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GENERALIZED SHEAR COORDINATES ON THE MODULI SPACES OF THREE-DIMENSIONAL SPACETIMES

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Abstract

We introduce coordinates on the moduli spaces of maximal globally hyperbolic constant curvature 3d spacetimes with cusped Cauchy surfaces S. They are derived from the parametrization of the moduli spaces by the bundle of measured geodesic laminations over Teichmüller space of S and can be viewed as analytic continuations of the shear coordinates on Teichmüller space. In terms of these coordinates, the gravitational symplectic structure takes a particularly simple form, which resembles the Weil–Petersson symplectic structure in shear coordinates, and is closely related to the cotangent bundle of Teichmüller space. We then consider the mapping class group action on the moduli spaces and show that it preserves the gravitational symplectic structure. This defines three distinct mapping class group actions on the cotangent bundle of Teichmüller space, corresponding to different values of the curvature.

1. Introduction

Moduli spaces of constant curvature spacetimes. Moduli spaces of three-dimensional constant curvature spacetimes classify the diffeomorphism classes of solutions of the Einstein equations on three-dimensional manifolds. As the Ricci curvature of a three-dimensional manifold determines its Riemann curvature tensor, all solutions of the Einstein equations with vanishing stress-energy tensor have constant curvature Λ, where Λ is the cosmological constant. This implies in particular that three-dimensional Einstein spacetimes, that is, Ricci constant spacetimes, are all locally isometric to one of three model Lorentzian geometries: three-dimensional Minkowski space, anti-de Sitter space, or de Sitter space for, respectively, $\Lambda = 0$, $\Lambda = -1$, and $\Lambda = 1$.

As a consequence, three-dimensional Einstein spacetimes can be classified completely under certain additional assumptions on their causality structure. This yields a classification of maximal globally hyperbolic

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(MGH) three-dimensional Einstein spacetimes of topology $\mathbb{R} \times S$, where S is an orientable surface, possibly with punctures. Remarkably, these three-dimensional structures are completely characterized in terms of two-dimensional structures and their associated moduli spaces—in particular, those related to hyperbolic geometry and Teichmüller theory.

Important results in this respect are the work by Mess [**30**] and Scannell [**40**], where the diffeomorphism classes of MGH Einstein spacetimes with a compact Cauchy surface S are classified by the cotangent bundle over Teichmüller space $T^*\mathcal{T}(S)$ for $\Lambda = 0$, two copies of Teichmüller space $\mathcal{T}(S) \times \mathcal{T}(S)$ for $\Lambda = -1$, and the space of complex projective structures $\mathcal{CP}(S)$ for $\Lambda = 1$. An explicit geometric construction of such spacetimes in terms of domains of dependence in the corresponding model spacetimes was later given by Benedetti and Bonsante [**8**]. These domains of dependence are described in terms of earthquakes and grafting along measured laminations, and the solutions for different values of curvature are related via so-called canonical Wick rotations and rescalings. This allows for a clear geometrical description of the moduli space of three-dimensional MGH Einstein spacetimes as the bundle $\mathcal{ML}(S)$ of measured geodesic laminations over Teichmüller space and generalizes the well-known description of the moduli space of three-dimensional hyperbolic manifolds by Thurston [**43**]. Recently, the relation between Einstein spacetimes for different values of the cosmological constant was explored in [**12**, **13**, **26**, **27**, **42**] from the perspective of hyperbolic geometry and Teichmüller theory, which resulted in a more detailed understanding of the geometry and their symplectic structure.

The fact that any three-dimensional MGH Einstein spacetime is locally isometric to one of the model Lorentzian geometries also gives rise to a classification of such spacetimes in terms of conjugacy classes of group homomorphism $\pi_1(S) \to G_\Lambda$, where G_Λ is the isometry group of the model spacetime. This identifies their moduli spaces with a certain subspace of the corresponding representation variety $\mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$ or, equivalently, of the moduli space of flat G_Λ-connections on S. This can be viewed as a direct generalisation of the realization of Teichmüller space $\mathcal{T}(S)$ as a connected component of the representation variety $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$.

From a physics perspective, this is related to the Chern–Simons formulation of 3d gravity developed by Achucarro–Townsend [**1**] and Witten [**46**] which describes three-dimensional gravity as a G_Λ-Chern–Simons theory on $\mathbb{R} \times S$. The representation variety $\mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$ then corresponds to the gauge invariant phase space of this Chern–Simons theory. The fact that only a certain subset of this phase space corresponds to gravity follows from the non-degeneracy of the metric, which imposes restrictions on the Chern–Simons connection.

3d spacetimes, Teichmüller theory, and quantization. This relation between three-dimensional Einstein geometry, two-dimensional hyperbolic geometry, and flat connections makes the moduli spaces of MGH Einstein spacetimes an interesting research topic from different perspectives. On one hand, Teichmüller theory is a very rich and well-developed theory with deep connections with complex analysis, algebraic topology, low-dimensional topology, and geometry and also symplectic geometry, to cite but a few; see [**34**, **21**]. The study of generalizations of Teichmüller space and its structures may thus lead to new insights and techniques for a wide variety of topics in mathematics.

From a physics viewpoint, moduli spaces of MGH Einstein spacetimes are interesting since they are the diffeomorphism invariant phase space of gravity in three dimensions. Three-dimensional gravity plays an important role as a toy model for the quantization of general relativity that allows one to investigate conceptual questions of quantum gravity and to develop new approaches to quantization; see [**11**] and references therein for an overview. Moreover, the quantization of moduli spaces of MGH Einstein spacetimes is also of mathematical interest due to their relation with the moduli spaces of flat connections and Chern–Simons gauge theory, which connects it directly to the construction of quantum invariants of three-manifolds and three-dimensional topological quantum field theory [**47**, **39**, **45**, **4**].

While quantization techniques for Chern–Simons theory are well established for compact, semisimple gauge groups, the case of non-compact groups remains a challenge due to their more complicated representation theory. A first step in the generalization of the definition of quantum invariants associated with non-compact Lie groups such as the group $\mathrm{PSL}(2,\mathbb{C})$ are the hyperbolic invariants of Baseilhac and Benedetti [**5**, **6**], which in turn are closely related with the theory of quantum Teichmüller spaces [**22**, **17**] via the quantum dilogarithm of Faddeev and Kashaev [**15**]; see also [**3**, **10**].

The challenges in the quantization of moduli spaces of flat connections for non-compact groups and the close relation with Teichmüller theory thus suggest to approach the quantization of the gravitational moduli spaces by applying or generalizing results from quantum Teichmüller theory. In particular, this provides a strong motivation to generalize the description of Teichmüller space in terms of Thurston's shear coordinates [**44**, **9**, **16**] to the context of the gravitational moduli spaces.

3d Einstein spacetimes in terms of generalized shear coordinates. The shear coordinate parametrisation of Teichmüller space has a direct geometrical interpretation and yields a simple description of the Weil–Petersson symplectic structure and the action of the mapping class group, which play a central role in the derivation of quantum Teichmüller theory. The present article introduces a set of coordinates on the moduli spaces $\mathcal{GH}_\Lambda(\mathbb{R}\times S)$ of MGH Einstein spacetimes that can be viewed as a natural generalization of Thurston's shear coordinates to the three-dimensional Lorentzian context. Similarly to the two-dimensional case, these generalized shear coordinates are defined by means of an ideal triangulation of a cusped surface and are obtained via the identification of the moduli space of MGH Einstein spacetimes with the bundle of measured geodesic laminations over Teichmüller space. They have a direct geometrical interpretation in terms of shearing and bending of hyperbolic structures along the ideal edges of the triangulation and allow one to directly determine the associated group homomorphisms $\pi_1(S) \to G_\Lambda$. This should also be compared with the work of Bonahon [**9**], where complex measured laminations are used to define complex-valued shear coordinates on the moduli space of hyperbolic 3-manifold.

From a more algebraic perspective, the coordinates introduced in this article can be understood as analytic continuations of the shear coordinates in Teichmüller space with values in a two-dimensional commutative real algebra R_Λ, which coincides with the complex numbers for $\Lambda = 1$, with the dual numbers for $\Lambda = 0$, and with the split complex numbers for $\Lambda = -1$. This algebra also provides a unified description of the isometry groups G_Λ and their Lie algebras in terms of matrices with entries in R_Λ. The R_Λ-valued shear coordinates then arise from the generalization of the shear coordinate parametrisation of the holonomy representations.

The gravitational symplectic structure on $\mathcal{GH}_\Lambda(\mathbb{R}\times S)$, which is the restriction of Goldman's symplectic form [**18**] on $\mathrm{Hom}(\pi_1(S), G_\Lambda)/G_\Lambda$, takes a particularly simple form in terms of these generalized shear coordinates. It is purely combinatorial and resembles the Weil–Petersson symplectic structure on Teichmüller space in shear coordinates. Moreover, for all values of Λ, it is also closely related to the cotangent bundle $T^*\mathcal{T}(S)$ of Teichmüller space. In fact, the spaces $\mathcal{GH}_\Lambda(\mathbb{R}\times S)$, for different values of Λ, are shown to be isomorphic as symplectic manifolds to $T^*\mathcal{T}(S)$.

The second part of the article investigates the action of the mapping class group $\mathrm{Mod}(S)$ on the moduli spaces of MGH Einstein spacetimes. Concrete expressions for mapping class group action on the generalized shear coordinates are derived in terms of Whitehead moves, which can be viewed as an analytic continuation of the corresponding expressions for shear coordinates on $\mathcal{T}(S)$. The Whitehead moves are shown to preserve the gravitational symplectic structure and thus induce a symplectic $\mathrm{Mod}(S)$-action on $\mathcal{GH}_\Lambda(\mathbb{R}\times S)$, for all values of Λ. This, in turn, induces three distinct symplectic actions on the cotangent bundle $T^*\mathcal{T}(S)$. These results are achieved via a simple decomposition of the Whitehead moves into terms generated via the symplectic structure and linear terms that implement the combinatorial transformation of the Poisson structure under Whitehead moves.

The Hamiltonians generating the non-linear part of the Whitehead moves are related to the "imaginary part" of the dilogarithm of the associated edge coordinate, which is known to be related to the hyperbolic volume of ideal tetrahedra. We show that this is a direct generalization of a corresponding result for Teichmüller space, in which the relevant Hamiltonian is the real dilogarithm. This gives a clear motivation for the appearance of the quantum dilogarithm in the quantum theory and suggests that a quantization of these moduli spaces could be used to define three-manifold invariants similar to those in [**5**, **6**].

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2. Teichmüller theory and shear coordinates

In this section, we summarize the relevant background on Teichmüller theory and Thurston's shear coordinates. We refer the reader to [**34**] for a general introductory overview, but discuss the aspects relevant to this article in detail to make it self-contained. In the following, S denotes an orientable surface of genus g with s punctures. We assume that the surface contains at least one puncture ($s > 0$) and that its universal cover is isometric to the hyperbolic plane ($2g - 2 + s > 0$).

2.1. Teichmüller space.

The Riemann moduli space and Teichmüller space. The Riemann moduli space $\mathcal{R}(S)$ parametrizes diffeomorphism classes of both hyperbolic and conformal structures on a given topological surface S. In the case of punctured Riemann surfaces ($s > 0$), there are different versions of this space, depending on the boundary conditions imposed on the hyperbolic metrics near each puncture [**16**, **37**]. In this article, we consider the Riemann moduli space of cusped hyperbolic structures. We thus define the orbifold $\mathcal{R}(S)$ as the space of finite-area complete hyperbolic metrics on S modulo orientation-preserving diffeomorphisms

$$\mathcal{R}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}^+(S).$$

The orbifold universal cover of the Riemann moduli space $\mathcal{R}(S)$ is the Teichmüller space $\mathcal{T}(S)$, which is the space of finite-area complete hyperbolic metrics on S modulo diffeomorphisms isotopic to the identity

$$\mathcal{T}(S) = \mathrm{Hyp}(S)/\mathrm{Diff}_0(S).$$

The group of deck transformations of the covering $\mathcal{T}(S) \to \mathcal{R}(S)$ is the mapping class group of S, which is given as the quotient of the group $\mathrm{Diff}^+(S)$ of orientation-preserving diffeomorphisms of S by its normal subgroup $\mathrm{Diff}_0(S)$ of diffeomorphisms isotopic to the identity

$$\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S).$$

Symplectic structure. Teichmüller space $\mathcal{T}(S)$ carries a canonical symplectic structure. This can be understood via its relation to the $\mathrm{PSL}(2,\mathbb{R})$-representation variety, which consists of conjugacy classes of group homomorphisms $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$

$$\mathrm{Rep}(S, \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

Via the uniformization theorem, points $h \in \mathcal{T}(S)$ are in one-to-one correspondence with Fuchsian representations $\rho \in \mathrm{Rep}(S, \mathrm{PSL}(2,\mathbb{R}))$ with fixed parabolic conjugacy classes around each puncture. Thus Teichmüller space can be viewed as a connected component of the representation variety $\mathrm{Rep}_{\mathcal{C}}(S, \mathrm{PSL}(2,\mathbb{R}))$, where the index $\mathcal{C}$ indicates the restriction to fixed parabolic conjugacy classes at the punctures.

For any Lie group G, the representation variety $\mathrm{Rep}(S, G)$ coincides with the moduli space of flat G-connections on S and carries a canonical non-degenerate closed 2-form, the Atiyah–Bott–Goldman symplectic form [**2**, **18**], which is determined by the choice of an Ad-invariant symmetric bilinear form on $\mathfrak{g} = \mathrm{Lie}\, G$. From a physics viewpoint, this moduli space is the (gauge invariant) phase space of Chern–Simons theory on the 3-manifold $\mathbb{R} \times S$. The Ad-invariant symmetric bilinear form that characterizes the Atiyah–Bott–Goldman symplectic structure enters in the definition of the Chern–Simons action and this symplectic structure can be seen as the associated physical symplectic structure on the phase space [**2**].

For the group $G = \mathrm{PSL}(2,\mathbb{R})$, the choice of an Ad-invariant symmetric bilinear form is unique up to rescaling, and it was shown by Goldman [**18**] that the restriction of the associated symplectic structure to Teichmüller space $\mathcal{T}(S) \subset \mathrm{Rep}(S, \mathrm{PSL}(2,\mathbb{R}))$ coincides with the Weil–Petersson Poisson structure on $\mathcal{T}(S)$.

Measured laminations and earthquakes. Measured geodesic laminations on two-dimensional hyperbolic surfaces and the associated operation of earthquakes play a prominent role in three-dimensional hyperbolic geometry [**43**, **14**, **9**, **29**] as well as in three-dimensional Lorentzian geometry [**30**, **40**, **8**, **31**]. Measured geodesic laminations can be viewed as generalizations of weighted simple closed geodesics, and earthquakes are defined via cutting and gluing operations along such geodesics. More precisely, a measured geodesic laminations on S associated to a hyperbolic metric $h \in \mathcal{T}(S)$ is a pair (λ, μ) formed by

1) a closed subset $\lambda \subset S$ that is foliated by non-self-intersecting disjoint complete geodesics, called the leaves of the lamination, which cannot be contracted to punctures; and
2) a positive measure μ on the set of arcs transverse to the leaves, which is invariant under homotopy through transverse arcs and additive under concatenation of arcs.

We denote by $\mathcal{ML}(S)$ the (total space of the) bundle of measured geodesic laminations over $\mathcal{T}(S)$, considered up to isotopy.

Note that although the definition of a measured geodesic lamination makes use of a reference hyperbolic metric $h \in \mathcal{T}(S)$, there is a canonical identification between measured laminations defined with respect to any pair of hyperbolic metrics $h, h' \in \mathcal{T}(S)$. This follows since any geodesic for a metric h can be smoothly deformed to a geodesic for a metric h', which gives a global identification between the fibers of the (trivial) bundle $\mathcal{ML}(S)$. In the following, however, such an identification of measured geodesic laminations for different metrics will not be convenient, and we shall not rely on any choice of trivialization.

An earthquake is an operation $Eq : \mathcal{ML}(S) \to \mathcal{T}(S)$ that associates to each point $h \in \mathcal{T}(S)$ and each measured geodesic lamination $\lambda \in \mathcal{ML}_h(S)$ another point $Eq^\lambda(h) \in \mathcal{T}(S)$, called the earthquake of h along λ. If λ is a simple closed geodesic with associated weight $\mu \in \mathbb{R}_+$, the hyperbolic metric $Eq^\lambda(h)$ is obtained by cutting the hyperbolic surface determined by h along λ and gluing the pieces back together after applying a (right) twist by $2\pi\mu$.

A more explicit description of the earthquake operation can be given in terms of the associated Fuchsian representations of the fundamental group of S. For a point $h \in \mathcal{T}(S)$, denote by $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ the corresponding Fuchsian representation, which is determined uniquely up to conjugation via the uniformization theorem. The earthquake $Eq^\lambda(h)$ of h along λ is then determined from a new Fuchsian representation $\rho^\lambda : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ constructed as follows. First, consider the lift $\tilde{\lambda} \in \mathcal{ML}(\mathbb{H}^2)$ of λ to the universal cover of (S,h). Each leaf $\tilde{l}$ of $\tilde{\lambda}$ is then a complete geodesic in the hyperbolic plane, and hence for each point $p \in \tilde{l}$ there is a unique hyperbolic isometry $A_p \in \mathrm{PSL}(2,\mathbb{R})$ mapping the imaginary axis to $\tilde{l}$, the point i to p, and preserving the orientation. This allows one to associate to the representation $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ and the lamination λ a ρ-cocycle $Z_E^\lambda : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ defined by

$$Z_E^\lambda(a) = \prod_{p \in \lambda \cap a} \mathrm{Ad}_{A_p} E(\mu_p) \qquad \forall a \in \pi_1(S), \tag{1}$$

where

$$E(\mu) = \begin{pmatrix} e^{\mu/2} & 0 \\ 0 & e^{-\mu/2} \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}) \tag{2}$$

is the hyperbolic translation of length μ along the imaginary axis and $\mu_p = \epsilon_p(\lambda, a)\mu$ is the measure of a at p multiplied by the oriented intersection number between λ and a. The fact that Z_E^λ is a ρ-cocycle

$$Z_E^\lambda(ab) = Z_E^\lambda(a)\mathrm{Ad}_{\rho(a)} Z_E^\lambda(b), \qquad \forall a, b \in \pi_1(S),$$

ensures that one obtains a representation $\rho^\lambda : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ by setting

$$\rho^\lambda(a) = Z_E^\lambda(a)\rho(a), \qquad \forall a \in \pi_1(S).$$

This representation is again Fuchsian and therefore it determines a unique hyperbolic metric $Eq^\lambda(h) \in \mathcal{T}(S)$, the earthquake of h along λ.

It was shown by Thurston (see [**24**] for the proof) that any two hyperbolic metrics are related via an earthquake. More precisely, Thurston's earthquake theorem states that for any pair of hyperbolic metrics $h, h' \in \mathcal{T}(S)$ there exists a unique measured geodesic lamination $\lambda \in \mathcal{ML}_h(S)$ such that $Eq^\lambda(h) = h'$. In particular, for a given point $h \in \mathcal{T}(S)$ there is a bijection $\mathcal{ML}_h(S) \to \mathcal{T}(S)$ between the fiber of measured geodesic laminations over h and Teichmüller space.

2.2. Shear coordinates on Teichmüller space.

Definition of shear coordinates on $\mathcal{T}(S)$. A very effective tool in Teichmüller theory, in particular in the study of its Poisson geometry and subsequent quantization, is a special set of global coordinates on $\mathcal{T}(S)$ associated with ideal triangulations of the punctured surface S. These coordinates were first introduced by Thurston [**44**] and further developed by Bonahon [**9**] and Fock [**16**] and have a direct geometrical interpretation. They measure the hyperbolic displacement, or shear, of adjacent ideal triangles — see also [**35**] for a related interpretation in terms of distances between horocycles at each puncture.

Let τ be an ideal triangulation of the surface S, that is, a triangulation of S whose set of vertices coincides with the set of punctures of the surface. Note that being ideal is a rather restrictive condition on the triangulation, which determines the number of its vertices, edges, and faces uniquely. From the formula $v - e + f = 2 - 2g$ for the Euler characteristic of S together with the relations $2e = 3f$ and $v = s$, it follows that the number of edges and faces of the ideal triangulation are given by $e = 6g - 6 + 3s$ and $f = 4g - 4 + 2s$.

In the following we will also consider the dual graph Γ of an ideal triangulation τ. Note that for a general combinatorial graph the notion of face is not defined a priori. However, graphs dual to a triangulation of an oriented surface S carry additional structure—namely, a cyclic ordering of the incident edges at each vertex induced by the orientation of the underlying surface. A graph with such a cyclic ordering of the incident edges at each vertex is called a fat graph. The notion of face can then be defined as certain closed edge paths on Γ. In the case of a trivalent fat graph dual to a triangulation, a face is a closed edge path of the graph that takes the "same turn," either left or right, at each vertex and which does not pass through an edge twice in the same direction.

An ideal triangulation τ of a surface S thus determines uniquely (up to isotopy) an embedded trivalent dual fat graph Γ. While different ideal triangulations of S lead to different embedded fat graphs, the associated fat graphs are always related by sequences of Whitehead moves (see Section 2.3). Conversely, for a trivalent fat graph Γ there is a unique (up to diffeomorphism) oriented surface with an ideal triangulation, which is obtained by gluing punctured discs along the faces of the graph. Graphs related by a sequence of Whitehead moves give rise to the same topological surface, and punctured surfaces are therefore in one-to-one correspondence with trivalent fat graphs modulo Whitehead moves.

Given a surface S and a corresponding embedded trivalent fat graph Γ, one defines Thurston's shear coordinates on Teichmüller space $\mathcal{T}(S)$ as follows. Denote by $V(\Gamma)$, $E(\Gamma)$, and $F(\Gamma)$, respectively, the sets of vertices, edges, and faces of Γ. A point $h \in \mathcal{T}(S)$ corresponds to an equivalence class of hyperbolic structures on S and determines an ideal geodesic triangulation of S dual to Γ. Each edge $\alpha \in E(\Gamma)$ corresponds to an ideal hyperbolic square on S or, equivalently, to a $\pi_1(S)$-equivalence class of hyperbolic squares in the universal cover. The shear coordinate $x^\alpha = x^\alpha(h)$ assigned to α is then defined as the logarithm of the cross-ratio associate with the ideal square determined by α.

Working with the upper-half plane model of the hyperbolic plane, we may normalize this ideal square in such a way that one of the two adjacent triangles has vertices at $-1, 0, \infty$ and the other at $\infty, 0, t$ as shown in Figure 1. The cross-ratio is then given by the coordinate $t \in \mathbb{R}$ of the fourth vertex, and the shear coordinate takes the form $x^\alpha(h) = \log t$.

The shear coordinate x^α has a simple geometric interpretation as the shear between the two adjacent ideal triangles. This follows from the fact that the triangle with vertices $\infty, 0, t$ is obtained from a reference triangle with vertices $0, 1, \infty$ by applying the hyperbolic transformation $E(x^\alpha)$ in (2). This transformation preserves the imaginary axis, which

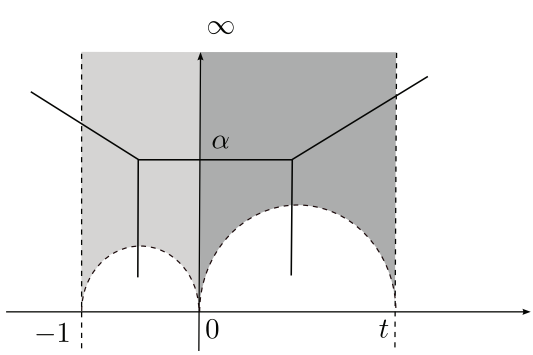

Figure 1. Ideal square determined by the edge α of the fat graph dual to an ideal triangulation.

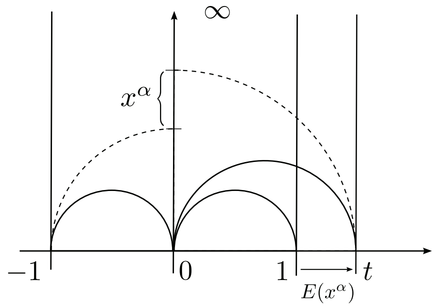

Figure 2. Earthquake with weight x^α along the imaginary axis.

is the lift of the ideal geodesic dual to the edge α, and the signed hyperbolic distance between a point on the imaginary axis and its image is exactly x^α. This allows one to interpret the coordinate x^α as the weight parameter for an earthquake along the imaginary axis, as shown in Figure 2.

Besides their rather natural interpretation, an important feature of shear coordinates is their relation to the holonomies of simple closed curves on S, which are the images of elements of $\pi_1(S)$ under the associated Fuchsian representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$. For a given vertex $v \in V(\Gamma)$, any simple closed curve a on S is homotopic to a unique closed-edge path along the fat graph Γ starting and ending at v. Such a closed-edge path corresponds to a sequence $(\alpha_1, \dots, \alpha_n)$ of edges of Γ, and the holonomy of a is given by

$$\rho(a) = P_n^a E(x^{\alpha_n}) \cdots P_1^a E(x^{\alpha_1}), \tag{3}$$

where $E(x^\alpha)$ is given by (2) and P_k^a is either

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

depending on whether α_k comes before or after α_{k+1} with respect to the ordering of incident edges at their common vertex. The former corresponds to a left turn and the latter to a right turn at the vertex between α_k and α_{k+1}. Note that this prescription determines the holonomies only up to the choice of a basepoint, and different choices give rise to holonomies that are related by conjugation. Also note that our parametrization of the holonomies is equivalent to the one in [**17**], although the matrices L and R are different from the matrices L and R used there.

As they correspond to deck transformations in the universal cover, the holonomies encode key geometric properties of the hyperbolic surface S. For instance, any element $a \in \pi_1(S)$ is homotopic to a unique closed geodesic of S, and its geodesic length $l(a)$ is given by the trace of the associated holonomy

$$\mathrm{Tr}\rho(a) = 2\cosh(l(a)/2).$$

The parametrization (3) of the holonomies thus gives rise to a simple description of the geometric properties of the hyperbolic surface in terms of the shear coordinates x^α.

It is also directly apparent from (3) that the shear coordinates $x^\alpha : \mathcal{T}(S) \to \mathbb{R}$ cannot be all independent but must satisfy certain constraints associated with the boundary conditions at the punctures. By definition, each face of the graph Γ corresponds to a puncture and the holonomy of the associated edge path must be parabolic. On the other hand, for an edge path $(\alpha_1, \ldots, \alpha_n)$ around a face, the matrices P_k^a in (3) are necessarily all equal to L or all equal to R. A simple computation then shows that the parabolicity condition is equivalent to imposing the constraint that the sum of shear coordinates associated to the edges of a given face identically vanishes. More explicitly, for each face $i \in F(\Gamma)$ we have

$$c^i(x) = \sum_{\alpha \in E(\Gamma)} {\theta^i}_\alpha x^\alpha = 0, \tag{4}$$

where ${\theta^i}_\alpha$ denotes the multiplicity of the edge α in the face i. Note that these constraints are linear in the coordinates x^α. Thus, denoting by V, E, and F the number of vertices, edges, and faces of Γ, one can interpret the constraints as a linear map $c : \mathbb{R}^E \to \mathbb{R}^F$, which identifies $\mathcal{T}(S)$ with a linear subspace of $\mathbb{R}^E$ of codimension F.

Theorem 2.1 (Fock and Chekhov [**17**])**.** *The functions $x^\alpha : \mathcal{T}(S) \to \mathbb{R}$ define an embedding $x : \mathcal{T}(S) \hookrightarrow \mathbb{R}^E$ whose image is the kernel of the linear map $c : \mathbb{R}^E \to \mathbb{R}^F$ whose components are given by* (4)*.*

Symplectic structure in shear coordinates. Another remarkable property of the shear coordinates is that they give rise to a very simple description of the Weil–Petersson symplectic structure on $\mathcal{T}(S)$ in terms of a Poisson structure on $\mathbb{R}^E$. This Poisson structure is given in terms of combinatorial constants associated with the graph Γ [**16**, **36**]. Denoting by $\partial/\partial x^\alpha \in T\mathbb{R}^E$ the basis of coordinate vector fields on $\mathbb{R}^E$, one can characterize this Poisson structure by the Poisson bivector

$$\pi_{WP} = \frac{1}{2} \sum_{\alpha \in E(\Gamma)} \frac{\partial}{\partial x^\alpha} \wedge \Big(\frac{\partial}{\partial x^\beta} - \frac{\partial}{\partial x^\gamma} + \frac{\partial}{\partial x^\delta} - \frac{\partial}{\partial x^\epsilon} \Big). \tag{5}$$

Here, the sum is taken over all edges $\alpha \in E(\Gamma)$ and $\beta, \gamma, \delta, \epsilon$ are the incident edges at the source and target vertices of α, ordered as in Figure 3. Note that this expression is also valid in the case where some of the edges $\beta, \gamma, \delta, \epsilon$ are equal.

The corresponding Poisson bracket $\{\ ,\ \}_{WP}$ on $\mathbb{R}^E$ is given by

$$\{f_1, f_2\}_{WP} = (df_1 \otimes df_2)(\pi_{WP}) \qquad \forall f_1, f_2 \in C^\infty(\mathbb{R}^E).$$

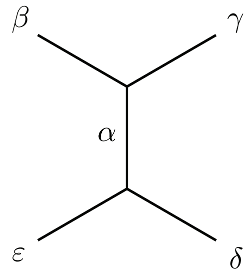

Figure 3. Edge α in the trivalent fat graph Γ and adjacent edges.

In particular, one obtains for the coordinate functions $x^\alpha : \mathbb{R}^E \to \mathbb{R}$ the following expression:

$$\{x^\alpha, x^{\alpha'}\}_{WP} = (dx^\alpha \otimes dx^{\alpha'})(\pi_{WP}) = \pi_{WP}^{\alpha\alpha'} = \delta^{\beta\alpha'} - \delta^{\gamma\alpha'} + \delta^{\delta\alpha'} - \delta^{\epsilon\alpha'}.$$

The Poisson bracket between the components c^i of the linear constraint defined in (4) and a general function $f \in C^\infty(\mathbb{R}^E)$ is also easily computed and takes the form

(6)
$$\{f, c^i\}_{WP} = \sum_{\alpha\in E(\Gamma)} \frac{\partial f}{\partial x^\alpha}\{x^\alpha, c^i\}_{WP} = \sum_{\alpha\in E(\Gamma)} \frac{\partial f}{\partial x^\alpha}\Big(\theta^i{}_\beta - \theta^i{}_\gamma + \theta^i{}_\delta - \theta^i{}_\epsilon\Big).$$

This vanishes identically for all $f \in C^\infty(\mathbb{R}^E)$, since every face $i \in F(\Gamma)$ involves only left or right turns, as can be seen from the fact that a face containing any of the edges $\alpha, \beta, \gamma, \delta, \epsilon$ in Figure 3 must involve (a combination of) the edge paths (β, γ), (γ, α, δ), (δ, ϵ), or $(\epsilon, \alpha, \beta)$ or their inverses. For all of these paths, the corresponding sum of multiplicities in (6) vanishes and hence the bracket is trivial.

This demonstrates that the components of the linear constraint (4) are Casimir functions for the Poisson bivector (5) and implies that the Poisson bivector π_{WP} restricts to a Poisson bivector for the constraint surface $\mathrm{Ker}c \subset \mathbb{R}^E$. The induced symplectic structure on Teichmüller space coincides with the Weil–Petersson symplectic structure as the following theorem states.

Theorem 2.2 (Fock and Chekhov [**17**], Penner [**36**, **37**])**.** *The linear constraint $c : \mathbb{R}^E \to \mathbb{R}^F$ (4) is Casimir with respect to the Poisson bivector π_{WP} on $\mathbb{R}^E$ and the symplectic quotient determined by c is Poisson isomorphic to Teichmüller space with the Weil–Petersson structure.*

2.3. The mapping class group action.

Combinatorial description of the mapping class group. Recall from the previous subsection that mapping class group $\mathrm{Mod}(S)$ is the orbifold fundamental group of the Riemann moduli space $\mathcal{R}(S)$ and is given as the quotient of the group $\mathrm{Diff}^+(S)$, of orientation, preserving diffeomorphisms of S, by its normal subgroup $\mathrm{Diff}_0(S)$, of diffeomorphisms isotopic to the identity. It acts on Teichmüller space via pullback, and a classical result shows this action is properly discontinuous, although not free.

The choice of an embedded trivalent fat graph Γ, on S gives rise to a combinatorial description of the mapping class group and of its action on Teichmüller space in terms of shear coordinates. As a first step, we consider the action of the mapping class group on embedded fat graphs. Given an element of the mapping class group $\varphi \in \mathrm{Mod}(S)$ and an embedded fat graph Γ one obtains another, combinatorially equivalent, embedded fat graph $\Gamma' = \varphi(\Gamma)$ as the image of Γ under φ. Clearly, if φ is a non-trivial element of $\mathrm{Mod}(S)$, the isotopy classes of Γ and $\varphi(\Gamma)$ are necessarily distinct. Conversely, for any two isotopy classes Γ, Γ' of embeddings of the same combinatorial trivalent fat graph, there is a unique element of $\varphi \in \mathrm{Mod}(S)$ such that $\Gamma' = \varphi(\Gamma)$. Mapping class group elements can therefore be characterized as pairs of isotopy classes of embeddings of a given combinatorial trivalent fat graph. In fact, a result of Penner [**36**, **37**] allows one to decompose elements of the mapping class group into sequences of elementary graph transformations between any such a pair.

Two embedded trivalent fat graphs Γ and Γ' are said to be related by a Whitehead move $W_\alpha : \Gamma \mapsto \Gamma' = \Gamma_\alpha$ along an edge α if Γ' is obtained from Γ by collapsing the edge α into a four-valent vertex and then expanding it in the opposite direction as shown in Figure 4.

Similarly, two trivalent fat graphs Γ and Γ' with ordered edges are said to be related by a transposition of the edges α and β if Γ' is obtained from Γ by exchanging the order of α and β. Such a transformation will be denoted by $\sigma = (\alpha\ \beta) : \Gamma \mapsto \Gamma'$, where $(\alpha\ \beta) \in S_E$ is interpreted as an element of the symmetric group S_E.

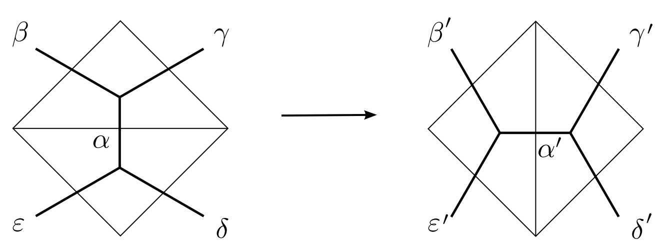

Figure 4. Whitehead move along α.

The result in [**36**, **37**] states that any two embeddings Γ, Γ' of a combinatorial edge-ordered trivalent fat graph are related by a sequence of Whitehead moves and transposition of the edges whose interaction is characterized by a set of simple relations. In particular, this provides a presentation of the mapping class group(oid) in terms of generators and relations.

Theorem 2.3 (Penner [**36**, **37**]). *Elements of* $\mathrm{Mod}(S)$ *are in bijection with finite sequences of elementary graph transformations between embeddings* Γ, Γ' *of a combinatorial edge-ordered trivalent fat graph, modulo the following relations:*

1) *(Involutivity) For every edge* $\alpha \in E(\Gamma)$
$$W_\alpha^2 = \mathrm{id};$$
2) *(Naturality) For every edge* α *and every transposition* σ *of edges*
$$\sigma \circ W_\alpha = W_{\sigma(\alpha)};$$
3) *(Commutativity) For edges* $\alpha, \beta \in E(\Gamma)$ *that do not share a common vertex*
$$W_\alpha \circ W_\beta = W_\beta \circ W_\alpha;$$
4) *(Pentagon) for edges* $\alpha, \beta \in E(\Gamma)$ *sharing exactly one vertex,*
$$W_\alpha \circ W_\beta \circ W_\alpha \circ W_\beta \circ W_\alpha = (\alpha\ \beta).$$

The mapping class group action in shear coordinates. The description of the mapping class group in terms of elementary graph transformations gives rise to simple and explicit expressions for its action on Teichmüller space in terms of shear coordinates. Consider again an embedded trivalent fat graph Γ, and denote by $x^\alpha : \mathcal{T}(S) \to \mathbb{R}$ the coordinate function associated to an edge $\alpha \in E(\Gamma)$. For each element $\varphi \in \mathrm{Mod}(S)$, denote by $\Gamma' = \varphi(\Gamma)$ the image of Γ under φ and by $x'^\alpha : \mathcal{T}(S) \to \mathbb{R}$ the coordinate function for the edge $\varphi(\alpha) \in E(\Gamma')$. The mapping class group action $\mathrm{Mod}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$ extends to an equivariant action $\mathrm{Mod}(S) \times \mathbb{R}^E \to \mathbb{R}^E$. This extended action is determined uniquely by the condition that the coordinates of $\varphi^* h \in \mathcal{T}(S)$ with respect to the embedded graph $\Gamma = \varphi^*(\Gamma')$ agree with the coordinates of $h \in \mathcal{T}(S)$ with respect to the embedded graph Γ'. In other words, the coordinate functions $x^\alpha : \mathcal{T}(S) \to \mathbb{R}$ and $x'^\alpha : \mathcal{T}(S) \to \mathbb{R}$ are related by
$$x'^\alpha = \varphi(x^\alpha) = x^\alpha \circ \varphi^*.$$
From this, it follows that the corresponding group action $\mathrm{Mod}(S) \times \mathbb{R}^E \to \mathbb{R}^E$ is simply given by the change of coordinates $x^\alpha \mapsto x'^\alpha$ determined by the two distinct embeddings Γ, Γ' of the given combinatorial fat graph.

As a consequence of Theorem 2.3, it is then sufficient to consider the transformation of coordinates $x^\alpha \mapsto x'^\alpha$ under Whitehead moves, as their transformation under edge transpositions is immediate. Choosing a point $h \in \mathcal{T}(S)$ and comparing its shear coordinates before and after the move, one obtains an expression for the transformation of the shear coordinates [**17**]. For the Whitehead move along the edge $\alpha \in E(\Gamma)$ depicted in Figure 4, the relation between the shear coordinates for Γ and those for $\Gamma' = \Gamma_\alpha$ is given by

$$W_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = -x^\alpha \\ x^{\beta,\delta} \mapsto x'^{\beta,\delta} = x^{\beta,\delta} + \log(1+e^{x^\alpha}) \\ x^{\gamma,\epsilon} \mapsto x'^{\gamma,\epsilon} = x^{\gamma,\epsilon} - \log(1+e^{-x^\alpha}), \end{cases} \tag{7}$$

while all other edge coordinates remain unchanged. This formula allows one to determine properties of the Whitehead moves via direct computations. In particular, it follows that they satisfy the conditions in Theorem 2.3; see [**17**]. It is also straightforward to show that the constraints (4) and the Poisson bivector (5) are preserved by (7). More precisely, the pull-back of the constraint c', defined with respect to the fat graph $\Gamma' = \Gamma_\alpha$, coincides with the constraint c, defined with respect to Γ, and the push-forward of the bivector π_{WP}, defined with respect to Γ, agrees with the bivector π'_{WP}, defined with respect to Γ':

$$c = c' \circ W_\alpha, \qquad (W_\alpha)_* \pi_{WP} = \pi'_{WP}.$$

This, therefore, proves that the Whitehead moves induce a symplectic mapping class group action on Teichmüller space.

Theorem 2.4 (Fock and Chekhov [**17**], Penner [**36, 37**]). *The Whitehead moves $W_\alpha : \mathbb{R}^E \to \mathbb{R}^E$ (7) satisfy the relations of Theorem 2.3. Furthermore, they preserve the constraints $c : \mathbb{R}^E \to \mathbb{R}^F$ (4) and the Weil–Petersson Poisson bivector π_{WP} (5).*

3. Moduli spaces of 3d gravity

In this section, we consider moduli spaces of geometric structures that arise in the context of 3d gravity—namely, the moduli spaces of maximal globally hyperbolic (MGH) Einstein spacetimes. These moduli spaces are higher-dimensional generalizations of Teichmüller space and classify the diffeomorphism classes of constant curvature Lorentzian metrics on a three-dimensional manifold M. In the following, M denotes a three-dimensional manifold of topology $\mathbb{R} \times S$, where S is a compact orientable genus g surface with s punctures satisfying $2g-2+s > 0$. We also restrict attention to Einstein metrics that are globally hyperbolic with Cauchy surface S and maximal in the sense that any isometric embedding of M into another globally hyperbolic spacetime N is a global isometry. See [**7**] for details on causality of Lorentzian manifolds.

3.1. The isometry groups in 3d gravity.

The isometry groups and their Lie algebras. The simplicity of 3d gravity is a consequence of the vanishing of the traceless part of the Riemann tensor, the Weyl tensor, in three dimensions. It implies that the Riemann curvature tensor and therefore the sectional curvature of a three-dimensional manifold are determined uniquely by its Ricci tensor. It then follows that any Einstein spacetime, a solution of Einstein equations with vanishing stress–energy tensor, is locally isometric to one of three model Lorentzian manifolds with sectional curvature given by the cosmological constant Λ.

These model spacetimes are three-dimensional Minkowski space M_3 for $\Lambda = 0$, anti-de Sitter space AdS_3 for $\Lambda = -1$, and de Sitter space dS_3 for $\Lambda = 1$. In the following, we denote these model spacetimes by X_Λ, their isometry groups by $G_\Lambda = \mathrm{Isom}(X_\Lambda)$, and the associated Lie algebras by $\mathfrak{g}_\Lambda = \mathrm{Lie}(G_\Lambda)$. As solutions for different values of the curvature Λ can be obtained by simple rescalings of the metrics, we restrict attention to the cases $\Lambda = 0, -1, 1$.

The three model spacetimes have a simple description in terms of the group $\mathrm{PSL}(2,\mathbb{R})$, which is outlined in Appendix A, and their isometry groups are given by

$$G_\Lambda = \begin{cases} \mathrm{PSL}(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R}) & \Lambda = 0 \\ \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) & \Lambda = -1 \\ \mathrm{PSL}(2,\mathbb{C}) & \Lambda = 1. \end{cases}$$

In all cases, the associated Lie algebra $\mathfrak{g}_\Lambda$ is a six-dimensional real Lie algebra and can be described in terms of a common basis for which the cosmological constant plays the role of a structure constant [**46**]. This basis involves a basis $\{J_i\}_{i=0,1,2}$ of $\mathfrak{sl}(2,\mathbb{R})$ and three additional basis vectors $\{P_i\}_{i=0,1,2}$ such that the Lie bracket is given by

(8)
$$[J_i, J_j] = \sum_{k=0}^{2} \epsilon_{ij}{}^k J_k, \quad [J_i, P_j] = \sum_{k=0}^{2} \epsilon_{ij}{}^k P_k, \quad [P_i, P_j] = -\Lambda \sum_{k=0}^{2} \epsilon_{ij}{}^k J_k,$$

where ϵ_{ijk} is the totally antisymmetric tensor in three dimensions with $\epsilon_{012} = 1$ and indices are raised and lowered with the three-dimensional Minkowski metric $\eta = \mathrm{diag}(-1,1,1)$. For $\Lambda = 0$, this is simply the Poincaré algebra in three dimensions. For $\Lambda = 1$ and $\Lambda = -1$, one can introduce the alternative basis $\{J_i^\pm\}_{i=0,1,2}$ with $J_i^\pm = \frac{1}{2}\left(J_i \pm P_i/\sqrt{-\Lambda}\right)$, in which the Lie bracket of $\mathfrak{g}_\Lambda$ reads

$$[J_i^\pm, J_j^\pm] = \epsilon_{ij}{}^k J_k^\pm, \qquad [J_i^\pm, J_j^\mp] = 0.$$

This shows that the Lie algebra (8) is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$ for $\Lambda = 0$, to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ for $\Lambda = -1$, and to $\mathfrak{sl}(2,\mathbb{C})$ for $\Lambda = 1$. In the following, we also need to consider Ad-invariant symmetric bilinear forms on $\mathfrak{g}_\Lambda$. In all three cases, the real vector space of such bilinear forms is two-dimensional, with a basis given by the forms $(\ ,\ ), \langle\ ,\ \rangle : \mathfrak{g}_\Lambda \times \mathfrak{g}_\Lambda \to \mathbb{R}$ defined by

$$\begin{aligned}(J_i, P_j) &= 0, & (J_i, J_j) &= \eta_{ij}, & (P_i, P_j) &= -\Lambda\eta_{ij},\\ \langle J_i, P_j\rangle &= \eta_{ij}, & \langle J_i, J_j\rangle &= 0, & \langle P_i, P_j\rangle &= 0.\end{aligned} \tag{9}$$

A unified description of the Lie algebras. A convenient description of the isometry groups G_Λ and their Lie algebras $\mathfrak{g}_\Lambda$ is obtained by exteding $\mathfrak{sl}(2,\mathbb{R})$ to a Lie algebra over a commutative real algebra R_Λ; see [**31**]. As a vector space, this algebra R_Λ is isomorphic to $\mathbb{R}^2$ and its multiplication law is given by

$$(x, y) \cdot (u, v) = (xu - \Lambda yv, xv + yu) \qquad \forall x, y, u, v \in \mathbb{R}.$$

Writing $1 = (1, 0)$ and $\ell = (0, 1)$, one obtains a parametrization of R_Λ analogous to the complex numbers, and consequently we use the notation $\mathrm{Re}_\ell(x + \ell y) = x$, $\mathrm{Im}_\ell(x + \ell y) = y$ for all $x, y \in \mathbb{R}$. A direct computation shows that R_Λ is isomorphic to $\mathbb{C}$ with $\ell = \mathrm{i}$ for $\Lambda = 1$, to the split-complex numbers for $\Lambda = -1$, and to the dual numbers for $\Lambda = 0$.

Note that, for $\Lambda = 0, -1$, the algebra R_Λ has zero divisors. For $\Lambda = -1$, these are of the form $\frac{1}{2}(1 \pm \ell)x$ with $x \in \mathbb{R}$, and one has

$$\tfrac{1}{2}(1 \pm \ell) \cdot \tfrac{1}{2}(1 \pm \ell) = \tfrac{1}{2}(1 \pm \ell), \qquad \tfrac{1}{2}(1 \pm \ell) \cdot \tfrac{1}{2}(1 \mp \ell) = 0.$$

This allows one to extend analytic functions $f : \mathbb{R} \to \mathbb{R}$ to analytic functions $f : R_{-1} \to R_{-1}$:

$$f(x + \ell y) = \tfrac{1}{2}(1 + \ell)f(x + y) + \tfrac{1}{2}(1 - \ell)f(x - y).$$

For $\Lambda = 0$, the zero divisors in R_Λ are of the form ℓy with $y \in \mathbb{R}$. Analytic functions $f : \mathbb{R} \to \mathbb{R}$ can thus be extended to functions $f : R_0 \to R_0$ via

$$f(x + \ell y) = f(x) + \ell f'(x)y.$$

Note that these expressions generalize the extension of real analytic functions $f : \mathbb{R} \to \mathbb{R}$ to complex analytic functions $f : \mathbb{C} \to \mathbb{C}$ and also give rise to the following generalization of the Cauchy–Riemann differential equations:

$$\frac{\partial \mathrm{Re}_\ell f}{\partial x} = \frac{\partial \mathrm{Im}_\ell f}{\partial y}, \qquad \frac{\partial \mathrm{Re}_\ell f}{\partial y} = -\Lambda\,\frac{\partial \mathrm{Im}_\ell f}{\partial x}.$$

The algebra R_Λ allows one to identify the Lie algebras $\mathfrak{g}_\Lambda$ with the Lie algebra of traceless 2×2-matrices with entries in R_Λ [**31**]. By considering a basis $\{J_i\}_{i=0,1,2}$ of $\mathfrak{sl}(2,\mathbb{R})$ with Lie bracket $[J_i, J_j] = \epsilon_{ij}{}^k J_k$ and setting $P_i = \ell J_i$, one obtains the Lie algebras $\mathfrak{g}_\Lambda$ with Lie bracket (8). This description of the Lie algebras $\mathfrak{g}_\Lambda$ in terms of R_Λ also gives rise to a common description of the bilinear forms (9). They are obtained as the real and imaginary parts of the bilinear extension of the Killing form κ on $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}_\Lambda$:

$$(\ ,\ ) = 2\mathrm{Re}_\ell\,\kappa \qquad \langle\ ,\ \rangle = 2\mathrm{Im}_\ell\,\kappa, \tag{10}$$

where $\kappa(X,Y) = \mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{g}_\Lambda$.

3.2. Einstein spacetimes and their moduli spaces.

Classification of Einstein spacetimes. Generalizing the results summarized in Section 2, we now consider the moduli spaces of maximal globally hyperbolic (MGH) Einstein metrics of curvature Λ on M that induce a complete metric of finite area on the Cauchy surface S modulo orientation-preserving diffeomorphisms

$$\mathcal{M}_\Lambda(M) = \mathrm{Ein}_\Lambda(M)/\mathrm{Diff}^+(M).$$

As in the case of the Riemann moduli space, it is convenient to consider the universal covering space of $\mathcal{M}_\Lambda(S)$ by identifying only those metrics that are related by the subgroup of diffeomorphisms isotopic to the identity. This leads to the Teichmüller-like moduli spaces

$$\mathcal{GH}_\Lambda(M) = \mathrm{Ein}_\Lambda(M)/\mathrm{Diff}_0(M).$$

One approach to the classification of MGH Einstein spacetimes in three dimensions is based on their description as quotients of regions in the model spacetimes by a discrete group of isometries [**46**, **30**, **40**, **8**]. The resulting classification is analogous to the uniformization theorem for hyperbolic surfaces in two dimensions and states that MGH Einstein spacetimes are largely determined by their holonomy representation $\rho : \pi_1(M) \cong \pi_1(S) \to G_\Lambda$, which defines the action of $\pi_1(S)$ on the universal cover of M. More precisely, for $\Lambda = 0, -1$, a MGH Einstein metric $g \in \mathcal{GH}_\Lambda(M)$ can be described as follows. First, consider the universal cover $\tilde{S}$ of the Cauchy surface S. It is shown in [**30**] that $\tilde{S}$ isometrically embeds in X_Λ and that the universal cover $\tilde{M}$ of M is obtained from this embedding. For $\Lambda = -1$, $\tilde{M}$ coincides with the domain of dependence of $\tilde{S}$, and for $\Lambda = 0$ it is the chronological future of this domain of dependence. In other words, the universal cover of M isometrically embeds in X_Λ and the group of deck transformations provides a representation of $\pi_1(M) \cong \pi_1(S)$ into G_Λ.

Due to the non-degeneracy of the three-dimensional metric, not all representations $\rho : \pi_1(S) \to G_\Lambda$ arise as holonomy representations of MGH spacetimes. The allowed representations are also described in [**30**] and can be characterized as follows. For $\Lambda = -1$, the allowed holonomy representations $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ are the ones that decompose into two Fuchsian components $\rho_{l,r} = \mathrm{pr}_{1,2} \circ \rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. For $\Lambda = 0$, they are representations $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$ that decompose into a Fuchsian part $\rho_0 = \mathrm{pr}_1 \circ \rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ and a ρ_0-cocycle $\tau = \mathrm{pr}_2 \circ H : \pi_1(S) \to \mathfrak{sl}(2,\mathbb{R})$ with

$$\tau(ab) = \tau(a) + \rho_0(a)\tau(b)\rho_0(a)^{-1}, \qquad \forall a, b \in \pi_1(S).$$

The moduli spaces for $\Lambda = -1, 0$ are thus shown to be in one-to-one correspondence with certain components of the representation variety $\mathrm{Rep}_{\mathcal{C}}(S, \mathrm{G}_\Lambda)$.

For $\Lambda = 1$, the correspondence between holonomy representations and MGH spacetimes is only locally injective, which means that the holonomy data is not sufficient to distinguish certain MGH de Sitter spacetimes. This is a consequence of the fact that the universal cover of M is in general only immersed in X_1. The suitable data for the classification of such spacetimes is obtained by grafting of hyperbolic surfaces along measured geodesic laminations, which also provides an alternative description for the flat and AdS case.

Grafting parametrization. As explained in Appendix A, all three model spacetimes X_Λ are equipped with certain embeddings of the hyperbolic plane, either in X_Λ itself ($\Lambda = -1, 0$) or in an appropriate dual space ($\Lambda = 1$). For a spacetime with purely Fuchsian holonomy $\rho_0 : \pi(S) \to \mathrm{PSL}(2, \mathbb{R}) \subset G_\Lambda$, the action of the holonomy group on X_Λ induces an action on these embedded hyperbolic planes and thus defines a hyperbolic surface $h \in \mathcal{T}(S)$. Applying earthquakes along measured laminations for h, one then obtains a description of all such Fuchsian spacetimes in terms of the fiber $\mathcal{ML}_h(S)$.

General MGH spacetimes, whose holonomies are not restricted to the subgroup $\mathrm{PSL}(2, \mathbb{R}) \subset G_\Lambda$, are obtained as deformations of these Fuchsian spacetimes via grafting. Grafting is an operation $Gr : \mathcal{ML}(S) \to \mathcal{GH}_\Lambda(M)$ that associates to each point $h \in \mathcal{T}(S)$ and each measured geodesic lamination $\lambda \in \mathcal{ML}_h(S)$ a MGH metric $Gr^\lambda(h) \in \mathcal{GH}_\Lambda(M)$, called the grafting of h along λ. In terms of representations, the construction is similar to the one of earthquakes on Riemann surfaces and can be described as follows.

Consider a point $h \in \mathcal{T}(S)$ with associated Fuchsian representation $\rho_0 : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$. Then the grafting of h along $\lambda \in \mathcal{ML}_h(S)$ corresponds to another representation $\rho : \pi_1(S) \to G_\Lambda$ that is the product $\rho = Z_G^\lambda \cdot \rho_0$ of the Fuchsian representation ρ_0 and a grafting cocycle $Z_G^\lambda : \pi_1(S) \to G_\Lambda$. Such cocycles are defined in exactly the same way as the cocycles for earthquakes in (1), only now all measures are multiplied by the R_Λ-imaginary unit ℓ. Let $\tilde{\lambda} \in \mathcal{ML}(\mathbb{H}^2)$ be the lift of λ to the universal cover of (S, h). As the hyperbolic plane is embedded in X_Λ (or in the associated dual space), there is a well defined notion of rotation around any leaf $\tilde{l}$ of $\tilde{\lambda}$. For any point $p \in \tilde{l}$, this rotation is given by $\mathrm{Ad}_{A_p} E(\ell \mu_p)$, where A_p is the hyperbolic isometry mapping the imaginary axis to $\tilde{l}$, i to p and preserving the orientation, μ_p is the associated oriented weight, as in (1), which determines the angle of rotation and

$$E(\ell\mu) = \begin{cases} (\mathrm{id}, \mu J_1) & \Lambda = 0 \\ (E(\mu), E(-\mu)) & \Lambda = -1 \\ E(\mathrm{i}\mu) & \Lambda = 1. \end{cases}$$

Here, J_1 denotes the generator of hyperbolic translations along the imaginary axis on $\mathbb{H}^2$; see (39). Note that this rotation is a direct generalization of the hyperbolic translation along a geodesic in (2) and that the cocycle $Z_G^\lambda : \pi_1(S) \to G_\Lambda$ is obtained as a direct generalization of the cocycle (1)

$$Z_G^\lambda(a) = Z_E^{\ell\lambda}(a) = \prod_{p \in \lambda \cap a} \mathrm{Ad}_{A_p} E(\ell\mu_p). \tag{11}$$

In particular, this expression makes it clear that for $\Lambda = 1$ the correspondence between measured laminations and de Sitter holonomy representations can only be locally injective, since laminations with the same support whose measures differ by multiples of 2π give rise to the same cocycle and hence to the same holonomy representation. This correspondence between MGH Einstein spacetimes and measured laminations allows one to identify the former with the bundle of measured geodesic laminations over Teichmüller space.

Theorem 3.1 (Mess [**30**], Scannell [**40**], Benedetti–Bonsante [**8**]). *Let S be a closed orientable surface of genus g and with s punctures satisfying $2g - 2 + s > 0$. Then the Teichmüller-like moduli spaces of MGH Einstein spacetimes on $M = \mathbb{R} \times S$ are homeomorphic to the bundle of measured geodesic laminations over Teichmüller space*

$$\mathcal{GH}_\Lambda(M) \cong \mathcal{ML}(S).$$

The gravitational symplectic structure. From a physics viewpoint, the parametrization of MGH spacetimes in terms of holonomies is closely related to the Chern–Simons formulation of 3d gravity developed in [**1**, **46**]. In this formulation, the spacetime metric is first decomposed into a (co-)frame field e and an associated spin connection ω, which are then combined into a G_Λ-connection

$$A = \omega^i J_i + e^i P_i = (\omega^i + \ell e^i) J_i,$$

where J_i and P_i denote the basis of $\mathfrak{g}_\Lambda$ introduced in (8). The requirements of flatness and vanishing torsion on e and ω translate into a flatness condition $F = dA + A \wedge A = 0$ for the G_Λ-connection. This allows one to relate the moduli spaces of MGH Einstein spacetimes of curvature Λ to the moduli space of flat G_Λ on the Cauchy surface S [**46**]. In particular, the gravitational symplectic structure on $\mathcal{GH}_\Lambda(M)$ can be characterized in terms of the Chern–Simons symplectic structure, which agrees with the Atiyah–Bott symplectic structure on $\mathrm{Rep}_{\mathcal{C}}(S, G_\Lambda)$ [**2**].

We start by summarizing the relevant results on this symplectic structure for a general structure group G. Given an Ad-invariant, non-degenerate symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$, one obtains a canonical symplectic structure on the moduli space $\mathrm{Rep}_{\mathcal{C}}(S, G)$ [**2**, **18**]. It was shown by Goldman [**19**] that the corresponding Poisson structure can be expressed in terms of the bilinear form B and the intersection behavior of curves representing elements of $\pi_1(S)$ as follows. To each class function $f \in C_G(G)$ and each element $a \in \pi_1(S)$, one associates a function $f_a : \mathrm{Rep}_{\mathcal{C}}(S, G) \to \mathbb{R}$ defined by $f_a(\rho) = f(\rho(a))$. Then, the Goldman Poisson bracket between two such functions f_a, g_b is defined by

$$\{f_a, g_b\}_G(\rho) = \sum_{p \in a \cap b} \epsilon_p(a, b)\, B(F_{a_p}(\rho), G_{b_p}(\rho)), \tag{12}$$

where the sum is over the intersection points of $a, b \in \pi_1(S)$ and $\epsilon_p(a, b)$ denotes their oriented intersection number at p. The indices a_p, b_p on the right-hand side of (12) stand for representatives of a and b based at the point p, and the functions $F_{a_p}, G_{b_p} : \mathrm{Rep}_{\mathcal{C}}(S, G) \to \mathfrak{g}$ are defined by

$$B(F_{a_p}(\rho), X) = \frac{d}{dt}\bigg|_{t=0} f(\rho(a_p)\, e^{tX}), \qquad \forall X \in \mathfrak{g}.$$

In the following, it will be convenient to express this Poisson structure (12) in terms of a basis $\{T_k\}$ of $\mathfrak{g}$. Denoting by $B_{kl} = B(T_k, T_l)$ the coefficients of the Ad-invariant symmetric bilinear form B with respect to this basis and by B^{kl} the entries of the inverse of the coefficient matrix, one find that the bracket (12) is given by

$$B(F_{a_p}(\rho), G_{b_p}(\rho)) = \sum_{k,l=1}^{\dim(\mathfrak{g})} B^{kl} B(F_{a_p}, T_k) B(G_{b_p}, T_l). \tag{13}$$

Note that although the definition of this bracket involves a choice of paths on S that represent the elements $a, b \in \pi_1(S)$, it is shown in [**19**] that the bracket does not depend on this choice and induces a symplectic structure on $\mathrm{Rep}_{\mathcal{C}}(S, G)$.

Theorem 3.2 (Goldman [**18**, **19**])**.** *Formula* (12) *defines a symplectic structure on* $\mathrm{Rep}_{\mathcal{C}}(S, G)$ *that coincides with the Atiyah–Bott symplectic structure.*

In particular, Goldman's symplectic structure can be used to describe the Weil–Petersson structure on Teichmüller space, by realising the latter as a connected component of $\mathrm{Rep}_{\mathcal{C}}(S, \mathrm{PSL}(2, \mathbb{R}))$. In other words, for the structure group $\mathrm{PSL}(2, \mathbb{R})$ and the the Killing form κ on $\mathfrak{sl}(2, \mathbb{R})$, the restriction of the Goldman Poisson bracket (12) to the Teichmüller component of $\mathrm{Rep}_{\mathcal{C}}(S, \mathrm{PSL}(2, \mathbb{R}))$ induces the Weil–Petersson symplectic structure on Teichmüller space.

Similarly, the symplectic structures on the moduli spaces of 3d MGH spacetimes are closely related to the Goldman bracket on $\mathrm{Rep}_{\mathcal{C}}(S, G_\Lambda)$. However, unlike in the case of $\mathrm{PSL}(2, \mathbb{R})$, the space of Ad-invariant symmetric bilinear forms on $\mathfrak{g}_\Lambda$ is two dimensional, and there are inequivalent versions of the Goldman bracket on these moduli spaces, corresponding to different linear combinations of the bilinear forms $(\, , \,)$ and $\langle \, , \, \rangle$ defined in (9). It is shown in [**46**] that the bilinear forms on $\mathfrak{g}_\Lambda$ that are relevant for 3d gravity are the forms $\langle \, , \, \rangle$, which according to equation (10) can be interpreted as the imaginary part of the bilinear extension of the Killing form κ to $\mathfrak{g}_\Lambda$.

Theorem 3.3 (Witten [**46**])**.** *The gravitational Poisson structure on the Teichmüller-like moduli spaces of MGH spacetimes agrees with the restriction of the imaginary part of the Atiyah–Bott–Goldman Poisson structure on the G_Λ-representation variety.*

Only for this choice of the bilinear form, the Chern–Simons action agrees with the Einstein–Hilbert action for 3d gravity in Cartan's formulation. Other choices of the Ad-invariant symmetric bilinear form on $\mathfrak{g}_\Lambda$ yield a different action that gives rise to the same equations of motion but induces a different symplectic structure on the moduli space. It should, therefore, be expected that the choice of the correct bilinear form has important consequences for the resulting quantum theory.

4. Generalized shear coordinates

We are now ready to introduce generalized shear-bending coordinates on the Teichmüller-like moduli spaces $\mathcal{GH}_\Lambda(M)$. This will be achieved by parametrizing the deformation cocycles in terms of analytic shear coordinates on $\mathcal{ML}^{R_\Lambda}(S)$, the bundle of R_Λ-valued measured geodesic laminations. Using Thurston's earthquake theorem, we first define shear coordinates on the bundle $\mathcal{ML}(S)$ of measured geodesic laminations and, using the fact that the earthquake cocycles depend analytically on the set of measures on a lamination with fixed support, we then define an analytic extension of these coordinates to $\mathcal{ML}^{R_\Lambda}(S)$. This construction gives rise to coordinates on the moduli spaces $\mathcal{GH}_\Lambda(M)$ that have a clear geometric interpretation in terms of grafting along ideal edges of an ideal triangulation of S. We then derive an expression for the gravitational symplectic structure on $\mathcal{GH}_\Lambda(M)$ in terms of these coordinates and describe its relation to the Weil–Petersson symplectic structure and to the cotangent bundle over Teichmüller space.

4.1. Definition of shear-bending coordinates.

Shear coordinates for $\mathcal{ML}(S)$ via Thurston's theorem. To construct generalized shear coordinates, we first show how Thurston's shear coordinates on Teichmüller space provide a global parametrization of the bundle $\mathcal{ML}(S)$ of measured geodesic laminations via Thurston's earthquake theorem (see end of Subsection 2.1).

Let $h \in \mathcal{T}(S)$ be a point in Teichmüller space, $\lambda \in \mathcal{ML}_h(S)$ a measured geodesic lamination for h and $Eq^\lambda(h) \in \mathcal{T}(S)$ the earthquake of h along λ. For an embedded trivalent fat graph Γ, denote by $x^\alpha = x^\alpha(h)$ and $x'^\alpha = x^\alpha(Eq^\lambda(h))$ the shear coordinates of h and $Eq^\lambda(h)$ associated to an edge $\alpha \in E(\Gamma)$. Comparing the holonomy representations (3) of h and $Eq^\lambda(h)$ in terms of shear coordinates, we obtain the following parametrization of the associated earthquake cocycle $Z_E^\lambda : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ defined in (1):

(14)
$$Z_E^\lambda(a) = \rho^\lambda(a)\rho(a)^{-1} = \mathrm{Ad}_{A_n^a(x)} E(x'^{\alpha_n} - x^{\alpha_n}) \cdots \mathrm{Ad}_{A_1^a(x)} E(x'^{\alpha_1} - x^{\alpha_1}).$$

Here, $(\alpha_1, \ldots, \alpha_n)$ is the sequence of edges of Γ representing $a \in \pi_1(S)$ and $A_k^a(x)$ is the hyperbolic isometry that maps the imaginary axis on $\mathbb{H}^2$ to the lift of the ideal geodesic dual to α_k,

$$A_k^a(x) = P_n^a E(x^{\alpha_n}) P_{n-1}^a \cdots E(x^{\alpha_{k+1}}) P_k^a. \tag{15}$$

This expression for the earthquake cocycle Z_E^λ in terms of the difference between the shear coordinates of $Eq^\lambda(h)$ and h then allows us to define coordinates $u^\alpha : \mathcal{ML}_h(S) \to \mathbb{R}$ parametrizing the fibers of $\mathcal{ML}(S)$ via

$$u^\alpha(\lambda) = x^\alpha(Eq^\lambda(h)) - x^\alpha(h), \qquad \forall \lambda \in \mathcal{ML}_h(S). \tag{16}$$

Clearly, these coordinates are not all independent but satisfy the same constraints as the shear coordinates x^α—namely, for each face $i \in F(\Gamma)$

$$c^i(u) = \sum_{\alpha \in E(\Gamma)} \theta^i{}_\alpha u^\alpha = 0. \tag{17}$$

Interpreting the constraints for the different faces $i \in F(\Gamma)$ as components of a linear map $c : \mathbb{R}^E \to \mathbb{R}^F$, we thus obtain an explicit description of measured geodesic lamination in terms of shear coordinates on S, which characterizes $\mathcal{ML}(S)$ as a linear subspace of $\mathbb{R}^E \times \mathbb{R}^E$.

Proposition 4.1. *The coordinate functions $x^\alpha, u^\alpha : \mathcal{ML}(S) \to \mathbb{R}$ define an embedding $(x, u) : \mathcal{ML}(S) \hookrightarrow \mathbb{R}^E \times \mathbb{R}^E$ whose image is the kernel of the linear map $c \oplus c : \mathbb{R}^E \times \mathbb{R}^E \to \mathbb{R}^F \times \mathbb{R}^F$ whose components are given by* (4) *and* (17).

Note also that the coordinate functions u^α satisfy certain cocycle conditions reminiscent from the properties of the cocycle (14). For two measured geodesic laminations $\lambda_1, \lambda_2 \in \mathcal{ML}_h(S)$, denote by $h_1 = Eq^{\lambda_1}(h)$ and $h_2 = Eq^{\lambda_2}(h)$ the images of h under the associated earthquakes, and let $\lambda' \in \mathcal{ML}_{h_1}(S)$ be the measured geodesic lamination with $h_2 = Eq^{\lambda'}(h_1)$. Then the definition of u^α directly implies the cocycle condition

$$u^\alpha(\lambda') = u^\alpha(\lambda_2) - u^\alpha(\lambda_1).$$

Analytic extension to $\mathcal{ML}^{R_\Lambda}(S)$. An analogous description of grafting construction is obtained by considering geodesic laminations with ℓ-imaginary measures and analytic continuation of the coordinates u^α defined above. In fact, we may consider more general R_Λ-valued measured laminations, defined as pairs $(\lambda, \mu + \ell\nu)$ with μ and ν real transverse measures supported on λ. Given a point $h \in \mathcal{T}(S)$ and an R_Λ-valued measured lamination $\lambda \in \mathcal{ML}_h^{R_\Lambda}(S)$, we define the ℓ-complexified earthquake of h along λ in terms of the cocycle $Z^\lambda_{EG} : \pi_1(S) \to G_\Lambda$,

$$Z^\lambda_{EG}(a) = \prod_{p \in \lambda \cap a} \mathrm{Ad}_{A_p} E(\mu_p + \ell\nu_p), \qquad \forall a \in \pi_1(S),$$

with the same notation as in (1) and

$$E(\mu + \ell\nu) = \begin{cases} (E(\mu), \nu J_1) & \Lambda = 0 \\ (E(\mu+\nu), E(\mu-\nu)) & \Lambda = -1 \\ E(\mu + \mathrm{i}\nu) & \Lambda = 1. \end{cases} \tag{18}$$

Note that earthquake and grafting cocycles in (1) and (11) are obtained as particular cases of this R_Λ-valued cocycle for purely real or purely imaginary R_Λ-measures.

For an embedded trivalent fat graph Γ on S, denote by $u^\alpha : \mathcal{ML}(S) \to \mathbb{R}$ the shear coordinates for $\mathbb{R}$-valued measured laminations, as defined in (16). We now wish to analytically extend the coordinate functions u^α to $\mathcal{ML}^{R_\Lambda}(S)$. This is indeed possible since the earthquake map $Eq : \mathcal{ML}(S) \to \mathcal{T}(S)$ depends analytically on the measure of laminations with fixed support [**29**], and the shear coordinates $x^\alpha : \mathcal{T}(S) \to \mathbb{R}$ on Teichmüller space are also analytic [**38**]. Together, these imply the following result.

Proposition 4.2. *The coordinates $u^\alpha : \mathcal{ML}(S) \to \mathbb{R}$ are analytic on the measure of laminations with fixed support and therefore admit a unique analytic extension $w^\alpha : \mathcal{ML}^{R_\Lambda}(S) \to R_\Lambda$ satisfying $w^\alpha|_{\mathcal{ML}(S)} = u^\alpha$.*

This allows us to describe the associated shear-bending cocycle $Z^\lambda_{EG} : \pi_1(S) \to G_\Lambda$ via

$$Z^\lambda_{EG}(a) = \mathrm{Ad}_{A^a_n(x)} E(u^{\alpha_n} + \ell v^{\alpha_n}) \cdots \mathrm{Ad}_{A^a_1(x)} E(u^{\alpha_1} + \ell v^{\alpha_1}),$$

with the same notation as in (14), $u^\alpha = \mathrm{Re}_\ell(w^\alpha)$, $v^\alpha = \mathrm{Im}_\ell(w^\alpha)$, and $E(u + \ell v)$ as in (18).

Shear-bending coordinates on $\mathcal{GH}_\Lambda(M)$. The analytic extension w^α of the coordinates u^α in (16) now allows us to define generalized shear coordinates on $\mathcal{GH}_\Lambda(M)$ as follows. Consider a point $g \in \mathcal{GH}_\Lambda(M)$, let $h \in \mathcal{T}(S)$ be the hyperbolic metric determined by the Fuchsian part of its holonomy representation, and, $\lambda \in \mathcal{ML}_h(S)$, the measured lamination associated with its grafting cocycle. Denoting by $x^\alpha(h)$ the shear coordinate of h and by $w^\alpha(\ell\lambda)$ the shear coordinate of $\ell\lambda$ for each edge $\alpha \in E(\Gamma)$, we define the generalized shear coordinate of g as

$$z^\alpha(g) = x^\alpha(h) + w^\alpha(\ell\lambda) = x^\alpha(g) + \ell y^\alpha(g). \tag{19}$$

Note that $\mathrm{Re}_\ell(w^\alpha(\ell\lambda))$ may in general be non-zero and, consequently, that the real part $x^\alpha(g)$ of the generalized shear coordinates does not necessarily agree with the real part of the corresponding shear coordinates $x^\alpha(h)$.

As a direct consequence of the definition of generalized shear coordinates, one finds that the holonomy representation of g can be parametrized exactly as in (3) by

$$\rho(a) = P_n^a E(z^{\alpha_n}) \cdots P_1^a E(z^{\alpha_1}), \tag{20}$$

where the real matrices $E(x^{\alpha_k})$ in (3) are replaced by R^Λ-valued matrices $E(z^{\alpha_k})$. These terms can be interpreted as a combination of earthquakes and grafting along the ideal edges of the triangulation dual to Γ. We therefore refer to the generalized shear coordinates z^α as shear-bending coordinates. Note that these coordinates can also be viewed as generalizations of the shear-bending coordinates of Bonahon [**9**] to the Lorentzian context.

As in the real case, the parametrization (20) of the holonomies in terms of shear-bending coordinates gives rise to constraints associated with the faces of Γ. A direct computation yields

$$c^i_\Lambda(z) = \sum_{\alpha \in E(\Gamma)} \theta^i{}_\alpha z^\alpha = 0, \tag{21}$$

for each face $i \in F(\Gamma)$. These constraints again impose the tracelessness of the holonomies around the punctures and allow one to realise the moduli spaces $\mathcal{GH}_\Lambda(M)$ of 3d spacetimes as linear subspaces of R^E_Λ, thus generalizing Theorem 2.1.

Theorem 4.3. *The coordinate functions $z^\alpha : \mathcal{GH}_\Lambda(S) \to R_\Lambda$ in* (19) *define an embedding $z : \mathcal{GH}_\Lambda(S) \hookrightarrow R^E_\Lambda$ whose image agrees with the kernel of the linear map $c_\Lambda : R^E_\Lambda \to R^F_\Lambda$ whose components are given by* (21).

4.2. The gravitational symplectic structures.

Symplectic structure in terms of shear-bending coordinates. We now describe how the gravitational symplectic structure on the moduli space $\mathcal{GH}_\Lambda(M)$ of 3d spacetimes can be expressed in terms of shear-bending coordinates. First, recall that the gravitational symplectic structure is given by Goldman's symplectic structure (12) with structure group G_Λ and the Ad-invariant symmetric bilinear form $\langle\,,\rangle$ from (9). As the latter is the imaginary part $\mathrm{Im}_\ell(\kappa)$ of the bilinear extension of the Killing form on $\mathfrak{sl}(2,\mathbb{R})$ and that the generalized shear coordinates can be interpreted as an R_Λ-analytic continuation of the shear coordinates on $\mathcal{T}(S)$, it is natural to expect that Goldman's symplectic structure on $\mathcal{GH}_\Lambda(M)$ is given by a Poisson structure on R^E_Λ that resembles the Weil–Petersson Poisson structure.

In the following, we shall prove that the gravitational symplectic structure on $\mathcal{GH}_\Lambda(M)$ is induced from the Poisson bivector

$$\pi_\Lambda = \frac{1}{2} \sum_{\alpha,\beta \in E(\Gamma)} \pi^{\alpha\beta}_{WP} \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial y^\beta}, \tag{22}$$

where the coordinates x^α and y^β denote, respectively, the real and imaginary parts of the generalized shear coordinates z^α in (19). As the Poisson bracket of the variables x^α, y^β is a combinatorial constant, it is immediate that the Jacobi identity is satisfied and that this bivector defines a Poisson structure on R^Λ_E. Moreover, the bivector π_Λ induces a Poisson structure on the constraint surface $\mathrm{Ker} c_\Lambda \subset R^E_\Lambda$, where $c_\Lambda : R^E_\Lambda \to R^F_\Lambda$ is the linear constraint map from Theorem 4.3. This follows directly from the combinatorics of π_Λ, via its relation to π_{WP}, and from the combinatorics of the constraints. A simple computation shows that the Poisson bracket of a function $f \in C^\infty(R^E_\Lambda)$ with a component $c^i_\Lambda : R^E_\Lambda \to R_\Lambda$ of the constraint map in Theorem 4.3 is given by

$$\{f, c^i_\Lambda\}_\Lambda = (df \otimes dc^i_\Lambda)(\pi_\Lambda) = \frac{1}{2} \sum_{\alpha \in E(\Gamma)} \left(\ell \frac{\partial f}{\partial x^\alpha} + \frac{\partial f}{\partial y^\alpha} \right) ({\theta^i}_\beta - {\theta^i}_\gamma + {\theta^i}_\delta - {\theta^i}_\epsilon).$$

By an argument similar to the one following equation (6), one finds that this expression vanishes identically for all $f \in C^\infty(R^E_\Lambda)$. This follows since every face $i \in F(\Gamma)$ involves only left or right turns, and hence the linear combinations of the multiplicities θ^i on the right-hand-side cancel. Consequently, the constraint $c_\Lambda : R^E_\Lambda \to R^F_\Lambda$ is Casimir with respect to the Poisson structure (22), and the Poisson bivector π_Λ restricts to a Poisson bivector on $\mathrm{Ker} c_\Lambda \cong \mathcal{GH}_\Lambda(M)$. It is then easy to see that this Poisson structure is symplectic.

We will now prove that this Poisson structure on $\mathcal{GH}_\Lambda(M)$ agrees with the gravitational symplectic structure, that is, Goldman's symplectic structure (12) for the group G_Λ and the Ad-invariant symmetric bilinear form $\langle\,,\,\rangle$ in (9). This yields the following theorem.

Theorem 4.4. *The linear constraint $c_\Lambda : R^E_\Lambda \to R^F_\Lambda$ defined in* (21) *is Casimir with respect to the Poisson bivector π_Λ on R^E_Λ and induces a symplectic structure on* $\mathrm{Ker} c_\Lambda \cong \mathcal{GH}_\Lambda(M)$*. This symplectic structure coincides with the gravitational Poisson structure on $\mathcal{GH}_\Lambda(M)$.*

Proof. The general idea of the proof is to compare the Poisson structure on R^E_Λ induced by (22) with Goldman's symplectic structure (12) for a pair of class functions $f, g : G_\Lambda \to \mathbb{R}$. Given $a, b \in \pi_1(S)$, we consider the coordinate expressions of the associated functions f_a, g_b :

$\mathcal{GH}_\Lambda(M) \to \mathbb{R}$, defined by $f_a(\rho) = f(\rho(a))$, $g_b(\rho) = g(\rho(b))$, where $\mathcal{GH}_\Lambda(M)$ is viewed as a component of $\mathrm{Rep}(S, G_\Lambda)$ and $\rho : \pi_1(S) \to G_\Lambda$ is a group homomorphism. With the parametrization (20) of this group homomorphism ρ in terms of generalized shear coordinates, we are able compute the Poisson bracket

$$\{f_a, g_b\}_\Lambda = (df_a \otimes dg_b)(\pi_\Lambda).$$

The combinatorial structure of the Poisson bivector π_Λ then allows one to interpret each non-trivial contribution in terms of the essential intersection points of closed edge paths representing a and b.

For this, consider an embedded fat graph Γ dual to an ideal triangulation of S, and let $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_m)$ be closed edge paths in Γ that are freely homotopic to, respectively, $a \in \pi_1(S)$ and $b \in \pi_1(S)$. Expression (20) allows one to interpret their holonomies as analytic functions $\rho(a), \rho(b) : R^E_\Lambda \to G_\Lambda$. The associated functions f_a, g_b and F_a, G_b in (12) can then be expressed in shear coordinates as

$$\begin{aligned} &f_a(z) = f(P^a_n E(z^{\alpha_n}) \cdots P^a_1 E(z^{\alpha_1})) \\ &\mathrm{Im}_\ell\, \kappa(F_a(z), X) = \frac{d}{dt}\Big|_{t=0} f(P^a_n E(z^{\alpha_n}) \cdots P^a_1 E(z^{\alpha_1}) e^{tX}). \end{aligned}$$

The conjugation invariance of f then yields the identities

$$\frac{\partial f_a}{\partial x^\alpha} = \sum_{k=1}^n \delta^{\alpha_k}_\alpha \mathrm{Im}_\ell\, \kappa(F_a(z), J^a_k(z)), \quad \frac{\partial f_a}{\partial y^\alpha} = \sum_{k=1}^n \delta^{\alpha_k}_\alpha \mathrm{Re}_\ell\, \kappa(F_a(z), J^a_k(z)),$$

where $J^a_k : R^E_\Lambda \to \mathfrak{g}_\Lambda$ is given by

$$J^a_k(z) = \mathrm{Ad}_{A^a_k(z)} J_1 \tag{23}$$

with A^a_k defined as in (15) and J_1 as in (39). This allows us to directly compute the Poisson bracket of the functions $f_a, g_b : R^E_\Lambda \to \mathbb{R}$ induced by the bivector π_Λ in (22):

$$\{f_a, g_b\}_\Lambda = \mathrm{Im}_\ell \Big(\sum_{\alpha_k \in a, \beta_l \in b} \pi^{\alpha_k \beta_l}_{WP} \kappa(F_a, J^a_k) \kappa(G_b, J^b_l) \Big). \tag{24}$$

To show that this agrees with Goldman's symplectic structure (12), note that (24) is obtained as the imaginary part of the contraction of $F_a \otimes G_b \in \mathfrak{g}_\Lambda \otimes \mathfrak{g}_\Lambda$ with the bivector

$$\sum_{\alpha_k \in a, \beta_l \in b} \pi^{\alpha_k \beta_l}_{WP} J^a_k \otimes J^b_l \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda \tag{25}$$

with respect to the Ad-invariant bilinear form κ on $\mathfrak{g}_\Lambda$. Similarly, Goldman's symplectic structure (12) is obtained by contracting $F_a \otimes G_b$ with the bivector

$$\sum_{p \in a \cap b} 2\epsilon_p(a, b) \mathrm{Ad}_{A^a_p \otimes A^b_p} \Big(\sum_{i,j=0}^2 \eta^{ij} J_i \otimes J_j \Big) \in \mathfrak{g}_\Lambda \wedge \mathfrak{g}_\Lambda \tag{26}$$

where J_i is given by (39). We now compare the bivectors (26) and (25), making use of the combinatorial structure of π_{WP}.

First, note that $\pi_{WP}^{\alpha_k\beta_l} = 0$ unless the edges α_k and β_l are distinct edges sharing a common vertex. As all vertices of Γ are trivalent, this implies non-zero contributions only arise for either $\alpha_{k-1} = \beta_{l-1}$, $\alpha_{k-1} = \beta_{l+1}$, $\alpha_{k+1} = \beta_{l-1}$, or $\alpha_{k+1} = \beta_{l+1}$. We may thus organize the sum in (25) as a sum over edge path segments in the intersection of edge paths a and b. Up to cyclic relabeling of the edges and orientation reversal, each contribution involves segments of the form $(\alpha_2, \ldots, \alpha_{s-1}) = (\beta_2, \ldots, \beta_{s-1})$ with distinct initial and final edges, $\alpha_1 \neq \beta_1$ and $\alpha_s \neq \beta_s$.

We thus obtain

$$\begin{aligned}\sum_{\alpha_k\in a,\beta_l\in b} \pi_{WP}^{\alpha_k\beta_l} J_k^a \otimes J_l^b = \sum_{\text{segm.}\in a\cap b} \Big[&\pi_{WP}^{\alpha_1\beta_2}\Big(J_1^a\otimes J_2^b + J_2^a\otimes J_1^b - J_1^a\otimes J_1^b\Big)\\ &+\sum_{k=2}^{s-2}\pi_{WP}^{\alpha_k\beta_{k+1}}\Big(J_k^a\otimes J_{k+1}^b - J_{k+1}^a\otimes J_k^b\Big)\\ &+\pi_{WP}^{\alpha_{s-1}\beta_s}\Big(J_{s-1}^a\otimes J_s^b + J_s^a\otimes J_{s-1}^b - J_s^a\otimes J_s^b\Big)\Big].\end{aligned}$$

To simplify this expression, we use the following relations between the coefficients of the Weil–Petersson bivector:

$$\pi_{WP}^{\alpha_k\alpha_{k+1}} = \begin{cases} 1 & \text{if } P_k = L,\\ -1 & \text{if } P_k = R,\end{cases}$$

$$\pi_{WP}^{\beta_k\beta_{k+1}} = \begin{cases} -\pi_{WP}^{\alpha_1\alpha_2} = \pi_{WP}^{\alpha_1\beta_1} & \text{for } k=1\\ \pi_{WP}^{\alpha_k\alpha_{k+1}} & \text{for } k=2,\ldots,s-2\\ -\pi_{WP}^{\alpha_{s-1}\alpha_s} = -\pi_{WP}^{\alpha_s\beta_s} & \text{for } k=s-1,\end{cases}$$

and the following recursion relation satisfied by the Lie algebra–valued functions J_k^a

$$J_k^a - J_{k+1}^a = \mathrm{Ad}_{A_k^a}(J_0 - \pi_{WP}^{\alpha_k\alpha_{k+1}} J_2),$$

as well as its counterpart for J_k^b. These recursion relations are derived from (23) by direct computation using the commutators

$$[P_k^a, J_1] = J_0 - \pi^{\alpha_k\alpha_{k+1}} J_2.$$

With these identities, one finds that the Lie algebra–valued bivector (25) can be expressed as

$$\sum_{\alpha_k\in a,\beta_l\in b} \pi_{WP}^{\alpha_k\beta_l} J_k^a \otimes J_l^b$$
$$= \sum_{\text{segm.}\in a\cap b} \Big[\mathrm{Ad}_{A_2^a\otimes A_2^b}\Big(J_2\otimes J_0 - J_0\otimes J_2 + \pi_{WP}^{\alpha_1\beta_2}\sum_{i,j=0}^{2}\eta^{ij}J_i\otimes J_j\Big)$$
$$+ \mathrm{Ad}_{A_{s-1}^a\otimes A_{s-1}^b}\Big(J_0\otimes J_2 - J_2\otimes J_0 + \pi_{WP}^{\alpha_{s-1}\beta_s}\sum_{i,j=0}^{2}\eta^{ij}J_i\otimes J_j\Big)$$
$$+ \sum_{k=2}^{s-2}\mathrm{Ad}_{A_k^a\otimes A_k^b}\Big(J_1\otimes J_2 - J_2\otimes J_1 + \pi^{\alpha_k\beta_{k+1}}(J_0\otimes J_1 - J_1\otimes J_0)\Big)\Big].$$

A simple computation based on the relations

$$[P_k, J_0] = J_1P_k + \frac{1}{2}(J_0 - \pi^{\alpha_k\alpha_{k+1}}J_2)$$

$$[P_k, \pi^{\alpha_k\alpha_{k+1}}J_2] = J_1P_k + \frac{1}{2}(J_0 - \pi^{\alpha_k\alpha_{k+1}}J_2)$$

yields the identities

$$\mathrm{Ad}_{A_{k+1}^a\otimes A_{k+1}^b}\Big(J_2\otimes J_0 - J_0\otimes J_2\Big) - \mathrm{Ad}_{A_k^a\otimes A_k^b}\Big(J_2\otimes J_0 - J_0\otimes J_2\Big)$$
$$= \mathrm{Ad}_{A_k^a\otimes A_k^b}\Big(J_1\otimes J_2 - J_2\otimes J_1 + \pi^{\alpha_k\beta_{k+1}}(J_0\otimes J_1 - J_1\otimes J_0)\Big)$$
$$\mathrm{Ad}_{A_{k+1}^a\otimes A_{k+1}^b}\Big(\sum_{i,j=0}^{2}\eta^{ij}J_i\otimes J_j\Big) = \mathrm{Ad}_{A_k^a\otimes A_k^b}\Big(\sum_{i,j=0}^{2}\eta^{ij}J_i\otimes J_j\Big).$$

These identities allow one to rewrite the bivector (25) as

$$\sum_{\alpha_k\in a,\beta_l\in b} \pi_{WP}^{\alpha_k\beta_l} J_k^a \otimes J_l^b = \sum_{\text{segm.}\in a\cap b}(\pi_{WP}^{\alpha_1\beta_2} + \pi_{WP}^{\alpha_{s-1}\beta_s})\mathrm{Ad}_{A_{s-1}^a\otimes A_{s-1}^b}\Big(\sum_{i,j=0}^{2}\eta^{ij}J_i\otimes J_j\Big),$$

and the associated Poisson bracket (24) takes the form

$$\{f_a, g_b\}_\Lambda = \sum_{\text{segm.}\in a\cap b}(\pi_{WP}^{\alpha_1\beta_2} + \pi_{WP}^{\alpha_{s-1}\beta_s})\mathrm{Im}_\ell\Big(\sum_{i,j=0}^{2}\eta^{ij}\kappa(F_{a_p}, J_i)\kappa(G_{b_p}, J_j)\Big).$$

It remains to relate the factor $\pi_{WP}^{\alpha_1\beta_2} + \pi_{WP}^{\alpha_{s-1}\beta_s}$ for each edge path segment in $a\cap b$ to the oriented intersection numbers of each essential intersection point. For this, note that when $\pi_{WP}^{\alpha_1\beta_2} + \pi_{WP}^{\alpha_{s-1}\beta_s} = 0$, the edge path segments $(\alpha_1, \dots, \alpha_s)$ and $(\beta_1, \dots, \beta_s)$ do not give rise to an essential intersection point, as shown in Figure 5 (a). On the other hand, when $\pi_{WP}^{\alpha_1\beta_2} = \pi_{WP}^{\alpha_{s-1}\beta_s}$, the edge path segments $(\alpha_1, \dots, \alpha_s)$ and $(\beta_1, \dots, \beta_s)$

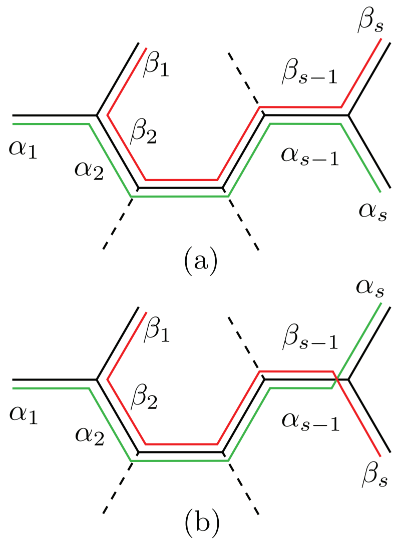

Figure 5. Edge path segments contributing to the Poisson bracket.

correspond to exactly one essential intersection point p, as shown in Figure 5(b). This proves the identity

$$\pi_{WP}^{\alpha_1\beta_2} + \pi_{WP}^{\alpha_{s-1}\beta_s} = 2\epsilon_p(a,b),$$

and shows that the bivectors (25) and (26) agree. q.e.d.

Note that the proof of Theorem 4.4 is largely combinatorial. It only makes use of properties of the Weil–Petersson bivector (5) and of the expression (3) for the holonomies in terms of shear coordinate and the matrices E, L, and R, defined in Section 2. It therefore directly generalizes the proof that the Weil–Petersson bivector (5) induces the Weil–Petersson symplectic structure on Teichmüller space. In fact, this proof is obtained directly from the proof of Theorem 4.4 by replacing R_Λ^E with $\mathbb{R}^E$, replacing G_Λ with $\mathrm{PSL}(2,\mathbb{R})$, and omitting the expressions Im_ℓ and Re_ℓ throughout the proof.

As a final remark on the symplectic structures on the moduli spaces $\mathcal{GH}_\Lambda(M)$, we consider Goldman's symplectic structure for the groups $G = G_\Lambda$ and for a general real linear combination $B = \mu(\,,\,) + \nu\langle\,,\,\rangle$ of the Ad-invariant symmetric bilinear forms $(\,,\,)$ and $\langle\,,\,\rangle$ in (10). A short computation shows that the bilinear form B is non-degenerate if and only if $\Lambda\mu^2 + \nu^2 \neq 0$. While this condition is satisfied for all non-vanishing linear combinations if $\Lambda = 1$, it is violated if $\mu = \pm\nu$ and $\Lambda = -1$ or $\nu = \Lambda = 0$. This shows that Goldman's symplectic structure (12) is well defined for all values of Λ if $\mu = 0$ and $\nu = 1$, which is the case considered above. In contrast, the choice $\mu = 1$ and $\nu = 0$ yields $B = (\,,\,) = \mathrm{Re}_\ell(\kappa)$ and is defined only for $\Lambda \neq 0$. In this case, expression

(13) becomes

$$B(F_{a_p}(\rho), G_{b_p}(\rho)) = \frac{1}{2\Lambda} \sum_{i,j=0}^{2} \kappa^{ij} \Big((F_{a_p}, J_i)(G_{b_p}, J_j) + (F_{a_p}, P_i)(G_{b_p}, P_j) \Big).$$

A direct computation along the lines of the proof of Theorem 4.4 shows that this corresponds to the choice of the following Poisson bivector on R^E_Λ:

$$\pi = \frac{1}{2} \sum_{\alpha \in E(\Gamma)} \pi^{\alpha\beta}_{WP} \left(\frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta} + \frac{1}{\Lambda} \frac{\partial}{\partial y^\alpha} \wedge \frac{\partial}{\partial y^\beta} \right).$$

This provides an additional motivation for the choice of the symmetric bilinear form $\langle\, , \,\rangle = \mathrm{Im}_\ell(\kappa)$ on $\mathfrak{g}_\Lambda$ when defining the symplectic structure on $\mathcal{GH}_\Lambda(M)$. Besides the physical considerations discussed in Section 3.2—in particular, Theorem 3.3—this choice of bilinear form is the only one that is non-degenerate for all values $\Lambda \in \mathbb{R}$. This allows one to interpret the cosmological constant Λ as a deformation parameter in the description of the symplectic structure on the moduli spaces $\mathcal{GH}_\Lambda(M)$.

Geometrical interpretation in terms of earthquake and grafting. To give a geometrical interpretation to the gravitational symplectic structure, it is instructive to determine the transformation of the the edge coordinates z^α generated via Poisson brackets by the traces of the G_Λ-valued holonomies. Thus, let $(\alpha_1, \ldots, \alpha_n)$ be a closed edge path freely homotopic to $a \in \pi_1(S)$, and denote by $\rho(a)$ the associated holonomy defined as in (20) and interpreted as a G_Λ-valued function over $\mathcal{GH}_\Lambda(M)$. Using the identities

$$\ell \frac{\partial \rho(a)}{\partial x^{\alpha_k}} = \frac{\partial \rho(a)}{\partial y^{\alpha_k}} = \ell J^a_k \rho(a), \tag{27}$$

which follow directly from (20) and (23), one obtains the expressions for the Poisson bracket between the real and imaginary parts of the holonomies and the shear-bending coordinates

$$\{\mathrm{Re}_\ell \mathrm{Tr}\rho(a), z^\beta\} = \frac{1}{2} \sum_{k=1}^{n} \pi^{\alpha_k\beta}_{WP}\, \mathrm{Tr}(\ell J^a_k \rho(a)),$$

$$\{\mathrm{Im}_\ell \mathrm{Tr}\rho(a), z^\beta\} = \frac{1}{2} \sum_{k=1}^{n} \pi^{\alpha_k\beta}_{WP}\, \mathrm{Tr}(J^a_k \rho(a)).$$

This shows that the flows generated by, respectively, $\mathrm{Re}_\ell \mathrm{Tr}\rho(a)$ and $\mathrm{Im}_\ell \mathrm{Tr}\rho(a)$, take the form

$$\Phi^{Re}_t(z^\beta) = z^\beta + \frac{\ell t}{2} \sum_{k=1}^{n} \pi^{\alpha_k\beta}_{WP} \mathrm{Tr}(J^a_k \rho(a)) + O(t^2),$$

$$\Phi_t^{Im}(z^\beta) = z^\beta + \frac{t}{2}\sum_{k=1}^{n} \pi_{WP}^{\alpha_k\beta}\mathrm{Tr}(J_k^a\rho(a)) + O(t^2).$$

Computing the transformation of the holonomies (20) generated by these Hamiltonian functions then shows that they are related by shear-bending cocycles along a with measures ℓt and t, respectively. The transformations generated by the real and imaginary part of $\mathrm{Tr}\rho(a)$ can therefore be interpreted in terms of grafting and earthquake along the unique geodesic homotopic to a.

The description in terms of shear-bending coordinates thus generalizes the result in [**31**], where it was shown that grafting and earthquake transformations along closed simple geodesics are generated via the Atiyah-Bott-Goldman symplectic structure (12) by the real and imaginary part of the traces of the associated holonomy. This can also be viewed as a generalization of the well-known fact that the earthquake map in Teichmüller space along a closed simple geodesic $a \in \pi_1(S)$ is generated by the geodesic length $l(a)$, which is given by the trace of the associated holonomy $\rho(a) \in \mathrm{PSL}(2,\mathbb{R})$.

4.3. Cotangent bundle over Teichmüller space.

Shear coordinates description. To relate the symplectic structure on $\mathcal{GH}_\Lambda(M)$ to the cotangent bundle structure on $T^*\mathcal{T}(S)$, we describe the latter as a constrained submanifold of $T^*\mathbb{R}^E$ with the symplectic structure induced by symplectic reduction. For this, note that the global parametrization of Teichmüller space $\mathcal{T}(S)$ by shear coordinates also provides a global parametrization of its tangent and cotangent bundles. Given a trivalent fat graph Γ dual to an ideal triangulation S, we may describe the tangent and cotangent bundles over $\mathcal{T}(S)$ in terms of the coordinate vector fields $\partial/\partial x^\alpha$ and coordinate 1-forms dx^α on $\mathbb{R}^E$. A tangent vector $\xi = \sum_\alpha \xi^\alpha\, \partial/\partial x^\alpha$ to $\mathbb{R}^E$ at a point in $\mathrm{Ker}\, c$ determines a tangent vector to $\mathcal{T}(S)$ if and only if its coefficient functions $\xi^\alpha \in C^\infty(\mathbb{R}^E)$ satisfy the constraints (4)

$$c^i(\xi) = \sum_{\alpha\in E(\Gamma)} \theta^i{}_\alpha \xi^\alpha = 0,$$

for every face $i \in F(\Gamma)$. By duality, 1-forms on $\mathcal{T}(S)$ correspond to equivalence classes of 1-forms on $\mathbb{R}^E$ modulo translations by linear combinations of the differentials of the constraints

$$p = \sum_{\alpha\in E(\Gamma)} p_\alpha dx^\alpha \quad \text{with} \quad p \sim p + \sum_{\alpha\in E(\Gamma)}\sum_{i\in F(\Gamma)} p_i\theta^i_\alpha\, dx^\alpha. \tag{28}$$

This implies that the cotangent bundle $T^*\mathcal{T}(S) \cong T^*\mathrm{Ker}\, c$ is given as the direct product of $\mathrm{Ker}\, c \subset \mathbb{R}^E$, which parametrizes the base space, and of the quotient $(\mathbb{R}^E)^*/\mathrm{Ann}(\mathrm{Ker}\, c)$, which parametrizes its fibers in terms of equivalence classes of 1-forms on $\mathbb{R}^E$. Here and in the following, $\mathrm{Ann}(\mathrm{Ker}\, c) = \mathrm{Span}\{c^1, \ldots, c^F\}$ denotes the annihilator subspace of $\mathrm{Ker}\, c \subset \mathbb{R}^E$.

From the viewpoint of constrained mechanical systems (see, for instance, [**20**]), this quotient can be interpreted as a gauge freedom in the definition of coordinates on the fibers of $T^*\mathcal{T}(S)$, which may be then eliminated via an appropriate gauge-fixing condition. This is convenient since it allows one to describe the cotangent bundle $T^*\mathcal{T}(S)$ as a constrained submanifold of $T^*\mathbb{R}^E$. Thus, note that the constraints $c^i : \mathbb{R}^E \to \mathbb{R}$ in (4) are first-class with respect to the cotangent bundle symplectic structure defined by the Poisson bivector

$$\pi_{T^*} = \sum_{\alpha \in E(\Gamma)} \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial p_\alpha}. \tag{29}$$

This means the constraints satisfy the relations $\{c^i, c^j\}_{T^*} = dc^i \otimes dc^j (\pi_{T^*}) = 0$ for all $i, j \in F(\Gamma)$ and that the equivalence relation in (28) is generated by these constraints via the cotangent bundle Poisson bracket $\sum_i p_i \theta^i_\alpha = \{\sum_i p_i c^i, p_\alpha\}_{T^*}$. This allows one to interpret the equivalence classes of 1-forms in (28) as gauge orbits generated by the constraint $c : \mathbb{R}^E \to \mathbb{R}^F$.

To fix the arbitrary parameters p_i, we may then impose gauge fixing conditions, which can be chosen as linear constraints on the coordinates p_α:

$$\tilde{c} : (\mathbb{R}^E)^* \to (\mathbb{R}^F)^*, \qquad \tilde{c}_i(p) = \sum_{\alpha \in E(\Gamma)} \tilde{\theta}_i{}^\alpha p_\alpha = 0.$$

The condition that $\mathrm{Ker}\, \tilde{c}$ contains exactly one representative in each equivalence class is equivalent to the invertibility of the matrix $M \in \mathrm{Mat}(F, \mathbb{R})$ with entries

$$M_i{}^j = \{\tilde{c}_i, c^j\}_{T^*} = \sum_{\alpha \in E(\Gamma)} \tilde{\theta}^\alpha_i \theta^j_\alpha. \tag{30}$$

Note that there is a particularly natural choice for $\tilde{c}$ given by $\tilde{\theta} = \theta^T$, the transpose of the matrix θ in (4). In the following, however, we consider more general gauge fixings that satisfy this condition and refer to such gauge fixing conditions as admissible gauge fixings. We thus have the following statement.

Proposition 4.5. *The cotangent bundle $T^*\mathcal{T}(S)$ is isomorphic to the quotient*

$$T^*\mathcal{T}(S) \cong \mathrm{Ker} c \times \frac{(\mathbb{R}^E)^*}{\mathrm{Ann}(\mathrm{Ker} c)}, \tag{31}$$

and for any admissible gauge-fixing map $\tilde{c} : (\mathbb{R}^E)^* \to (\mathbb{R}^F)^*$, the coordinate functions $x^\alpha, p_\alpha : T^*\mathcal{T}(S) \to \mathbb{R}$ define an embedding (x, p) :

$T^*\mathcal{T}(S) \hookrightarrow \mathbb{R}^E \times (\mathbb{R}^E)^*$ whose image agrees with the kernel of the linear map $c \oplus \tilde{c} : \mathbb{R}^E \times (\mathbb{R}^E)^* \to \mathbb{R}^F \times (\mathbb{R}^F)^*$.

We now show that the cotangent bundle symplectic structure on $T^*\mathbb{R}^E \cong \mathbb{R}^E \times (\mathbb{R}^E)^*$ induces the cotangent bundle symplectic structure on $T^*\mathcal{T}(S)$. In terms of the quotient (31), this follows from symplectic reduction of $\mathbb{R}^E \times (\mathbb{R}^E)^*$ with respect to the constraint $c : \mathbb{R}^E \to \mathbb{R}^F$ and means that (31) inherits a symplectic structure that coincides with the cotangent bundle symplectic structure on $T^*\mathcal{T}(S)$. Equivalently, in terms of gauge fixing, the proof amounts to the construction of the Dirac bracket on $\mathbb{R}^E \times (\mathbb{R}^E)^*$. This is a Poisson structure on $\mathbb{R}^E \times (\mathbb{R}^E)^*$ for which all components of the constraint $c : \mathbb{R}^E \to \mathbb{R}^F$ and the gauge-fixing conditions $\tilde{c} : (\mathbb{R}^E)^* \to (\mathbb{R}^F)^*$ are Casimir functions and which coincides with the original Poisson structure (29) for all functions that Poisson-commute with the constraints and gauge-fixing conditions.

Proposition 4.6.

1) *The cotangent bundle symplectic structure on $\mathbb{R}^E \times (\mathbb{R}^E)^*$ induces a symplectic structure on the quotient* (31) *that coincides with the cotangent symplectic structure on $T^*\mathcal{T}(S) \cong T^*\mathrm{Ker}\, c$.*
2) *For any admissible gauge fixing $\tilde{c} : (\mathbb{R}^E)^* \to (\mathbb{R}^F)^*$, the linear constraint $c \oplus \tilde{c}$ is second-class with respect to the cotangent bundle symplectic structure on $\mathbb{R}^E \times (\mathbb{R}^E)^*$, and the associated Dirac bracket induces the cotangent bundle symplectic structure on $T^*\mathcal{T}(S) \cong \mathrm{Ker}(c \oplus \tilde{c})$.*

Proof. The first point is a direct consequence of the theory of linear symplectic reduction; for an accessible overview, see [**28**]. The linear subspace $\mathrm{Ker} c \times (\mathbb{R}^E)^* \subset \mathbb{R}^E \times (\mathbb{R}^E)^*$ is coisotropic with respect to the cotangent bundle symplectic structure (29) with symplectic complement $\mathrm{Ann}(\mathrm{Ker} c) = \mathrm{Span}\{c^1, \ldots, c^F\} \subset (\mathbb{R}^E)^*$. Hence, the associated symplectic quotient is given by (31), and it is immediate from the discussion above that the induced symplectic structure is the cotangent bundle symplectic structure on $T^*\mathcal{T}(S)$.

For the second point, a direct computation shows that the Dirac matrix for the constraint function $c \oplus \tilde{c}$ in Theorem 4.5 takes the form

$$D = (\{(c \oplus \tilde{c})_i, (c \oplus \tilde{c})_j\}_{T^*}) = \begin{pmatrix} 0 & -M^T \\ M & 0 \end{pmatrix}, \tag{32}$$

with M given by (30). If the gauge-fixing condition $\tilde{c}$ is admissible, the matrix M is invertible, which implies that D is invertible and the constraint function $c \oplus \tilde{c}$ is second-class. This defines a Poisson structure on $\mathbb{R}^E \times (\mathbb{R}^E)^*$, the Dirac bracket, that is given by

$$\{f, g\}_D = \{f, g\}_{T^*} + \sum_{i,j=1}^{2F} D^{-1}_{ij} \{f, (c \oplus \tilde{c})_i\}_{T^*} \{g, (c \oplus \tilde{c})_j\}_{T^*},$$

for every $f, g \in C^\infty(\mathbb{R}^E \times (\mathbb{R}^E)^*)$. All constraint components $(c \oplus \tilde{c})_i : \mathbb{R}^E \times (\mathbb{R}^E)^* \to \mathbb{R}$ are Casimir functions for this Poisson structure. It induces a symplectic structure on $\mathrm{Ker}(c \oplus \tilde{c}) \cong T^*\mathcal{T}(S)$ that coincides with the symplectic structure (29) if f or g Poisson commute with all constraint functions $(c \oplus \tilde{c})_i$. Moreover, it is easy to see from the block diagonal form of the Dirac matrix that this symplectic structure coincides with the cotangent bundle symplectic structure on $T^*\mathcal{T}(S)$. q.e.d.

Symplectomorphisms between $T^*\mathcal{T}(S)$ and $\mathcal{GH}_\Lambda(M)$. For all values of the cosmological constant Λ, the realization of the cotangent bundle $T^*\mathcal{T}(S)$ given in Proposition 4.6 allows one to relate the moduli spaces $\mathcal{GH}_\Lambda(M)$ of three-dimensional MGH Einstein spacetimes with their gravitational symplectic structures to the cotangent bundle $T^*\mathcal{T}(S)$ with the cotangent bundle structure. From a physics perspective, this is motivated by another formulation of 3d gravity as a Hamiltonian system on Teichmüller space [**32**, **25**] and is quite natural mathematically in view of the common parametrization of the gravitational moduli spaces by measured laminations [**30**, **40**] related to the so-called canonical Wick rotation-rescaling theory [**8**]. See also [**41**] for a description of the symplectomorphism $T^*\mathcal{T}(S) \to \mathcal{GH}_{-1}(M)$ in the more general context of universal Teichmüller theory, and [**42**] for a geometric description of the symplectic properties of Wick rotations between the moduli spaces of three-dimensional geometric structures in relation to earthquakes and harmonic maps between surfaces.

To construct a symplectomorphism between $T^*\mathcal{T}(S)$ and $\mathcal{GH}_\Lambda(M)$ in terms of shear-bending coordinates, note that the expression (22) for the gravitational Poisson bivector π_Λ on R^E_Λ can be readily related to the Poisson bivector π_{T^*} given in (29). Thus, we consider the following map between $T^*\mathbb{R}^E$ and R^E_Λ:

$$\begin{aligned} \pi^\sharp_{WP} : \mathbb{R}^E \times (\mathbb{R}^E)^* &\to \mathbb{R}^E \times \mathbb{R}^E, \\ (x^\alpha, p_\alpha) &\mapsto (x^\alpha, y^\alpha) = (x^\alpha, \Sigma_\beta \pi^{\alpha\beta}_{WP} p_\beta). \end{aligned} \tag{33}$$

It is easy to see that this map is a Poisson map up to a multiplicative constant

$$\begin{aligned} (\pi^\sharp_{WP})_* \pi_{T^*} &= \sum_{\alpha \in E(\Gamma)} (\pi^\sharp_{WP})_* \frac{\partial}{\partial x^\alpha} \wedge (\pi^\sharp_{WP})_* \frac{\partial}{\partial p_\alpha} \\ &= -\sum_{\alpha \in E(\Gamma)} \pi^{\alpha\beta}_{WP} \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial y_\beta} = -2\pi_\Lambda, \end{aligned}$$

and, together with Proposition 4.6, that it descends to a symplectomorphism between $T^*\mathcal{T}(S)$ and $\mathcal{GH}_\Lambda(M)$.

Theorem 4.7. *The linear Poisson map $\pi^\sharp_{WP} : T^*\mathbb{R}^E \to R^E_\Lambda$ defined in* (33) *induces a symplectomorphism $\pi^\sharp_{WP} : T^*\mathcal{T}(S) \to \mathcal{GH}_\Lambda(M)$.*

Proof. By Propositions 4.5 and 4.6, the cotangent bundle $T^*\mathcal{T}(S)$ can be identified with the symplectic quotient (31). By Theorem 4.4, the moduli space $\mathcal{GH}_\Lambda(M)$ is given by the restriction of the Poisson structure (22) to the kernel constraint $c_\Lambda : R_\Lambda^E \to R_\Lambda^F$ in (21). It is therefore sufficient to show that the linear map $\pi^\sharp_{WP}$ annihilates the linear subspace $\mathrm{Ann}(\mathrm{Ker} c) = \mathrm{Span}\{c^1, \ldots, c^F\} \subset (\mathbb{R}^E)^*$ and maps the linear subspace $\mathrm{Ker} c \times (\mathbb{R}^E)^* \subset \mathbb{R}^E \times (\mathbb{R}^E)^*$ to $\mathrm{Ker} c_\Lambda \subset \mathbb{R}^E_\Lambda$.

Both statements follow directly from the fact that the constraints on the shear coordinates for each face $i \in F(\Gamma)$ are Casimir functions for the Weil–Petersson Poisson bivector; see Theorem 2.2 and the preceding discussion. This proves that the map $\pi^\sharp_{WP}$ descends to a symplectomorphism $\pi^\sharp_{WP} : T^*\mathcal{T}(S) \to \mathcal{GH}_\Lambda(M)$. q.e.d.

It is interesting to compare these results to the recent work by Loustau [**26**, **27**], which extends the earlier work by Kawai [**23**]. These works relate the symplectic structure of the space of complex projective structures on a surface S of genus $g \geq 2$ to Goldman's symplectic structure (13) on the representation variety $\mathrm{Rep}_\mathcal{C}(S, \mathrm{SL}(2,\mathbb{C}))$, to the cotangent bundle $T^*\mathcal{T}(S)$ and to Taubes' space of hyperbolic germs. In particular, it is shown in [**27**] that the canonical symplectic structure on the latter coincides with the imaginary part of Goldman's symplectic form on the space of almost Fuchsian structures, and in [**26**] that complex Fenchel–Nielsen coordinates are Darboux coordinates for the canonical symplectic form on the space of quasi-Fuchsian structures. It seems plausible that by passing from generalized shear coordinates to complex Fenchel–Nielsen coordinates, one could relate these results explicitly and, possibly, extend the description in Fenchel–Nielsen coordinates to the G_Λ representation varieties for $\Lambda = 0, -1$.

5. Mapping class group actions

In this section, we investigate the mapping class group action on the moduli spaces $\mathcal{GH}_\Lambda(M)$. We start by describing the action of $\mathrm{Mod}(S)$ on the space of measured geodesic laminations and its R_Λ-extensions, making use of formula (16) and the Whitehead moves (7) for shear coordinates on $\mathcal{T}(S)$. We then show that the Whitehead moves take a particularly simple form in shear-bending coordinates on $\mathcal{GH}_\Lambda(M)$, which can be viewed as an analytic continuation of formula (7). We then prove that these Whitehead moves induce three different actions of the mapping class group on the cotangent bundle $T^*\mathcal{T}(S)$ of Teichmüller space, corresponding to the different values of Λ. Finally, we prove that all these mapping class group actions are symplectic, making use of simple decomposition of the Whitehead move transformation into a non-linear term generated by the Poisson structure and a linear term that implements the combinatorial transformation of the Poisson structure under the Whitehead move.

5.1. Mapping class group action on $\mathcal{GH}_\Lambda(M)$.

Whitehead moves for measured laminations. We now describe the action of the mapping class group $\mathrm{Mod}(S)$ on $\mathcal{ML}(S)$ in terms of the parametrization (16) by differences of shear coordinates on $\mathcal{T}(S)$. More generally, we look at the coordinates w^α on $\mathcal{ML}^\Lambda(S)$ obtained from (16) via analytic continuation. Similarly to the derivation in Subsection 2.3 of the shear coordinate expression (7) for the mapping class group action on Teichmüller space, we extend the action on $\mathcal{ML}^\Lambda(S)$ to $\mathbb{R}^E \times \mathbb{R}^E$ by equivariance:

$$(x'^\alpha, w'^\alpha) = \varphi(x^\alpha, w^\alpha) = (x^\alpha, w^\alpha) \circ \varphi^*.$$

Here, x^α and w^α are the shear coordinates on the base Teichmüller space and on the fibers of R_Λ-valued measured laminations associated with an embedded trivalent fat graph Γ on S, and x'^α and w'^α are the corresponding coordinates associated with $\Gamma' = \varphi(\Gamma)$ for $\varphi \in \mathrm{Mod}(S)$.

For a Whitehead move along an edge $\alpha \in E(\Gamma)$, it is easy to compute the coordinate transformation, as the coordinates w^α are analytic extensions of differences of coordinates x^α. This yields

$$W_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = -x^\alpha \\ x^{\beta,\delta} \mapsto x'^{\beta,\delta} = x^{\beta,\delta} + \log(1 + e^{x^\alpha}) \\ x^{\gamma,\epsilon} \mapsto x'^{\gamma,\epsilon} = x^{\gamma,\epsilon} - \log(1 + e^{-x^\alpha}) \\ w^\alpha \mapsto w'^\alpha = -w^\alpha \\ w^{\beta,\delta} \mapsto w'^{\beta,\delta} = w^{\beta,\delta} + \log\left(\frac{1+e^{x^\alpha+w^\alpha}}{1+e^{x^\alpha}}\right) \\ w^{\gamma,\epsilon} \mapsto w'^{\gamma,\epsilon} = w^{\gamma,\epsilon} - \log\left(\frac{1+e^{-x^\alpha-w^\alpha}}{1+e^{-x^\alpha}}\right), \end{cases} \tag{34}$$

while the coordinates of all other edges are preserved. A direct computation, which again makes use of the definition of the coordinates w^α as an analytic continuation of differences of shear coordinates, shows that the Whitehead moves (34) satisfy all relations of Theorem 2.3 and also preserve the analytic continuation of the constraints (17). This proves the following proposition.

Proposition 5.1. *The transformations* (34) *induce an action of the mapping class group* $\mathrm{Mod}(S)$ *on the bundle* $\mathcal{ML}(S)$ *of measured geodesic laminations on* S.

The mapping class group action in shear-bending coordinate. We now consider the moduli spaces of MGH Einstein spacetimes. The definition (19) of the shear-bending coordinates on $\mathcal{GH}_\Lambda(M)$ now allows one to derive their transformation under Whitehead moves directly from the action (34) by adding the terms in (34) for the real and imaginary part of the coordinates z^α:

$$W_\alpha^\Lambda : \begin{cases} z^\alpha \mapsto z'^\alpha = -z^\alpha \\ z^{\beta,\delta} \mapsto z'^{\beta,\delta} = z^{\beta,\delta} + \log(1+e^{z^\alpha}) \\ z^{\gamma,\epsilon} \mapsto z'^{\gamma,\epsilon} = z^{\gamma,\epsilon} - \log(1+e^{-z^\alpha}). \end{cases} \tag{35}$$

Note that for $\Lambda = 1$ the logarithms are not well defined, due to the presence of branching points. These, however, does not affect the holonomies where only the exponentials of shear coordinates appear.

Clearly, the transformation (35) is simply the analytic continuation of the associated Whitehead move $W_\alpha : \mathbb{R}^E \to \mathbb{R}^E$ in (7). As an immediate generalization of Theorem 2.4, we thus have the following result.

Theorem 5.2. *The Whitehead moves $W_\alpha^\Lambda : R_\Lambda^E \to R_\Lambda^E$ satisfy the relations of Theorem* 2.3*. Furthermore, they preserve the constraints* (21) *and induce an action of the mapping class group* $\mathrm{Mod}(S)$ *on* $\mathcal{GH}_\Lambda(M)$*.*

Proof. This follows immediately from Theorem 2.4 as the Whitehead moves (35) and the constraints (21) are the analytic continuation of (7) and (4). The computation proving the pentagon identity for Whitehead moves (7) is given in Appendix B for the convenience of the reader. The other relations of Theorem 2.3 are easily verified. q.e.d.

Mapping class group actions on $T^*\mathcal{T}(S)$. In view of the symplectomorphism between $T^*\mathcal{T}(S)$ and $\mathcal{GH}_\Lambda(M)$ obtained in Theorem 4.7, it is natural to study the mapping class group action on $T^*\mathcal{T}(S)$ induced by (35) via pull-back. We will now show that the induced actions on $T^*\mathcal{T}(S)$ are all symplectic but are all distinct for different signs of the curvature Λ. This should have interesting consequences for the quantum theory as it provides a common representation for the corresponding quantum operators in terms of the Weyl algebra such that the algebra of quantum symmetries, e.g., the quantum mapping class group action, is the only distinguishing feature between the theories for different values of Λ. This is further evidence for the importance of the mapping class group in the quantization.

We start by computing the pull-back of the Whitehead move $W_\alpha^\Lambda : R_\Lambda^E \to R_\Lambda^E$ (35) with the linear Poisson map $\pi_{WP}^\sharp : T^*\mathbb{R}^E \to R_\Lambda^E$ in (33):

$$W_\alpha^\Lambda \circ \pi_{WP}^\sharp : \begin{cases} (x^\alpha, p_\alpha) \mapsto z'^\alpha = -x^\alpha - \ell\pi_{WP}^{\alpha\zeta} p_\zeta \\ (x^{\beta,\delta}, p_{\beta,\delta}) \mapsto z'^{\beta,\delta} = x^{\beta,\delta} + \ell\pi_{WP}^{\beta,\delta\zeta} p_\zeta + \log\left(1 + e^{x^\alpha + \ell\pi_{WP}^{\alpha\zeta} p_\zeta}\right) \\ (x^{\gamma,\epsilon}, p_{\gamma,\epsilon}) \mapsto z'^{\gamma,\epsilon} = x^{\gamma,\epsilon} + \ell\pi_{WP}^{\gamma,\epsilon\zeta} p_\zeta - \log\left(1 + e^{-x^\alpha - \ell\pi_{WP}^{\alpha\zeta} p_\zeta}\right). \end{cases}$$

Here and in the following, we use Einstein's summation convention and omit the sum over the edge label ζ for better legibility. Comparing this expression with the map $\pi'^\sharp_{WP} : T^*\mathbb{R}^E \to R_\Lambda^E$ defined by the Weil–Petersson bivector π'_{WP} for the fat graph $\Gamma' = \Gamma_\alpha$, it is easy to identify the associated maps $\tilde{W}_\alpha^\Lambda : T^*\mathbb{R}^E \to T^*\mathbb{R}^E$ for which the following diagram commutes:

$$\begin{tikzcd} T^*\mathbb{R}^E \arrow[r, "\pi^\sharp_{WP}"] \arrow[d, "\widetilde{W}_\alpha^\Lambda"'] & R_\Lambda^E \arrow[d, "W_\alpha^\Lambda"] \\ T^*\mathbb{R}^E \arrow[r, "\pi'^\sharp_{WP}"'] & R_\Lambda^E. \end{tikzcd}$$

Note that since $\pi'^\sharp_{WP}$ is not injective, such maps are unique only up to translations by elements of $\operatorname{Ker} \pi'^\sharp_{WP}$. A direct computation shows that the following map satisfies this condition:

$$\tilde{W}_\alpha^\Lambda : \begin{cases} x^\alpha \mapsto x'^\alpha = -x^\alpha \\ x^{\beta,\delta} \mapsto x'^{\beta,\delta} = x^{\beta,\delta} + \mathrm{Re}_\ell \log\left(1 + e^{x^\alpha + \ell\pi_{WP}^{\alpha\zeta} p_\zeta}\right) \\ x^{\gamma,\epsilon} \mapsto x'^{\gamma,\epsilon} = x^{\gamma,\epsilon} - \mathrm{Re}_\ell \log\left(1 + e^{-x^\alpha - \ell\pi_{WP}^{\alpha\zeta} p_\zeta}\right) \\ p_\alpha \mapsto p'_\alpha = -p_\alpha + p_\gamma + p_\epsilon + \mathrm{Im}_\ell \log\left(1 + e^{x^\alpha + \ell\pi_{WP}^{\alpha\zeta} p_\zeta}\right) \\ p_{\beta,\gamma,\delta,\epsilon} \mapsto p'_{\beta,\gamma,\delta,\epsilon} = p_{\beta,\gamma,\delta,\epsilon}. \end{cases} \tag{36}$$

Using this expression for the map $\tilde{W}_\alpha^\Lambda : T^*\mathbb{R}^E \to T^*\mathbb{R}^E$, one can verify its properties by direct computations, which yields the following theorem.

Theorem 5.3. *The Whitehead moves $\widetilde{W}_\alpha^\Lambda : T^*\mathbb{R}^E \to T^*\mathbb{R}^E$ satisfies the relations of Theorem* 2.3*. Furthermore, they preserve the constraints* (4) *and their gauge orbits and induce an action of the mapping class group* $\mathrm{Mod}(S)$ *on* $T^*\mathcal{T}(S)$*.*

Proof. That map $\tilde{W}_\alpha^\Lambda : T^*\mathbb{R}^E \to T^*\mathbb{R}^E$ in (36) satisfy the relations of Theorem 2.3 follows by direct computations. The only non-trivial case is that of the pentagon relation, which is analogous to that for (7); see Appendix B.

That the constraints (4) and their gauge orbits are preserved can be seen from the combinatorial relation between Γ and $\Gamma' = \Gamma_\alpha$. Consider a face $i \in F(\Gamma)$ containing the sequence of edges $(\beta, \alpha, \epsilon)$ and its transformation under a Whitehead move as in Figures 3 and 4. The corresponding face $i \in F(\Gamma')$ then necessarily contains the sequence (β, ϵ). As the constraints are given by a sum over the coordinates of edges in the face, it is clear from (36) that the pull-back of c'^i can be written as

$$\begin{aligned} c'^i \circ \tilde{W}^\Lambda_\alpha &= x^\beta + x^\epsilon + \mathrm{Re}_\ell \log\left(\frac{1+e^{x^\alpha+\ell\pi^{\alpha\zeta}_{WP}p_\zeta}}{1+e^{-x^\alpha-\ell\pi^{\alpha\zeta}_{WP}p_\zeta}}\right)+\cdots \\ &= x^\beta + x^\epsilon + x^\alpha + \cdots = c^i, \end{aligned}$$

where the dots stand for the contribution of the other edges in the face i, which is invariant under $\tilde{W}^\Lambda_\alpha$. On the other hand, the gauge orbits are the orbits of points in $\mathbb{R}^E \times (\mathbb{R}^E)^*$ under the translations of 1-forms by differentials of the constraints as in (28). So consider the transformation of a 1-form $p + qdc^i$ under (36). For the coordinate of the edge α we have

$$\begin{aligned} \tilde{W}^\Lambda_\alpha(p_\alpha + q{\theta^i}_\alpha) &= -p_\alpha - q{\theta^i}_\alpha + p_\gamma + q{\theta^i}_\gamma + p_\epsilon + q{\theta^i}_\epsilon \\ &\qquad + \mathrm{Im}_\ell \log\left(1+e^{x^\alpha+\ell\pi^{\alpha\zeta}p_\zeta+\ell\pi^{\alpha\zeta}q{\theta^i}_\zeta}\right) \\ &= \tilde{W}^\Lambda_\alpha(p_\alpha) + q(-{\theta^i}_\alpha + {\theta^i}_\gamma + {\theta^i}_\epsilon) = \tilde{W}^\Lambda_\alpha(p_\alpha) + q{\theta'^i}_\alpha, \end{aligned}$$

since ${\theta^i}_\alpha = 1, {\theta^i}_\gamma = 0, {\theta^i}_\epsilon = 1$ and ${\theta'^i}_\alpha = 0$. For any other edge, including the neighboring edges of α, we have

$$\tilde{W}^\Lambda_\alpha(p_\eta + q{\theta^i}_\eta) = p_\eta + q{\theta^i}_\eta = \tilde{W}^\Lambda_\alpha(p_\eta) + q{\theta'^i}_\eta,$$

since the multiplicity of η on the face i does not change under a Whitehead move. This shows the gauge orbits of c^i are mapped to the gauge orbits of c'^i or, equivalently,

$$\tilde{W}^\Lambda_\alpha(p + qdc^i) = \tilde{W}^\Lambda_\alpha(p) + qdc'^i.$$

The same arguments can be applied to other (combinations of) edge sequences involved in the Whitehead move. This shows that $c' \circ \tilde{W}^\Lambda_\alpha = c$ and $\tilde{W}^\Lambda_\alpha([p]) = [\tilde{W}^\Lambda_\alpha(p)]$ and completes the proof. q.e.d.

5.2. The mapping class group action is symplectic.

Hamiltonians for the Whitehead move. In this section, we show that the transformations (35) and (36) are Poisson maps with respect to the gravitational and cotangent bundle Poisson structure and that the induced mapping class group actions $\mathcal{GH}_\Lambda(M)$ and $T^*\mathcal{T}(S)$ are all symplectic. This is achieved by decomposing the Whitehead moves into terms generated via the Poisson structure by a Hamiltonian and an additional term, which is linear, purely combinatorial, and independent of Λ. We show that the latter corresponds to the transformation of the Poisson bivector. Such a decomposition of the Whitehead moves is already possible in the Teichmüller context, so the proof presented here can be viewed as a generalization of the corresponding result for shear coordinates.

We therefore start by considering the Whitehead move for shear coordinates on Teichmüller space. Let Γ be an embedded trivalent fat graph on S, and denote by x^α the corresponding shear coordinate for an edge $\alpha \in E(\Gamma)$. For each edge $\alpha \in E(\Gamma)$, consider the Hamiltonian function $H_\alpha : \mathbb{R}^E \to \mathbb{R}$ given by

$$H_\alpha(x) = H(x^\alpha) = \frac{(x^\alpha)^2}{4} + \mathrm{Li}_2(-e^{x^\alpha}), \tag{37}$$

where Li_2 denotes Euler's dilogarithm. A short computation using the combinatorial structure of π_{WP} and the expression for the derivative of H_α

$$\frac{\partial H_\alpha}{\partial x^\alpha} = \tfrac{1}{2}x^\alpha - \log(1 + e^{x^\alpha})$$

then shows that the Whitehead move $W_\alpha : \mathbb{R}^E \to \mathbb{R}^E$ in (7) is given by

$$W_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = -x^\alpha \\ x^{\beta,\gamma,\delta,\epsilon} \mapsto x'^{\beta,\gamma,\delta,\epsilon} = x^{\beta,\gamma,\delta,\epsilon} + \frac{1}{2}x^\alpha + \left\{x^{\beta,\gamma,\delta,\epsilon}, H_\alpha\right\}_{WP}. \end{cases}$$

This allows us to decompose it as $W_\alpha = A_\alpha \circ B_\alpha$ with maps $A_\alpha, B_\alpha : \mathbb{R}^E \to \mathbb{R}^E$ defined by

$$A_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = -x^\alpha, \\ x^{\beta,\gamma,\delta,\epsilon} \mapsto x'^{\beta,\gamma,\delta,\epsilon} = x^{\beta,\gamma,\delta,\epsilon} + \frac{1}{2}x^\alpha, \end{cases}$$

$$B_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = x^\alpha, \\ x^{\beta,\gamma,\delta,\epsilon} \mapsto x'^{\beta,\gamma,\delta,\epsilon} = x^{\beta,\gamma,\delta,\epsilon} + \{x^{\beta,\gamma,\delta,\epsilon}, H_\alpha\}_{WP}. \end{cases}$$

It is then immediate that the transformation B_α is a Poisson map $B_\alpha : (\mathbb{R}^E, \pi_{WP}) \to (\mathbb{R}^E, \pi_{WP})$, as it can be interpreted as the Hamiltonian flow generated by the Hamiltonian H_α via the Weil–Petersson Poisson bracket. In contrast, the combinatorial linear transformation $A_\alpha : \mathbb{R}^E \to \mathbb{R}^E$ transforms the Poisson bivector π_{WP} associated with the fat graph Γ into the Poisson bivector π'_{WP} associated with $\Gamma' = \Gamma_\alpha$. This follows by a simple computation of the push-forward of π_{WP} by A_α, which yields $(A_\alpha)_*\pi_{WP} = \pi'_{WP}$. This gives a simple proof that the Whitehead moves (7) indeed induces a symplectic action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$, as stated in Theorem 2.4.

Similarly, in the context of the moduli spaces $\mathcal{GH}_\Lambda(M)$, we may decompose the Whitehead moves $W^\Lambda_\alpha : R^E_\Lambda \to R^E_\Lambda$ in (35) using the analytic extension $H^\Lambda_\alpha : R_\Lambda \to R_\Lambda$ of the Hamiltonian function (37). More precisely, for each edge $\alpha \in E(\Gamma)$ consider the real functions $\mathrm{Im}_\ell H^\Lambda_\alpha : R_\Lambda \to \mathbb{R}$ given by the imaginary part of H^Λ_α,

$$\begin{aligned} \mathrm{Im}_\ell\, H^\Lambda_\alpha(z) &= \mathrm{Im}_\ell \left(H^\Lambda(z^\alpha)\right) \\ &= \tfrac{1}{2}x^\alpha y^\alpha + \begin{cases} -y^\alpha \log(1 + e^{x^\alpha}), & \Lambda = 0 \\ \frac{1}{2}\mathrm{Li}_2(-e^{x^\alpha + y^\alpha}) - \frac{1}{2}\mathrm{Li}_2(-e^{x^\alpha - y^\alpha}), & \Lambda = -1 \\ \mathrm{Im}(\mathrm{Li}_2(-e^{x^\alpha + \mathrm{i}y^\alpha})), & \Lambda = 1, \end{cases} \end{aligned}$$

and their derivatives with respect to the real and imaginary parts of z^α

$$\frac{\partial \mathrm{Im}_\ell H^\Lambda_\alpha}{\partial x^\alpha} = \mathrm{Im}_\ell\Big(\tfrac{1}{2}z^\alpha - \log(1+e^{z^\alpha})\Big),$$

$$\frac{\partial \mathrm{Im}_\ell H^\Lambda_\alpha}{\partial y^\alpha} = \mathrm{Re}_\ell\Big(\tfrac{1}{2}z^\alpha - \log(1+e^{z^\alpha})\Big).$$

The Whitehead move $W^\Lambda_\alpha : R^E_\Lambda \to R^E_\Lambda$ can then be written as

$$W^\Lambda_\alpha : \begin{cases} z^\alpha \mapsto z'^\alpha = -z^\alpha, \\ z^{\beta,\gamma,\delta,\epsilon} \mapsto z'^{\beta,\gamma,\delta,\epsilon} = z^{\beta,\gamma,\delta,\epsilon} + \frac{1}{2}z^\alpha + \{z^{\beta,\gamma,\delta,\epsilon}, \mathrm{Im}_\ell H^\Lambda_\alpha\}_\Lambda, \end{cases}$$

and can be decomposed as $W^\Lambda_\alpha = A^\Lambda_\alpha \circ B^\Lambda_\alpha$, with $A^\Lambda_\alpha, B^\Lambda_\alpha : R^E_\Lambda \to R^E_\Lambda$ given by

$$A^\Lambda_\alpha : \begin{cases} z^\alpha \mapsto z'^\alpha = -z^\alpha, \\ z^{\beta,\gamma,\delta,\epsilon} \mapsto z'^{\beta,\gamma,\delta,\epsilon} = z^{\beta,\gamma,\delta,\epsilon} + \frac{1}{2}z^\alpha, \end{cases}$$

$$B^\Lambda_\alpha : \begin{cases} z^\alpha \mapsto z'^\alpha = z^\alpha, \\ z^{\beta,\gamma,\delta,\epsilon} \mapsto z'^{\beta,\gamma,\delta,\epsilon} = z^{\beta,\gamma,\delta,\epsilon} + \{z^{\beta,\gamma,\delta,\epsilon}, \mathrm{Im}_\ell H^\Lambda_\alpha\}_\Lambda. \end{cases}$$

The transformation B^Λ_α is again a Hamiltonian flow—namely, the one generated by $\mathrm{Im}_\ell H_\alpha$ via the gravitational Poisson structure. The combinatorial transformation A^Λ_α sends the gravitational bivector π_Λ for Γ to the gravitational bivector π'_Λ for $\Gamma' = \Gamma_\alpha$, which means $(A^\Lambda_\alpha)_*\pi_\Lambda = \pi'_\Lambda$. When combined with Theorem 5.2, this generalizes Theorem 2.4 from the context of Teichmüller space to the moduli spaces of MGH Einstein spacetimes and proves that the mapping class group action on $\mathcal{GH}_\Lambda(M)$ is symplectic.

Theorem 5.4. *The Whitehead moves $W^\Lambda_\alpha : (R^E_\Lambda, \pi_\Lambda) \to (R^E_\Lambda, \pi'_\Lambda)$ in* (35) *are Poisson maps, and the induced mapping class group action on $\mathcal{GH}_\Lambda(M)$ is symplectic.*

Another benefit of this decomposition of the Whitehead moves is that it has a geometrical interpretation. For this, note that the imaginary part of the complex dilogarithm is closely related to the Bloch–Wigner function $D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z|\arg(1-z)$, which describes the volume of an ideal hyperbolic tetrahedron in three-dimensional hyperbolic space. The volume of such an ideal hyperbolic tetrahedron is given by the Bloch–Wigner function of the cross-ratio of its vertices (see, for instance, [**48**]), and there are similar results for the spherical tetrahedra [**33**]. As a Whitehead move corresponds to gluing an ideal tetrahedron on two adjacent ideal triangles in an ideal triangulation, it is natural that the Hamiltonian generating this transformation is related to the volume of an ideal tetrahedron. In the context of three-dimensional Einstein manifolds, it seems plausible that the Hamiltonian obtained for different values of Λ could be related to the volumes of certain tetrahedra in three-dimensional Minkowski, anti-de Sitter and de Sitter space.

Hamiltonians for the cotangent bundle. We now show that a similar decomposition of the Whitehead move is possible for the mapping class group action (36) on the cotangent bundle $T^*\mathcal{T}(S)$. For this, we construct transformations $\tilde{A}^\Lambda_\alpha, \tilde{B}^\Lambda_\alpha : T^*\mathbb{R}^E \to T^*\mathbb{R}^E$, which are defined uniquely up to translations generated by the constraints by the requirement that the following diagrams commute:

(38)
$$\begin{tikzcd} T^*\mathbb{R}^E \arrow[r, "\pi^\sharp_{WP}"] \arrow[d, "\tilde{A}^\Lambda_\alpha"'] & R^E_\Lambda \arrow[d, "A^\Lambda_\alpha"] \\ T^*\mathbb{R}^E \arrow[r, "\pi'^\sharp_{WP}"'] & R^E_\Lambda \end{tikzcd} \qquad \begin{tikzcd} T^*\mathbb{R}^E \arrow[r, "\pi^\sharp_{WP}"] \arrow[d, "\tilde{B}^\Lambda_\alpha"'] & R^E_\Lambda \arrow[d, "B^\Lambda_\alpha"] \\ T^*\mathbb{R}^E \arrow[r, "\pi^\sharp_{WP}"'] & R^E_\Lambda . \end{tikzcd}$$

For this, we first pull-back A^Λ_α and B^Λ_α via the map $\pi^\sharp_{WP} : T^*\mathbb{R}^E \to R^E_\Lambda$ in (33) and then compare the results with $\pi'^\sharp_{WP}$ and $\pi^\sharp_{WP}$, respectively. A similar computation to that of A^Λ_α and B^Λ_α together with the fact the map $\pi^\sharp_{WP}$ is Poisson shows that these transformations are given by

$$\tilde{A}^\Lambda_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = -x^\alpha \\ x^{\beta,\gamma,\delta,\epsilon} \mapsto x'^{\beta,\gamma,\delta,\epsilon} = x^{\beta,\gamma,\delta,\epsilon} + \frac{1}{2}x^\alpha \\ p_\alpha \mapsto p'_\alpha = -p_\alpha + \frac{1}{2}(p_\beta + p_\gamma + p_\delta + p_\epsilon) \\ p_{\beta,\gamma,\delta,\epsilon} \mapsto p'_{\beta,\gamma,\delta,\epsilon} = p_{\beta,\gamma,\delta,\epsilon}, \end{cases}$$

$$\tilde{B}^\Lambda_\alpha : \begin{cases} x^\alpha \mapsto x'^\alpha = x^\alpha \\ x^{\beta,\gamma,\delta,\epsilon} \mapsto x'^{\beta,\gamma,\delta,\epsilon} = x^{\beta,\gamma,\delta,\epsilon} + \{x^{\beta,\gamma,\delta,\epsilon}, \mathrm{Im}_\ell H^\Lambda_\alpha \circ \pi^\sharp_{WP}\}_{T^*} \\ p_\alpha \mapsto p'_\alpha = p_\alpha + \{p_\alpha, \mathrm{Im}_\ell H^\Lambda_\alpha \circ \pi^\sharp_{WP}\}_{T^*} \\ p_{\beta,\gamma,\delta,\epsilon} \mapsto p'_{\beta,\gamma,\delta,\epsilon} = p_{\beta,\gamma,\delta,\epsilon}. \end{cases}$$

This defines a decomposition of the Whitehead moves in (36) as $\tilde{W}^\Lambda_\alpha = \tilde{A}^\Lambda_\alpha \circ \tilde{B}^\Lambda_\alpha$. The map $\tilde{B}^\Lambda_\alpha$ is again a Hamiltonian flow—namely, the one generated by the Hamiltonian $\mathrm{Im}_\ell H^\Lambda_\alpha \circ \pi^\sharp_{WP} : T^*\mathbb{R}^E \to \mathbb{R}$ via the cotangent bundle Poisson structure. The map $\tilde{A}^\Lambda_\alpha$ is again a linear combinatorial transformation, which does not depend on Λ and maps the cotangent bundle bivector π_{T^*} for the graph Γ to the cotangent bundle bivector π'_{T^*} for the graph $\Gamma' = \Gamma_\alpha$, e.g., $(\tilde{A}^\Lambda_\alpha)_* \pi_{T^*} = \pi'_{T^*}$. This proves the following theorem which, combined with Theorem 5.3, shows that the induced mapping class group actions on $T^*\mathcal{T}(S)$ are symplectic for all values of Λ.

Theorem 5.5. *The Whitehead moves* $\tilde{W}^{\Lambda}_{\alpha} : (T^*\mathbb{R}^E, \pi_{T^*}) \to (T^*\mathbb{R}^E, \pi'_{T^*})$ (36) *are Poisson maps, and the induced mapping class group actions on* $T^*\mathcal{T}(S)$ *are symplectic for all values of* Λ.

Appendix A. Model spacetimes for 3d gravity

In this appendix, we describe the model spacetimes of 3d gravity and their description in terms of the groups $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$. These model spacetimes are three-dimensional Minkowski space M_3 for $\Lambda = 0$, anti-de Sitter space AdS_3 for $\Lambda < 0$, and de Sitter space dS_3 for $\Lambda > 0$. All of these model spacetimes are of constant curvature, which is given by the cosmological constant Λ, and admit a simple description in terms of the Lie group $\mathrm{PSL}(2,\mathbb{R})$ and its complexification $\mathrm{PSL}(2,\mathbb{C})$.

The group $\mathrm{SL}(2,\mathbb{R})$ **and its Lie algebra.** In order to exhibit the similarities between the model spacetimes, it is helpful to first consider the Lie group $\mathrm{SL}(2,\mathbb{R})$ and its Lie algebra. For this, we introduce the following basis of $\mathfrak{sl}(2,\mathbb{R})$,

(39)
$$J_0 = \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad J_1 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_2 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which diagonalizes the Killing form on $\mathfrak{sl}(2,\mathbb{R})$ and relates it to the three-dimensional Minkowski metric $\eta = \mathrm{diag}(-1,1,1)$:

$$\kappa(J_i, J_j) = \mathrm{Tr}(J_i J_j) = \tfrac{1}{2}\eta_{ij}.$$

Minkowski geometry. Three-dimensional Minkowski space M_3 is an affine space over the vector space $\mathbb{R}^3$ with the Lorentzian metric $\eta = \mathrm{diag}(-1,1,1)$. Elements of $(\mathbb{R}^3, \eta)$ can be identified with the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ via the map

$$\phi_0 :\ (x^0, x^1, x^2) \mapsto X = 2x^a J_a = \begin{pmatrix} x^1 & x^2 - x^0 \\ x^2 + x^0 & -x^1 \end{pmatrix},$$

such that the metric is given by minus the determinant:

$$\eta_0(x,x) = -\det X = \tfrac{1}{2}\mathrm{Tr}X^2 - \tfrac{1}{2}(\mathrm{Tr}X)^2.$$

The group of orientation–preserving and time orientation-preserving isometries of M_3 is the Poincaré group in three dimensions $G_0 = \mathrm{ISO}(2,1) \cong \mathrm{PSL}(2,\mathbb{R}) \ltimes \mathfrak{sl}(2,\mathbb{R})$, with the group multiplication

$$(A, X) \cdot (B, Y) = (AB, X + AYA^{-1}),$$

where $A, B \in \mathrm{PSL}(2,\mathbb{R})$, $X, Y \in \mathfrak{sl}(2,\mathbb{R})$. With the identification $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^3$ from above, its action on M_3 is given by

$$(A, X) \cdot Y = AYA^{-1} + X.$$

Two-dimensional hyperbolic space $\mathbb{H}^2$ embeds in Minkowski space as the space of future-oriented timelike unit vectors with the induced metric. In terms of the matrix realization of M_3 this can be described as the subspace of matrices with determinant 1. The subgroup of G_0 that preserves this embedding of $\mathbb{H}^2$ is $\mathrm{PSL}(2,\mathbb{R}) \cong \{(A,0) : A \in \mathrm{PSL}(2,\mathbb{R})\}$.

Anti-de Sitter geometry. Three-dimensional anti-de Sitter space is defined as a quadric in $\mathbb{R}^4$:

$$\mathrm{AdS}_3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 : -(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2 = -1\}$$

with the Lorentzian metric induced by the flat pseudo-Riemannian metric $\mathrm{diag}(-1,1,1,-1)$. Topologically, anti-de Sitter space is a product $\mathbb{S}^1 \times \mathbb{R}^2$ presenting a closed timelike direction. This is somewhat irrelevant for our considerations since one can always unwrap such closed time direction by going to the universal covering space. On the other hand, it is sometimes also convenient to consider certain quotient of anti-de Sitter space by the group $\mathbb{Z}_2$ and to work with its image in three-dimensional projective space:

$$\mathrm{X}_{-1} = \{[x^0 : x^1 : x^2 : x^3] \in \mathbb{RP}^3 : -(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2 = -1\}.$$

As in the case of Minkowski space, three-dimensional anti-de Sitter space can be described in terms of the group $\mathrm{PSL}(2,\mathbb{R})$. The map

$$\phi_{-1} : (x^0, x^1, x^2, x^3) \mapsto X = \begin{pmatrix} x^3 + x^1 & x^2 - x^0 \\ x^2 + x^0 & x^3 - x^1 \end{pmatrix}$$

identifies AdS_3 with the group SL(2,$\mathbb{R}$) and X_{-1} with the group PSL(2,$\mathbb{R}$). In both cases, the metric is the one induced by minus the determinant:

$$\eta_{-1}(x,x) = -\det X = \tfrac{1}{2}\mathrm{Tr}X^2 - \tfrac{1}{2}(\mathrm{Tr}X)^2.$$

The isometry group of AdS_3 is the group SO(2,2)$\cong$SL(2,$\mathbb{R}$)$\times$SL(2,$\mathbb{R}$)$/\mathbb{Z}_2$, whose action on the realization above is given by the following action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$:

$$(A,B) \cdot X = AXB^{-1}.$$

As the kernel of this group action is $\{(\mathbb{1},\mathbb{1}), (-\mathbb{1},-\mathbb{1})\}$, it induces an action of $\mathrm{SO}(2,2)$ on AdS_3. Similarly, the isometry group of X_{-1} is the group $G_{-1} = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

Two-dimensional hyperbolic space $\mathbb{H}^2$ embeds in AdS_3 and in X_{-1} as a totally geodesic surface with the induced metric. In terms of the matrix realization, $\mathbb{H}^2$ can be described as the subspace characterized by the condition $x^3 = 0$. The subgroup of G_{-1} that preserves $\mathbb{H}^2$ is given by the diagonal embedding $\mathrm{PSL}(2,\mathbb{R}) \cong \{(A,A) : A \in \mathrm{PSL}(2,\mathbb{R})\}$.

de Sitter geometry. Three-dimensional de Sitter space is also given as a quadric in $\mathbb{R}^4$:

$$\mathrm{X}_1 = \mathrm{dS}_3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^4;\ -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\},$$

with the Lorentzian metric induced by the flat metric $\mathrm{diag}(-1, 1, 1, 1)$. The map

$$\phi_1 : (x^0, x^1, x^2, x^3) \mapsto \mathrm{i}X = \mathrm{i}\begin{pmatrix} \mathrm{i}x^3 + x^1 & x^2 - x^0 \\ x^2 + x^0 & \mathrm{i}x^3 - x^1 \end{pmatrix}$$

identifies dS_3 with a subset of $\mathrm{SL}(2, \mathbb{C})$. The image of ϕ_1 consists of those matrices of $\mathrm{SL}(2, \mathbb{C})$ that are invariant under the involution

$$(\mathrm{i}X)^\circ = J_0(\mathrm{i}X)^\dagger J_0^{-1}.$$

In this case, the induced metric is given by the determinant

$$\eta_1(x, x) = \det(\mathrm{i}X) = \tfrac{1}{2}\mathrm{Tr}X^2 - \tfrac{1}{2}(\mathrm{Tr}X)^2.$$

The associated isometry group is the Lorentz group in four dimensions $G_1 = \mathrm{SO}(3, 1) \cong \mathrm{PSL}(2, \mathbb{C})$, which acts on the matrix realization of dS_3 via

$$A \cdot (\mathrm{i}X) = A(\mathrm{i}X)A^\circ.$$

Although the two-dimensional hyperbolic space $\mathbb{H}^2$ also embeds in de Sitter space as a totally geodesic surface, the relevant embedding of $\mathbb{H}^2$ is the one in the dual hyperbolic 3-space. The duality correspondence between X_1 and $\mathbb{H}^3$ is given via the duality correspondence between one-dimensional and three-dimensional hyperplanes through the origin in $\mathbb{R}^{3,1}$. It maps points, geodesics, and geodesic planes in X_1, respectively, to geodesic planes, geodesics, and points in $\mathbb{H}^3$. In terms of the matrix realization above, this embedding of $\mathbb{H}^2$ can be described as the dual plane to the point $x^3 = 1$. The subgroup of G_1 that preserves $\mathbb{H}^2$ is then the subgroup $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{PSL}(2, \mathbb{C})$.

Appendix B. The pentagon relation for the Whitehead moves

Theorem B.1. *The Whitehead moves* (35) *satisfies the pentagon relation.*

Proof. This follows by a direct computation from the expression (35) for the Whitehead moves in terms of shear-bending coordinates $z^\alpha : \mathcal{GH}_\Lambda(M) \to R_\Lambda$. Explicitly, the transformation of coordinates under

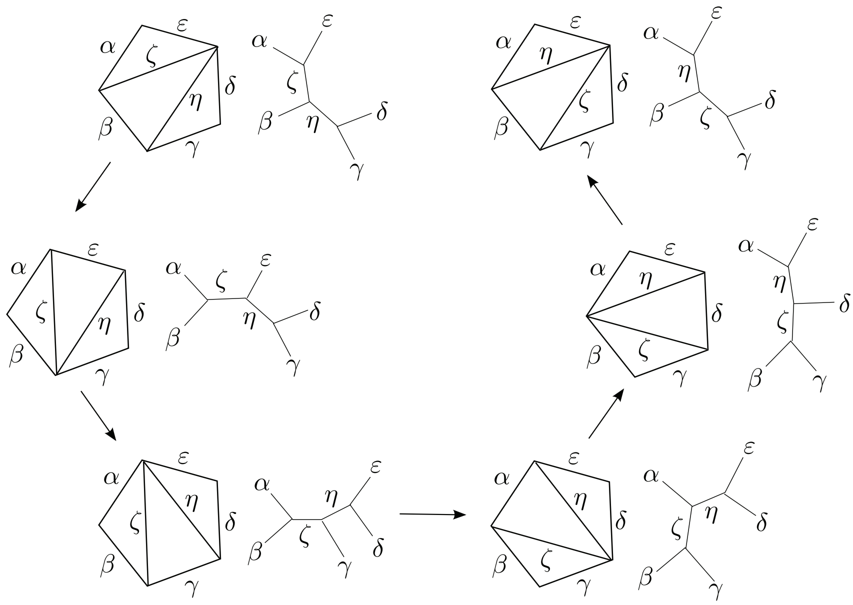

Figure 6. The pentagon relation for the Whitehead moves.

the sequence of Whitehead moves in Figure 6 is given by

$$\begin{bmatrix} z^\alpha \\ z^\beta \\ z^\gamma \\ z^\delta \\ z^\epsilon \\ z^\zeta \\ z^\eta \end{bmatrix} \mapsto \begin{bmatrix} z^\alpha + \log(1+e^{z^\zeta}) \\ z^\beta - \log(1+e^{-z^\zeta}) \\ z^\gamma \\ z^\delta \\ z^\epsilon - \log(1+e^{-z^\zeta}) \\ -z^\zeta \\ z^\eta + \log(1+e^{z^\zeta}) \end{bmatrix} \mapsto \begin{bmatrix} z^\alpha + \log(1+e^{z^\zeta}) \\ z^\beta - \log(1+e^{-z^\zeta}) \\ z^\gamma + z^\eta - \log(\frac{1+e^{z^\eta}+e^{z^\zeta+z^\eta}}{1+e^{z^\zeta}}) \\ z^\delta + \log(1+e^{z^\eta}+e^{z^\zeta+z^\eta}) \\ z^\epsilon + z^\zeta + z^\eta - \log(1+e^{z^\eta}+e^{z^\zeta+z^\eta}) \\ -z^\zeta + \log(1+e^{z^\eta}+e^{z^\zeta+z^\eta}) \\ -z^\eta - \log(1+e^{z^\zeta}) \end{bmatrix}$$

$$\mapsto \begin{bmatrix} z^\alpha + \log(\frac{1+e^{z^\eta}+e^{z^\eta+z^\zeta}}{1+e^{z^\eta}}) \\ z^\beta + \log(1+e^{z^\eta}) \\ z^\gamma - \log(1+e^{-z^\eta}) \\ z^\delta + \log(1+e^{z^\eta}+e^{z^\zeta+z^\eta}) \\ z^\epsilon + z^\zeta + z^\eta - \log(1+e^{z^\eta}+e^{z^\zeta+z^\eta}) \\ z^\zeta - \log(1+e^{z^\eta}+e^{z^\zeta+z^\eta}) \\ -z^\zeta + \log(1+e^{-z^\eta}) \end{bmatrix} \mapsto \begin{bmatrix} z^\alpha \\ z^\beta + \log(1+e^{z^\eta}) \\ z^\gamma - \log(1+e^{-z^\eta}) \\ z^\delta + \log(1+e^{z^\eta}) \\ z^\epsilon \\ -z^\eta \\ z^\zeta - \log(1+e^{-z^\eta}) \end{bmatrix} \mapsto \begin{bmatrix} z^\alpha \\ z^\beta \\ z^\gamma \\ z^\delta \\ z^\epsilon \\ z^\eta \\ z^\zeta \end{bmatrix}.$$

This proof is essentially identical to the corresponding proof for shear coordinates on Teichmüller space and is included only for the reader's convenience. q.e.d.

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