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THE CENTRO-AFFINE MINKOWSKI PROBLEM FOR POLYTOPES

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Abstract

The centro-affine Minkowski problem in affine differential geometry is considered. Existence for the solution of the discrete centro-affine Minkowski problem is proved.

1. Introduction

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A convex body in $\mathbb{R}^n$ is a compact convex set that has non-empty interior. Let K be a convex body in $\mathbb{R}^n$ whose boundary ∂K is a C^2 closed convex hypersurface with positive Gauss curvature. If K contains the origin in its interior, then the affine support function of K (also called the affine distance, see, e.g., [**30**, pp. 62–63]), is defined by

$$\tilde{h} = h\kappa^{-\frac{1}{n+1}},$$

where, as functions of the unit outer normal, h is the support function and κ is the Gauss curvature.

It is known that the affine support function of a convex body is invariant when the convex body undergoes an $SL(n)$ transformation. In particular, the affine support function is constant if the convex body is an ellipsoid centered at the origin. Conversely, for a convex body with C^∞ boundary if the affine support function of the convex body is a positive constant, then the convex body is an ellipsoid (see, e.g., Tzitséica [**35**], Loewner and Nirenberg [**23**], Calabi [**6**], and Leichtweiss [**22**]).

The function

$$\tilde{\kappa} = \tilde{h}^{-n-1} = h^{-n-1}\kappa$$

is called the centro-affine Gauss curvature (see, e.g., [**10**, pp. 76]).

Characterizing the centro-affine Gauss curvature (or the affine support function) is of great interest. The problem (posed explicitly by Chou and Wang in [**10**, pp. 76]; see also Jian and Wang [**19**, pp. 432]) is as follows:

Centro-affine Minkowski problem: *Given a positive function f on*

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the unit sphere S^{n-1}, find necessary and sufficient conditions for f so that f is the centro-affine Gauss curvature of a convex body in $\mathbb{R}^n$.

Obviously, the centro-affine Minkowski problem is equivalent to the following Monge–Ampère type equation:

$$h^{1+n}\det(h_{ij}+h\delta_{ij}) = 1/f, \tag{1.1}$$

where h_{ij} is the covariant derivative of h with respect to an orthonormal frame on S^{n-1} and δ_{ij} is the Kronecker delta.

In [**10**], Chou and Wang posed the centro-affine Minkowski problem and established a necessary condition for the existence of solutions to this problem. For the case where the data is rotationally symmetric, existence for the centro-affine Minkowski problem was proved by Lu and Wang [**24**].

The centro-affine Minkowski problem is a special case of the L_p Minkowski problem (posed by Lutwak [**26**]).

If $p \in \mathbb{R}$ and K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the L_p surface area measure, $S_p(K,\cdot)$, of K is a Borel measure on S^{n-1} defined for a Borel $\omega \subset S^{n-1}$ by

$$S_p(K,\omega) = \int_{x\in\nu_K^{-1}(\omega)} (x\cdot\nu_K(x))^{1-p}\,d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of boundary points of K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

Obviously, $S_1(K,\cdot)$ is the classical surface area measure of K. In addition, $\frac{1}{n}S_0(K,\cdot)$ is the cone-volume measure of K. In recent years, the L_p surface area measure appeared in, e.g., [**17**, **25**, **26**, **31**].

In [**26**], Lutwak posed the following L_p Minkowski problem:

L_p **Minkowski problem:** *Find necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the L_p surface area measure of a convex body in $\mathbb{R}^n$.*

Obviously, the centro-affine Minkowski problem is a special case of the L_p Minkowski problem when $p = -n$ and μ has a density. For this reason, the L_{-n} surface area measure and the L_{-n} Minkowski problem in this paper will be called, respectively, the centro-affine surface area measure and the general centro-affine Minkowski problem.

General centro-affine Minkowski problem: *Find necessary and sufficient conditions on a finite Borel measure μ on S^{n-1} so that μ is the centro-affine surface area measure of a convex body in $\mathbb{R}^n$.*

Besides the general centro-affine Minkowski problem, there are two other important cases for the L_p Minkowski problem. The case $p = 1$ of the L_p Minkowski problem is of course the classical Minkowski problem, which is completely solved (see, e.g., Alexandrov [**1**], Cheng and Yau [**8**], and Schneider [**32**]). The case $p = 0$ of the L_p Minkowski problem is called the logarithmic Minkowski problem. Very recently, a major breakthrough in the logarithmic Minkowski problem was made by Böröczky, Lutwak, Yang, and Zhang [**4**].

Today, the L_p Minkowski problem is one of the central problems in convex geometric analysis, and is studied in, e.g., [**5**, **7**, **11**, **16**, **20**, **21**, **24**, **27**, **29**, **33**], and especially by Lutwak [**26**], Chou and Wang [**10**], Guan and Lin [**15**], and Hug, Lutwak, Yang, and Zhang [**18**]. The solutions to the Minkowski problem and the L_p Minkowski problem are connected to some important flows (see, e.g., [**2**, **3**, **9**, **12**]) and play key roles in establishing the affine Sobolev–Zhang inequality [**36**] and the L_p affine Sobolev inequality [**28**].

The centro-affine Minkowski problem is the continuous case of the general centro-affine Minkowski problem when μ has a density. Another important case of the general centro-affine Minkowski problem is the polytopal case, i.e., μ is a discrete measure.

A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided that the convex hull has positive n-dimensional volume. The convex hull of a subset of these points is called a *facet* of the polytope if the convex hull lies entirely on the boundary of the polytope and has positive $(n-1)$-dimensional volume. If a polytope P contains the origin in its interior with N facets whose outer unit normals are $u_1, \dots, u_N$, and if the facet with outer unit normal u_k has area a_k and distance from the origin h_k for all $k \in \{1, \dots, N\}$, then the centro-affine surface area measure of P is

$$\sum_{k=1}^{N} h_k^{1+n} a_k \delta_{u_k}(\cdot),$$

where δ_{u_k} denotes the delta measure that is concentrated at the point u_k.

Centro-affine Minkowski problem for polytopes: *Find necessary and sufficient conditions on a discrete measure μ on S^{n-1} so that μ is the centro-affine surface area measure of a convex polytope in $\mathbb{R}^n$.*

The Minkowski problem and the L_p Minkowski problem for polytopes are of great importance. One reason that the problem for polytopes is so important is that the Minkowski problem and the L_p Minkowski problem (for $p > 1$) for arbitrary measures can be solved by an approximation argument by first solving the polytopal case (see, e.g., [**18**] or [**32**, pp. 392–393]). It is the aim of this paper to solve the centro-affine Minkowski problem for polytopes.

A finite subset U of S^{n-1} is said to be *in general position* if any k elements of U, $1 \le k \le n$, are linearly independent.

It is the aim of this paper to solve the general centro-affine Minkowski problem for the case of discrete measures whose supports are in general position:

Theorem. *Let μ be a discrete measure on the unit sphere S^{n-1}. Then μ is the centro-affine surface area measure of a polytope whose outer unit normals are in general position if and only if the support of μ is in general position and not contained in a closed hemisphere.*

Our theorem is a necessary and sufficient condition on the class of polytopes whose outer unit normals are in general position. However, the condition that the support of μ is in general position is not necessary for general discrete measures (e.g., when μ is the centro-affine surface area measure of the unit cube). The condition that μ is not contained in a closed hemisphere is necessary for general measures. Otherwise, the corresponding general centro-affine Minkowski problem will not have bounded solution.

For the case where the measure μ has a positive density, it is easy to construct discrete measures μ_i whose supports are in general position that converge weakly to μ. This might provide a possible way to solve the centro-affine Minkowski problem in affine differential geometry by using an approximation argument and the solution to the discrete centro-affine Minkowski problem.

2. Preliminaries

In this section, we collect some notation regarding convex bodies. For general references regarding convex bodies, see, e.g., [**13**, **14**, **32**, **34**].

The sets in this paper are subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we write $x \cdot y$ for the standard inner product of x and y, $|x|$ for the Euclidean norm of x, and S^{n-1} for the unit sphere of $\mathbb{R}^n$.

For convex bodies K_1, K_2 in $\mathbb{R}^n$ and $c_1, c_2 \ge 0$, the Minkowski combination is defined by

$$c_1K_1 + c_2K_2 = \{c_1x_1 + c_2x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a convex body K is defined, for $x \in \mathbb{R}^n$, by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for $c \ge 0$ and $x \in \mathbb{R}^n$,

$$h(cK, x) = h(K, cx) = ch(K, x).$$

The *Hausdorff distance* of two convex bodies K_1, K_2 in $\mathbb{R}^n$ is defined by

$$\delta(K_1, K_2) = \inf\{t \geq 0 : K_1 \subset K_2 + tB^n, K_2 \subset K_1 + tB^n\},$$

where B^n is the unit ball.

If K is a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$, then the *support set* $F(K, u)$ of K in direction u is defined by

$$F(K, u) = K \cap \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\}.$$

The diameter of a convex body K in $\mathbb{R}^n$ is defined by

$$d(K) = \max\{|x - y| : x, y \in K\}.$$

Let $\mathcal{P}$ be the set of polytopes in $\mathbb{R}^n$. If the unit vectors $u_1, \ldots, u_N$ ($N \geq n + 1$) are in general position and not contained in a closed hemisphere, let $\mathcal{P}(u_1, \ldots, u_N)$ be the subset of $\mathcal{P}$ such that a polytope $P \in \mathcal{P}(u_1, \ldots, u_N)$ if

$$P = \bigcap_{k=1}^{N} \{x : x \cdot u_k \leq h(P, u_k)\}.$$

Obviously, if $P \in \mathcal{P}(u_1, \ldots, u_N)$, then P has at most N facets, and the outer unit normals of P are a subset of $\{u_1, \ldots, u_N\}$. Let $\mathcal{P}_N(u_1, \ldots, u_N)$ be the subset of $\mathcal{P}(u_1, \ldots, u_N)$ such that a polytope $P \in \mathcal{P}_N(u_1, \ldots, u_N)$ if $P \in \mathcal{P}(u_1, \ldots, u_N)$, and P has exactly N facets.

The following lemmas will be needed (see, [**37**, Lemma 4.1 and Theorem 4.3]).

Lemma 2.1. *If the unit vectors $u_1, \ldots, u_N$ ($N \geq n + 1$) are in general position and not contained in a closed hemisphere, and $P \in \mathcal{P}(u_1, \ldots, u_N)$, then $F(P, u_i)$ is either a point or a facet of P for all $1 \leq i \leq N$.*

Lemma 2.2. *If the unit vectors $u_1, \ldots, u_N$ ($N \geq n+1$) are in general position and not contained in a closed hemisphere, $P_i \in \mathcal{P}(u_1, \ldots, u_N)$ (with $o \in P_i$) is a sequence of polytopes, and $V(P_i) = 1$, then P_i is bounded.*

3. An extremal problem related to the centro-affine Minkowski problem

In this section, we solve an extremal problem. Its solution also solves the centro-affine Minkowski problem.

If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ ($N \geq n + 1$) are in general position and not contained in a closed hemisphere, and $P \in$

$\mathcal{P}(u_1, \ldots, u_N)$, then define $\Phi_P : \text{Int } (P) \to \mathbb{R}$ by

$$\Phi_P(\xi) = \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi \cdot u_k)^{-n},$$

where Int (P) is the interior of P.

Lemma 3.1. *If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ ($N \geq n+1$) are in general position and not contained in a closed hemisphere, and $P \in \mathcal{P}(u_1, \ldots, u_N)$, then there exists a unique point $\xi(P) \in$ Int (P) such that*

$$\Phi_P(\xi(P)) = \inf_{\xi \in \mathit{Int}\ (P)} \Phi_P(\xi).$$

Proof. Obviously, t^{-n} is strictly convex on $(0, +\infty)$. Thus, for $0 < \lambda < 1$ and $\xi_1, \xi_2 \in \text{Int } (P)$,

$$\begin{aligned}
\lambda \Phi_P(\xi_1) + (1-\lambda)\Phi_P(\xi_2) &= \lambda \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_1 \cdot u_k)^{-n} \\
&\qquad + (1-\lambda) \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi_2 \cdot u_k)^{-n} \\
&= \sum_{k=1}^{N} \alpha_k \big[\lambda (h(P, u_k) - \xi_1 \cdot u_k)^{-n} \\
&\qquad + (1-\lambda)(h(P, u_k) - \xi_2 \cdot u_k)^{-n}\big] \\
&\geq \sum_{k=1}^{N} \alpha_k \left[h(P, u_k) - (\lambda \xi_1 + (1-\lambda)\xi_2) \cdot u_k\right]^{-n} \\
&= \Phi_P(\lambda \xi_1 + (1-\lambda)\xi_2),
\end{aligned}$$

with equality if and only if $\xi_1 \cdot u_k = \xi_2 \cdot u_k$ for all $k = 1, \ldots, N$. Since $u_1, \ldots, u_N$ are in general position, $\xi_1 = \xi_2$. Thus, Φ_P is strictly convex on Int (P).

Since $P \in \mathcal{P}(u_1, \ldots, u_N)$, for any boundary point $x \in \partial P$, there exists a $u_{i_0} \in \{u_1, \ldots, u_N\}$ such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Thus, $\Phi_P(\xi)$ goes to $+\infty$ whenever $\xi \in \text{Int } (P)$ and $\xi \to \partial P$. Therefore, there exists a unique interior point $\xi(P)$ of P such that

$$\Phi_P(\xi(P)) = \min_{\xi \in \text{Int } (P)} \Phi_P(\xi).$$

q.e.d.

By definition, for $\lambda > 0$ and $P \in \mathcal{P}(u_1, \ldots, u_N)$,

$$\xi(\lambda P) = \lambda \xi(P). \tag{3.1}$$

Obviously, if $P_i \in \mathcal{P}(u_1, \dots, u_N)$ and P_i converges to a polytope P, then $P \in \mathcal{P}(u_1, \dots, u_N)$.

Lemma 3.2. *If $\alpha_1, \dots, \alpha_N$ are positive, the unit vectors $u_1, \dots, u_N$ ($N \geq n+1$) are in general position and not contained in a closed hemisphere, $P_i \in \mathcal{P}(u_1, \dots, u_N)$, and P_i converges to a polytope P, then $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and*

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)),$$

where $\Phi_P(\xi) = \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi \cdot u_k)^{-n}$.

Proof. Let $a_0 = \min_{u \in S^{n-1}} \{h(P, u) - \xi(P) \cdot u\} > 0$. Since P_i converges to P and $\xi(P) \in \mathrm{Int}\,(P)$, there exists an $N_0 > 0$ such that

$$h(P_i, u_k) - \xi(P) \cdot u_k > \frac{a_0}{2},$$

for all $k = 1, \dots, N$, whenever $i > N_0$. Thus,

$$\Phi_{P_i}(\xi(P_i)) \leq \Phi_{P_i}(\xi(P)) < \big(\sum_{k=1}^N \alpha_k\big)\big(\frac{a_0}{2}\big)^{-n}, \tag{3.2}$$

whenever $i > N_0$.

From the conditions, $\xi(P_i)$ is bounded. Suppose that $\xi(P_i)$ does not converge to $\xi(P)$; then there exists a subsequence P_{i_j} of P_i such that P_{i_j} converges to P, $\xi(P_{i_j}) \to \xi_0$ but $\xi_0 \neq \xi(P)$. Obviously, $\xi_0 \in P$. We claim that ξ_0 is not a boundary point of P; otherwise, $\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) = +\infty$, and this contradicts (3.2). If ξ_0 is an interior point of P with $\xi_0 \neq \xi(P)$, then

$$\begin{aligned}\lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) &= \Phi_P(\xi_0) \\ &> \Phi_P(\xi(P)) \\ &= \lim_{j\to\infty} \Phi_{P_{i_j}}(\xi(P)).\end{aligned}$$

This contradicts

$$\Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \Phi_{P_{i_j}}(\xi(P)).$$

Therefore, $\lim_{i\to\infty} \xi(P_i) = \xi(P)$ and thus,

$$\lim_{i\to\infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

q.e.d.

The following lemma will be needed.

Lemma 3.3. *If P is a polytope, $u_{i_0} \in S^{n-1}$, $F(P, u_{i_0})$ is a point, and*

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\},$$

then there exists a positive δ_0 such that when $0 < \delta < \delta_0$, $P \setminus P_\delta$ is a cone and

$$V(P \setminus P_\delta) = c_0 \delta^n,$$

where c_0 is a constant that depends on P and u_{i_0}.

Proof. Since P is a polytope and $F(P, u_{i_0})$ is a point, there exists a positive δ_0 (depends on P and u_{i_0}) such that when $0 < \delta \le \delta_0$, $P \setminus P_\delta$ is a cone and $F(P, u_{i_0})$ is the apex. Then, when $0 < \delta \le \delta_0$,

$$\frac{V_{n-1}\big(P \cap \{x : x \cdot u_{i_0} = h(P, u_{i_0}) - \delta\}\big)}{V_{n-1}\big(P \cap \{x : x \cdot u_{i_0} = h(P, u_{i_0}) - \delta_0\}\big)} = \left(\frac{\delta}{\delta_0}\right)^{n-1}.$$

Therefore, when $0 < \delta < \delta_0$,

$$V(P \setminus P_\delta) = \int_0^\delta V_{n-1}(P \cap \{x : x \cdot u_{i_0} = h(P, u_{i_0}) - t\})dt = c_0 \delta^n,$$

where c_0 is a constant depends on P and u_{i_0}. q.e.d.

Lemma 3.4. *If $\alpha_1, \dots, \alpha_N$ are positive, and the unit vectors $u_1, \dots, u_N$ ($N \ge n+1$) are in general position and not contained in a closed hemisphere, then there exists a $P \in \mathcal{P}_N(u_1, \dots, u_N)$ with $\xi(P) = o$ and $V(P) = 1$ such that*

$$\Phi_P(o) = \sup\{\min_{\xi \in Int\ (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, \dots, u_N) \text{ and } V(Q) = 1\},$$

where $\Phi_Q(\xi) = \sum_{k=1}^N \alpha_k (h(Q, u_k) - \xi \cdot u_k)^{-n}$.

Proof. Obviously, for $P, Q \in \mathcal{P}_N(u_1, \dots, u_N)$, if there exists an $x \in \mathbb{R}^n$ such that $P = Q + x$, then

$$\Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).$$

Thus, we can choose a sequence $P_i \in \mathcal{P}_N(u_1, \dots, u_N)$ with $\xi(P_i) = o$ and $V(P_i) = 1$ such that $\Phi_{P_i}(o)$ converges to

$$\sup\{\min_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, \dots, u_N) \text{ and } V(Q) = 1\}.$$

Because of Lemma 2.2, P_i is bounded. Thus, from Lemma 3.2 and the Blaschke selection theorem, there exists a subsequence of P_i that converges to a polytope P such that $P \in \mathcal{P}(u_1, \dots, u_N)$, $V(P) = 1$, $\xi(P) = o$, and

$$\Phi_P(o) = \sup\{\min_{\xi \in \text{Int } (Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, \dots, u_N) \text{ and } V(Q) = 1\}. \tag{3.3}$$

We next prove that $F(P, u_i)$ are facets for all $i = 1, \dots, N$. Otherwise, from Lemma 2.1, there exist $1 \le i_0 < \dots < i_m \le N$ with $m \ge 0$ such that

$$F(P, u_i)$$

is a point for $i \in \{i_0, \dots, i_m\}$ and is a facet of P for $i \notin \{i_0, \dots, i_m\}$.

Choose $\delta > 0$ small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \le h(P, u_{i_0}) - \delta\}$$

has exactly $(N-m)$ facets and

$$P \cap \{x : x \cdot u_{i_0} \geq h(P, u_{i_0}) - \delta\}$$

is a cone. From this and Lemma 3.3, we have

$$h(P_\delta, u_k) = h(P, u_k) \tag{3.4}$$

for $k \neq i_0$,

$$h(P_\delta, u_{i_0}) = h(P, u_{i_0}) - \delta, \tag{3.5}$$

and

$$V(P_\delta) = 1 - c_0 \delta^n,$$

where $c_0 > 0$ is a constant that depends on P and direction u_{i_0}.

Because of Lemma 3.2, for any $\delta_i \to 0$, it is always true that $\xi(P_{\delta_i}) \to o$. We have

$$\lim_{\delta \to 0} \xi(P_\delta) = o.$$

Let δ be small enough so that

$$h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta \tag{3.6}$$

for all $k \in \{1, \ldots, N\}$, and let

$$\lambda = V(P_\delta)^{-\frac{1}{n}} = \left(\frac{1}{1 - c_0 \delta^n}\right)^{\frac{1}{n}}. \tag{3.7}$$

From (3.6), (3.1), (3.4), and (3.5), we have

(3.8)

$$\begin{aligned}
\Phi_{\lambda P_\delta}(\xi(\lambda P_\delta)) &= \sum_{k=1}^{N} \alpha_k \big(h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta)\cdot u_k\big)^{-n} \\
&= \lambda^{-n} \sum_{k=1}^{N} \alpha_k \big(h(P_\delta, u_k) - \xi(P_\delta)\cdot u_k\big)^{-n} \\
&= \lambda^{-n} \sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^{-n} \\
&\qquad + \alpha_{i_0}\lambda^{-n}\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^{-n} \\
&\qquad\qquad - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^{-n}\Big] \\
&= \Phi_P(\xi(P_\delta)) + (\lambda^{-n} - 1)\sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^{-n} \\
&\qquad + \alpha_{i_0}\lambda^{-n}\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^{-n} \\
&\qquad\qquad - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^{-n}\Big] \\
&= \Phi_P(\xi(P_\delta)) + B(\delta),
\end{aligned}$$

where

$$\begin{aligned}
B(\delta) &= (\lambda^{-n} - 1)\sum_{k=1}^{N} \alpha_k \big(h(P, u_k) - \xi(P_\delta)\cdot u_k\big)^{-n} \\
&\qquad + \alpha_{i_0}\lambda^{-n}\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^{-n} \\
&\qquad\qquad - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^{-n}\Big] \\
&= -c_0\delta^n \sum_{k=1}^{N} \alpha_k (h(P, u_k) - \xi(P_\delta)\cdot u_k)^{-n} \\
&\qquad + \alpha_{i_0}(1 - c_0\delta^n)\Big[\big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta\big)^{-n} \\
&\qquad\qquad - \big(h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0}\big)^{-n}\Big].
\end{aligned}$$

Let d_0 be the diameter of P. Since

$$d_0 > h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} > h(P, u_{i_0}) - \xi(P_\delta)\cdot u_{i_0} - \delta > 0,$$

$\big(h(P,u_{i_0})-\xi(P_\delta)\cdot u_{i_0}-\delta\big)^{-n}-\big(h(P,u_{i_0})-\xi(P_\delta)\cdot u_{i_0}\big)^{-n} > (d_0-\delta)^{-n}-d_0^{-n}$.
Then

$$B(\delta) > -c_0\delta^n \sum_{k=1}^{N} \alpha_k (h(P,u_k) - \xi(P_\delta)\cdot u_k)^{-n} + \alpha_{i_0}(1-c_0\delta^n)\big[(d_0-\delta)^{-n} - d_0^{-n}\big]. \tag{3.9}$$

On the other hand, for $0 < \delta < d_0$,

$$(d_0-\delta)^{-n} - d_0^{-n} > 0, \tag{3.10}$$

$$\lim_{\delta\to 0} \sum_{k=1}^{N} \alpha_k \big(h(P,u_k) - \xi(P_\delta)\cdot u_k\big)^{-n} = \sum_{k=1}^{N} \alpha_k h(P,u_k)^{-n}, \tag{3.11}$$

and

$$\lim_{\delta\to 0} \frac{-c_0\delta^n}{(d_0-\delta)^{-n} - d_0^{-n}} = \lim_{\delta\to 0} \frac{-nc_0\delta^{n-1}}{(-n)(d_0-\delta)^{-n-1}(-1)} = 0. \tag{3.12}$$

From Equations (3.9), (3.10), (3.11), and (3.12), we have $B(\delta) > 0$ for small enough $\delta > 0$. From Equation (3.8), there exists a small $\delta_0 > 0$ such that P_{δ_0} has exactly $(N-m)$ facets and

$$\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) > \Phi_P(\xi(P_{\delta_0})) \geq \Phi_P(\xi(P)) = \Phi_P(o),$$

where $\lambda_0 = V(P_{\delta_0})^{-\frac{1}{n}}$. Let $P_0 = \lambda_0 P_{\delta_0} - \xi(\lambda_0 P_{\delta_0})$, then $P_0 \in \mathcal{P}(u_1,\ldots,u_N)$, $V(P_0) = 1$, $\xi(P_0) = o$, and

$$\Phi_{P_0}(o) > \Phi_P(o). \tag{3.13}$$

If $m = 0$, then (3.13) and (3.3) yield a contradiction. If $m \geq 1$, choose positive δ_i so that $\delta_i \to 0$ as $i \to \infty$,

$$P_{\delta_i} = P_0 \cap \big(\cap_{j=1}^m \{x : x\cdot u_{i_j} \leq h(P_0,u_{i_j}) - \delta_i\}\big)$$

and $\lambda_i P_{\delta_i} \in \mathcal{P}_N(u_1,\ldots,u_N)$, where $\lambda_i = V(P_{\delta_i})^{-\frac{1}{n}}$. Obviously, $\lambda_i P_{\delta_i}$ converges to P_0. From Lemma 3.2, Equation (3.13), and Equation (3.3), we have

$$\begin{aligned}
\lim_{n\to\infty} & \Phi_{\lambda_i P_{\delta_i}}(\xi(\lambda_i P_{\delta_i})) \\
&= \Phi_{P_0}(o) \\
&> \Phi_P(o) \\
&= \sup\{\min_{\xi\in \mathrm{Int}\,(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1,\ldots,u_N), V(Q) = 1\}.
\end{aligned}$$

This contradicts (3.3). Therefore,

$$P \in \mathcal{P}_N(u_1,\ldots,u_N).$$

q.e.d.

4. The centro-affine Minkowski problem for polytopes

In this section, we prove the main theorem. Obviously, if a finite subset U of S^{n-1} has no more than n elements, then U must be contained in a closed hemisphere. Thus, $N \geq n+1$ is always true if U is not contained in a closed hemisphere. We only need prove the following theorem:

Theorem 4.1. *If $\alpha_1, \ldots, \alpha_N \in \mathbb{R}^+$, the unit vectors $u_1, \ldots, u_N$ ($N \geq n+1$) are in general position and not contained in a closed hemisphere, then there exists a polytope P_0 such that*

$$S_{-n}(P_0, \cdot) = \sum_{k=1}^{N} \alpha_k \delta_{u_k}(\cdot).$$

Proof. From Lemma 3.4, there exists a polytope $P \in \mathcal{P}_N(u_1, \ldots, u_N)$ with $\xi(P) = o$ and $V(P) = 1$ such that

$$\Phi_P(o) = \sup\{\min_{\xi \in \text{Int }(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}_N(u_1, \ldots, u_N) \text{ and } V(Q) = 1\},$$

where $\Phi_Q(\xi) = \sum_{k=1}^{N} \alpha_k (h(Q, u_k) - \xi \cdot u_k)^{-n}$.

For $\delta_1, \ldots, \delta_N \in \mathbb{R}$, choose $|t|$ small enough so that the polytope P_t defined by

$$P_t = \bigcap_{i=1}^{N} \{x : x \cdot u_i \leq h(P, u_i) + t\delta_i\}$$

has exactly N facets. Then

$$V(P_t) = V(P) + t\left(\sum_{i=1}^{N} \delta_i a_i\right) + o(t),$$

where a_i is the $(n-1)$-dimensional volume of $F(P, u_i)$. Thus,

$$\lim_{t \to 0} \frac{V(P_t) - V(P)}{t} = \sum_{i=1}^{N} \delta_i a_i.$$

Let $\lambda(t) = V(P_t)^{-\frac{1}{n}}$; then $\lambda(t)P_t \in \mathcal{P}_N(u_1, \ldots, u_N)$, $V(\lambda(t)P_t) = 1$, and

$$\lambda'(0) = -\frac{1}{n} \sum_{i=1}^{N} \delta_i a_i. \tag{4.1}$$

For convenience, let $\xi(t) = \xi(\lambda(t)P_t)$ and

$$\begin{aligned} \Phi(t) &= \min_{\xi \in \text{Int }(\lambda(t)P_t)} \sum_{k=1}^{N} \alpha_k (\lambda(t)h(P_t, u_k) - \xi \cdot u_k)^{-n} \\ &= \sum_{k=1}^{N} \alpha_k (\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k)^{-n}. \end{aligned} \tag{4.2}$$

From Equation (4.2) and the fact that $\xi(t)$ is an interior point of $\lambda(t)P_t$, we have

$$\sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k]^{1+n}} = 0, \tag{4.3}$$

for $i = 1, \dots, n$, where $u_k = (u_{k,1}, \dots, u_{k,n})^T$. As a special case when $t = 0$, we have

$$\sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{h(P, u_k)^{1+n}} = 0,$$

for $i = 1, \dots, n$. Therefore,

$$\sum_{k=1}^{N} \alpha_k \frac{u_k}{h(P, u_k)^{1+n}} = 0. \tag{4.4}$$

Let

$$F_i(t, \xi_1, \dots, \xi_n) = \sum_{k=1}^{N} \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \cdots + \xi_n u_{k,n})]^{1+n}}$$

for $i = 1, \dots, n$. Then

$$\left.\frac{\partial F_i}{\partial \xi_j}\right|_{(0,\dots,0)} = \sum_{k=1}^{N} \frac{(1+n)\alpha_k}{h(P, u_k)^{2+n}} u_{k,i} u_{k,j}.$$

Thus,

$$\left(\left.\frac{\partial F}{\partial \xi}\right|_{(0,\dots,0)} \right)_{n\times n} = \sum_{k=1}^{N} \frac{(1+n)\alpha_k}{h(P, u_k)^{2+n}} u_k \cdot u_k^T,$$

where $u_k \cdot u_k^T$ is an $n \times n$ matrix.

For any $x \in \mathbb{R}^n$ with $x \neq 0$, from the fact that $u_1, \dots, u_N$ are in general position there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Then

$$\begin{aligned} x^T \cdot \left(\sum_{k=1}^{N} \frac{(1+n)\alpha_k}{h(P, u_k)^{2+n}} u_k \cdot u_k^T \right) \cdot x &= \sum_{k=1}^{N} \frac{(1+n)\alpha_k}{h(P, u_k)^{2+n}} (x \cdot u_k)^2 \\ &\geq \frac{(1+n)\alpha_{i_0}}{h(P, u_{i_0})^{2+n}} (x \cdot u_{i_0})^2 > 0. \end{aligned}$$

Thus, $(\frac{\partial F}{\partial \xi}|_{(0,\dots,0)})$ is positive definite. From this, the fact $\xi(0) = 0$, Equations (4.3), and the implicit function theorem, we have

$$\xi'(0) = (\xi_1'(0), \dots, \xi_n'(0))$$

exists.

From the fact that $t=0$ is an extreme point of $\Phi(t)$ (in Equation (4.2)), Equation (4.1), and Equation (4.4), we have

$$\begin{aligned}
0 &= \Phi'(0)/(-n)\\
&= \sum_{k=1}^{N} \alpha_k h(P,u_k)^{-n-1}\left(\lambda'(0)h(P,u_k)+\delta_k-\xi'(0)\cdot u_k\right)\\
&= \sum_{k=1}^{N} \alpha_k h(P,u_k)^{-n-1}\left[-\frac{1}{n}\left(\sum_{i=1}^{N} a_i\delta_i\right)h(P,u_k)+\delta_k\right]\\
&\qquad -\xi'(0)\cdot\left[\sum_{k=1}^{N}\alpha_k\frac{u_k}{h(P,u_k)^{1+n}}\right]\\
&= \sum_{k=1}^{N} \alpha_k h(P,u_k)^{-n-1}\delta_k-\left(\sum_{i=1}^{N} a_i\delta_i\right)\frac{\sum_{k=1}^{N}\alpha_k h(P,u_k)^{-n}}{n}\\
&= \sum_{k=1}^{N}\left(\alpha_k h(P,u_k)^{-n-1}-\frac{\sum_{j=1}^{N}\alpha_j h(P,u_j)^{-n}}{n}a_k\right)\delta_k.
\end{aligned}$$

Since $\delta_1,\ldots,\delta_N$ are arbitrary,

$$\frac{\sum_{j=1}^{N}\alpha_j h(P,u_j)^{-n}}{n}h(P,u_k)^{1+n}a_k=\alpha_k,$$

for all $k=1,\ldots,N$. Thus, for

$$P_0=\left(\frac{\sum_{j=1}^{N}\alpha_j h(P,u_j)^{-n}}{n}\right)^{\frac{1}{2n}}P,$$

we have

$$S_{-n}(P_0,\cdot)=\sum_{k=1}^{N}\alpha_k\delta_{u_k}(\cdot).$$

q.e.d.

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