that $N$ is irreducible). Sela also explicitly bounds, by the number of faces of the Thurston norm ball of $H_2(N, \partial N - T)$, the number of slopes $b$ for which Equation (‡) may fail for classes $z = \rho_b(\widehat{z}) \in H_2(N, \partial N - T)$ when $\mathrm{wind}_{K_b}(\widehat{z}) = 0$. We appeal to his result to handle the classes in $H_2(N, \partial N - T)$.

In the same vein as Gabai's and Sela's results, Lackenby [Lac97b, Theorem 1.4b] (under additional hypotheses and a change of notation[2]) showed that if $\widehat{Q}$ is a compact connected surface in $M' = N_T(a)$ which cannot be isotoped to be disjoint from $K_a$ and if there is a norm-reducing class under a filling of slope $b$ with $\Delta = \Delta(a,b) \geq 2$, then $\widehat{Q}$ can be isotoped so that

$$|K_a \cap \widehat{Q}|(\Delta - 1) \leq -\chi(\widehat{Q}).$$

If, in Lackenby's setup, $\widehat{Q}$ is taken to be a taut representative of a non-zero class $\widehat{y} \in H_2(M', \partial M')$, then we have (after rearranging the inequality):

$$\Delta \leq 1 + \frac{x(\widehat{y})}{|K_a \cap \widehat{Q}|}.$$

Our Corollary 4.3, gives a version of this result for the situation when $H_2(N, \partial N)$, and not just $H_2(N, \partial N - T)$, has a norm-reducing class with respect to the slope $b$.

In addition to considering a fixed component $T$ of $\partial N$ and studying the dependency of the Thurston norm on the filling slope, we can also consider a 3-manifold $M$ and consider how the Thurston norm of manifolds $M'$ obtained by surgery on an oriented knot $K$ in $M$ depends on the dual Thurston norm $x^*([K])$ of the class $\alpha = [K] \in H_1(M; \mathbb{Z})$.

**Theorem 4.7.** *Let $M$ be a compact, orientable 3–manifold whose boundary is a union of tori, $\Delta \in \mathbb{N}$, and $\alpha \in H_1(M; \mathbb{Z})$. Assume that every sphere, disk, annulus, and torus in $M$ separates. If*

$$(\Delta - 1)x^*(\alpha) > 1,$$

[2] In Lackenby's paper, see Assumptions 1.1 and Remark 1.3. To convert the notation from ours to Lackenby's make the following changes: $\gamma = \varnothing$, $M' \to M$, $K_a \to L$, $N \to M - \mathrm{int}(N(L))$, $\widehat{Q} \to F$. The class whose norm is reduced is called $z_1$ by Lackenby.

*then every irreducible, $\partial$–irreducible 3–manifold obtained by a Dehn surgery of distance $\Delta$ on a knot $K$ representing $\alpha$ has no norm-reducing classes with respect to the knot which is surgery dual to $K$.*

The contrapositive is also a useful formulation, as it shows that knots resulting from non-longitudinal surgery on a knot with a norm-reducing class have bounded dual norm.

Finally, we give an application to the genus of knots in twist families. A *twist family* of knots $\{K_n\}$ is obtained by performing $-1/n$–Dehn surgery on an unknot $c$ that links a given knot $K = K_0$. When $\ell k(K, c) = 0$, it is a fundamental consequence of [Gab87a, Corollary 2.4] that $g(K_n)$ is constant for all integers $n$ except at most one where the genus decreases. Using the multivariable Alexander polynomial, the first author and Motegi showed that if $|\ell k(K, c)| \geq 2$, then $g(K_n) \to \infty$ as $n \to \infty$ [BM15]. When $|\ell k(K, c)| = 1$, this fails if $c$ is a meridian of $K$ since $K_n = K$ for all $K$. Here we answer [BM15, Question 2.2] by showing this is the only exception.

**Theorem 5.1.** *If $\omega = |\ell k(K, c)| > 0$, then $\lim_{n\to\infty} g(K_n) = \infty$ unless $c$ is a meridian of $K$.*

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## 2. Preliminaries

**2.1. Notation and conventions.** The following notation is used throughout the article. We take $N$ to be a compact, connected, irreducible oriented 3–manifold where $\partial N$ is a non-empty union of tori and focus upon a particular component $T \subset \partial N$. Given two slopes $a, b \subset T$, we set the results of Dehn filling $N$ along these slopes to be the two 3–manifolds $M = N_T(b)$ and $M' = N_T(a)$. Furthermore, we let $K = K_b \subset M$ and $K' = K_a \subset M'$ denote the core knots of the two filling solid tori.

The **distance** $\Delta = \Delta(a, b)$ between two slopes $a, b \subset T$ is the minimal number of points of intersection between simple closed curves in $T$ representing $a$ and $b$.

Given a surface $S$ properly embedded in $N$, the union of the boundary components of $S$ in $T$ is $\partial_T S = \partial S \cap T$. If the slope of each component of $\partial_T S$ in $T$ is $b$ (as an unoriented curve), then we set $\widehat{S} \subset M$ to be the surface obtained by capping off the components of $\partial_T S$ with meridian disks of the filling solid torus. Observe that by construction, $|K \cap \widehat{S}| = |\partial_T S|$.

In this article, a **lens space** is a closed 3–manifold with a genus 1 Heegaard splitting other than $S^3$ and $S^1 \times S^2$. In particular, the fundamental group of a lens space is a non-trivial, finite, cyclic group.

**2.2. Thurston norm.** Thurston introduced two norms on the homology groups of a compact, orientable 3–manifold $W$ [Thu86], now commonly known as the **Thurston norm** and the **dual Thurston norm**:

$$x\colon H_2(W, \partial W; \mathbb{R}) \to [0, \infty) \quad \text{and} \quad x^*\colon H_1(W; \mathbb{R}) \to [0, \infty),$$

which we may write as $x_W$ and $x^*_W$ to emphasize the 3–manifold $W$.

On an integral class $\sigma \in H_2(W, \partial W; \mathbb{Z})$, the Thurston norm is defined by

$$x(\sigma) = \min_S \sum_{i=1}^{n} \max\{0, -\chi(S_i)\},$$

where the minimum is taken over all embedded surfaces $S$ representing $\sigma$ with connected components $S_1, \ldots, S_n$. The function $x$ is linear on rays and convex. These properties enable it to be extended first to rational homology classes and then to real homology classes.

In general, the function $x$ is only a pseudo-norm; $x$ is a norm if $W$ contains no non-separating sphere, disk, torus, or annulus. Nevertheless, $x$ is reasonably well behaved even in the presence of non-separating tori and annuli, it is non-separating spheres and disks that complicate the norm:

> *If an integral class $\sigma \in H_2(W, \partial W; \mathbb{Z})$ cannot be represented by a surface with a non-separating sphere or disk component, then $x(\sigma)$ is just the minimum of $-\chi(S)$ among surfaces representing $\sigma$.*

It is for such integral classes that Inequality (†) holds. Assuming $W$ has no $S^1 \times S^2$ or $S^1 \times D^2$ summand ensures this is the case for all classes, as does the more heavy-handed assumption that $W$ is irreducible and $\partial$–irreducible. In particular, we can now prove Lemma 1.1.

*Proof of Lemma 1.1.* Recall that $N$ is a compact, orientable, irreducible 3–manifold with $\partial N$ the union of tori and $T \subset \partial N$ a component. Let $b$ be a slope in $T$ and assume that $N_T(b)$ has no $S^1 \times D^2$ or $S^1 \times S^2$ summands. Let $\partial_T\colon H_2(N, \partial N) \to H_1(T)$ be the boundary map restricted to $T$. We will show that for all classes $\widehat{z} \in H_2(N_T(b), \partial N_T(b))$,

$$x(z) \geq x(\widehat{z}) + \mathrm{wind}_{K_b}(\widehat{z}). \tag{†}$$

As usual, it suffices to prove the inequality for integral classes. In which case, there exists a properly embedded oriented surface $S \subset N$ such that $S$ has no separating component, $[S] = z$, and all components of $\partial_T S$ are coherently oriented curves, each of slope $b$, and $x(S) = x(z)$. If some component of $S$ is a sphere or disk, then it would persist into $N_T(b)$ as a non-separating sphere or disk, contrary to our hypotheses. Hence $S$ has no sphere or disk component and $x(S) = -\chi(S)$.

Cap off the components of $\partial_T(S)$ in $N_T(b)$ with disks to obtain the surface $\widehat{S}$. Observe that

$$|\partial_T S| = |\widehat{S} \cap K_b| = \mathrm{wind}_{K_b}(\widehat{z})$$

since the components of $\partial_T S$ are coherently oriented. Since $M$ contains no non-separating sphere or disk, $-\chi(\widehat{S}) \geq x(\widehat{z})$. Consequently,

$$x(z) = -\chi(S) = -\chi(\widehat{S}) + \mathrm{wind}_{K_b}(\widehat{z}) \geq x(\widehat{z}) + \mathrm{wind}_{K_b}(\widehat{z}).$$

q.e.d.

Finally, on a class $\alpha \in H_1(W;\mathbb{R})$, the dual Thurston norm is defined by

$$x^*(\alpha) = \sup_{x(\sigma)\leq 1} |\alpha \cdot \sigma|,$$

where $\cdot$ denotes the intersection product. The function $x^*\colon\ H_1(W;\mathbb{R}) \to [0,\infty)$ is continuous.

**2.3. Wrapping numbers.** Having defined the winding number, we now turn to wrapping number. A compact, oriented, properly embedded surface $S$ in a 3–manifold $W$ is **taut** (or $\varnothing$–taut) if it is incompressible (i.e. does not admit a compressing disk), and minimizes the Thurston norm among embedded surfaces representing the class $[S, \partial S] \in H_2(W, \partial S)$ [Sch89, Def. 1.2]. Observe that if a surface $S \subset N$ is taut and has the property that $x(S) = x([S])$, then the surface $S'$ obtained by discarding all separating components of $S$ (which are necessarily spheres, disks, annuli, and tori) is also taut and has the properties that $[S] = [S'] \in H_2(N, \partial N)$ and $x(S') = x([S]) = x([S'])$.

We define the **wrapping number** of $K$ about an integral homology class $\widehat{z} \in H_2(M, \partial M;\mathbb{Z})$ to be

$$\mathrm{wrap}_K(\widehat{z}) = \min_{\widehat{S}} |K \cap \widehat{S}|,$$

where the minimum is taken over all *taut* representatives $\widehat{S}$ of $\widehat{z}$.

> *Since discarding separating components of $\widehat{S}$ will not increase $|K \cap \widehat{S}|$, we will henceforth assume that whenever we discuss a taut surface realizing the Thurston norm of a homology class in the second homology group of a 3-manifold relative to the boundary of that 3-manifold, we have discarded all separating components.*

We may extend the wrapping number to $H_2(M, \partial M;\mathbb{Q})$. Assume $\widehat{S}$ is a taut surface realizing $\mathrm{wrap}_K(\widehat{z})$ for an integral class $\widehat{z} \in H_2(M, \partial M;\mathbb{Z})$. Then, following [Thu86, Lemma 1], $n$ parallel copies of $\widehat{S}$ is a taut surface realizing $\mathrm{wrap}_K(n\,\widehat{z}) = n\,\mathrm{wrap}_K(\widehat{z})$ for positive integers $n$. Thus for a rational class $\widehat{q}$ we define $\mathrm{wrap}_K(\widehat{q}) = \frac{1}{n}\mathrm{wrap}_K(n\,\widehat{q})$ where $n$ is a positive integer such that $n\widehat{q}$ is an integral class. Since algebraic intersection numbers give lower bounds for geometric intersection numbers,

$\mathrm{wrap}_K(\widehat{q}) \geq \mathrm{wind}_K(\widehat{q})$ for all $\widehat{q} \in H_2(M, \partial M; \mathbb{Q})$. Observe that if $M$ has no norm-reducing classes with respect to $K$, then $\mathrm{wrap}_K = \mathrm{wind}_K$ is a pseudo-norm. However, we believe that, in general, the triangle inequality will not hold for $\mathrm{wrap}_K$.

**Question 2.1.** Must the wrapping number satisfy the triangle inequality?

A class $\widehat{z} \in H_2(M, \partial M)$ is **exceptional** with respect to a knot $K$ [Tay14] if the winding number and wrapping number are not equal; that is $\widehat{z}$ is exceptional with respect to $K$ if

$$\mathrm{wind}_K(\widehat{z}) < \mathrm{wrap}_K(\widehat{z}).$$

This definition takes root in the practical difference between the Thurston norm and Scharlemann's $\beta$–norm. As discussed in [Tay14], a class $\widehat{z}$ is *exceptional* with respect to $K$ if and only if no representative of $\widehat{z}$ is both $\varnothing$–taut and $K$–taut. (Here, $K$ is playing the role of $\beta$. See [Sch89] for the definitions of the $\beta$–norm and $\beta$–taut surfaces.)

For our present purposes, we observe that *norm-reducing* classes and *exceptional* classes are equivalent in the absence of non-separating spheres and disks. This allows us to parlay technical results about exceptional classes into results about norm-reduction.

**Lemma 2.2.** *Suppose that $M$ contains no non-separating sphere or disk. Then, with respect to a knot $K$ in $M$, a class $\widehat{z} \in H_2(M, \partial M)$ is exceptional if and only if it is norm-reducing.*

*Proof.* Assume $M = N_T(b)$ where $K = K_b$. For a class $\widehat{z} \in H_2(M, \partial M)$, let $z = \rho_b(\widehat{z}) \in H_2(N, \partial N)$.

First, we claim that if $S$ is a taut representative of a class $[S] \in \operatorname{im} \rho_b$, then

$$x([S]) = x(S) = -\chi(S).$$

To see this, let $S \subset N$ be taut and have each component of $\partial_T S$ of slope $b$. By definition, $x([S]) = x(S)$. Suppose that $x(S) \neq -\chi(S)$. Then $S$ contains a component $P$ which is a sphere or disk. Since $S$ is taut, $P$ is non-separating. Capping off $\partial_T P$ in $M$, if necessary, creates a non-separating sphere or disk in $M$, contrary to hypothesis.

We now embark on the proof. The claim is trivially satisfied for the 0 class, so assume that $0 \neq \widehat{z} \in H_2(M, \partial M; \mathbb{Z})$ is not an exceptional class for $K$. Then there is a taut representative $\widehat{S} \subset M$ of $\widehat{z}$ for which $\mathrm{wrap}_K(\widehat{S}) = \mathrm{wind}_K(\widehat{S})$. Thus

$$\begin{aligned} x_N(z) &\leq x_N(S) \\ &= -\chi(S) \\ &= -\chi(\widehat{S}) + \mathrm{wind}_K(\widehat{S}) \end{aligned}$$

$$= x_M(\widehat{z}) + \mathrm{wind}_K(\widehat{z})$$
$$\leq x_N(z),$$

where the last inequality is due to Inequality (†). Consequently $x_M(\widehat{z})+\mathrm{wind}_K(\widehat{z}) = x_N(z)$, and thus $\widehat{z}$ is not norm-reducing with respect to $K$.

Conversely, assume that $\widehat{z} \in H_2(M, \partial M)$ is exceptional with respect to $K$ so that $\mathrm{wrap}_K(\widehat{z}) > \mathrm{wind}_K(\widehat{z})$. Let $S$ be a taut surface in $N$ representing $z$, and let $\widehat{S} \subset M$ be the result of capping off $\partial_T S$ with disks so that $[\widehat{S}] = \widehat{z}$. Then

$$x_N(z) = -\chi(S) = -\chi(\widehat{S}) + |\widehat{S} \cap K| > x_M(\widehat{z}) + \mathrm{wind}_K(\widehat{z}),$$

because $|\widehat{S} \cap K| \geq |\widehat{S} \cdot K| = \mathrm{wind}_K(\widehat{z})$ and $-\chi(\widehat{S}) \geq x_M(\widehat{z})$. Thus, $\widehat{z}$ is norm-reducing with respect to $K$. q.e.d.

**2.4. Multi-$\partial$-compressing disks.** As is often the case in studies of Dehn filling, we will want use a surface $\widehat{Q}$ in one filling $M' = N_T(a)$ of $N$ to say something useful about a different filling $M = N_T(b)$. For us, the surface $\widehat{Q}$ will be most useful if it has no "multi-$\partial$-compressing disk."

Suppose that $\widehat{S} \subset M' = N_T(a)$ is a surface transversally intersecting $K' \subset M'$ non-trivially. A **multi-$\partial$-compressing disk** for $\widehat{S}$ (with respect to $K'$) is a disk $D \subset N$ such that there is a component $A \subset T - S$ such that:

- The interior of $D$ is disjoint from $\partial N \cup S$.
- The boundary of $D$ is a simple closed curve lying in $S \cup A$.
- After orienting $\partial D$, $\partial D \cap A$ is a non-empty, coherently oriented collection of spanning arcs of $A$.

Given a multi-$\partial$-compressing disk $D$ for $\widehat{S}$, then we may create a new surface $\widehat{S}'$ that is homologous to $\widehat{S}$ but intersects $K'$ in two fewer points: that is, $[\widehat{S}] = [\widehat{S}'] \in H_2(M', \partial M')$ and $|\widehat{S}' \cap K'| = |\widehat{S} \cap K'| - 2$. We create $\widehat{S}'$ by removing the open regular neighborhood of two points of $K' \cap \widehat{S}$, attaching the annulus $A$ (from the definition of "multi-$\partial$-compressing disk") and then compressing using $D$.

The next lemma allows us to know when we have a surface without a multi-$\partial$-compressing disk.

**Lemma 2.3.**

- *Suppose that $\widehat{S} \subset M'$ is a sphere transverse to $K'$ such that $S = \widehat{S} \cap N$ is incompressible and not $\partial$–parallel. Then either $M'$ has a lens space summand or $\widehat{S}$ does not have a multi-$\partial$-compressing disk with respect to $K'$.*
- *Suppose that $\widehat{S} \subset M'$ is a disk transverse to $K'$ such that $S = \widehat{S} \cap N$ is incompressible. Then either $M'$ has a lens space summand or $\widehat{S}$ does not have a multi-$\partial$-compressing disk with respect to $K'$.*

- *Suppose that $\widehat{S} \subset M'$ is a taut representative of some non-zero class in $H_2(M', \partial M'; \mathbb{Z})$ and that, out of all such taut surfaces representing that class, $\widehat{S}$ minimizes $|\widehat{S} \cap K'|$. Then either $M'$ contains a non-separating sphere or disk or $\widehat{S}$ does not have a multi-$\partial$-compressing disk with respect to $K'$.*

*Proof.* Suppose that $\widehat{S} \subset M'$ is a surface transverse to $K'$, such that $S$ is incompressible and not $\partial$-parallel. If $K'$ is disjoint from $\widehat{S}$, then trivially there is no multi-$\partial$-compressing disk. Hence we further assume $K'$ transversally intersects $\widehat{S}$ non-trivially.

Suppose that $D$ is an oriented multi-$\partial$-compressing disk for $\widehat{S}$. Then there is an annulus component $A \subset T \setminus S$ such $\partial D \cap A$ is a non-empty collection of coherently oriented spanning arcs of $A$. Let $\widehat{R}$ be the surface in $M'$ obtained from isotoping $S \cup A \subset N$ with support in a neighborhood of $A$ to be properly embedded in $N$ and then capping off the boundary components in $T$ with meridional disks of the filling solid torus; i.e. $\widehat{R}$ is the result of tubing $\widehat{S}$ along a particular arc of $K' \setminus \widehat{S}$. A further slight isotopy makes $\widehat{R}$ disjoint from $\widehat{S}$.

Now let $\widehat{S}'$ be the result of compressing $\widehat{R}$ using $D$, and slightly isotoping to be disjoint from $\widehat{R}$. Observe that $-\chi(\widehat{S}') = -\chi(\widehat{S})$ and that there is a natural bijection between the components of $\widehat{S}$ and $\widehat{S}'$.

First assume $\widehat{S}$ is a sphere. Then $\widehat{S}'$ must also be a sphere. If $\partial D$ runs just a single time across $A$, then $D$ provides a $\partial$–compression for $S$ in $N$. Since $N$ is irreducible, either $S$ is compressible or $S$ is a $\partial$-parallel annulus contrary to hypothesis. If $\partial D$ runs multiple times across $A$, then $\widehat{S}$ and $\widehat{S}'$ cobound a 3–manifold $W$ in which $\widehat{R}$ is a genus 1 Heegaard surface. Because $\widehat{S}$ and $\widehat{S}'$ are both spheres, $W$ is a twice-punctured lens space of finite order $|\partial D \cap A| > 1$. The complement of a neighborhood of an embedded arc in $W$ that connects both components of $\partial W$ is therefore a non-trivial lens space summand of $M'$.

When $\widehat{S}$ is a disk, we similarly obtain that $\widehat{S}'$ is also a disk. Along with an annulus in $\partial M'$, the disks $\widehat{S}$ and $\widehat{S}'$ bound a punctured lens space $W$ in which $\widehat{R}$ is a punctured Heegaard torus. Again, this lens space has finite order $|\partial D \cap A|$ which is non-trivial since $\widehat{S}$ is incompressible. Hence $W$ is a lens space summand of $M'$.

Now assume that $\widehat{S}$ is a taut representative of a class in $H_2(M', \partial M'; \mathbb{Z})$. If $\widehat{S}$ has a sphere, then the component must be non-separating since $\widehat{S}$ is taut. So we may further assume $\widehat{S}$ is not a sphere. By construction, the surface $\widehat{S}'$ represents the same class, has the same euler characteristic, and intersects $K'$ two fewer times than does $\widehat{S}$. Furthermore, since every component of $\widehat{S}$ is non-separating, every component of $\widehat{S}'$ is also non separating. If $\widehat{S}'$ is not taut, then since it is homologous to the taut surface $\widehat{S}$ and is also Thurston norm minimizing for this homology class, it must have a compressible component that is a non-separating torus or annulus. Compressing this torus or annulus creates a non-separating sphere or disk in $M'$. q.e.d.

## 3. A key theorem of Taylor

In [Tay14], the second author develops some classical results ([Sch89, Application III] and [Sch90]) from Scharlemann's combinatorial version [Sch89] of Gabai's sutured manifold theory [Gab83, Gab87a, Gab87b] in terms of surgeries on knots with exceptional classes. Here we adapt a key technical theorem for our purposes.

**Theorem 3.1** (Cf. [Tay14, Theorem 3.14]). *Assume that $N$ is irreducible and $\partial$–irreducible. Let $a, b$ be two distinct slopes in $T \subset \partial N$. Suppose that $M = N_T(b)$ is not a solid torus, has no proper summand which is a rational homology sphere, and $H_2(M, \partial M) \neq 0$. Suppose that $M' = N_T(a)$ contains a properly embedded, compact, orientable surface $\widehat{Q} \subset M'$ that transversally intersects $K'$ non-trivially, does not have a multi-$\partial$-compressing disk for $K'$, and restricts to an incompressible surface*[3] *$Q = \widehat{Q} \cap N$ in $N$.*

*If*

$$-\chi(\widehat{Q}) < |\widehat{Q} \cap K'|(\Delta(a, b) - 1),$$

*then $M$ is irreducible and $H_2(M, \partial M)$ has no exceptional classes with respect to $K$.*

For the proof, we content ourselves with explaining how the statement follows from [Tay14, Theorem 3.14]. We assume familiarity with the basic definitions regarding $\beta$–taut sutured manifold technology from [Sch89] (see also [Tay14]).

*Proof.* Our notation is very similar to that of [Tay14], except that we are using $K$ as the core knot of the filling $M = N(b)$ instead of $\beta$ and we consider classes $\widehat{y} \in H_2(M, \partial M)$ rather than classes $y$.

Our hypotheses immediately imply Conditions (1) and (3) of [Tay14, Theorem 3.14]. Since $N$ is irreducible and $\partial$–irreducible, we may consider it as a taut sutured manifold $(N, \varnothing, \varnothing)$, considering $\partial N$ as toroidal sutures. The filling $M = N_T(b)$ induces a sutured manifold $(M, \varnothing, K)$ that is then a $K$–taut sutured manifold, providing Condition (2).

Since $\widehat{Q} \cap K' \neq \varnothing$ and the curves of $\partial_T Q$ have slope $a$, the boundary of $Q$ is not disjoint from the slope $b$ in $T$. Sphere components of $\widehat{Q}$ that are disjoint from $K'$ are the sphere components of $Q$; however, since the irreducibility of $N$ implies that any sphere component of $Q$ must

[3] We use the convention that any sphere component of an incompressible surface does not bound a ball, and any disk component is not $\partial$–parallel.

bound a ball in $N$, the incompressibility of $Q$ prohibits the existence of such sphere components. Furthermore, no component of $Q$ is a disk with essential boundary since $N$ is $\partial$–irreducible and no component of $Q$ is a disk with inessential boundary due to the incompressibility of $Q$ and irreducibility of $N$. Thus Condition (4) is satisfied.

We may now apply [Tay14, Theorem 3.14]. Our hypothesis that $M$ has no proper summand that is a rational homology sphere immediately rules out Conclusion (4) of [Tay14, Theorem 3.14]. We proceed to show that Conclusions (3) and (2) also fail and that Conclusion (1) implies our stated result.

In the terminology of [Sch89, Section 7] and [Tay14, Section 2.2], the surface $Q$ is a *parameterizing surface* for the sutured manifold $(M, \varnothing, K)$. By definition (again, see [Sch89, Definition 7.4] and [Tay14, Section 2.2]), its *index* $I(Q)$ is given by

$$I(Q) = -2\chi(Q)$$

since (i) there are no annular sutures on $\partial M$ and (ii) $K$ is a knot (rather than a collection of properly embedded arcs). Without loss of generality, we may assume that the slope $b$ has been isotoped in $T$ to intersect $\partial Q$ minimally. Thus, $|\partial Q \cap b|$ is equal to $\Delta(a, b)|\widehat{Q} \cap K'|$. Our assumed inequality on the Euler characteristic of $\widehat{Q}$ can then be rearranged to yield

$$I(Q) < 2|\partial Q \cap b|.$$

Hence, Conclusion (3) of [Tay14, Theorem 3.14] does not hold.

A *Gabai disk* for $Q$ is a disk $D$ embedded in $M$ that $K$ non-trivially and coherently intersects, such that its restriction to $N$ is transverse to $Q$ and $|Q \cap \partial D| < \Delta(a, b)|\partial_T Q|$. It is shown in [CGLS87] (though without the language of Gabai disks), and further explained in [Sch90] and [Tay14], that a Gabai disk will contain a Scharlemann cycle. As $Q$ is incompressible and $N$ is irreducible, the interior of the Scharlemann cycle can be isotoped to be a multi-$\partial$-compressing disk for $\widehat{Q}$. See [Tay14, Section 4] for more details. (Although observe that [Tay14, Lemma 4.3] neglected to consider possible circles of intersection between the interior of the Scharlemann cycle and $Q$. We have added the incompressibility hypotheses to $Q$ to deal with this.) Since we are assuming that $\widehat{Q}$ has no multi-$\partial$-compressing disk, Conclusion (2) of [Tay14, Theorem 3.14] does not hold.

Consequently, the Conclusion (1) of [Tay14, Theorem 3.14] holds. Hence, given any non-zero class $\widehat{y} \in H_2(M, \partial M; \mathbb{Z})$, there is a $K$–taut hierarchy of $(M, \varnothing, K)$ which is also $\varnothing$–taut such that the first decomposing surface $\widehat{S} \subset M$ represents $\widehat{y}$. In particular, since sutured manifold decompositions yields a taut sutured manifold only if the decomposing surface is taut, the $K$–tautness and $\varnothing$–tautness of the hierarchy implies the surface $\widehat{S}$ must be both $K$–taut and $\varnothing$–taut (see e.g. [Sch89, Definition 4.18], [Sch90, Section 2], [Gab83, Lemma 3.5 and Section 4]). Since $(M, \varnothing, \varnothing)$ is $\varnothing$–taut, $M$ is irreducible. By the definition of $K$–taut, the knot $K$ always intersects $\widehat{S}$ with the same sign. That is, $\mathrm{wind}_K(\widehat{S}) = \mathrm{wrap}_K(\widehat{S})$. Since $\widehat{S}$ is $\varnothing$–taut, this implies that $\widehat{y}$ is not an exceptional class. Since this holds true for all non-zero classes in $H_2(M, \partial M; \mathbb{Z})$, so there are no exceptional classes in $H_2(M, \partial M; \mathbb{Z})$ with respect to $K$. q.e.d.

## 4. The Thurston norm and dual norm under Dehn filling

### 4.1. The Thurston norm.

**Theorem 4.1.** *Suppose that $N$ is irreducible and $\partial$–irreducible. Also assume that $M = N_T(b)$ is not a solid torus and has no proper rational homology sphere summand and that either $M$ is reducible or that $H_2(M, \partial M)$ has an exceptional class with respect to $K$. Then all of the following hold for $M' = N_T(a)$:*

- *Either $M'$ has a lens space summand or*
  - *$M'$ is irreducible and $\partial$–irreducible, and*
  - *$K' \subset M'$ is mp-small; that is, there is no essential, connected, properly embedded planar surface $Q \subset N$ such that $\partial Q = \partial_T Q \neq \varnothing$ and each component of $\partial Q$ has slope $b$ in $T$.*
- *For every $\widehat{y} \in H_2(M', \partial M')$,*

$$x(\widehat{y}) \geq \mathrm{wrap}_{K'}(\widehat{y})(\Delta(a, b) - 1).$$

**Remark 4.2.** The first conclusion of Theorem 4.1, that $M'$ is irreducible and $\partial$–irreducible, essentially follows from [Sch90].

*Proof.* Assume, for the moment, that either $M'$ is reducible or $\partial$–reducible or that $K'$ is not mp-small. Then there exists an essential, connected, properly embedded planar surface $Q \subset N$ such that $\partial Q$ has at most one component not in $T$, $\partial_T Q$ is non-empty (because $N$ is irreducible and $\partial$–irreducible), and every component of $\partial_T Q$ has slope $b$. Let $\widehat{Q} \subset M'$ be the sphere or disk that results from capping off $\partial_T Q$ with disks. Lemma 2.3 shows that there is no multi-$\partial$-compressing disk for $\widehat{Q}$. Then by Theorem 3.1, since either $M$ is reducible or $H_2(M, \partial M)$ has an exceptional class with respect to $K$, we have

$$0 > -\chi(\widehat{Q}) \geq |\widehat{Q} \cap K'|(\Delta(a, b) - 1) \geq 0,$$

which is a contradiction. Thus, $M'$ is irreducible, $\partial$–irreducible, and $K'$ is mp-small.

Because $M'$ is irreducible and $\partial$–irreducible, every sphere and disk in $M'$ separates. So consider a class $\widehat{y} \in H_2(M', \partial M')$. Among the taut surfaces in $M'$ representing $\widehat{y}$, let $\widehat{Q} \subset M'$ be chosen to minimize $|\widehat{Q} \cap K'|$. Tautness implies that no component of $\widehat{Q}$ is a sphere or disk, that $x(\widehat{y}) = -\chi(\widehat{Q})$, and that there is no compressing disk for $\widehat{Q}$ in $M'$. The minimality gives $\mathrm{wrap}_{K'}(\widehat{y}) = |\widehat{Q} \cap K'|$ while also implying that there can be no compressing disk for $Q = \widehat{Q} \cap N$ in $N$. Since every sphere and disk in $M'$ separates, Lemma 2.3 implies there are also no multi-$\partial$-compressing disks for $Q$ with respect to $K$.

If $\widehat{Q} \cap K' = \varnothing$, then $\mathrm{wrap}_{K'}(\widehat{y}) = 0$ and the desired inequality is trivially true. Thus, assume that $\widehat{Q} \cap K' \neq \varnothing$. Using Theorem 3.1 again, we then have

$$x(\widehat{y}) = -\chi(\widehat{Q}) \geq |\widehat{Q} \cap K'|(\Delta(a,b) - 1) = \mathrm{wrap}_{K'}(\widehat{y})(\Delta(a,b) - 1)$$

as desired. q.e.d.

The next corollary is a useful specialization.

**Corollary 4.3.** *Let $N$ be a compact, orientable, irreducible, $\partial$–irreducible 3–manifold such that $\partial N$ is a union of tori. Given distinct slopes $a$ and $b$ in a component $T$ of $\partial N$, let $M = N_T(b)$ and $M' = N_T(a)$ be the results of Dehn filling along these slopes, and let $K$ and $K'$ be the core knots of these fillings respectively.*

*Assume $M$ and $M'$ are irreducible, $\partial$–irreducible and $K'$ has non-zero wrapping number with respect to a class $\widehat{y} \in H_2(M', \partial M')$. If there exists a class of $H_2(M, \partial M)$ that is norm-degenerate with respect to $K$, then*

$$\Delta(a,b) \leq 1 + x(\widehat{y})/\mathrm{wrap}_{K'}(\widehat{y}) \leq 1 + x(\widehat{y}).$$

*Proof.* Since we may assume that both $H_2(M, \partial M)$ and $H_2(M', \partial M')$ are non-trivial, $N$ is not a solid torus. By the irreducibility and $\partial$–irreduciblity of $M$ and $M'$, every sphere and disk in $M$ and $M'$ must separate. Thus, according to Lemma 2.2 any class in $H_2(M, \partial M)$ that is norm-degenerate with respect to $K$ is also exceptional with respect to $K$. Then, due to Theorem 4.1, for every non-zero $\widehat{y} \in H_2(M', \partial M')$ we have $x(\widehat{y}) \geq \mathrm{wrap}_{K'}(\widehat{y})(\Delta(a,b) - 1)$. When the wrapping number is non-zero, we may obtain the stated inequalities. q.e.d.

We can now bound the number of slopes producing filled manifolds with norm-reducing classes (with respect to the filling).

**Corollary 4.4.** *Let $N$ be a compact, orientable, irreducible, and $\partial$–irreducible 3–manifold such that $\partial N$ is a union of tori. Assume for $i = 1, 2$, there is a slope $a_i$ in the component $T$ of $\partial N$ such that the manifold $M_i' = N_T(a_i)$ is irreducible and $\partial$–irreducible and the core $K_i'$ of the Dehn filling has non-zero wrapping number with respect to a class $\widehat{y}_i \in H_2(M_i', \partial M_i')$. If $\Delta(a_1, a_2) > 0$, then there are at most*

$$(1 + x(\widehat{y}_1))(1 + x(\widehat{y}_2)) + (\Delta(a_1, a_2) - 1)(1 + x(\widehat{y}_1))^2$$

*slopes $b \subset T$ distinct from $a_1$ and $a_2$ such that the 3–manifold $N_T(b)$ obtained by filling $T$ along $b$ is irreducible, $\partial$–irreducible, and has a norm-reducing class with respect to the filling.*

*Proof.* By Corollary 4.3, if $b$ is a slope in $T$ such that $N_T(b)$ is irreducible, $\partial$–irreducible, and has a norm-reducing slope for the core of the filling, then

$$\Delta(a_1, b) \leq 1 + x(\widehat{y}_1) \quad \text{and} \quad \Delta(a_2, b) \leq 1 + x(\widehat{y}_2).$$

Then Lemma 4.5 below gives that the number of slopes $b$ satisfying these constraints is at most

$$(1 + x(\widehat{y}_1))(1 + x(\widehat{y}_2)) + (\Delta(a_1, a_2) - 1)(1 + x(\widehat{y}_1))^2.$$

q.e.d.

**Lemma 4.5.** *Given slopes $b, c$ in $T$ with $\Delta(b, c) \geq 1$ and positive numbers $B, C$, then the number of slopes $a$ in $T$ such that $\Delta(a, b) \leq B$ and $\Delta(a, c) \leq C$ is at most $BC + (\Delta(b, c) - 1)B^2$.*

*Proof.* Let us regard slopes as being represented by oriented simple closed curves. We may choose a basis for $H_1(T)$ in which $[b] = (1, 0)$ and $[c] = (r, s)$ for coprime integers $0 \leq r < s$. Then $\Delta(b, c) = s$. For any slope $a$ in $T$, we may choose an orientation of the curve so that the constraints $\Delta(a, b) \leq B$ and $\Delta(a, c) \leq C$ and the orientation restrict its representatives in this homology basis to an element of the set $\Lambda$ of integer lattice point in the trapezoid $\{(x, y) \colon |y| \leq B, |ry - sx| \leq C, x \geq 0\}$. For points $(x, y) \in \Lambda$, one deduces that

$$\begin{aligned} 0 &\leq x \\ &\leq s|x| \\ &\leq |ry - sx| + r|y| \\ &\leq C + rB \\ &\leq C + (s-1)B \\ &= C + (\Delta(b, c) - 1)B. \end{aligned}$$

Thus $|\Lambda| \leq B \cdot (C + sB) = BC + (\Delta(b, c) - 1)B^2$, giving an upper bound on the number of slopes $a$ in $T$ satisfying the constraints. q.e.d.

**Theorem 4.6.** *Let $N$ be a compact, connected, orientable, irreducible, and $\partial$–irreducible 3–manifold whose boundary is a union of tori. Then either*

1) *$N$ is a product of a torus and an interval,*
2) *$N$ is a cable space, or*
3) *for each torus component $T \subset \partial N$ there is a finite set of slopes $\mathcal{R} = \mathcal{R}(N, T)$ in $T$ such that if $b \notin \mathcal{R}$ then $b$ is not norm-reducing.*

*Proof.* Let $T$ be a particular component of $\partial N$. By [HM02, GL96], $N_T(a)$ is a reducible for at most three slopes $a$. By [CGLS87, Corollary 2.4.4], unless $N \cong T \times [0,1]$ or $N$ is a cable space, $N_T(a)$ is $\partial$–reducible for at most three slopes $a$. Hence, we now assume $N$ is neither homeomorphic to $T \times [0,1]$ nor a cable space, so that there are at most 6 slopes in $T$ for which $N_T(a)$ is reducible or $\partial$–reducible.

Let $(\partial_T)_*\colon H_2(N, \partial N) \to H_1(T)$ be the composition of the boundary map on $H_2(N, \partial N)$ with the projection from $H_1(\partial N)$ to $H_1(T)$. For every slope $a$ in $T$ that generates a rank 1 subspace of the image of $(\partial_T)_*$ in $H_1(T)$, there is some class $\widehat{y} \in H_2(N_T(a), \partial N_T(a))$ such that $\mathrm{wind}_a(\widehat{y}) > 0$. Since $\mathrm{wind}_a$ gives a lower bound on $\mathrm{wrap}_a$, the core of the Dehn filling $N_T(a)$ has non-zero wrapping number with respect to the class $\widehat{y}$. Therefore, if $(\partial_T)_*$ surjects onto $H_1(T)$, the core of any Dehn filling of $N$ along $T$ will have non-zero wrapping number with respect to some class in the filled manifold. In this case we may find a pair of slopes satisfying the hypotheses of Corollary 4.4 so that the number of norm-reducing, but irreducible, and $\partial$–irreducible slopes is finite. Since the number of reducible or $\partial$–reducible slopes in $T$ is also finite, we have our conclusion.

On the other hand, if $(\partial_T)_*$ does not surject onto $H_1(T)$, its image must be a rank 1 subspace generated by a single slope, say $b$. For every other slope $a \neq b$, $\mathrm{wind}_a = 0$. Hence for all $a \neq b$, $\rho_a$ gives an isomorphism $H_2(N_T(a), \partial N_T(a)) \cong H_2(N, \partial N - T)$. Then it follows from [Sel90] (but using [Gab87a, Corollary 2.4] instead of just [Gab87a, Theorem 1.8] to avoid hypotheses of atoroidality, see also [Lac97a, Theorem A.21]) that there are finitely many norm reducing fillings. q.e.d.

**4.2. The dual norm.** As we observed in the introduction, Theorem 4.7 shows that, in general, there are no norm-reducing classes with respect to a knot that is surgery dual to a knot with "large" dual Thurston norm, quantified in terms of the distance of the surgery.

**Theorem 4.7.** *Assume that every sphere, disk, annulus, and torus in $M'$ separates. Given a class $\alpha \in H_1(M';\mathbb{Z})$ and an integer $\Delta$, if*

$$(\Delta - 1)x^*(\alpha) > 1,$$

*then no Dehn surgery of distance $\Delta$ on a knot representing $\alpha$ produces an irreducible, $\partial$–irreducible 3–manifold $M$ which has a norm-reducing class with respect to the core of the surgery.*

*Proof.* Assume $(\Delta - 1)x^*(\alpha) > 1$ so that $\Delta \geq 2$ and $x^*(\alpha) - 1/(\Delta - 1) > 0$.

Since $M'$ contains no non-separating sphere, disk, annulus, or torus, the Thurston norm on $M'$ is actually a norm and not just a pseudo-norm. Thus, the unit norm ball in $H_2(M', \partial M')$ is compact and $x^*(\alpha) = \sup_{x(\tau)=1} |\alpha \cdot \tau|$. Since $x^*$ is continuous, there exists a class $\sigma \in H_2(M',$

$\partial M';\mathbb{R})$ realizing this supremum, i.e. such that $x(\sigma) = 1$ and $x^*(\alpha) = |\alpha \cdot \sigma|$. For any $\epsilon > 0$, there is a rational class $\widehat{z}' \in H_2(M', \partial M'; \mathbb{Q})$ approximating $\sigma$ such that $x(\widehat{z}') = 1$ and

$$|\alpha \cdot \sigma| \geq |\alpha \cdot \widehat{z}'| > |\alpha \cdot \sigma| - \epsilon.$$

In particular, since $(\Delta - 1)x^*(\alpha) > 1$, let us choose $\epsilon$ so that $x^*(\alpha) - 1/(\Delta - 1) > \epsilon > 0$.

Since $|\alpha \cdot \tau|/x(\tau)$ is constant for non-zero multiples of any non-zero class $\tau \in H_2(M, \partial M; \mathbb{R})$, there exists an integral class $\widehat{z} \in H_2(M, \partial M; \mathbb{Z})$ that is a positive multiple of the rational class $\widehat{z}'$ for which

$$|\alpha \cdot \sigma| \geq \frac{|\alpha \cdot \widehat{z}|}{x(\widehat{z})} > |\alpha \cdot \sigma| - \epsilon.$$

Being an integral class, $\widehat{z}$ is represented by a surface. For any taut surface $\widehat{Q}$ representing $\widehat{z}$ we have $x(\widehat{z}) = -\chi(\widehat{Q})$ and $|\alpha \cdot \widehat{z}| = \mathrm{wind}_\alpha(\widehat{Q})$.

Now let $K'$ be any knot representing $\alpha$. Among the taut surfaces representing $\widehat{z}$, choose $\widehat{Q}$ to be one that minimizes $|\widehat{Q} \cap K'|$. Thus $\mathrm{wrap}_{K'}(\widehat{Q}) \geq \mathrm{wind}_{K'}(\widehat{Q}) = |K' \cdot \widehat{Q}| = |\alpha \cdot \widehat{z}|$.

Hence by the choice of $\sigma$,

$$x^*(\alpha) \geq \frac{\mathrm{wind}_\alpha(\widehat{Q})}{-\chi(\widehat{Q})} > x^*(\alpha) - \epsilon. \tag{$\circledast$}$$

Since $x^*(\alpha) - 1/(\Delta - 1) \geq \epsilon > 0$, we have $(\Delta - 1)(x^*(\alpha) - \epsilon) \geq 1$ and thus the right hand inequality of ($\circledast$) gives

$$(\Delta - 1)\frac{\mathrm{wind}_\alpha(\widehat{Q})}{-\chi(\widehat{Q})} > (\Delta - 1)(x^*(\alpha) - \epsilon) \geq 1.$$

Consequently,

$$(\Delta - 1)|K' \cap \widehat{Q}| = (\Delta - 1)\mathrm{wrap}_{K'}(\widehat{Q}) \geq (\Delta - 1)\mathrm{wind}_\alpha(\widehat{Q}) > -\chi(\widehat{Q}).$$

By the choice of $\widehat{Q}$ and Lemma 2.3, there is no multi-$\partial$-compressing disk for $\widehat{Q}$. Thus, by Theorem 3.1, if $M$ is obtained by a distance $\Delta$ Dehn surgery on $K'$, then $H_2(M, \partial M)$ cannot contain a norm-reducing class with respect to the core of the surgery. q.e.d.

## 5. Genus growth in twist families.

Let $Y$ be a closed, compact, connected, oriented, irreducible, 3–manifold with $H_2(Y) = 0$. Let $\{K_n\}$ be a twist family of null-homologous knots in $Y$ obtained by twisting a null-homologous knot $K = K_0$ along an unknot $c$. That is, $K_n$ is the knot in $Y = Y_c(-1/n)$ obtained by $-1/n$–surgery on $c$ for each integer $n$. Let $g(K_n)$ be the Seifert genus of $K_n$ and set $\omega = |\ell k(K, c)|$.

**Theorem 5.1.** *If $|\ell k(K, c)| > 0$, then $\lim_{n\to\infty} g(K_n) = \infty$ unless $c$ is a meridian of $K$.*

*Proof.* This follows as a corollary of the more precise Theorem 5.3 below which implies the limit is finite only if $\omega x([D]) = 0$. Here $x$ is the Thurston norm on the exterior of the link $K \cup c$ and $[D]$ is the homology class of a disk bounded by $c$, intersected by $K$, and restricted to this exterior. Since $\omega = |\ell k(K, c)| > 0$, the limit is finite only if $x([D]) = 0$. This, however, implies that $D$ is an annulus and hence $c$ is a meridian of $K$. q.e.d.

Let $N = Y - \mathcal{N}(K \cup c)$ be the exterior of the link $K \cup c$ with boundary components $T_K$ and $T_c$ corresponding to $K$ and $c$ respectively, and use the standard associated meridian-longitude bases relative to $K$ and $c$ for these tori. Then the exterior of $K_n$ is the manifold $Y - \mathcal{N}(K_n) = N_{T_c}(-1/n)$ which results from Dehn filling $N$ along the slope $-1/n$ in $T_c$; let $c_n$ be the core of this filling, setting $c = c_0$.

Let $\widehat{D}$ be a disk bounded by $c$ that is transverse to $K$ and set $D = \widehat{D} \cap N$. Let $\widehat{F}_n$ be a Seifert surface for $K_n$ that is transverse to $c_n$ and set $F_n = \widehat{F}_n \cap N$.

**Lemma 5.2.** $[F_{n+1}] = [F_n] + \omega[D]$ *for all integers* $n$.

*Proof.* Since $Y$ is a rational homology sphere by assumption, each knot $K_n$ (and $c$) has a unique homology class of Seifert surface up to sign. The formula then follows since $\omega = |\ell k(K, c)|$ and the surfaces $F_n$ and $D$ are the restrictions of Seifert surfaces for $K_n$ and $c$ to $N$. Indeed, $\partial[F_n]$ is homologous to one longitude of slope $-n\omega^2$ in $T_K$ and $\omega$ parallel curves of slope $-1/n$ in $T_c$ while $\partial[D]$ is homologous to $\omega$ meridians in $T_K$ and one longitude of slope 0 in $T_c$. It follows that (heeding orientations) $[F_n] + \omega[D]$ is represented by a properly embedded surface in $N$ that is the Haken sum of $F_n$ and $\omega$ parallel copies of $D$ which has boundary homologous to that of $\partial[F_{n+1}]$. If $[F_{n+1}] - [F_n] - \omega[D]$ were a non-zero class, it would be represented by a boundaryless surface in $N$ and thus represent a non-zero class in $H_2(Y)$ — a contradiction. Hence $[F_{n+1}] = [F_n] + \omega[D]$. q.e.d.

**Theorem 5.3.** *There is a constant* $G = G(K, c)$ *such that* $2g(K_n) = 2G + n\omega x([D])$ *for sufficiently large* $n > 0$.

*Proof.* Among disks bounded by $c$ in $Y$, let $\widehat{D}$ be one for which $|K \cap \widehat{D}| = p > 0$ is minimized and set $D = \widehat{D} \cap N$. Note that the minimality implies the punctured disk $D$ is incompressible and $\partial$–incompressible. Moreover $\partial D$ consists of one longitude of $c$ and $p$ meridional curves of $K$. In particular, if $p = 1$ then $D$ is an annulus so that $x([D]) = 0$ and $c$ is a meridian of $K$. Hence $K = K_n$ for all integers $n$ so the genus is constant and the theorem holds. Thus we assume $p \geq 2$. This further implies that $N$ is not the product of a torus and an interval.

If $N$ is a cable space, since $D$ is not an annulus but is a properly embedded, non-separating, incompressible and $\partial$–incompressible surface, it must be a fiber in a fibration of $N$ over $S^1$. (All classes in $H_2(N, \partial N; \mathbb{Z})$ other than multiples of the class of the cabling annulus are represented by fibers.) Therefore because $\partial D$ consists of a longitude of $c$ and meridians of $K$, it follows that $Y \cong S^3$ and $K$ is a torus knot in the solid torus exterior of the unknot $c$. In particular, this means that for some integer $q$ coprime to $p = |K \cap \widehat{D}|$, the knot $K_n$ is the $(p, q+np)$–torus knot and the theorem holds. Therefore we may assume that $N$ is not a cable space.

If $N$ is reducible, then there is a sphere in $N$ that does not bound a ball in $N$ and yet must bound a ball in $Y$ that contains either $K$ or $c$. If this sphere separates the two components of $\partial N$ then it separates $K$ and $c$ in $Y$ implying that $\ell k(K, c) = 0$, contrary to assumption. Thus $K \cup c$ must be contained in a ball in $Y$ and may be viewed as being contained in an $S^3$ summand of $Y$. Thus $N = N' \# Y$ where $N'$ is the irreducible exterior of $K \cup c$ in $S^3$. Since the summand will not affect the genera of the knots $K_n$, we may run the argument for $K \cup c$ in $S^3$. Thus we may assume $N$ is irreducible.

Let $\widehat{z}_n$ be the homology class of an oriented Seifert surface for $K_n$ in $Y - \mathcal{N}(K_n)$ for which $x(\widehat{z}_n) = 2g(K_n) - 1$. Then set $z_n = \rho_{-1/n}(\widehat{z}_n)$ to be the homology class of the restriction of the Seifert surface to $N = Y - \mathcal{N}(K \cup c)$. By Theorem 4.6, there is a finite set of integers $\mathcal{R}$ such that

$$x(z_n) = x(\widehat{z}_n) + \mathrm{wind}_{K_n}(\widehat{z}_n),$$

if $n \notin \mathcal{R}$. Since $\omega = \mathrm{wind}_{K_n}(\widehat{z}_n)$ for all integers $n$ and $2g(K_n) - 1 = x(\widehat{z}_n)$, then when $n \gg 0$ we have

$$2(g(K_{n+1}) - g(K_n)) = x(z_{n+1}) - x(z_n) = x(z_{n+1} - z_n).$$

By Lemma 5.2, $z_{n+1} - z_n = \omega[D]$ for all integers $n$. Hence for $n \gg 0$, $2(g(K_{n+1}) - g(K_n)) = \omega x([D])$. Therefore when $n$ is sufficiently large, $2g(K_n) = 2G + n\omega x([D])$ for some constant $G$ as desired. q.e.d.

**Remark 5.4.** At the expense of having to reckon with multiple homology classes of Seifert surfaces, one should be able to prove Theorem 5.3 without the hypothesis that $Y$ is a rational homology sphere.

**Remark 5.5.** One ought to be able to prove Theorem 4.6 and Theorem 5.3 using link Floer Homology.

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