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# HAMILTONIAN STABILITY OF THE GAUSS IMAGES OF HOMOGENEOUS ISOPARAMETRIC HYPERSURFACES. I

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## Abstract

The image of the Gauss map of any oriented isoparametric hypersurface in the standard unit sphere $S^{n+1}(1)$ is a minimal Lagrangian submanifold in the complex hyperquadric $Q_n(\mathbf{C})$. In this paper we show that the Gauss image of a compact oriented isoparametric hypersurface with $g$ distinct constant principal curvatures in $S^{n+1}(1)$ is a compact monotone and cyclic embedded Lagrangian submanifold with minimal Maslov number $2n/g$. We obtain the Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces of classical type with $g = 4$. Combining with our results in [**25**] and [**27**], we completely determine the Hamiltonian stability of the Gauss images of all homogeneous isoparametric hypersurfaces.

## Introduction

In the 1990s, Oh instigated the study of Hamiltonian minimality and Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds [**33**, **34**, **35**]. This provides a constrained volume variational problem for Lagrangian submanifolds in Kähler manifolds under Hamiltonian deformations. Thus it is natural to ask which Lagrangian submanifolds in specific Kähler manifolds are Hamiltonian stable (See Section 1 for the definitions). After Oh's pioneering work, there has been extensive research on Hamiltonian stability of minimal or Hamiltonian minimal Lagrangian submanifolds in various Kähler manifolds, such as complex Euclidean spaces, complex projective spaces, compact Hermitian symmetric spaces, certain toric Kähler manifolds, and so on. (see e.g., [**1**, **9**, **39**, **41**, **44**, **51**] and references therein.) In particular, a compact minimal Lagrangian submanifold $L$ in a compact homogeneous Einstein–Kähler manifold with positive Einstein constant $\kappa$ is Hamiltonian stable if and only if the first (positive) eigenvalue $\lambda_1$ of the Laplacian of $L$ with respect to the induced metric equals to $\kappa$. Hence, in this case,

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to determine the Hamiltonian stability becomes a problem of calculating the first eigenvalue of the Laplacian, which is an important subject in differential geometry. However, we do NOT know many examples of compact Hamiltonian stable Lagrangian submanifolds yet.

A hypersurface immersed in the standard sphere is called isoparametric if it has constant principal curvatures. Isoparametric hypersurfaces generalize geodesic spheres in the standard spheres. The theory was started by Élie Cartan and has been well developed since then. Particularly significant progress on the classification problem of isoparametric hypersurfaces in spheres was made in the recent works of Cecil-Chi-Jensen [**10**], Immervoll [**20**], Chi [**11**, **12**] and Miyaoka [**30**]. As the most fundamental result on isoparametric hypersurfaces in spheres, Münzner [**31**, **32**] showed that the number $g$ of distinct principal curvatures of an isoparametric hypersurface $N^n$ in $S^{n+1}(1)$ must be $g = 1, 2, 3, 4, 6$ and their multiplicities satisfy $m_1 = m_3 = \cdots \leq m_2 = m_4 = \cdots$. Moreover, $N^n$ is always real algebraic in the sense that $N^n$ is defined by a certain real homogeneous polynomial of degree $g$ called the "Cartan–Münzner polynomial."

We observed that the Gauss image—that is, the image of the Gauss map—of any compact oriented isoparametric hypersurface in the standard unit sphere is a smooth compact embedded minimal Lagrangian submanifold in the complex hyperquadric, and the Gauss map is a covering map over the Gauss images with covering transformation group $\mathbf{Z}_g$ [**25**, **37**]. Thus it can be expected that the Gauss images of isoparametric hypersurfaces in spheres provide a nice class of compact Lagrangian submanifolds embedded in complex hyperquadrics and moreover they should play certain roles in symplectic geometry. Note that the Gauss image is orientable if and only if $2n/g$ is even [**37**]. In this paper we show the following (see Theorem 2.1).

**Theorem.** *The Gauss image of a compact oriented isoparametric hypersurface with $g$ distinct constant principal curvatures in $S^{n+1}(1)$ is a compact monotone and cyclic embedded Lagrangian submanifold with minimal Maslov number $2n/g = m_1 + m_2$.*

Recall that all isoparametric hypersurfaces in the unit standard sphere are classified as either homogeneous or nonhomogeneous. An isoparametric hypersurface $N^n$ in the standard unit sphere $S^{n+1}(1)$ is called *homogeneous* if $N^n$ can be obtained as an orbit of a compact Lie subgroup of $SO(n+2)$. Every homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of a linear isotropy representation of a compact Riemannian symmetric pair $(U, K)$ of rank 2, as shown by Hsiang and Lawson [**18**] and Takagi and Takahashi [**46**]. Only in the case of $g = 4$ are there known to exist non-homogeneous isoparametric hypersurfaces, which were discovered first by Ozeki and

Takeuchi [**42**, **43**] and extensively generalized by Ferus, Karcher, and Münzner [**13**]. So it is interesting to consider the following.

**Problem.** Investigate the Hamiltonian stability of those compact minimal Lagrangian embedded submanifolds in $Q_n(\mathbf{C})$ obtained as the Gauss images of isoparametric hypersurfaces in $S^{n+1}(1)$.

This paper is a continuation of [**25**], where we have already treated the cases of $g = 1, 2$, and 3. Let $N^n$ be an oriented compact isoparametric hypersurface embedded in $S^{n+1}(1)$. In [**44**], Palmer showed that the Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion and that $\mathcal{G}$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}(1)$, which corresponds to the case $g = 1$. In the case when $g = 1$, $N^n = S^n$ is a great or small sphere and the Gauss image $\mathcal{G}(N^n) \cong S^n$ is totally geodesic and strictly Hamiltonian stable. More strongly, it is stable as a minimal submanifold [**47**]. When $n$ is even, it is homologically volume minimizing because it is a calibrated submanifold by an invariant $n$-form [**15**]. The recent result of [**21**] implies that it is Hamiltonian volume minimizing for general $n$. In the case when $g = 2$, $N^n = S^{m_1} \times S^{m_2}$ ($n = m_1 + m_2, 1 \leq m_1 \leq m_2$) is the Clifford hypersurface and the Gauss image $\mathcal{G}(N^n) = Q_{m_1+1,m_2+1}(\mathbf{R}) = (S^{m_1} \times S^{m_2})/\mathbf{Z}_2 \subset Q_n(\mathbf{C})$ is also totally geodesic. Then $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is NOT Hamiltonian stable if and only if $m_2 - m_1 \geq 3$, where the spherical harmonics of degree 2 on the sphere $S^{m_1} \subset \mathbf{R}^{m_1+1}$ of smaller dimension give volume decreasing Hamiltonian deformation of $\mathcal{G}(N^n)$. If $m_2 - m_1 = 2$, then it is Hamiltonian stable but not strictly Hamiltonian stable. If $m_2 - m_1 < 2$, then it is strictly Hamiltonian stable. In the case when $g = 3$, the Gauss image $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is strictly Hamiltonian stable [**25**].

Using harmonic analysis on compact homogeneous spaces and fibrations on homogeneous isoparametric hypersurfaces, we obtain the main result as follows:

**Theorem.** *Suppose that $N$ is a homogeneous isoparametric hypersurface in $S^{n+1}(1)$ given by the isotropy orbit of rank 2 Riemannian symmetric pair $(U, K)$ of classical type. Then the Gauss image $L = \mathcal{G}(N)$ is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$.*

Combining with our results in [**27**] on exceptional types, we obtain the following.

**Theorem.** *Suppose that $(U, K)$ is not of type* EIII*; that is, $(U, K) \neq (E_6, U(1) \cdot Spin(10))$. Then the Gauss image $L = \mathcal{G}(N)$ is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$. Moreover, if $(U, K)$ is of type* EIII*—namely, $g = 4$ and $(m_1, m_2) = (6, 9)$—then $L = \mathcal{G}(N)$ is strictly Hamiltonian stable.*

Hence we solve the above problem for ALL homogeneous isoparametric hypersurfaces, and our solution provides new examples of compact Hamiltonian stable minimal Lagrangian submanifolds embedded in $Q_n(\mathbf{C})$ and interesting relations between hypersurfaces in $S^{n+1}(1)$ and minimal Lagrangian submanifolds in $Q_n(\mathbf{C})$.

This paper is organized as follows: In Section 1 we review the notions and basic properties of Hamiltonian minimality, Hamiltonian stability and strict Hamiltonian stability of Lagrangian submanifolds in Kähler manifolds. In Section 2 we briefly explain properties of minimal Lagrangian submanifolds in complex hyperquadrics obtained as the Gauss images of isoparametric hypersurfaces in spheres. In Section 3 we explain the method of eigenvalue computations of our compact homogeneous spaces that are the Gauss images of compact homogeneous isoparametric hypersurfaces. The method is based on the fibrations on homogeneous isoparametric hypersurfaces by lower dimensional homogeneous isoparametric hypersurfaces. In Sections 4–8, we determine the strict Hamiltonian stability of the Gauss images of compact homogeneous isoparametric hypersurfaces with $g = 4$ obtained as principal orbits of the isotropic representations of Riemannian symmetric spaces of classical type.

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## 1. Hamiltonian minimality and Hamiltonian stability

Assume that $(M, \omega, J, g)$ is a Kähler manifold with the compatible complex structure $J$ and Kähler metric $g$. Let $\varphi : L \to M$ be a Lagrangian immersion, and let H denote the mean curvature vector field of $\varphi$. The corresponding 1-form $\alpha_{\mathrm{H}} := \omega(\mathrm{H}, \cdot) \in \Omega^1(L)$ is called the *mean curvature form* of $\varphi$. A smooth family of Lagrangian immersions $\varphi_t : L \to M$ is called a *Hamiltonian deformation* with $\varphi_0 = \varphi$, if the

1-form $\alpha_{V_t} := \omega(V_t, \cdot)$ is exact for each $t$, where $V_t := \frac{\partial \varphi_t}{\partial t}$ is the variational vector field. For simplicity, throughout this paper we assume that $L$ is compact without boundary.

**Definition 1.1.** Let $M$ be a Kähler manifold. A Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian minimal* (briefly, *H-minimal*) or *Hamiltonian stationary* if it is a critical point of the volume functional for all Hamiltonian deformations $\{\varphi_t\}$.

The corresponding Euler–Lagrange equation is $\delta\alpha_{\mathrm{H}} = 0$, where $\delta$ is the co-differential operator with the respect to the induced metric on $L$.

**Definition 1.2.** An H-minimal Lagrangian immersion $\varphi$ is called *Hamiltonian stable* (briefly, *H-stable*) if the second variation of the volume is nonnegative under every Hamiltonian deformation $\{\varphi_t\}$.

The second variational formula is given as follows ([**35**]):

$$\begin{aligned}&\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0}\\&= \int_L \left(\langle \Delta_L^1 \alpha, \alpha\rangle - \langle \overline{R}(\alpha), \alpha\rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_{\mathrm{H}}, S\rangle + \langle \alpha_{\mathrm{H}}, \alpha\rangle^2\right) dv,\end{aligned}$$

where $\Delta_L^1$ denotes the Laplace operator of $(L, \varphi^* g)$ acting on the vector space $\Omega^1(L)$ of smooth 1-forms on $L$ and $\alpha := \omega(V, \cdot) \in B^1(L)$ is the exact 1-form corresponding to an infinitesimal Hamiltonian deformation $V$. Here

$$\langle \overline{R}(\alpha), \alpha\rangle := \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j)$$

for a local orthonormal frame $\{e_i\}$ on $L$ and

$$S(X, Y, Z) := \omega(B(X, Y), Z)$$

for each $X, Y, Z \in C^\infty(TL)$, which is a symmetric 3-tensor field on $L$ defined by the second fundamental form $B$ of $L$ in $M$. The *index* of $\varphi$ is defined as the dimension of the maximal vector subspace of $B^1(L)$ on which the second variation is negative definite.

For an H-minimal Lagrangian immersion $\varphi : L \to M$, we denote by $E_0(\varphi)$ the null space of the second variation on $B^1(L)$, or equivalently the solution space to the linearized H-minimal Lagrangian submanifold equation, and we call $n(\varphi) := \dim E_0(\varphi)$ the *nullity* of $\varphi$.

If $H^1(M, \mathbf{R}) = \{0\}$, then any holomorphic Killing vector field on $M$ is a Hamiltonian vector field, and thus it generates a volume-preserving Hamiltonian deformation of $\varphi$. Namely,

$$\begin{aligned}&\{\varphi^*\alpha_X \mid X \text{ is a holomorphic Killing vector field on } M\}\\&\quad\subset E_0(\varphi) \subset B^1(L).\end{aligned}$$

Set $n_{hk}(\varphi) := \dim\{\varphi^*\alpha_X \mid X$ is a holomorphic Killing vector field on $M\}$, which is called the *holomorphic Killing nullity* of $\varphi$.

**Definition 1.3.** An H-minimal Lagrangian immersion $\varphi$ is called *strictly Hamiltonian stable* (briefly, *strictly H-stable*) if $\varphi$ is Hamiltonian stable and $n_{hk}(\varphi) = n(\varphi)$.

Note that if $L$ is strictly Hamiltonian stable, then $L$ has locally minimum volume under each Hamiltonian deformation.

In the case when $L$ is a compact minimal Lagrangian submanifold in an Einstein–Käher manifold $M$ with Einstein constant $\kappa$, the second variational formula becomes much simpler. We see that $L$ is H-stable if and only if the first (positive) eigenvalue $\lambda_1$ of the Laplacian of $L$ acting on smooth functions satisfies $\lambda_1 \geq \kappa$ [**33**]. On the other hand, it is known that the first eigenvalue $\lambda_1$ of the Laplacian of any compact minimal Lagrangian submanifold $L$ in a compact homogeneous Einstein–Kähler manifold with positive Einstein constant $\kappa$ has the upper bound $\lambda_1 \leq \kappa$ [**38**, **39**]. In this case, $L$ is H-stable if and only if $\lambda_1 = \kappa$.

Assume that $(M, \omega, J, g)$ is a Kähler manifold and that $G$ is an analytic subgroup of its automorphism group $\mathrm{Aut}(M, \omega, J, g)$. A Lagrangian orbit $L = G \cdot x \subset M$ of $G$ is called a *homogeneous Lagrangian submanifold* of $M$. An easy but useful observation can be given as follows.

**Proposition 1.1.** *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is Hamiltonian minimal.*

*Proof.* Since $\alpha_{\mathrm{H}}$ is an invariant 1-form on $L$, $\delta\alpha_{\mathrm{H}}$ is a constant function on $L$. Hence by the divergence theorem we obtain $\delta\alpha_{\mathrm{H}} = 0$. q.e.d.

Set

$$\tilde{G} := \{a \in \mathrm{Aut}(M, \omega, J, g) \mid a(L) = L\}.$$

Then $G \subset \tilde{G}$ and $\tilde{G}$ is the maximal subgroup of $\mathrm{Aut}(M, \omega, J, g)$ preserving $L$. Moreover, we have $n_{hk}(\varphi) = \dim(\mathrm{Aut}(M, \omega, J, g)) - \dim(\tilde{G})$.

## 2. Gauss maps of isoparametric hypersurfaces in a sphere

**2.1. Gauss maps of oriented hypersurfaces in spheres.** Let $N^n$ be an oriented hypersurface immersed in the unit standard sphere $S^{n+1}(1) \subset \mathbf{R}^{n+2}$. Denote by $\mathbf{x}$ its position vector of a point $p$ of $N$, and denote $\mathbf{n}$ the unit normal vector field of $N$ in $S^{n+1}(1)$. It is a fundamental fact in symplectic geometry that the *Gauss map* defined by

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$$

is always a Lagrangian immersion in the complex hyperquadric $Q_n(\mathbf{C})$. Here the complex hyperquadric $Q_n(\mathbf{C})$ is identified with the real Grassmann manifold $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ of oriented 2-dimensional vector subspaces of $\mathbf{R}^{n+2}$, which has a symmetric space expression $SO(n+2)/(SO(2) \times SO(n))$.

Let $g^{std}_{Q_n(\mathbf{C})}$ be the standard Kähler metric of $Q_n(\mathbf{C})$ induced from the standard inner product of $\mathbf{R}^{n+2}$. Note that the Einstein constant of $g^{std}_{Q_n(\mathbf{C})}$ is equal to $n$. Let $\kappa_i$ $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$, and let H denote the mean curvature vector field of the Gauss map $\mathcal{G}$. Palmer showed the following mean curvature form formula [**44**]:

$$\alpha_{\mathrm{H}} = -d\left(\sum_{i=1}^{n} \operatorname{arc}\cot\kappa_i\right) = d\left(\operatorname{Im}\left(\log\prod_{i=1}^{n}(1+\sqrt{-1}\kappa_i)\right)\right). \tag{2.1}$$

Hence, if $N^n$ is an oriented austere hypersurface in $S^{n+1}(1)$, introduced by Harvey and Lawson [**17**], then its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. In particular, since any minimal surface in $S^3(1)$ is austere, its Gauss map is a minimal Lagrangian immersion in $Q_2(\mathbf{C}) \cong S^2 \times S^2$ [**9**]. Note that more minimal Lagrangian submanifolds of complex hyperquadrics can be obtained from Gauss maps of certain oriented hypersurfaces in spheres through Palmer's formula [**22**].

**2.2. Gauss maps of isoparametric hypersurfaces in spheres.** Now suppose that $N^n$ is a compact oriented hypersurface in $S^{n+1}(1)$ with constant principal curvatures—that is, an *isoparametric hypersurface*. By Münzner's result [**31**, **32**], the number $g$ of distinct principal curvatures must be 1, 2, 3, 4, or 6, and the distinct principal curvatures have the multiplicities $m_1 = m_3 = \cdots$, $m_2 = m_4 = \cdots$. We may assume that $m_1 \leq m_2$. It follows from (2.1) that its Gauss map $\mathcal{G} : N^n \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion. Moreover, the "Gauss image"of $\mathcal{G}$ is a compact minimal Lagrangian submanifold $L^n = \mathcal{G}(N^n) \cong N^n/\mathbf{Z}_g$ *embedded* in $Q_n(\mathbf{C})$ so that $\mathcal{G} : N^n \to \mathcal{G}(N^n) = L^n$ is a covering map with the Deck transformation group $\mathbf{Z}_g$ [**25**, **26**, **37**]. Note that the Gauss image $\mathcal{G}(N^n)$ is orientable if and only if $2n/g$ is even ([**37**]).

Here we mention the following symplectic topological properties of the Gauss images of isoparametric hypersurfaces.

**Theorem 2.1.** *The Gauss image $L = \mathcal{G}(N^n)$ is a compact monotone and cyclic Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ and its minimal Maslov number $\Sigma_L$ is given by*

$$\Sigma_L = \frac{2n}{g} = \begin{cases} m_1 + m_2, & \textit{if } g \textit{ is even,} \\ 2m_1, & \textit{if } g \textit{ is odd.} \end{cases}$$

We need to use the following result from H. Ono [**38**] which generalizes Oh's work [**36**].

**Lemma 2.1** ([**38**])**.** *Let $M$ be a simply connected Kähler–Einstein manifold with positive scalar curvature with a prequantization complex line bundle $E$. Then any compact minimal Lagrangian submanifold $L$ in $M$ is monotone and cyclic. Moreover, the minimal Maslov number $\Sigma_L$ of $L$ satisfies*

$$n_L\,\Sigma_L \;=\; 2\,\gamma_{c_1}, \tag{2.2}$$

*where*

$$\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M;\mathbb{Z}), c_1(M)(A) > 0\} \in \mathbf{Z}$$

*is called the index of a Kähler manifold $M$ and*

$$n_L := \min\{k \in \mathbf{Z} \mid k \geq 1, \otimes^k(E,\nabla)_{|L} \textit{ is trivial}\}.$$

Using this lemma and the properties of isoparametric hypersurfaces in a sphere, we shall prove Theorem 2.1.

*Proof.* It follows from Lemma 2.1 and the minimality of the Gauss image $L = \mathcal{G}(N^n)$ that $L$ is a monotone and cyclic Lagrangian submanifold in $Q_n(\mathbf{C})$. Note that the index of $Q_n(\mathbf{C})$ is known as follows [**6**]: $\gamma_{c_1} = n$ if $n \geq 2$ and $\gamma_{c_1} = 2$ if $n = 1$. So in order to find the minimal Maslov number $\Sigma_L$ of $L$, we only need to compute $n_L$. Let $\tilde{N}^n$ be the Legendrian lift of $N^n$ to the unit tangent sphere bundle $UTS^{n+1}(1) = V_2(\mathbf{R}^{n+2})$. Then $\pi : V_2(\mathbf{R}^{n+2})|_L \to L = \mathcal{G}(N^n)$ is a flat principal fiber bundle with structure group $SO(2)$ and the covering map $\pi : \tilde{N}^n \to \mathcal{G}(N^n)$ with Deck transformation group $\mathbf{Z}_g$ coincides with its holonomy subbundle with the holonomy group $\mathbf{Z}_g$. Let $E$ be a complex line bundle over $Q_n(\mathbf{C})$ associated with the principal fiber bundle $\pi : V_2(\mathbf{R}^{n+2}) \to \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C})$ by the standard action of $SO(2) \cong U(1)$ on $\mathbf{C}$. Then $E|_L$ is a flat complex line bundle over $\mathcal{G}(N^n)$ associated with the principal fiber bundle $\pi : V_2(\mathbf{R}^{n+2})|_L \to \mathcal{G}(N^n)$ by the standard action of $SO(2) \cong U(1)$ on $\mathbf{C}$. The tautological complex line bundle $\mathcal{W}$ over $Q_n(\mathbf{C}) \subset \mathbf{C}P^{n+1}$ is defined by $\mathcal{W}_x := \mathbf{C}(\mathbf{a} + \sqrt{-1}\mathbf{b})$ for each $[\mathbf{a} + \sqrt{-1}\mathbf{b}] \in Q_n(\mathbf{C})$. Then $E = \mathcal{W}$ if $n \geq 2$ and $\otimes^2 E = \mathcal{W}$ if $n = 1$. Indeed, $c_1(\mathcal{W})(\mathbf{C}P^1) = 1$ if $n \geq 2$. Here $\mathbf{C}P^1$ denotes the set of 1-dimensional complex vector subspaces in a 2-dimensional isotropic vector subspace of $\mathbf{C}^{n+2}$. For $k = 1, \cdots, g$, the generator $e^{\sqrt{-1}\frac{2\pi}{g}}$ of the holonomy group $\mathbf{Z}_g$ on $E|_L$ induces the multiplication by $e^{\sqrt{-1}\frac{2\pi k}{g}}$ on $\otimes^k E|_L$. Thus the holonomy group of $\otimes^k E|_L$ is generated by $e^{\sqrt{-1}\frac{2\pi k}{g}}$ of $\mathbf{Z}_g$. Hence $\otimes^k E|_L$ has nontrivial holonomy for $k = 1, \cdots, g-1$, and $\otimes^g E|_L$ has trivial holonomy. Therefore, $n_L = g$ if $n \geq 2$ and $n_L = 2$ if $n = 1$. Thus the conclusion follows from (2.2). q.e.d.

A hypersurface $N^n$ in $S^{n+1}(1)$ is *homogeneous* if it is obtained as an orbit of a compact connected subgroup $G$ of $SO(n+2)$. Obviously any homogeneous hypersurface in $S^{n+1}(1)$ is an isoparametric hypersurface. It turns out that $N^n$ is homogeneous if and only if its Gauss image $\mathcal{G}(N^n)$ is homogeneous [**25**].

Consider

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \subset \bigwedge^2 \mathbf{R}^{n+2}.$$

Here $\bigwedge^2 \mathbf{R}^{n+2} \cong \mathfrak{o}(n+2)$ can be identified with the Lie algebra of all (holomorphic) Killing vector fields on $S^{n+1}(1)$ or $\widetilde{Gr}_2(\mathbf{R}^{n+2})$. Let $\tilde{\mathfrak{k}}$ be the Lie subalgebra of $\mathfrak{o}(n+2)$ consisting of all Killing vector fields tangent to $N^n$ or $\mathcal{G}(N^n)$, and let $\tilde{K}$ be a compact connected Lie subgroup of $SO(n+2)$ generated by $\tilde{\mathfrak{k}}$. Take the orthogonal direct sum

$$\bigwedge^2 \mathbf{R}^{n+2} = \tilde{\mathfrak{k}} + \mathcal{V},$$

where $\mathcal{V}$ is a vector subspace of $\mathfrak{o}(n+2)$. The linear map

$$\mathcal{V} \ni X \longmapsto \alpha_X|_{\mathcal{G}(N^n)} \in E_0(\mathcal{G}) \subset B^1(\mathcal{G}(N^n))$$

is injective, and $n_{hk}(\mathcal{G}) = \dim \mathcal{V}$. Then $\mathcal{G}(N^n) \subset \mathcal{V}$, and thus

$$\mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}.$$

Indeed, for each $X \in \tilde{\mathfrak{k}}$ and each $p \in N^n$, $\langle X, \mathbf{x}(p) \wedge \mathbf{n}(p)\rangle = \langle X\mathbf{x}(p), \mathbf{n}(p)\rangle - \langle \mathbf{x}(p), X\mathbf{n}(p)\rangle = 2\langle X\mathbf{x}(p), \mathbf{n}(p)\rangle = 0$.

Note that $\mathcal{G}(N^n)$ is a compact minimal submanifold embedded in the unit hypersphere of $\mathcal{V}$ and that by the theorem of Tsunero Takahashi each coordinate function of $\mathcal{V}$ restricted to $\mathcal{G}(N^n)$ is an eigenfunction of the Laplace operator with eigenvalue $n$. Then we observe the following.

**Lemma 2.2.** *The number $n$ is just the first (positive) eigenvalue of $\mathcal{G}(N^n)$ if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian stable. Moreover the dimension of the vector space $\mathcal{V}$ is equal to the multiplicity of the (resp. first) eigenvalue $n$ if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is Hamiltonian rigid (resp. strictly Hamiltonian stable).*

Next we mention a relationship between the Gauss images $\mathcal{G}(N^n)$ of isoparametric hypersurfaces and the intersection $\widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$. In [**26**] we showed that if $N^n$ is homogeneous, then $\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$.

Define a map $\mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \to \bigwedge^2 \mathbf{R}^{n+2}$ by

$$\mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \ni [W] \longmapsto \mathbf{a} \wedge \mathbf{b} \in \bigwedge^2 \mathbf{R}^{n+2} \cong \mathfrak{o}(n+2) = \tilde{\mathfrak{k}} + \mathcal{V}.$$

The moment map of the action $\tilde{K}$ on $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ is given by $\mu_{\tilde{\mathfrak{k}}} := \pi_{\tilde{\mathfrak{k}}} \circ \mu : \widetilde{Gr}_2(\mathbf{R}^{n+2}) \to \tilde{\mathfrak{k}}$, where $\pi_{\tilde{\mathfrak{k}}} : \mathfrak{o}(n+2) \to \tilde{\mathfrak{k}}$ denotes the orthogonal projection onto $\tilde{\mathfrak{k}}$. For any $p \in N^n$, we have

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subset \mathcal{G}(N^n) \subset \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

It is obvious that $N^n$ is homogeneous if and only if $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$. On the other hand, assume that $\mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V}$. Then $\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) = \mathcal{G}(N^n)$, that is, $N^n$ is homogeneous. Therefore we obtain (see [**26**]) that $N^n$ is not homogeneous if and only if

$$\tilde{K}(\mathbf{x}(p) \wedge \mathbf{n}(p)) \subsetneqq \mathcal{G}(N^n) \subsetneqq \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cap \mathcal{V} = \mu_{\tilde{\mathfrak{k}}}^{-1}(0).$$

All isoparametric hypersurfaces in spheres are classified as either homogeneous or nonhomogeneous. By Hsiang and Lawson [**17**] and Takagi and Takahashi [**46**], any homogeneous isoparametric hypersurface in a sphere can be obtained as a principal orbit of the isotropy representation of a compact Riemannian symmetric pair $(U, K)$ of rank 2 (see Table 1).

Compact homogeneous minimal Lagrangian submanifolds obtained as the Gauss images of homogeneous isoparametric hypersurfaces are constructed in the following way (cf. [**25**]). Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ as a symmetric Lie algebra of a symmetric pair $(U, K)$ of rank 2, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Define an $\mathrm{Ad}U$-invariant inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ of $\mathfrak{u}$ from the Killing–Cartan form of $\mathfrak{u}$. Then the vector space $\mathfrak{p}$ equipped with the inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$ can be identified with the Euclidean space $\mathbf{R}^{n+2}$ and $S^{n+1}(1)$ denotes the $(n+1)$-dimensional unit standard sphere in $\mathfrak{p}$. The linear isotropy action $\mathrm{Ad}_{\mathfrak{p}}$ of $K$ on $\mathfrak{p}$ and thus on $S^{n+1}(1)$ induces the group action of $K$ on $\widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbf{C})$. For each *regular* element $H$ of $\mathfrak{a} \cap S^{n+1}(1)$, we get a homogeneous isoparametric hypersurface in the unit sphere

$$N^n = (\mathrm{Ad}_{\mathfrak{p}}K)H \subset S^{n+1}(1) \subset \mathfrak{p} \cong \mathbf{R}^{n+2}.$$

Its Gauss image is

$$L^n = \mathcal{G}(N^n) = K \cdot [\mathfrak{a}] = [(\mathrm{Ad}_{\mathfrak{p}}K)\mathfrak{a}] \subset \widetilde{Gr}_2(\mathfrak{p}) \cong Q_n(\mathbf{C}).$$

Here $N$ and $\mathcal{G}(N^n)$ have homogeneous space expressions $N \cong K/K_0$ and $\mathcal{G}(N^n) \cong K/K_{[\mathfrak{a}]}$, where we define

$$\begin{aligned}
K_0 &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H\} \\
&\quad = \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(H) = H \text{ for each } H \in \mathfrak{a}\}, \\
K_{\mathfrak{a}} &:= \{k \in K \mid \mathrm{Ad}_{\mathfrak{p}}(k)(\mathfrak{a}) = \mathfrak{a}\}, \\
K_{[\mathfrak{a}]} &:= \{k \in K_{\mathfrak{a}} \mid \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \longrightarrow \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}.
\end{aligned}$$

The deck transformation group of the covering map $\mathcal{G} : N \to \mathcal{G}(N^n)$ is equal to $K_{[\mathfrak{a}]}/K_0 = W(U, K)/\mathbf{Z}_2 \cong \mathbf{Z}_g$, where $W(U, K) = K_{\mathfrak{a}}/K_0$ is the Weyl group of $(U, K)$.

Since we know that $\mathrm{Ad}_{\mathfrak{p}}K$ is the maximal compact subgroup of $SO(n+2)$ preserving $N$ and/or $\mathcal{G}(N^n)$ [**18**, **25**], in this case its nullity is given as

$$n_{hk}(\mathcal{G}) = n_{hk}(\mathcal{G}(N^n)) = \dim SO(n+2) - \dim K.$$

## 3. The method of eigenvalue computations for our compact homogeneous spaces

**3.1. Basic results from harmonic analysis on compact homogeneous spaces.** Now we review the basic theory of harmonic analysis on general compact homogeneous spaces (cf. [**48**]). Let $\mathcal{D}(G)$ be the

**Table 1.** Homogeneous isoparametric hypersurfaces in spheres

| $g$ | $Type$ | $(U, K)$ | dimN | $m_1, m_2$ | $K/K_0$ |
|---|---|---|---|---|---|
| 1 | $S^1\times$ BDII | $(S^1 \times SO(n+2), SO(n+1))$ $(n \geq 1)$ $[\mathbf{R} \oplus A_1]$ | $n$ | $n$ | $S^n$ |
| 2 | BDII$\times$ BDII | $(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $(1 \leq p \leq n-1)$ $[A_1 \oplus A_1]$ | $n$ | $p, n-p$ | $S^p \times S^{n-p}$ |
| 3 | AI$_2$ | $(SU(3), SO(3))$ $[A_2]$ | 3 | $1, 1$ | $\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$ |
| 3 | $\mathfrak{a}_2$ | $(SU(3) \times SU(3), SU(3))$ $[A_2]$ | 6 | $2, 2$ | $\frac{SU(3)}{T^2}$ |
| 3 | AII$_2$ | $(SU(6), Sp(3))$ $[A_2]$ | 12 | $4, 4$ | $\frac{Sp(3)}{Sp(1)^3}$ |
| 3 | EIV | $(E_6, F_4)$ $[A_2]$ | 24 | $8, 8$ | $\frac{F_4}{Spin(8)}$ |
| 4 | $\mathfrak{b}_2$ | $(SO(5) \times SO(5), SO(5))$ $[B_2]$ | 8 | $2, 2$ | $\frac{SO(5)}{T^2}$ |
| 4 | AIII$_2$ | $(SU(m+2), S(U(2) \times U(m)))$ $(m \geq 2)$ $[BC_2](m \geq 3), [B_2](m = 2)$ | $4m-2$ | $2,$ $2m-3$ | $\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$ |
| 4 | BDI$_2$ | $(SO(m+2), SO(2) \times SO(m))$ $(m \geq 3)$ $[B_2]$ | $2m-2$ | $1,$ $m-2$ | $\frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$ |
| 4 | CII$_2$ | $(Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$ $[BC_2](m \geq 3), [B_2](m = 2)$ | $8m-2$ | $4,$ $4m-5$ | $\frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$ |
| 4 | DIII$_2$ | $(SO(10), U(5))$ $[BC_2]$ | 18 | $4, 5$ | $\frac{U(5)}{SU(2)\times SU(2)\times U(1)}$ |
| 4 | EIII | $(E_6, U(1) \cdot Spin(10))$ $[BC_2]$ | 30 | $6, 9$ | $\frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$ |
| 6 | $\mathfrak{g}_2$ | $(G_2 \times G_2, G_2)$ $[G_2]$ | 12 | $2, 2$ | $\frac{G_2}{T^2}$ |
| 6 | G | $(G_2, SO(4))$ $[G_2]$ | 6 | $1, 1$ | $\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$ |

complete set of all inequivalent irreducible unitary representations of a compact connected Lie group $G$. For a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, let $\Sigma(G)$ be the set of all roots of $\mathfrak{g}$ and $\Sigma^+(G)$ be its subset of all positive root $\alpha \in \Sigma(G)$ relative to a linear order fixed on $\mathfrak{t}$. Set

$$\begin{aligned}
&\Gamma(G) := \{\xi \in \mathfrak{t} \mid \exp(\xi) = e\},\\
&Z(G) := \{\Lambda \in \mathfrak{t}^* \mid \Lambda(\xi) \in 2\pi\mathbf{Z} \text{ for each } \xi \in \Gamma(G)\},\\
&D(G) := \{\Lambda \in Z(G) \mid \langle \Lambda, \alpha\rangle \geq 0 \text{ for each } \alpha \in \Sigma^+(G)\}.
\end{aligned}$$

Then there is a bijective correspondence between $D(G) \ni \Lambda \longmapsto (V_\Lambda, \rho_\Lambda) \in \mathcal{D}(G)$, where $(V_\Lambda, \rho_\Lambda)$ denotes an irreducible unitary representation of $G$ with the highest weight $\Lambda$ equipped with a $\rho_\Lambda(G)$-invariant Hermitian inner product $\langle\ ,\ \rangle_{V_\Lambda}$. Let $\langle\ ,\ \rangle_{\mathfrak{g}}$ be an Ad$G$-invariant inner product of $\mathfrak{g}$. For a compact Lie subgroup $H$ of $G$ with Lie subalgebra $\mathfrak{h}$, we take the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ relative to $\langle\ ,\ \rangle_{\mathfrak{g}}$. Set

$$D(G, H) := \{\Lambda \in D(G) \mid (V_\Lambda)_H \neq \{0\}\}, \tag{3.1}$$

where

$$(V_\Lambda)_H := \{w \in V_\Lambda \mid \rho_\Lambda(a)w = w \ (\forall a \in H)\}. \tag{3.2}$$

Let $\Lambda \in D(G,H)$. For each $\bar{w}\otimes v \in (V_\Lambda)^*_H \otimes V_\Lambda$, we define a real analytic function $f_{\bar{w}\otimes v}$ on $G/H$ by

$$(f_{\bar{w}\otimes v})(aH) := \langle v, \rho_\Lambda(a)w\rangle_{V_\Lambda} \tag{3.3}$$

for all $aH \in G/H$. By virtue of the Peter–Weyl theorem and the Frobenius reciprocity law, we have a linear injection

$$(V_\Lambda)^*_H \otimes V_\Lambda \ni \bar{w}\otimes v \longmapsto f_{\bar{w}\otimes v} \in C^\infty(G/H, \mathbf{C}) \tag{3.4}$$

and the decomposition

$$C^\infty(G/H, \mathbf{C}) = \bigoplus_{\Lambda \in D(G,H)} (V_\Lambda)^*_H \otimes V_\Lambda \tag{3.5}$$

in the sense of $C^\infty$-topology. Via the natural homogeneous projection $\pi : G \to G/H$, the vector space $C^\infty(G/H, \mathbf{C})$ of all complex valued smooth functions on $G/H$ can be identified with the vector space $C^\infty(G, \mathbf{C})_H$ of all complex valued smooth functions on $G$ invariant under the right action of $H$. Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of Lie algebra $\mathfrak{g}$, which is identified with the algebra of all left-invariant linear differential operators on $C^\infty(G, \mathbf{C})$. Let

$$\mathrm{U}(\mathfrak{g})_H := \{D \in \mathrm{U}(\mathfrak{g}) \mid \mathrm{Ad}(h)D = R_h \circ D \circ R_{h^{-1}} = D \text{ for each } h \in H\}$$

be the subalgebra of $\mathrm{U}(\mathfrak{g})$ consisting of elements fixed by the adjoint action of $H$. Here $(R_h\tilde{f})(u) := \tilde{f}(uh)$ for $\tilde{f} \in C^\infty(G, \mathbf{C})$. For each $D \in \mathrm{U}(\mathfrak{g})_H$, we have $D(C^\infty(G, \mathbf{C})_H) \subset C^\infty(G, \mathbf{C})_H$. The *Casimir operator* $\mathcal{C}_{G/H,\langle\ ,\ \rangle_\mathfrak{g}}$ of $(G,H)$ relative to $\langle\ ,\ \rangle_\mathfrak{g}$ is defined by $\mathcal{C} = \mathcal{C}_{G/H,\langle\ ,\ \rangle_\mathfrak{g}} := \sum_{i=1}^n (X_i)^2$, where $\{X_i \mid i = 1, \cdots, n\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle_\mathfrak{g}$. Then $\mathcal{C}_{G/H,\langle\ ,\ \rangle_\mathfrak{g}} \in \mathrm{U}(\mathfrak{g})_H$ and by the $\mathrm{Ad}G$-invariance of $\langle\ ,\ \rangle_\mathfrak{g}$ and Schur's Lemma there is a nonpositive real constant $c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g})$ such that

$$\mathcal{C}_{G/H,\langle\ ,\ \rangle_\mathfrak{g}}(f_{\bar{w}\otimes v}) = c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g}) f_{\bar{w}\otimes v} \tag{3.6}$$

for each $\bar{w}\otimes v \in (V_\Lambda)^*_H \otimes V_\Lambda$. The eigenvalue $c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g})$ is given by the Freudenthal's formula

$$c(\Lambda, \langle\ ,\ \rangle_\mathfrak{g}) = -\langle \Lambda, \Lambda + 2\delta\rangle_\mathfrak{g}, \tag{3.7}$$

where $2\delta = \sum_{\alpha \in \Sigma^+(G)} \alpha$.

Now we shall consider our compact homogeneous spaces $N^n = K/K_0$ and $L^n = \mathcal{G}(N^n) = K/K_{[\mathfrak{a}]}$ ([**25**]). Let $\Sigma(U,K)$ be the set of (restricted) roots of $(\mathfrak{u}, \mathfrak{k})$, and let $\Sigma^+(U,K)$ be its subset of positive roots. We have the following root decompositions of $\mathfrak{k}$ and $\mathfrak{p}$ as follows:

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{k}_\gamma, \qquad \mathfrak{p} = \mathfrak{a} + \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{p}_\gamma,$$

where

$$\begin{aligned}\mathfrak{k}_0 :=&\{X \in \mathfrak{k} \mid [X, \mathfrak{a}] \subset \mathfrak{a}\}\\ =&\{X \in \mathfrak{k} \mid [X, H] = 0 \quad \text{for each } H \in \mathfrak{a}\},\\ \mathfrak{k}_\gamma :=&\{X \in \mathfrak{k} \mid (\mathrm{ad}H)^2 X = (\gamma(H))^2 X \text{ for each } H \in \mathfrak{a}\},\\ \mathfrak{p}_\gamma :=&\{Y \in \mathfrak{p} \mid (\mathrm{ad}H)^2 Y = (\gamma(H))^2 Y \text{ for each } H \in \mathfrak{a}\}.\end{aligned}$$

For each $\gamma \in \Sigma^+(U, K)$, set $m(\gamma) := \dim \mathfrak{k}_\gamma = \dim \mathfrak{p}_\gamma$. Define

$$\mathfrak{m} := \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{k}_\gamma \quad \text{and} \quad \mathfrak{a}^\perp := \sum_{\gamma \in \Sigma^+(U,K)} \mathfrak{p}_\gamma. \tag{3.8}$$

Then the tangent vector spaces $T_{eK_0}(K/K_0)$ and $T_{eK_{[\mathfrak{a}]}}(K/K_{[\mathfrak{a}]})$ can be identified with the vector subspace $\mathfrak{m}$ of $\mathfrak{k}$. We can choose an orthonormal basis of $\mathfrak{m}$ and $\mathfrak{a}^\perp$ with respect to $\langle\ ,\ \rangle_\mathfrak{u}$

$$\{X_{\gamma,i} \in \mathfrak{k}_\gamma \mid \gamma \in \Sigma^+(U, K), i = 1, 2, \cdots, m(\gamma)\}$$

and

$$\{Y_{\gamma,i} \in \mathfrak{p}_\gamma \mid \gamma \in \Sigma^+(U, K), i = 1, 2, \cdots, m(\gamma)\} \tag{3.9}$$

such that

$$[H, X_{\gamma,i}] = \sqrt{-1}\gamma(H)Y_{\gamma,i}, \quad [H, Y_{\gamma,i}] = -\sqrt{-1}\gamma(H)X_{\gamma,i} \tag{3.10}$$

for each $H \in \mathfrak{a}$. Let $\langle\ ,\ \rangle$ denote the $\mathrm{Ad}_\mathfrak{m}(K_0)$-invariant inner product of $\mathfrak{m}$ corresponding to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ on $K/K_0$. Thus we know (see [**25**]) that

$$\left\{ \frac{1}{\|\gamma\|_\mathfrak{u}} X_{\gamma,i} \mid \gamma \in \Sigma^+(U, K), i = 1, 2, \cdots, m(\gamma) \right\}$$

is an orthonormal basis of $\mathfrak{m}$ relative to $\langle\ ,\ \rangle$.

The Laplace operator $\Delta^0_{L^n} = \delta d$ acting on $C^\infty(K/K_0, \mathbf{C})$ with respect to the induced metric $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ corresponds to the linear differential operator $-\mathcal{C}_{L^n}$ on $C^\infty(K, \mathbf{C})_{K_0}$, where $\mathcal{C}_{L^n} \in \mathrm{U}(\mathfrak{k})$ is the Casimir operator relative to the $\mathrm{Ad}_\mathfrak{m}(K_0)$-invariant inner product $\langle\ ,\ \rangle$ of $\mathfrak{m}$ defined by

$$\mathcal{C}_{L^n} := \sum_{\gamma \in \Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} \frac{1}{\|\gamma\|^2_\mathfrak{u}} (X_{\gamma,i})^2. \tag{3.11}$$

Note that $\mathcal{C}_{L^n} \in \mathrm{U}(\mathfrak{k})_{K_0}$ because of the $\mathrm{Ad}_\mathfrak{m}(K_0)$-invariance of $\langle\ ,\ \rangle$.

Suppose that $\Sigma(U, K)$ is irreducible. Let $\gamma_0$ denote the highest root of $\Sigma(U, K)$. For $g = 3, 4$, or $6$, the restricted root system $\Sigma(U, K)$ is of type $A_2$, $B_2$, $BC_2$, or $G_2$. Then we know that for each $\gamma \in \Sigma^+(U, K)$,

$$\frac{\|\gamma\|^2_\mathfrak{u}}{\|\gamma_0\|^2_\mathfrak{u}} = \begin{cases} 1 & \text{if } \Sigma(U, K) \text{ is of type } A_2,\\ 1 \text{ or } 1/3 & \text{if } \Sigma(U, K) \text{ is of type } G_2,\\ 1 \text{ or } 1/2 & \text{if } \Sigma(U, K) \text{ is of type } B_2,\\ 1, 1/2 \text{ or } 1/4 & \text{if } \Sigma(U, K) \text{ is of type } BC_2.\end{cases}$$

Set

$$\Sigma_1^+(U,K) := \{\gamma \in \Sigma^+(U,K) \mid \|\gamma\|_{\mathfrak{u}}^2 = \|\gamma_0\|_{\mathfrak{u}}^2\}. \tag{3.12}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_1, \mathfrak{k}_1)$ by

$$\begin{aligned} &\mathfrak{k}_1 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma_1^+(U,K)} \mathfrak{k}_\gamma, \quad \mathfrak{p}_1 := \mathfrak{a} + \sum_{\gamma \in \Sigma_1^+(U,K)} \mathfrak{p}_\gamma, \\ &\mathfrak{u}_1 := \mathfrak{k}_1 + \mathfrak{p}_1. \end{aligned}$$

Let $K_1$ and $U_1$ denote connected compact Lie subgroups of $K$ and $U$ generated by $\mathfrak{k}_1$ and $\mathfrak{u}_1$.

Suppose that $\Sigma^+(U,K)$ is of type $BC_2$. Define

$$\Sigma_2^+(U,K) := \{\gamma \in \Sigma^+(U,K) \mid \|\gamma\|_{\mathfrak{u}}^2 = \|\gamma_0\|_{\mathfrak{u}}^2 \text{ or } \|\gamma_0\|_{\mathfrak{u}}^2/2\}. \tag{3.13}$$

Define a symmetric Lie subalgebra $(\mathfrak{u}_2, \mathfrak{k}_2)$ by

$$\begin{aligned} &\mathfrak{k}_2 := \mathfrak{k}_0 + \sum_{\gamma \in \Sigma_2^+(U,K)} \mathfrak{k}_\gamma, \quad \mathfrak{p}_2 := \mathfrak{a} + \sum_{\gamma \in \Sigma_2^+(U,K)} \mathfrak{p}_\gamma, \\ &\mathfrak{u}_2 := \mathfrak{k}_2 + \mathfrak{p}_2. \end{aligned}$$

Let $K_2$ and $U_2$ denote connected compact Lie subgroups of $K$ and $U$ generated by $\mathfrak{k}_2$ and $\mathfrak{u}_2$. We have the following subgroups of $K$ in each case:

$$\begin{array}{ll} K_0 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } A_2, \\ K_0 \subset K_1 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } B_2 \text{ or } G_2, \\ K_0 \subset K_1 \subset K_2 \subset K, & \text{if } \Sigma(U,K) \text{ is of type } BC_2. \end{array}$$

Set

$$\begin{aligned} \mathcal{C}_{K/K_0, \langle\ ,\ \rangle_{\mathfrak{u}}} &:= \sum_{\gamma \in \Sigma^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2, \\ \mathcal{C}_{K_1/K_0, \langle\ ,\ \rangle_{\mathfrak{u}}} &:= \sum_{\gamma \in \Sigma_1^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2, \qquad (3.14) \\ \mathcal{C}_{K_2/K_0, \langle\ ,\ \rangle_{\mathfrak{u}}} &:= \sum_{\gamma \in \Sigma_2^+(U,K)} \sum_{i=1}^{m(\gamma)} (X_{\gamma,i})^2. \end{aligned}$$

Then $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_1/K_0}, \mathcal{C}_{K_2/K_0} \in \mathrm{U}(\mathfrak{k})_{K_0}$, and the Casimir operator $\mathcal{C}_{L^n}$ can be decomposed as follows:

**Lemma 3.1.**

$$\mathcal{C}_{L^n} = \begin{cases} \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \textit{if } \Sigma(U,K) \textit{ is of type } A_2, \\[2ex] \dfrac{3}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \textit{if } \Sigma(U,K) \textit{ is of type } G_2, \\[2ex] \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \textit{if } \Sigma(U,K) \textit{ is of type } B_2, \\[2ex] \dfrac{4}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \dfrac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} & \\ \qquad\qquad \textit{if } \Sigma(U,K) \textit{ is of type } BC_2. \end{cases}$$

Moreover, by direct computations we obtain the following.

**Lemma 3.2.** *The Casimir operators $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_1/K_0}$ (and $\mathcal{C}_{K_2/K_0}$) commute with each other.*

See also Theorem 1.5 and Theorem 3.6 in [**5**] for more general results. The commuting property implies the existence of simultaneous eigenfunctions for the Casimir operators. The choice of such eigenfunctions will be performed concretely in our settings.

**3.2. Fibrations on homogeneous isoparametric hypersurfaces by homogeneous isoparametric hypersurfaces.** For $g = 4$ or $6$, $(U,K)$ is of type $G_2$, $B_2$, or $BC_2$ as indicated in the 3rd column of Table 1.

In the case when $(U,K)$ is of type $B_2$ or $G_2$, we have one fibration as follows:

$$\begin{tikzcd} N^n = K/K_0 \arrow[d, "K_1/K_0"] \\ K/K_1 \end{tikzcd}$$

In the case when $(U,K)$ is of type $BC_2$, we have the following two fibrations:

$$\begin{CD} N^n = K/K_0 @>{=}>> K/K_0 \\ @V{K_1/K_0}VV @VV{K_2/K_0}V \\ K/K_1 @>{K_2/K_1}>> K/K_2 \end{CD}$$

**3.2.1. In the case $g = 6$ and $(U, K) = (G_2, SO(4))$, $(m_1, m_2) = (1, 1)$.**

$$\begin{CD} N^6 = K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2) \\ @VV{K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2)}V \\ K/K_1 = SO(4)/SO(3) \cong S^3 \end{CD}$$

Here $U_1/K_1 = SU(3)/SO(3)$ is a maximal totally geodesic submanifold of $U/K = G_2/SO(4)$. $K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2)$ is a homogeneous isoparametric hypersurface with $g = 6$, $m_1 = m_2 = 1$, and $K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2)$ is a homogenous isoparametric hypersurface with $g = 3$, $m_1 = m_2 = 1$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $G_2/SO(4)$ are classified as $SU(3)/SO(3)$, $\mathbf{C}P^2$, $S^2 \cdot S^2$.

**3.2.2. In the case $g = 6$ and $(U, K) = (G_2 \times G_2, G_2)$, $(m_1, m_2) = (2, 2)$.**

$$\begin{CD} N^{12} = K/K_0 = G_2/T^2 \\ @VV{K_1/K_0 = SU(3)/T^2}V \\ K/K_1 = G_2/SU(3) \cong S^6 \end{CD}$$

Here $U_1/K_1 = (SU(3) \times SU(3))/SU(3)$ is a maximal totally geodesic submanifold of $U/K = (G_2 \times G_2)/G_2$. $K/K_0 = G_2/T^2$ is a homogenous isoparametric hypersurface with $g = 6$, $m_1 = m_2 = 2$, and $K_1/K_0 = SU(3)/T^2$ is a homogenous isoparametric hypersurface with $g = 3$, $m_1 = m_2 = 2$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $G_2$ are classified as $G_2/SO(4)$, $SU(3)$, $S^3 \cdot S^3$.

**3.2.3. In the case $g = 4$ and $(U, K) = (SO(5) \times SO(5), SO(5))$, $(m_1, m_2) = (2, 2)$.**

$$\begin{array}{c} N^8 = K/K_0 = SO(5)/T^2 \\ \Big\downarrow \; K_1/K_0 = SO(4)/T^2 \\ K/K_1 = SO(5)/SO(4) \cong S^4 \end{array}$$

Here $U_1/K_1 = (SO(4) \times SO(4))/SO(4) \cong SO(4) \cong S^3 \cdot S^3$ is a maximal totally geodesic submanifold of $U/K = (SO(5) \times SO(5))/SO(5) \cong SO(5)$. $K/K_0 = SO(5)/T^2$ is a homogeneous isoparametric hypersurface with $g = 4$, $m_1 = m_2 = 2$, and $K_1/K_0 = SO(4)/T^2 \cong S^2 \times S^2$ is a homogeneous isoparametric hypersurface with $g = 2$, $m_1 = m_2 = 2$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $Sp(2) \cong Spin(5)$ are classified as $\widetilde{Gr}_2(\mathbf{R}^5)$, $S^1 \cdot S^3$, $S^3 \times S^3$, $S^4$.

**3.2.4. In the case $g = 4$ and $(U, K) = (SO(10), U(5))$, $(m_1, m_2) = (4, 5)$.**

$$\begin{tikzcd} N^{18} = \frac{U(5)}{SU(2)\times SU(2)\times U(1)} \arrow[r, "="] \arrow[d, "K_1/K_0 \cong S^1\times S^1"] & K/K_0 = \frac{U(5)}{SU(2)\times SU(2)\times U(1)} \arrow[d, "K_2/K_0 \cong \frac{U(4)}{SU(2)\times SU(2)}"] \\ K/K_1 = \frac{U(5)}{U(2)\times U(2)\times U(1)} \arrow[r, "K_2/K_1 \cong Gr_2(\mathbf{C}^4)"] & K/K_2 = \frac{U(5)}{U(4)\times U(1)} \end{tikzcd}$$

Here $U_2/K_2 = \frac{SO(8)\times SO(2)}{U(4)\times U(1)} \cong \frac{SO(8)}{U(4)} \cong \frac{SO(8)}{SO(2)\times SO(6)} \cong \widetilde{Gr}_2(\mathbf{R}^8)$ is a maximal totally geodesic submanifold of $U/K = SO(10)/U(5)$, but $U_1/K_1 = \frac{SO(4)\times SO(4)\times SO(2)}{U(2)\times U(2)\times U(1)} \cong \widetilde{Gr}_2(\mathbf{R}^4)$ is not a maximal totally geodesic submanifold of $U_2/K_2$. Notice that $K/K_0 = \frac{U(5)}{SU(2)\times SU(2)\times U(1)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 5)$, $K_2/K_0 = \frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)} \cong \frac{SO(2)\times SO(6)}{\mathbf{Z}_2\times SO(4)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (1, 4)$, and $K_1/K_0 = \frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)} \cong \frac{U(2)}{SU(2)} \times \frac{U(2)}{SU(2)} \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $\dfrac{SO(10)}{U(5)}$ are classified as $\widetilde{Gr}_2(\mathbf{R}^8)$, $Gr_2(\mathbf{C}^5)$, $SO(5)$, $S^2 \times \mathbf{C}P^3$, $\mathbf{C}P^4$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $\widetilde{Gr}_2(\mathbf{R}^8)$ are classifed as $\widetilde{Gr}_2(\mathbf{R}^7)$, $S^p \cdot S^q$ $(p+q=6)$, $\mathbf{C}P^3$.

**3.2.5. In the case $g=4$ and $(U,K)=(SO(m+2), SO(2)\times SO(m))$ $(m\geq 3)$, $(m_1,m_2)=(1,m-2)$.**

$$\begin{array}{l}
N^{2m-2} = K/K_0 = \frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)} \\
\quad\Big\downarrow \quad K_1/K_0 = \frac{SO(2)\times SO(2)\times SO(m-2)}{\mathbf{Z}_2\times SO(m-2)} \cong \frac{SO(2)\times SO(2)}{\mathbf{Z}_2} \cong S^1\times S^1 \\
K/K_1 = \frac{SO(2)\times SO(m)}{SO(2)\times SO(2)\times SO(m-2)} \cong \frac{SO(m)}{SO(2)\times SO(m-2)} \cong \widetilde{Gr}_2(\mathbf{R}^m)
\end{array}$$

Here $U_1/K_1 = \frac{SO(4)\times SO(m-2)}{SO(2)\times SO(2)\times SO(m-2)} \cong \widetilde{Gr}_2(\mathbf{R}^4) \cong S^2\times S^2$ is not a maximal totally geodesic submanifold of $U/K = \frac{SO(m+2)}{SO(2)\times SO(m)} \cong \widetilde{Gr}_2(\mathbf{R}^{m+2})$. Notice that $K/K_0 = \frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$ is a homogeneous isoparametric hypersurface with $g=4$, $(m_1,m_2)=(1,m-2)$, and $K_1/K_0 = \frac{SO(2)\times SO(2)\times SO(m-2)}{\mathbf{Z}_2\times SO(m-2)} \cong \frac{SO(2)\times SO(2)}{\mathbf{Z}_2} \cong S^1\times S^1$ is a homogeneous isoparametric hypersurface with $g=2$, $(m_1,m_2)=(1,1)$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $\widetilde{Gr}_2(\mathbf{R}^{m+2})$ $(m\geq 3)$ are classified as $\widetilde{Gr}_2(\mathbf{R}^{m+1})$, $S^p\cdot S^q (p+q=m)$, $\mathbf{C}P^{[\frac{m}{2}]}$.

**3.2.6. In the case $g=4$ and $(U,K)=(SU(m+2), S(U(2)\times U(m))$ $(m\geq 2)$, $(m_1,m_2)=(2,2m-3)$.**

(i) $m=2$, $(U,K)=(SU(4), S(U(2)\times U(2))$, $(m_1,m_2)=(2,1)$

$$\begin{array}{l}
N^6 = K/K_0 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))} \\
\quad\Big\downarrow \quad K_1/K_0 = \frac{S(U(1)\times U(1)\times U(1)\times U(1))}{S(U(1)\times U(1))} \cong S^1\times S^1 \\
K/K_1 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong S^2\times S^2
\end{array}$$

Here $U_1/K_1 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1)\times U(1)\times U(1))} \cong S^2\times S^2$ is not a maximal totally geodesic submanifold in $U/K = \frac{SU(4)}{S(U(2)\times U(2))} \cong Gr_2(\mathbf{C}^4) \cong \widetilde{Gr}_2(\mathbf{R}^6)$. Notice that $K/K_0 = \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (2, 1)$, and $K_1/K_0 \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

(ii) $m \geq 3$

$$\begin{tikzcd}
N^{4m-2} = \frac{K}{K_0} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0} \cong \frac{S(U(1)\times U(1)\times U(1)\times U(1))}{S(U(1)\times U(1))}"] & \frac{K}{K_0} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))} \arrow[d, "\frac{K_2}{K_0} \cong \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))}"] \\
\frac{K}{K_1} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))} \arrow[r, "\frac{K_2}{K_1} \cong \mathbf{C}P^1 \times \mathbf{C}P^1"] & \frac{K}{K_2} = \frac{S(U(2)\times U(m))}{S(U(2)\times U(2)\times U(m-2))}
\end{tikzcd}$$

Here $U_2/K_2 \cong Gr_2(\mathbf{C}^4)$ is not a maximal totally geodesic submanifold of $U/K = \frac{SU(m+2)}{S(U(2)\times U(m))} \cong Gr_2(\mathbf{C}^{m+2})$ and $U_1/K_1 = \frac{S(U(2)\times U(2)\times U(m-2))}{S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))} \cong \mathbf{C}P^1 \times \mathbf{C}P^1$ is not a maximal totally geodesic submanifold of $U_2/K_2$. Notice that $K/K_0 = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (2, 2m-3)$, $K_2/K_0 \cong \frac{S(U(2)\times U(2))}{S(U(1)\times U(1))}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (2, 1)$, and $K_1/K_0 \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

Remark. ([**24**]) Maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{C}^{m+2})$ $(m \geq 3)$ are classified as $Gr_2(\mathbf{C}^{m+1})$, $Gr_2(\mathbf{R}^{m+2})$, $\mathbf{C}P^p \times \mathbf{C}P^q$ $(p+q = m)$, $\mathbf{H}P^{[\frac{m}{2}]}$.

**3.2.7. In the case $g = 4$ and $(U, K) = (Sp(m+2), Sp(2)\times Sp(m))$ $(m \geq 2)$, $(m_1, m_2) = (4, 4m-5)$.**

(i) In the case $g = 4$ and $(U, K) = (Sp(4), Sp(2) \times Sp(2))$ $(m = 2)$, $(m_1, m_2) = (4, 3)$

$$\begin{tikzcd}
N^{14} = K/K_0 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)} \arrow[d, "K_1/K_0 = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \cong S^3 \times S^3"] \\
K/K_1 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1 \times \mathbf{H}P^1 \cong S^4 \times S^4
\end{tikzcd}$$

Here $U_1/K_1 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)} \cong \mathbf{H}P^1 \times \mathbf{H}P^1$ is a maximal totally geodesic submanifold of $U/K = \frac{Sp(4)}{Sp(2)\times Sp(2)} \cong Gr_2(\mathbf{H}^4)$.

Notice that $K/K_0 = \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 3)$, and $K_1/K_0 = \frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)} \cong S^3 \times S^3$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (3, 3)$.

(ii) $m \geq 3$

$$\begin{tikzcd}
N^{8m-2} = \frac{K}{K_0} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0}\cong\frac{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)}{Sp(1)\times Sp(1)}"] & \frac{K}{K_0} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)} \arrow[d, "\frac{K_2}{K_0}\cong\frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}"] \\
\frac{K}{K_1} = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)} \arrow[r, "\frac{K_2}{K_1}\cong\mathbf{H}P^1\times\mathbf{H}P^1"] & \frac{K}{K_2} = \frac{Sp(2)\times Sp(m)}{Sp(2)\times Sp(2)\times Sp(m-2)}
\end{tikzcd}$$

Here $U_2/K_2 = \frac{Sp(4)\times Sp(m-2)}{Sp(2)\times Sp(2)\times Sp(m-2)} \cong Gr_2(\mathbf{H}^4)$ is not a maximal totally geodesic submanifold of $U/K = \frac{Sp(m+2)}{Sp(2)\times Sp(m)} \cong Gr_2(\mathbf{H}^{m+2})$, but $U_1/K_1 = \frac{Sp(2)\times Sp(2)\times Sp(m-2)}{Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)\times Sp(m-2)} \cong \mathbf{H}P^1 \times \mathbf{H}P^1$ is a maximal totally geodesic submanifold of $U_2/K_2$. Notice that $K/K_0 = \frac{Sp(2)\times Sp(m)}{Sp(1)\times Sp(1)\times Sp(m-2)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 4m-5)$, $K_2/K_0 \cong \frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (4, 3)$, and $K_1/K_0 \cong S^3 \times S^3$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (3, 3)$.

Remark. ([**24**]) Maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{H}^4)$ are classified as $Sp(2)$, $\mathbf{H}P^2$, $S^1 \cdot S^5$, $S^4 \times S^4$, $Gr_2(\mathbf{C}^4)$.

Maximal totally geodesic submanifolds embedded in $Gr_2(\mathbf{H}^{m+2})$ ($m \geq 3$) are classified as $Gr_2(\mathbf{H}^{m+1})$, $Gr_2(\mathbf{C}^{m+2})$, $\mathbf{H}P^p \times \mathbf{H}P^q$ ($p + q = m$).

**3.2.8. In the case $g = 4$ and $(U, K) = (E_6, U(1)\cdot Spin(10))$, $(m_1, m_2) = (6, 9)$.**

$$\begin{tikzcd}
N^{30} = \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \arrow[r, "="] \arrow[d, "\frac{K_1}{K_0}=\frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)}"] & \frac{K}{K_0} = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)} \arrow[d, "\frac{K_2}{K_0}=\frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)}"] \\
\frac{K}{K_1} = \frac{U(1)\cdot Spin(10)}{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))} \arrow[r, "\frac{K_2}{K_1}\cong\widetilde{Gr}_2(\mathbf{R}^8)"] & \frac{K}{K_2} = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))}
\end{tikzcd}$$

Here $U_2/K_2 = \frac{U(1)\cdot Spin(10)}{U(1)\cdot(Spin(2)\cdot Spin(8))} \cong \widetilde{Gr}_2(\mathbf{R}^{10})$ is a maximal totally geodesic submanifold of $U/K = \frac{E_6}{U(1)\cdot Spin(10)}$, but $U_1/K_1 = \frac{S^1\cdot Spin(4)\cdot Spin(6)}{S^1\cdot(Spin(2)\cdot Spin(2)\cdot Spin(6))} \cong S^2 \times S^2$ is not a maximal totally geodesic submanifold in $U_2/K_2$. Notice that $K/K_0 = \frac{U(1)\cdot Spin(10)}{S^1\cdot Spin(6)}$ is a homogeneous isoparametric hypersurface with $g = 4$, $(m_1, m_2) = (6, 9)$, $K_2/K_0 = \frac{U(1)\cdot(Spin(2)\cdot Spin(8))}{S^1\cdot Spin(6)} \cong \frac{Spin(2)\cdot Spin(8)}{Spin(6)} \cong \frac{SO(2)\times SO(8)}{\mathbf{Z}_2\times SO(6)}$ is a homogeneous isoparametric hypersurface with $g = 4, (m_1, m_2) = (1, 6)$, and $K_1/K_0 = \frac{S^1\cdot(Spin(2)\cdot(Spin(2)\cdot Spin(6)))}{S^1\cdot Spin(6)} \cong S^1 \times S^1$ is a homogeneous isoparametric hypersurface with $g = 2$, $(m_1, m_2) = (1, 1)$.

Remark ([**24**]). Maximal totally geodesic submanifolds embedded in $E_6/U(1)\cdot Spin(10)$ are classified as $Gr_2(\mathbf{H}^4)/\mathbf{Z}_2$, $\mathbf{O}P^2$, $S^2 \times \mathbf{C}P^2$, $SO(10)/U(5)$, $Gr_2(\mathbf{C}^6)$, $\widetilde{Gr}_2(\mathbf{R}^{10})$.

The cases described in 3.2.1, 3.2.2, and 3.2.8 are treated in [**27**].

## 4. The case $(U, K) = (SO(5) \times SO(5), SO(5))$

Now $(U, K)$ is of type $B_2$, and $U = SO(5) \times SO(5)$, $K = \{(x, x) \in U \mid x \in SO(5)\}$. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition, where $\mathfrak{u} = \mathfrak{o}(5) \oplus \mathfrak{o}(5)$, $\mathfrak{k} = \{(X, X) \mid X \in \mathfrak{o}(5)\} \cong \mathfrak{o}(5)$, and $\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{o}(5)\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ given by

$$\mathfrak{a} = \left\{ (H, -H) \mid H = H(\xi_1, \xi_2) = \begin{pmatrix} 0 & -\xi_1 & 0 & 0 & 0 \\ \xi_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi_2 & 0 \\ 0 & 0 & \xi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\}$$
$$\cong \mathfrak{t} = \{H(\xi_1, \xi_2) \mid \xi_1, \xi_2 \in \mathbf{R}\} \subset \mathfrak{o}(5).$$

Then the centralizer $K_0$ of $\mathfrak{a}$ in $K$ is given by

$$K_0 = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid A, B \in SO(2) \right\} \cong T^2,$$

which is a maximal torus of $SO(5)$, and $N = K/K_0 \cong SO(5)/T^2$ is a maximal flag manifold of dimension $n = 8$. Moreover, $K_{[\mathfrak{a}]}$ is described as

$$K_{[\mathfrak{a}]} = \begin{pmatrix} \mathrm{I}_2 & 0 & 0 \\ 0 & \mathrm{I}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot T^2 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & -1 \end{pmatrix} \cdot T^2$$
$$\cup \begin{pmatrix} 1 & 0 & & & \\ 0 & -1 & & & \\ & & 1 & 0 & \\ & & 0 & -1 & \\ & & & & 1 \end{pmatrix} \cdot T^2 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & -1 & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & & & -1 \end{pmatrix} \cdot T^2.$$

The deck transformation group of the covering map $\mathcal{G} : N^8 \to \mathcal{G}(N^8)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

**4.1. Description of the Casimir operator.** Choose $\langle X, Y \rangle_{\mathfrak{k}} := -\mathrm{tr}(XY)$ for each $X, Y \in \mathfrak{k} = \mathfrak{so}(5)$. The restricted root system $\Sigma(U, K)$ of type $B_2$, can be described as follows (cf. [**7**]):

$$\begin{aligned} \Sigma(U,K) =&\{\pm(\epsilon_1 - \epsilon_2) = \pm\alpha_1, \pm\epsilon_2 = \pm\alpha_2, \pm(\epsilon_1 + \epsilon_2) = \pm(\alpha_1 + 2\alpha_2), \\ &\pm \epsilon_1 = \pm(\alpha_1 + \alpha_2)\}. \end{aligned}$$

Then the square length of each $\gamma \in \Sigma(U, K)$ relative to $\langle\ ,\ \rangle_{\mathfrak{k}}$ is

$$\|\gamma\|_{\mathfrak{u}}^2 = \begin{cases} \dfrac{1}{4} & \text{if } \gamma \text{ is short,} \\[2ex] \dfrac{1}{2} & \text{if } \gamma \text{ is long.} \end{cases}$$

In this case, $K = SO(5) \supset K_1 = SO(4) \supset K_0 = T^2$. The Casimir operator $\mathcal{C}_L$ of $L^n$ relative to the induced metric from $g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ becomes

$$\begin{aligned} \mathcal{C}_L &= \frac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\, ,\, \rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\, ,\, \rangle_{\mathfrak{u}}} \\ &= 4\,\mathcal{C}_{K/K_0,\langle\, ,\, \rangle_{\mathfrak{u}}} - 2\,\mathcal{C}_{K_1/K_0,\langle\, ,\, \rangle_{\mathfrak{u}}} \\ &= 2\,\mathcal{C}_{K/K_0} - \mathcal{C}_{K_1/K_0} \\ &= \ \mathcal{C}_{K/K_0} + \mathcal{C}_{K/K_1}, \end{aligned} \tag{4.1}$$

where $\mathcal{C}_{K/K_0}$ and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operators of $K/K_0$ and $K_1/K_0$ relative to $\langle\ ,\ \rangle_{\mathfrak{k}}$ and $\langle\ ,\ \rangle_{\mathfrak{k}}|_{\mathfrak{k}_1}$, respectively.

**4.2. Descriptions of $D(K)$ and $D(K_1)$.** Since the maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ can be given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in \mathbf{R} \right\} \subset \mathfrak{k}_1 \subset \mathfrak{k},$$

we have

$$
\begin{aligned}
&\Gamma(K) = \Gamma(K_1)\\
&= \left\{ \xi = \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \mid \xi_1, \xi_2 \in 2\pi\mathbf{Z} \right\}.
\end{aligned}
$$

Denote by $\varepsilon_i$ $(i = 1, 2)$ a linear function $\epsilon_i : \mathfrak{t} \ni \xi \mapsto \xi_i \in \mathbf{R}$. Then

$$
\begin{aligned}
D(K) &= D(SO(5)) = \{\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \mid k_1, k_2 \in \mathbf{Z}, k_1 \geq k_2 \geq 0\},\\
D(K_1) &= D(SO(4)) = \{\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \mid k_1, k_2 \in \mathbf{Z}, k_1 \geq |k_2|\}.
\end{aligned}
$$

**4.3. Branching law of** $(SO(5), SO(4))$**.**

**Lemma 4.1** (Branching law of $(SO(5), SO(4))$ [**19**])**.** *Let* $\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \in D(SO(5))$ *be the highest weight of an irreducible* $SO(5)$*-module* $V_\Lambda$*, where* $k_1, k_2 \in \mathbf{Z}$ *and* $k_1 \geq k_2 \geq 0$*. Then* $V_\Lambda$ *contains an irreducible* $SO(4)$*-module* $W_{\Lambda'}$ *with the highest weight* $\Lambda' = k'_1\epsilon_1 + k'_2\epsilon_2 \in D(SO(4))$*, where* $k'_1, k'_2 \in \mathbf{Z}$*,* $k'_1 \geq |k'_2|$*, if and only if*

$$k_1 \geq k'_1 \geq k_2 \geq |k'_2|. \tag{4.2}$$

**4.4. Descriptions of** $D(K, K_0)$ **and** $D(K_1, K_0)$**.** Define an $\mathrm{Ad}(K)$-invariant inner product of $\mathfrak{k}$ by $\langle X, Y\rangle_{\mathfrak{k}} := -\mathrm{tr}(XY)$ $(X, Y \in \mathfrak{k} = \mathfrak{o}(5))$.

Let $\{\alpha'_1 = \epsilon_1 - \epsilon_2, \alpha'_2 = \epsilon_1 + \epsilon_2\}$ be the fundamental root system of $SO(4)$, and let $\{\Lambda'_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2), \Lambda'_2 = \frac{1}{2}(\epsilon_1 + \epsilon_2)\}$ be the fundamental weight system of $SO(4)$. Then:

**Lemma 4.2** ([**52**])**.**

$$
\begin{aligned}
D(K_1, K_0) =\ & D(SO(4), T^2)\\
=&\Big\{\Lambda' = k'_1\epsilon_1 + k'_2\epsilon_2 = m'_1\Lambda'_1 + m'_2\Lambda'_2 = p'_1\alpha'_1 + p'_2\alpha'_2 \mid\\
&\quad k'_i \in \mathbf{Z},\ k'_1 \geq |k'_2|, m'_i \in \mathbf{Z}, m'_i \geq 0, p'_i \in \mathbf{Z}, p'_i \geq 1,\\
&\quad m'_1 = k'_1 - k'_2 = 2p'_1 \geq 0, m'_2 = k'_1 + k'_2 = 2p'_2 \geq 0\Big\}.
\end{aligned} \tag{4.3}
$$

*The eigenvalue formula of the Casimir operator* $\mathcal{C}_{K_1/K_0}$ *relative to* $\langle X, Y\rangle_{\mathfrak{k}}|_{\mathfrak{k}_1}$ *is*

$$-c_{\Lambda'} = \frac{1}{2}((k'_1)^2 + (k'_2)^2 + 2k'_1),$$

*for each* $\Lambda' = k'_1\epsilon_1 + k'_2\epsilon_2 \in D(K_1, K_0)$*.*

Let $\{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2\}$ be the fundamental root system of $SO(5)$, and let $\{\Lambda_1 = \epsilon_1, \Lambda_2 = \frac{1}{2}(\epsilon_1 + \epsilon_2)\}$ be the fundamental weight system of $SO(5)$. Then:

**Lemma 4.3** ([**52**]).

(4.4)

$$
\begin{aligned}
&D(K,K_0)= D(SO(5),T^2)\\
=&\Big\{\Lambda = k_1\epsilon_1 + k_2\epsilon_2 = m_1\Lambda_1 + m_2\Lambda_2 = p_1\alpha_1 + p_2\alpha_2 \mid \\
&\quad k_i \in \mathbf{Z},\, k_1 \geq k_2 \geq 0,\, m_i \in \mathbf{Z},\, m_i \geq 0,\, p_i \in \mathbf{Z},\, p_i \geq 1,\\
&\quad m_1 = 2p_1 - p_2 \geq 0,\, m_2 = -2p_1 + 2p_2 \geq 0,\, p_1 = k_1,\, p_2 = k_1 + k_2\Big\}.
\end{aligned}
$$

*The eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ with respect to the inner product $\langle X, Y\rangle_{\mathfrak{k}}$ is*

$$-c_\Lambda = \frac{1}{2}(k_1^2 + k_2^2 + 3k_1 + k_2),$$

*for each $\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \in D(K,K_0)$.*

**4.5. Eigenvalue computation.** By Lemmas 4.2 and 4.3, we have the following eigenvalue formula for $\mathcal{C}_L$:

$$
\begin{aligned}
-c_L =& \, -2c_{K/K_0} + c_{K_1/K_0}\\
&= (k_1^2 + k_2^2 + 3k_1 + k_2) - \frac{1}{2}((k_1')^2 + (k_2')^2 + 2k_1').
\end{aligned}
$$

Since

$$-\mathcal{C}_L = -\mathcal{C}_{K/K_0} - \mathcal{C}_{S^4} \geq -\mathcal{C}_{K/K_0},$$

the condition $-c_L \leq n = 8$ implies that $-c_\Lambda \leq 8$. We have the following.

**Lemma 4.4.** $\Lambda = k_1\epsilon_1 + k_2\epsilon_2 \in D(SO(5),T^2)$ *has eigenvalue* $-c_L \leq 8$ *if and only if* $(k_1,k_2)$ *is one of* $\{(0,0),(1,0),\ (1,1),\ (2,0),\ (2,1),\ (2,2)\}$.

*Proof.* Assume that $-c_\Lambda = \frac{1}{2}(k_1^2 + k_2^2 + 3k_1 + k_2) \leq 8$. Then it follows from Lemma 4.3 that $k_1 \leq 2$. Moreover, if $k_1 = 1$, then $k_2 = 0$ or $1$. If $k_1 = 2$, then $k_2 = 0, 1$ or $2$. q.e.d.

Suppose that $(k_1,k_2) = (1,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 5$. It follows from Lemma 4.1 that $(k_1',k_2') = (0,0)$ or $(1,0)$. By Lemma 4.2, we have $(p_1',p_2') = (0,0)$ or $(\frac{1}{2},\frac{1}{2})$, but $\Lambda'|_{(p_1',p_2')=(0,0)},\ \Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{1}{2})} \notin D(SO(4),T^2)$. Hence $\Lambda = (1,0) \notin D(SO(5),T^2) = D(K,K_0)$.

Suppose that $(k_1,k_2) = (1,1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 10$, $V_\Lambda \cong \mathfrak{o}(5,\mathbf{C})$ and $K_{[\mathfrak{a}]}/K_0$ acts on $(V_\Lambda)_{K_0} \cong (\mathfrak{t}^2)^{\mathbf{C}} \cong \mathfrak{a}^{\mathbf{C}}$ via the action of Weyl group $W(U,K)$. Thus it must be $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$. Hence $\Lambda|_{(k_1,k_2)=(1,1)} \notin D(K,K_{[\mathfrak{a}]})$.

Suppose that $(k_1,k_2) = (2,0)$. Then $(m_1,m_2) = (2,0)$ and $\dim_{\mathbf{C}} V_{2\Lambda_1} = 14$. It follows from Lemma 4.1 that $(k_1',k_2') = (0,0),(1,0)$, or $(2,0)$. By Lemma 4.2, we have $(p_1',p_2') = (0,0),(\frac{1}{2},\frac{1}{2})$, or $(1,1)$. Note that $\Lambda'|_{(p_1',p_2')=(0,0)},\ \Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{1}{2})} \notin D(SO(4),T^2)$. If $(p_1',p_2') = (1,1)$, then $(m_1',m_2') = (2,2)$ and $-c_\Lambda = 5$, $-c_{\Lambda'} = 4$, and thus

$$-c_L = -2c_\Lambda + c_{\Lambda'} = 10 - 4 = 6 < 8.$$

On the other hand, we observe that

$$\begin{aligned}V_{2\Lambda_1} \cong&\mathrm{Sym}_0(\mathbf{C}^5)\\ =&\mathbf{C}\cdot\begin{pmatrix}-\frac{1}{4}I_4 & 0\\ 0 & 1\end{pmatrix}\oplus\left\{\begin{pmatrix}X & 0\\ 0 & 0\end{pmatrix}\mid X\in \mathrm{Sym}_0(\mathbf{C}^4)\right\}\\ &\oplus\left\{\begin{pmatrix}0 & Z\\ {}^tZ & 0\end{pmatrix}\mid Z\in M(4,1;\mathbf{C})\right\}\\ =&W_{|\Lambda'=0}\oplus W_{2\Lambda'_1+2\Lambda'_2}\oplus W_{\Lambda'_1+\Lambda'_2},\end{aligned}$$

and

$$(V_{2\Lambda_1})_{K_0}=\left\{\begin{pmatrix}c_1I_2 & & \\ & c_2I_2 & \\ & & c_3\end{pmatrix}\mid c_1,c_2,c_3\in\mathbf{C}, 2c_1+2c_2+c_3=0\right\}.$$

As

$$\begin{aligned}&\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&-1\end{pmatrix}\begin{pmatrix}c_1I_2 & & \\ & c_2I_2 & \\ & & c_3\end{pmatrix}\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&-1\end{pmatrix}\\ =&\begin{pmatrix}c_2I_2 & & \\ & c_1I_2 & \\ & & c_3\end{pmatrix},\end{aligned}$$

we get

$$(V_{2\Lambda_1})_{K_{[\mathfrak{a}]}}=\left\{\begin{pmatrix}-\frac{c}{4}\mathrm{I}_4 & \\ & c\end{pmatrix}\mid c\in\mathbf{C}\right\}=W_{|\Lambda'=0}.$$

Thus

$$W'_{2\Lambda'_1+2\Lambda'_2}\cap(V_{2\Lambda_1})_{K_{[\mathfrak{a}]}}=\{0\}.$$

Suppose that $(k_1,k_2)=(2,1)$. Then $(m_1,m_2)=(2,1)$ and $\dim_{\mathbf{C}}V_{2\Lambda_1+\Lambda_2}=35$. It follows from Lemma 4.1 that $(k'_1,k'_2)=(1,0)$, $(1,-1)$, $(1,1)$, $(2,0)$, $(2,-1)$, or $(2,1)$—that is, $(m'_1,m'_2)=(1,1)$, $(2,0)$, $(0,2)$, $(2,2)$, $(3,1)$, or $(1,3)$, and thus

$$V_{2\Lambda_1+\Lambda_1}=W_{\Lambda'_1+\Lambda'_2}\oplus W_{2\Lambda'_1}\oplus W_{2\Lambda'_2}\oplus W_{2\Lambda'_1+2\Lambda'_2}\oplus W_{3\Lambda'_1+\Lambda'_2}\oplus W_{\Lambda'_1+3\Lambda'_2}.$$

By Lemma 4.3, we have $(p'_1,p'_2)=(\frac{1}{2},\frac{1}{2})$, $(1,0)$, $(0,1)$, $(1,1)$, $(\frac{3}{2},\frac{1}{2})$, or $(\frac{1}{2},\frac{3}{2})$. Then by Lemma 4.2 we see that $\Lambda'|_{(p'_1,p'_2)=(\frac{1}{2},\frac{1}{2})}$, $\Lambda'|_{(p'_1,p'_2)=(1,0)}$, $\Lambda'|_{(p'_1,p'_2)=(0,1)}$, $\Lambda'|_{(p'_1,p'_2)=(\frac{3}{2},\frac{1}{2})}$, $\Lambda'|_{(p'_1,p'_2)=(\frac{1}{2},\frac{3}{2})}\notin D(SO(4),T^2)$. If $(p'_1,p'_2)=(1,1)$—that is, $(m'_1,m'_2)=(2,2)$—then $-c_\Lambda=6$, $-c_{\Lambda'}=4$, and thus

$$-c_L=-2\,c_\Lambda+c_{\Lambda'}=12-4=8.$$

So we need to determine the dimension of $(W_{2\Lambda'_1+2\Lambda'_2})_{K_{[\mathfrak{a}]}}\neq\{0\}$.

Since $W_{2\Lambda'_1+2\Lambda'_2}\cong\mathfrak{sl}(2,\mathbf{C})\boxtimes\mathfrak{sl}(2,\mathbf{C})$ and

$$(W_{2\Lambda'_1+2\Lambda'_2})_{K_0}\cong(\mathfrak{sl}(2,\mathbf{C})\boxtimes\mathfrak{sl}(2,\mathbf{C}))_{K_0}=\mathbf{C}\boxtimes\mathbf{C},$$

we have $\dim_{\mathbf{C}}(W_{2\Lambda_1'+2\Lambda_2'})_{K_0} = 1$. Let $\wedge^2\mathbf{R}^{10} = \mathfrak{so}(10) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{so}(5)) + \mathcal{V}$. Then $\wedge^2\mathbf{C}^{10} = (\wedge^2\mathbf{R}^{10})^{\mathbf{C}} = \mathfrak{so}(10,\mathbf{C}) = \mathrm{ad}(\mathfrak{so}(5))^{\mathbf{C}} + \mathcal{V}^{\mathbf{C}} \cong \mathfrak{so}(5,\mathbf{C}) + \mathcal{V}^{\mathbf{C}}$, where $\{0\} \neq \mathcal{V}^{\mathbf{C}} \subset V_{2\Lambda_1+\Lambda_2}$. By the irreducibility of $V_{2\Lambda_1+\Lambda_2}$, we see that $\mathcal{V}^{\mathbf{C}} = V_{2\Lambda_1+\Lambda_2}$. Since

$$\{0\} \neq (\mathcal{V}^{\mathbf{C}})_{K_{[\mathfrak{a}]}} = (W_{2\Lambda_1'+2\Lambda_2'})_{K_{[\mathfrak{a}]}} \subset (W_{2\Lambda_1'+2\Lambda_2'})_{K_0}$$

and $\dim_{\mathbf{C}}(W_{2\Lambda_1'+2\Lambda_2'})_{K_0} = 1$, we get

$$\{0\} \neq (\mathcal{V}^{\mathbf{C}})_{K_{[\mathfrak{a}]}} = (W_{2\Lambda_1'+2\Lambda_2'})_{K_{[\mathfrak{a}]}} = (W_{2\Lambda_1'+2\Lambda_2'})_{K_0}$$

and $\dim_{\mathbf{C}}(W_{2\Lambda_1'+2\Lambda_2'})_{K_{[\mathfrak{a}]}} = 1$. Hence $2\Lambda_1 + \Lambda_2 \in D(K, K_{[\mathfrak{a}]})$ and its multiplicity is equal to 1.

Suppose that $(k_1, k_2) = (2, 2)$. It follows from Lemma 4.1 that $(k_1', k_2') = (2, 0)$, $(2, 1)$, $(2, 2)$, $(2, -1)$, or $(2, -2)$. By Lemma 4.2, we have $(p_1', p_2') = (1, 1)$, $(\frac{1}{2}, \frac{3}{2})$, $(0, 2)$, $(\frac{3}{2}, \frac{1}{2})$, or $(2, 0)$, and thus $\Lambda'|_{(p_1',p_2')=(\frac{1}{2},\frac{3}{2})}$, $\Lambda'|_{(p_1',p_2')=(0,2)}$, $\Lambda'|_{(p_1',p_2')=(\frac{3}{2},\frac{1}{2})}$, $\Lambda'|_{(p_1',p_2')=(2,0)} \notin D(SO(4), T^2)$. If $(p_1', p_2') = (1, 1)$, then $-c_\Lambda = 8$, $-c_{\Lambda'} = 4$, and hence

$$-c_L = -2c_\Lambda + c_{\Lambda'} = 16 - 4 = 12 > 8.$$

Now we obtain that the Gauss image $L^8 = \mathcal{G}(SO(5)/T^2) \subset Q_8(\mathbf{C})$ is Hamiltonian stable. Moreover, it also follows that

$$n(L^8) = \dim_{\mathbf{C}}(V_{2\Lambda_1+\Lambda_2}) = 35 = \dim(SO(10)) - \dim(SO(5)) = n_{hk}(L^8).$$

Hence the Gauss image $L^8 = \mathcal{G}(SO(5)/T^2) \subset Q_8(\mathbf{C})$ is Hamiltonian rigid.

From theses results we conclude the following.

**Theorem 4.1.** *The Gauss image* $L^8 = \mathcal{G}(SO(5)/T^2) = \frac{SO(5)}{T^2\cdot\mathbf{Z}_2} \subset Q_8(\mathbf{C})$ *is strictly Hamiltonian stable.*

## 5. The case $(U, K) = (SO(10), U(5))$

In this case, $(U, K)$ is of $BC_2$ type and $K = U(5) \subset U = SO(10)$. Here each $A+\sqrt{-1}B \in U(5)$ can be identified with an element $\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(10)$ with $A, B \in \mathfrak{g}l(5,\mathbf{R})$. The canonical decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{u}$ and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ are given by $\mathfrak{u} = \mathfrak{so}(10)$,

$$\begin{aligned}
\mathfrak{k} &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathfrak{so}(10) \mid -X^t = X, Y^t = Y \right\} \\
&\cong \mathfrak{u}(5) = \left\{ T = X + \sqrt{-1}Y \in \mathfrak{gl}(5,\mathbf{C}) \mid T^* = -T \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{so}(10) \mid X, Y \in \mathfrak{so}(5) \right\}
\end{aligned}$$

and

$$\mathfrak{a} = \left\{ \begin{pmatrix} H_1 & 0 \\ 0 & -H_1 \end{pmatrix} \mid H_1 = \begin{pmatrix} 0 & -\xi_1 & & & \\ \xi_1 & 0 & & & \\ & & 0 & -\xi_2 & \\ & & \xi_2 & 0 & \\ & & & & 0 \end{pmatrix} \xi_1, \xi_2 \in \mathbf{R} \right\}.$$

Then the centralizer $K_0$ of $\mathfrak{a}$ in $K$ is as follows:

$$\begin{aligned} K_0 \\ = \left\{ \begin{pmatrix} a_{11}+\mathbf{i}b_{11} & a_{12}+\mathbf{i}b_{12} & 0 & 0 & 0 \\ -a_{12}+\mathbf{i}b_{12} & a_{11}-\mathbf{i}b_{11} & 0 & 0 & 0 \\ 0 & 0 & a_{22}+\mathbf{i}b_{22} & a_{21}+\mathbf{i}b_{21} & 0 \\ 0 & 0 & -a_{21}+\mathbf{i}b_{21} & a_{22}-\mathbf{i}b_{22} & 0 \\ 0 & 0 & 0 & 0 & a_{33}+\mathbf{i}b_{33} \end{pmatrix} \right. \\ \left. \in U(5) \right\} \cong SU(2) \times SU(2) \times U(1), \end{aligned}$$

and $N = K/K_0 \cong U(5)/SU(2) \times SU(2) \times U(1)$ is of dimension 18. Moreover,

$$\begin{aligned} K_{[\mathfrak{a}]} =& K_0 \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & 1 \end{pmatrix} \cdot K_0 \cup \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \cdot K_0 \\ & \cup \begin{pmatrix} & & 1 & 0 & \\ & & 0 & -1 & \\ 1 & 0 & & & \\ 0 & 1 & & & \\ & & & & 1 \end{pmatrix} \cdot K_0. \end{aligned}$$

This means that the deck transformation group of the covering map $\mathcal{G} : N \to \mathcal{G}(N^{18})$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

**5.1. Description of the Casimir operator.** Choose $\langle X, Y \rangle_{\mathfrak{u}} := -\mathrm{tr}(\mathrm{XY})$ for each $X, Y \in \mathfrak{u} = \mathfrak{so}(10)$. The restricted root system $\Sigma(U, K)$ of type $BC_2$ can be given as follows ([**7**]):

$$\begin{aligned} &\Sigma(U, K) \\ =&\{\pm\epsilon_2 = \pm\alpha_1, \pm(\epsilon_1 - \epsilon_2) = \pm\alpha_2, \pm\epsilon_1 = \pm(\alpha_1 + \alpha_2), \\ &\pm(\epsilon_1 + \epsilon_2) = \pm(2\alpha_1 + \alpha_2), \pm 2\epsilon_1 = \pm(2\alpha_1 + 2\alpha_2), \pm 2\epsilon_2 = \pm 2\alpha_1\}. \end{aligned}$$

Then the square length of each $\gamma \in \Sigma(U, K)$ relative to $\langle\ ,\ \rangle_{\mathfrak{u}}$ is

$$\|\gamma\|_{\mathfrak{u}}^2 = \frac{1}{4}, \frac{1}{2}, \text{ or } 1.$$

Hence the Casimir operator $\mathcal{C}_L$ of $L^n$ with respect to the induced metric from $g^{\mathrm{std}}_{Q_n(\mathbf{C})}$ can be expressed as follows:

(5.1)

$$\begin{aligned}\mathcal{C}_L &= \frac{4}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_1/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}} - \frac{2}{\|\gamma_0\|^2_{\mathfrak{u}}}\mathcal{C}_{K_2/K_0,\langle\ ,\ \rangle_{\mathfrak{u}}}\\ &= 4\,\mathcal{C}_{K/K_0,\langle,\rangle_{\mathfrak{u}}} - \mathcal{C}_{K_1/K_0,\langle,\rangle_{\mathfrak{u}}} - 2\,\mathcal{C}_{K_2/K_0,\langle,\rangle_{\mathfrak{u}}}\\ &= 2\,\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0},\end{aligned}$$

where $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_2/K_0}$, and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of $K/K_0$, $K_2/K_0$, and $K_1/K_0$ relative to $\langle\ ,\ \rangle|_{\mathfrak{k}}$, $\langle\ ,\ \rangle|_{\mathfrak{k}_2}$, and $\langle\ ,\ \rangle|_{\mathfrak{k}_1}$, respectively. Here, $\langle X, Y\rangle := -\mathrm{tr}(\mathrm{Re}(XY))$ for all $X, Y \in \mathfrak{k} = u(5)$.

**5.2. Descriptions of $D(K)$, $D(K_1)$ and $D(K_2)$.** Using a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ given by

$$\mathfrak{t} = \left\{ \sqrt{-1}\begin{pmatrix} y_1 & 0 & 0 & 0 & 0\\ 0 & y_2 & 0 & 0 & 0\\ 0 & 0 & y_3 & 0 & 0\\ 0 & 0 & 0 & y_4 & 0\\ 0 & 0 & 0 & 0 & y_5\end{pmatrix} \mid y_1, y_2, y_3, y_4, y_5 \in \mathbf{R} \right\} \subset \mathfrak{k},$$

we have

$$\begin{aligned}&\Gamma(K) = \Gamma(K_2) = \Gamma(K_1) = \Gamma(K_0)\\ &\quad = \left\{ \xi = \sqrt{-1}\begin{pmatrix} \xi_1 & 0 & 0 & 0 & 0\\ 0 & \xi_2 & 0 & 0 & 0\\ 0 & 0 & \xi_3 & 0 & 0\\ 0 & 0 & 0 & \xi_4 & 0\\ 0 & 0 & 0 & 0 & \xi_5\end{pmatrix} \mid \xi_1, \xi_2, \xi_3, \xi_4, \xi_5 \in 2\pi\mathbf{Z} \right\},\end{aligned}$$

$\Gamma(C(K)) = 2\pi\mathbf{Z}\mathrm{I}_5$.

Then $D(K)$, $D(K_1)$, and $D(K_2)$ are given as follows:

$$\begin{aligned} D(K) =& D(U(5))\\ =&\{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5\},\\ D(K_2) =& D(U(4) \times U(1))\\ =&\{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2 \geq p_3 \geq p_4\},\\ D(K_1) =& D(U(2) \times U(2) \times U(1))\\ =&\{\Lambda = p_1y_1 + \cdots + p_5y_5 \mid p_1, \cdots, p_5 \in \mathbf{Z}, p_1 \geq p_2, p_3 \geq p_4\}.\end{aligned}$$

**5.3. Branching laws of $(U(m+1), U(m) \times U(1))$.**

The branching laws for $(SU(m+1), S(U(1) \times U(m)))$ was shown by Ikeda and Taniguchi [**19**]. It can be reformulated to the branching laws for $(U(m+1), U(m) \times U(1))$ as follows:

**Lemma 5.1** (Branching laws for $(U(m+1), U(m) \times U(1))$). *Let $\Lambda = p_1 y_1 + \cdots + p_m y_m \in D(U(m))$ be the highest weight of an irreducible $U(m)$-module $V_\Lambda$, where $p_i \in \mathbf{Z}$ $(i = 1, \cdots, m)$ and $p_1 \geq p_2 \geq \cdots \geq p_m$. Then the irreducible decomposition of $V_\Lambda$ as a $U(m) \times U(1)$-module contains an irreducible $U(m) \times U(1)$-module $V_{\Lambda'}$ with the highest weight $V_{\Lambda'} = q_1 y_1 + \cdots + q_m y_m \in D(U(m) \times U(1))$, where $q_i \in \mathbf{Z}$ and $q_1 \geq q_2 \geq \cdots \geq q_m$, if and only if*

$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3 \geq q_3 \geq \cdots \geq p_{m-1} \geq q_{m-1} \geq p_m,$$
$$\sum_{i=1}^m p_i = \sum_{i=1}^m q_i.$$

*In particluar, the multiplicity of $V_{\Lambda'}$ is 1.*

In the next subsection we use the branching laws of $(U(m+1), U(m) \times U(1))$, and $(U(m), U(2) \times U(m-2))$ in the case of $m = 4$. The branching laws of $(U(m), U(2) \times U(m-2))$ are described in Lemma 7.1 of Section 7.

**5.4. Descriptions of $D(K, K_0)$, $D(K_2, K_0)$, and $D(K_1, K_0)$.**

Each $\Lambda \in D(K) = D(U(5))$ is expressed as

$$\Lambda = p_1 y_1 + \cdots p_5 y_5,$$

where $p_i \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$. Then by Lemma 5.1 in the case of $m = 4$, $V_\Lambda$ can be decomposed into irreducible $U(4) \times U(1)$-modules as

$$V_\Lambda = \bigoplus_{i=1}^s V'_{\Lambda'_i} = \bigoplus_{i=1}^s W'_{\Lambda'_{1i}} \boxtimes U_{q_5 y_5},$$

where $\Lambda'_i = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 \in D(K_2) = D(U(4) \times U(1))$, $\Lambda'_{1i} = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 \in D(U(4))$, $q_5 y_5 \in D(U(1))$, and $q_i \in \mathbf{Z}$ $(i = 1, 2, 3, 4, 5)$ satisfy

$$p_1 \geq q_1 \geq p_2 \geq q_2 \geq p_3 \geq q_3 \geq p_4 \geq q_4 \geq p_5,$$
$$\sum_{i=1}^5 p_i = \sum_{j=1}^5 q_j.$$

By the branching law for $(U(4), U(2) \times U(2))$ in Lemma 7.1, each $W'_{\Lambda'_{1i}}$ can be decomposed as

$$W'_{\Lambda'_{1i}} = \bigoplus W''_{\Lambda''} = \bigoplus W''_{\tilde{\Lambda}_\sigma} \boxtimes W''_{\tilde{\Lambda}_\rho},$$

where $\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4 \in D(U(2) \times U(2))$, $\tilde{\Lambda}_\sigma = k_1 y_1 + k_2 y_2 \in D(U(2))$, $\tilde{\Lambda}_\rho = k_3 y_3 + k_4 y_4 \in D(U(2))$, and $k_i \in \mathbf{Z}$ $(i = 1, 2, 3, 4)$ satisfy

(i) $\sum_{i=1}^4 k_i = \sum_{i=1}^4 q_i$;
(ii) $q_1 \geq k_1 \geq q_3$, $q_2 \geq k_2 \geq q_4$;

(iii) in the finite power series expansion in $X$ of $\dfrac{\prod_{i=1}^{3}(X^{r_i+1}-X^{-(r_i+1)})}{(X-X^{-1})^2}$,
where $r_i (i=1,2,3)$ are defined by

$$\begin{aligned} r_1 :=& q_1 - \max(k_1, q_2), \\ r_2 :=& \min(k_1, q_2) - \max(k_2, q_3), \\ r_3 :=& \min(k_2, q_3) - q_4, \end{aligned}$$

the coefficient of $X^{k_3-k_4+1}$ does not vanish. Moreover the value of this coefficient is the multiplicity of the $U(2)\times U(2)$-module $W''_{\Lambda''}$.

By the branching law of $(U(2), SU(2))$ (see Section 7), as $SU(2)$-modules they become

$$W''_{\tilde{\Lambda}_\sigma} = W''_{\Lambda_\sigma}, \quad W''_{\tilde{\Lambda}_\rho} = W''_{\Lambda_\rho},$$

where $\Lambda_\sigma = \dfrac{k_1-k_2}{2}(y_1-y_2) \in D(SU(2))$, $\Lambda_\rho = \dfrac{k_3-k_4}{2}(y_3-y_4) \in D(SU(2))$.

Hence one can decompose a $K$-module $V_\Lambda$ into the irreducible $K_0$-modules

$$V_\Lambda = \bigoplus\bigoplus W''_{\Lambda_\sigma} \boxtimes W''_{\Lambda_\rho} \boxtimes U_{q_5 y_5}.$$

Now assume that $\Lambda \in D(K, K_0)$. Then there exists at least one nonzero trivial irreducible $K_0$-module in the above decomposition for some $\sigma$ and $\rho$. So in this case, we have

$$k_1 - k_2 = 0, \quad k_3 - k_4 = 0, \quad q_5 = 0.$$

So we know that

$$\begin{aligned} & 2k_1 + 2k_3 = \sum_{i=1}^{4} q_i = \sum_{j=1}^{5} p_j, \\ & q_2 \geq k_1 = k_2 \geq q_3, \\ & r_1 = q_1 - q_2, \\ & r_2 = k_1 - k_2 = 0, \\ & r_3 = q_3 - q_4, \end{aligned}$$

and in the finite power series expansion in $X$ of

$$\frac{(X^{q_1-q_2+1}-X^{-(q_1-q_2+1)})(X^{q_3-q_4+1}-X^{-(q_3-q_4+1)})}{X-X^{-1}},$$

the coefficient of $X$ does not vanish. Moreover, the value of this coefficient is the multiplicity of the $U(2)\times U(2)$-module.

Therefore, in the above notations for each $\Lambda \in D(K, K_0)$ given by $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4+p_5y_5$, where $p_1, \cdots, p_5 \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$, each $\Lambda' \in D(K_2, K_0)$ is given by $\Lambda' = q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4$, where $q_1, \cdots, q_4 \in \mathbf{Z}$, $q_1 \geq q_2 \geq q_3 \geq q_4$, $\sum_{i=1}^{5} p_i = \sum_{j=1}^{4} q_j$. Moreover, each $\Lambda'' \in D(K_1, K_0)$ is given by $\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4$, where $k_1, \cdots, k_4 \in \mathbf{Z}$, $k_1 = k_2$, $k_3 = k_4$, $2k_1 + 2k_3 = \sum_{j=1}^4 q_j$.

**5.5. Eigenvalue computation.** For each $\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5 \in D(K, K_0)$, with $p_i \in \mathbf{Z}$, $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$, the eigenvalue formula of the Casimir operator $\mathcal{C}_{K/K_0}$ with respect to the inner product $\langle X, Y \rangle_{\mathfrak{k}} = -\mathrm{Tr}(\mathrm{Re}(\mathrm{XY}))$ for any $X, Y \in \mathfrak{k} = \mathfrak{u}(5)$ is given by

$$-c_\Lambda = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 4p_1 + 2p_2 - 2p_4 - 4p_5.$$

For each $\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 \in D(K_2, K_0)$ with $q_i \in \mathbf{Z}$ and $q_1 \geq q_2 \geq q_3 \geq q_4$, the eigenvalue formula of the Casimir operator $\mathcal{C}_{K_2/K_0}$ with respect to the inner product $\langle\, , \rangle_{\mathfrak{k}}|_{\mathfrak{k}_2}$ is given by

$$-c_{\Lambda'} = q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1 + q_2 - q_3 - 3q_4.$$

For each $\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4 \in D(K_1, K_0)$ with $k_1 = k_2$ and $k_3 = k_4$, the eigenvalue formula of the Casimir operator $\mathcal{C}_{K_1/K_0}$ with respect to the inner product $\langle\, , \rangle_{\mathfrak{k}}|_{\mathfrak{k}_1}$ is given by

$$\begin{aligned} -c_{\Lambda''} =& k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_1 - k_2 + k_3 - k_4 \\ =& k_1^2 + k_2^2 + k_3^2 + k_4^2. \end{aligned}$$

Hence we have the following eigenvalue formula:

$$\begin{aligned} -c_L =& -2c_\Lambda + c_{\Lambda'} + \frac{1}{2} c_{\Lambda''} \\ =& \, 2(p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + 4p_1 + 2p_2 - 2p_4 - 4p_5) \\ & - (q_1^2 + q_2^2 + q_3^2 + q_4^2 + 3q_1 + q_2 - q_3 - 3q_4) \\ & - \frac{1}{2}(k_1^2 + k_2^2 + k_3^2 + k_4^2). \end{aligned} \tag{5.2}$$

**Lemma 5.2.** $\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4 + p_5 y_5 \in D(K, K_0)$ *has eigenvalue* $-c_L \leq 18$ *if and only if* $(p_1, p_2, p_3, p_4, p_5)$ *is one of*

$$\Big\{ 0,\ (0,-1,-1,-1,-1),\ (1,1,1,1,0),\ (1,1,0,0,0),\ (0,0,0,-1,-1),$$
$$(1,0,0,0,-1),\ (2,1,1,0,0),\ (0,0,-1,-1,-2),\ (1,1,0,-1,-1) \Big\}.$$

*Proof.* Since

$$-\mathcal{C}_L = -\frac{1}{2}\mathcal{C}_{K/K_0} - \mathcal{C}_{K/K_2} - \frac{1}{2}\mathcal{C}_{K/K_1} \geq -\frac{1}{2}\mathcal{C}_{K/K_0},$$

the condition $-c_L \leq n = 18$ implies that $-c_\Lambda \leq 36$. From

$$-c_\Lambda = (p_1 + 2)^2 + (p_2 + 1)^2 + p_3^2 + (p_4 - 1)^2 + (p_5 - 2)^2 - 10 \leq 36,$$

it follows that $|p_i| \leq 2$. Using the eigenvalue formula (5.2), we obtain the result. q.e.d.

Denote by $\omega_1, \omega_2, \omega_3, \omega_4$ the fundamental weight system of $SU(5)$.

Suppose that $\Lambda = (1,1,1,1,0)$. Then $\dim V_\Lambda = 5$. By the branching law of $(U(5), U(4)\times U(1))$, $\Lambda' = (1,1,1,1,0)$ or $(1,1,1,0,1)$, where $\Lambda' = (1,1,1,1,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2)\times U(2))$, $\Lambda'' = (1,1,1,1) \in D(K_1, K_0)$. Thus $-c_\Lambda = 8$, $-c_{\Lambda'} = 4$, $-c_{\Lambda''} = 4$, and $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 10 < 18$.

On the other hand, $\Lambda = \Lambda_0 + \omega_4$, where $\Lambda_0 = \frac{4}{5}\sum_{i=1}^5 y_i$. The group $K = U(5) = C(U(5)) \cdot SU(5)$ acts on $\dim V_\Lambda = 5$ and $V_\Lambda \cong \mathbf{C} \otimes \bar{\mathbf{C}}^5$ by $\rho_{\Lambda_0} \boxtimes \bar{\mu}_5$, where $\bar{\mu}_5$ denotes the conjugate representation of the standard representation of $SU(5)$ on $\mathbf{C}^5$. For each element

$$g_0 = \begin{pmatrix} A & & \\ & B & \\ & & e^{\sqrt{-1}\theta} \end{pmatrix} \in K_0$$

and each element $u \otimes \mathbf{w} \in \mathbf{C} \otimes \bar{\mathbf{C}}^5$, where $A, B \in SU(2)$ and $\theta \in \mathbf{R}$,

$$\begin{aligned} \rho_\Lambda(g_0)(u \otimes \mathbf{w}) =& \rho_{\Lambda_0}(e^{\frac{\sqrt{-1}}{5}\theta} I_5)(u) \otimes \rho_{\omega_4}(e^{-\frac{\sqrt{-1}}{5}\theta} g_0)\mathbf{w} \\ =& e^{\frac{4\sqrt{-1}}{5}\theta} u \otimes \begin{pmatrix} e^{\frac{\sqrt{-1}}{5}\theta} \bar{A} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ e^{\frac{\sqrt{-1}}{5}\theta} \bar{B} \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} \\ e^{-\frac{4\sqrt{-1}}{5}\theta} w_5 \end{pmatrix}. \end{aligned}$$

Hence $(V_\Lambda)_{K_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$.

For a generator $g = \begin{pmatrix} & & 1 & 0 & \\ & & 0 & 1 & \\ 1 & 0 & & & \\ 0 & -1 & & & \\ & & & & 1 \end{pmatrix} \in K_{[\mathfrak{a}]} \subset K_2$ in $\mathbf{Z}_4$,

$$\begin{aligned} \rho_\Lambda(g)(u \otimes \mathbf{e}_5) &= \rho_{\Lambda_0}(e^{\sqrt{-1}\frac{\pi}{5}} I_5)(u) \otimes \rho_{\omega_4}(e^{-\sqrt{-1}\frac{\pi}{5}} g)(\mathbf{e}_5) \\ &= e^{\sqrt{-1}\frac{4\pi}{5}} u \otimes e^{\sqrt{-1}\frac{\pi}{5}} \mathbf{e}_5 = -u \otimes \mathbf{e}_5. \end{aligned}$$

So $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$, that is, $\Lambda = (1,1,1,1,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, we get $\Lambda = (0,-1,-1,-1,-1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,1,0,0,0)$. Then $\dim V_\Lambda = 10$. By the branching law of $(U(5), U(4) \times U(1))$, $\Lambda' = (1,1,0,0,0)$ or $(1,0,0,0,1)$, where $\Lambda' = (1,1,0,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$, $\Lambda'' = (1,1,0,0), (0,0,1,1),$ or $(1,0,1,0)$, where $\Lambda'' = (1,1,0,0)$

or $(0,0,1,1) \in D(K_1, K_0)$. Thus $-c_\Lambda = 8$, $-c_{\Lambda'} = 6$, $-c_{\Lambda''} = 2$ and $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 9 < 18$.

On the other hand, $\Lambda = \Lambda_0 + \omega_2$, where $\Lambda_0 = \frac{2}{5}\sum_{i=1}^{5} y_i$. $V_\Lambda \cong \mathbf{C} \oplus \wedge^2 \mathbf{C}^5$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\}$ be the standard basis of $\mathbf{C}^5$. For each element $g_0 \in K_0$ expressed as above and each element $u \otimes \mathbf{e}_i \wedge \mathbf{e}_j \in V_\Lambda$ $(1 \leq i < j \leq 5)$,

$$\begin{aligned}\rho_\Lambda(g_0)(u \otimes \mathbf{e}_i \wedge \mathbf{e}_j) &= \rho_{\Lambda_0}(e^{\frac{\sqrt{-1}}{5}\theta} I_5)(u) \otimes \rho_{\omega_2}(e^{-\frac{\sqrt{-1}}{5}\theta} g_0)(\mathbf{e}_i \wedge \mathbf{e}_j) \\ &= e^{\sqrt{-1}\frac{2}{5}\theta} u \otimes (e^{-\frac{\sqrt{-1}}{5}\theta} g_0 \mathbf{e}_i \wedge e^{-\frac{\sqrt{-1}}{5}\theta} g_0 \mathbf{e}_j).\end{aligned}$$

It follows from this that $(V_\Lambda)_{K_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2), 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4)\}$. For the generator $g \in K_{[\mathfrak{a}]}$ of $\mathbf{Z}_4$ given above, we have

$$\begin{aligned}\rho_\Lambda(g)(1 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2) &= -1 \otimes \mathbf{e}_3 \wedge \mathbf{e}_4, \\ \rho_\Lambda(g)(1 \otimes \mathbf{e}_3 \wedge \mathbf{e}_4) &= 1 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2.\end{aligned}$$

Hence $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$, that is, $\Lambda = (1,1,0,0,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, we get $\Lambda = (0,0,0,-1,-1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,0,0,0,-1)$. Then $\dim V_\Lambda = 24$. By the branching law of $(U(5), U(4) \times U(1))$, $\Lambda' = (1,0,0,0,-1)$, $(1,0,0,-1,0)$, $(0,0,0,0,0)$, or $(0,0,0,-1,1)$, where $\Lambda'_1 = (1,0,0,-1,0)$, $\Lambda'_2 = (0,0,0,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2) \times U(2))$, $\Lambda''_1 = (1,0,0,-1)$, $(1,-1,0,0)$, $(0,0,0,0)$, $(0,0,1,-1)$, or $(0,-1,1,0)$, where $\Lambda''_1 = (0,0,0,0) \in D(K_1, K_0)$. Also, $\Lambda''_2 = (0,0,0,0) \in D(K_1, K_0)$. Thus $-c_\Lambda = 10$, $-c_{\Lambda'_1} = 8$, $-c_{\Lambda''_1} = 0$, $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 12 < 18$ and $-c_{\Lambda'_2} = 0$, $-c_{\Lambda''_2} = 0$, $-c_L = 20 > 18$.

On the other hand, $\Lambda = \omega_1 + \omega_4$ corresponds to the adjoint representation of $SU(5)$:

$$\begin{aligned}V_\Lambda = \mathbf{C} \otimes \Bigg( &\mathbf{C} \cdot \begin{pmatrix} -\frac{1}{4}I_4 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbf{C} \cdot \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \\ &\oplus \mathbf{C} \cdot \begin{pmatrix} & & * \\ & & 0 \\ * & 0 & 0 \end{pmatrix} \oplus \mathbf{C} \cdot \begin{pmatrix} & & 0 \\ & & * \\ 0 & * & 0 \end{pmatrix} \Bigg) \\ &= V'_{(0,0,0,0,0)} \oplus V'_{(1,0,0,-1,0)} \oplus V'_{(1,0,0,0,-1)} \oplus V'_{(0,0,0,-1,1)};\end{aligned}$$

$$\begin{aligned}(V_\Lambda)_{K_0} = &\left\{ \begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbf{C}, 2c_1 + 2c_2 + c_3 = 0 \right\} \\ &\subset V'_{(0,0,0,0,0)} \oplus V'_{(1,0,0,-1,0)}.\end{aligned}$$

By direct calculations, we get that for a generator $g \in K_{[\mathfrak{a}]} \subset K_2$ in $\mathbf{Z}_4$ as above,

$$\mathrm{Ad}(g)\begin{pmatrix} c_1 I_2 & & \\ & c_2 I_2 & \\ & & c_3 \end{pmatrix} = \begin{pmatrix} c_2 I_2 & & \\ & c_1 I_2 & \\ & & c_3 \end{pmatrix}.$$

Hence

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} = \left\{ \begin{pmatrix} -\frac{c}{4}\mathrm{I}_4 & \\ & c \end{pmatrix} \mid c \in \mathbf{C} \right\} = V'_{(0,0,0,0,0)}.$$

But this 1-dimensional fixed vector space corresponds to the larger eigenvalue 20.

Suppose that $\Lambda = (2,1,1,0,0)$. Then $\dim V_\Lambda = 45$. By the branching law of $(U(5), U(4) \times U(1))$ that $V_\Lambda$ can be decomposed into the irreducible $K_2 = U(4) \times U(1)$-submodules

$$V_\Lambda = V'_{(2,1,1,0,0)} \oplus V'_{(1,1,1,0,1)} \oplus V'_{(2,1,0,0,1)} \oplus V'_{(1,1,0,0,2)},$$

where $\Lambda' = (2,1,1,0,0) \in D(K_2, K_0)$. By the branching law of $(U(4), U(2)\times U(2))$, $\Lambda'' = (2,1,1,0)$, $(2,0,1,1)$, $(1,1,2,0)$, $(1,1,1,1)$, or $(1,0,2,1)$, where $\Lambda'' = (1,1,1,1) \in D(K_1, K_0)$. Thus $-c_\Lambda = 16$, $-c_{\Lambda'} = 12$, $-c_{\Lambda''} = 4$, $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 18$.

On the other hand, since $V'_{(1,1,1,0,1)} \oplus V'_{(2,1,0,0,1)} \oplus V'_{(1,1,0,0,2)}$ has no nonzero vectors fixed by $K_0$, we see that $(V_\Lambda)_{K_0} \subset V'_{(2,1,1,0,0)}$. Note that $\Lambda' = 2y_1 + y_2 + y_3 = \sum_{i=1}^4 y_i + y_1 - y_4 \in D(K_2, K_0)$ corresponds to the tensor product of $C(U(4))$ representation with the highest weight $\sum_{i=1}^4 y_i$, the adjoint representation of $SU(4)$ with the highest weight $y_1 - y_4$, and the trivial representation of $U(1)$. Then for each element $g_0 \in K_0$ and each element $u \otimes X \otimes v \in \mathbf{C} \otimes \mathfrak{su}(4) \otimes \mathbf{C} \cong V_{\Lambda'}$,

$$\rho_{\Lambda'}(g_0)(u \otimes X \otimes v) = u \otimes \mathrm{Ad}\begin{pmatrix} A & \\ & B \end{pmatrix}(X) \otimes v.$$

Thus $(V_\Lambda)_{K_0} = \mathrm{span}\{1 \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \otimes 1\}$. For the element $g \in K_{[\mathfrak{a}]} \subset K_2$,

$$\rho_{\Lambda'}(g)(u \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix} \otimes v) = e^{\sqrt{-1}\pi} u \otimes \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix} \otimes v.$$

It follows that $(V_\Lambda)_{K_{[\mathfrak{a}]}} = (V_\Lambda)_{K_0}$, that is, $\Lambda = (2,1,1,0,0) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1. Similarly, $\Lambda = (0,0,-1,-1,-2) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 1 and it also gives the eigenvalue 18.

Suppose that $\Lambda = (1,1,0,-1,-1)$. Then $\dim V_\Lambda = 75$. By the branching law of $(U(4), U(2)\times U(2))$, $V_\Lambda$ can be decomposed into the irreducible $K_1 = U(4) \times U(1)$-submodules:

$$V_\Lambda = V'_{(1,1,0,-1,-1)} \oplus V'_{(1,1,-1,-1,0)} \oplus V'_{(1,0,0,-1,0)} \oplus V'_{(1,0,-1,-1,1)},$$

where $\Lambda'_1 = (1,1,-1,-1,0)$ and $\Lambda'_2 = (1,0,0,-1,0) \in D(K_2,K_0)$. For $\Lambda'_2$, by the branching law of $(U(4),U(2)\times U(2))$, $\Lambda''_2 = (1,0,0,-1)$, $(1,-1,0,0)$, $(0,0,-1,-1)$, $(0,0,0,0)$, or $(0,-1,1,0)$, where $\Lambda''_2 = (0,0,0,0) \in D(K_1,K_0)$. Therefore, $-c_\Lambda = 16$, $-c_{\Lambda'_2} = 8$, $-c_{\Lambda''_2} = 0$, and $-c_L = -2c_\Lambda + c_{\Lambda'_2} + \frac{1}{2}c_{\Lambda''_2} = 24 > 18$. For $\Lambda'_1$, by the branching law of $(U(4),U(2)\times U(2))$, $\Lambda'' = (1,1,-1,-1)$, $(1,0,0,-1)$, $(1,-1,1,-1)$, $(0,0,0,0)$, $(0,-1,1,0)$ or $(-1,-1,1,1)$, where $\Lambda''_{11} = (1,1,-1,-1)$, $\Lambda''_{12} = (-1,-1,1,1)$, $\Lambda''_{13} = (0,0,0,0) \in D(K_1,K_0)$. Thus $-c_\Lambda = 16$, $-c_{\Lambda'} = 12$, $-c_{\Lambda''_{11}} = -c_{\Lambda''_{12}} = 4$, $-c_{\Lambda''_{13}} = 0$, $-c_L = -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} = 18, 18$, or $20$. Moreover, from the above irreducible $K_2$-decomposition of $V_\Lambda$ and eigenvalue calculations, we only need to determine $\dim(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap (V''_{11} \oplus V''_{12})$ since the fixed vectors in this subspace by $K_{[\mathfrak{a}]}$ give the eigenvalue 18. Here we set $V''_{11} := V''_{\Lambda''_{11}}$ and $V''_{12} := V''_{\Lambda''_{12}}$.

Recall that the irreducible representation of $SU(4)$ with the highest weight $\Lambda'_1 = y_1 + y_2 - y_3 - y_4 = 2\omega_2$ can be described as follows ([**14**]):

$$\mathrm{Sym}^2(\wedge^2\mathbf{C}^4) = I(Gr_2(\mathbf{C}^4))_2 \oplus V'_{\Lambda'_1},$$

where $I(Gr_2(\mathbf{C}^4))_2$, the ideal of the Grassmannian $Gr_2(\mathbf{C}^4)$, denotes the space of all homogeneous polynomials of degree 2 on $\mathbf{P}(\wedge^2\mathbf{C}^{4*})$ that vanish on $Gr_2(\mathbf{C}^4)$. Here $I(Gr_2(\mathbf{C}^4))_2 \cong \wedge^4\mathbf{C}^4 \cong \mathbf{C}$ can be written down explicitly in terms of a basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ of $\mathbf{C}^4$:

$$\begin{aligned} I(Gr_2(\mathbf{C}^4))_2 = \mathrm{span}\{&(\mathbf{e}_1\wedge\mathbf{e}_2)\cdot(\mathbf{e}_3\wedge\mathbf{e}_4) + (\mathbf{e}_1\wedge\mathbf{e}_4)\cdot(\mathbf{e}_2\wedge\mathbf{e}_3) \\ &- (\mathbf{e}_1\wedge\mathbf{e}_3)\cdot(\mathbf{e}_2\wedge\mathbf{e}_4)\}. \end{aligned}$$

Thus a basis for $V'_{\Lambda'_1}$ can be given explicitly. For any element $g_0 \in K_0$, denote $g'_0 = \begin{pmatrix} A & \\ & B \end{pmatrix} \in SU(2)\times SU(2) \subset U(4)$. The representation of $K_0$ on any element $u\otimes X\otimes w \in \mathbf{C}\otimes V'_{\Lambda'_1}\otimes\mathbf{C}$ is

$$\rho_\Lambda(g)(u\otimes X\otimes w) = \rho_0(1)(u)\otimes\rho_{\Lambda'_1}(g'_0)(X)\otimes\rho_0(e^{\sqrt{-1}\theta})(w).$$

By direct computations, we obtain

$$\begin{aligned} (V_\Lambda)_{K_0} \cap V'_{\Lambda'_1} = \mathrm{span}_{\mathbf{C}}\{\, &1\otimes(\mathbf{e}_1\wedge\mathbf{e}_2)\cdot(\mathbf{e}_1\wedge\mathbf{e}_2)\otimes 1, \\ &1\otimes(\mathbf{e}_3\wedge\mathbf{e}_4)\cdot(\mathbf{e}_3\wedge\mathbf{e}_4)\otimes 1,\ 1\otimes(\mathbf{e}_1\wedge\mathbf{e}_2)\cdot(\mathbf{e}_3\wedge\mathbf{e}_4)\otimes 1\}, \end{aligned}$$

where $(\mathbf{e}_1\wedge\mathbf{e}_2)\cdot(\mathbf{e}_1\wedge\mathbf{e}_2) \in V''_{11}$, $(\mathbf{e}_3\wedge\mathbf{e}_4)\cdot(\mathbf{e}_3\wedge\mathbf{e}_4) \in V''_{12}$ and $(\mathbf{e}_1\wedge\mathbf{e}_2)\cdot(\mathbf{e}_3\wedge\mathbf{e}_4) \in V''_{13}$. For the generator $g \in K_{[\mathfrak{a}]} \subset K_2$, denote $g' = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}$. The representation of $g$ on $u\otimes X\otimes w$ is

$$\rho_\Lambda(g)(u\otimes X\otimes w) = \rho_0(e^{\frac{\sqrt{-1}}{4}\pi}I_4)(u)\otimes\rho_{\Lambda'_1}(e^{-\frac{\sqrt{-1}}{4}\pi}g')(X)\otimes\rho_0(1)(w).$$

It follows that

$$(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap V'_{\Lambda'_1} = \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1,$$
$$1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1 - 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}.$$

In particular, $\Lambda = (1, 1, 0, -1, -1) \in D(K, K_{[\mathfrak{a}]})$ and

$$\begin{aligned}&(V_\Lambda)_{K_{[\mathfrak{a}]}} \cap (V''_{11} \oplus V''_{12}) \\ &= \mathrm{span}_{\mathbf{C}}\{1 \otimes (\mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) \otimes 1 - 1 \otimes (\mathbf{e}_3 \wedge \mathbf{e}_4) \cdot (\mathbf{e}_3 \wedge \mathbf{e}_4) \otimes 1\}\end{aligned}$$

with dimension 1, which corresponds to the eigenvalue 18.

Now we obtain that the Gauss image $L^{18}$ is Hamiltonian stable. Moreover,

$$\begin{aligned}n(L^{18}) =& \dim V_{(0,0,-1,-1,-2)} + \dim V_{(2,1,1,0,0)} + \dim V_{(1,1,0,-1,-1)} \\ =& 45 + 45 + 75 = 165 = \dim SO(20) - \dim U(5) = n_{hk}(L^{18}).\end{aligned}$$

Hence the Gauss image $L^{18}$ is Hamiltonian rigid.

Therefore, we conclude that the Gauss image $L^{18}$ is Hamiltonian stable.

**Theorem 5.1.** *The Gauss image* $L^{18} = \mathcal{G}\left(\frac{U(5)}{(SU(2)\times SU(2)\times U(1))}\right) = \frac{U(5)}{(SU(2)\times SU(2)\times U(1))\cdot \mathbf{Z}_4} \subset Q_{18}(\mathbf{C})$ *is strictly Hamiltonian stable.*

## 6. The case $(U, K) = (SO(m+2), SO(2) \times SO(m))$ $(m \geq 3)$

.

In this case $(U, K)$ is of type $B_2$. The canonical decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{u} = \mathfrak{o}(m+2)$ and a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ are given as

$$\mathfrak{k} = \left\{\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \mid T_1 \in \mathfrak{o}(2), T_2 \in \mathfrak{o}(m)\right\} = \mathfrak{o}(2) + \mathfrak{o}(m),$$
$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} \mid X \in M(m, 2; \mathbf{R})\right\},$$
$$\mathfrak{a} = \left\{H = H(\xi_1, \xi_2) = \begin{pmatrix} 0 & -{}^t\xi & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R}\right\}.$$

Then

$$\begin{aligned}K_0 &= \left\{\begin{pmatrix} \pm \mathrm{I}_4 & 0 \\ 0 & T \end{pmatrix} \mid T \in SO(m-2)\right\} \\ &\cong \mathbf{Z}_2 \times SO(m-2).\end{aligned}$$

Moreover

$$K_{[\mathfrak{a}]} \cong (\mathbf{Z}_2 \times SO(m-2)) \cdot \mathbf{Z}_4$$

consists of all elements

$$a = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B' \end{pmatrix} \in K = SO(2) \times SO(m),$$

where

$$\begin{aligned}(A,B) = &\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \\ &\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right), \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right), \\ &\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \\ &\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right), \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right).\end{aligned}$$

Here note that $K_{[\mathfrak{a}]} \not\subset K_1 = SO(2) \times SO(2) \times SO(m-2)$. Thus the deck transformation group of the covering map $\mathcal{G} : N^{2m-2} \to \mathcal{G}(N^{2m-2})$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$. The element

$$g = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & B' \end{pmatrix} \in K_{[\mathfrak{a}]}$$

represents a generator of $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

**6.1. Description of the Casimir operator.** Denote $\langle X, Y \rangle_{\mathfrak{u}} := -\frac{1}{2}\mathrm{tr}XY$ for each $X, Y \in \mathfrak{u} = \mathfrak{o}(m+2)$. The restricted root system $\Sigma(U,K)$ of type $B_2$ can be given as follows ([**7**]):

$$\Sigma^+(U,K) = \{\varepsilon_1 - \epsilon_2 = \alpha_1,\ \varepsilon_2 = \alpha_2,\ \varepsilon_1 + \epsilon_2 = \alpha_1 + 2\alpha_2,\ \varepsilon_1 = \alpha_1 + \alpha_2\}.$$

Then, relative to the above inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$, the square length of any restrict root $\gamma \in \Sigma(U,K)$ is $\|\gamma\|_{\mathfrak{u}}^2 = 1$ or 2. Hence the Casimir operator $\mathcal{C}_L$ of $L$ with respect to the induced metric from $Q_{2m-2}(\mathbf{C})$ is given as follows:

$$\begin{aligned}\mathcal{C}_L &= \frac{2}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} - \frac{1}{\|\gamma_0\|_{\mathfrak{u}}^2}\mathcal{C}_{K_1/K_0,\langle\,,\,\rangle_{\mathfrak{u}}} \\ &= \mathcal{C}_{K/K_0} - \frac{1}{2}\,\mathcal{C}_{K_1/K_0},\end{aligned} \tag{6.1}$$

where $K = SO(2) \times SO(m) \supset K_1 = SO(2) \times SO(2) \times SO(m-2) \supset K_0 = \mathbf{Z}_2 \times SO(m-2)$ and $\mathcal{C}_{K/K_0}$, $\mathcal{C}_{K_1/K_0}$ denote the Casimir operators of $K/K_0$ and $K_1/K_0$ relative to $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}}$ and $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_1}$, respectively.

**6.2. Branching laws for** $(SO(n+2), SO(2) \times SO(n))$**.** We need the branching laws for $(SO(n+2), SO(2) \times SO(n))$ by Tsukamoto [**49**].

**Lemma 6.1** (Branching laws for $(SO(2p+2), SO(2)\times SO(2p)), p \geq 1)$**.** *Let* $\Lambda = h_0\varepsilon_0 + h_1\varepsilon_1 + \cdots + h_{p-1}\varepsilon_{p-1} + \epsilon h_p\varepsilon_p \in D(SO(2p+2))$*, where* $\epsilon = 1$ *or* $-1$ *and* $h_0, h_1, \cdots, h_p$ *are integers satisfying*

$$h_0 \geq h_1 \geq \cdots \geq h_p \geq 0 \tag{6.2}$$

*and* $\Lambda' = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_{p-1}\varepsilon_{p-1} + \epsilon' k_p\varepsilon_p \in D(SO(2) \times SO(2p))$*, where* $\epsilon' = 1$ *or* $-1$ *and* $k_0, k_1, \cdots, k_p$ *are integers satisfying*

$$k_1 \geq \cdots \geq k_p \geq 0. \tag{6.3}$$

*The irreducible decomposition of* $V_\Lambda$ *as a* $SO(2) \times SO(2p)$*-module contains an irreducible* $SO(2) \times SO(2p)$*-module* $V'_{\Lambda'}$ *if and only if*

$$\begin{aligned}&h_{i-1} \geq k_i \geq h_{i+1} \quad (1 \leq i \leq p-1),\\ &h_{p-1} \geq k_p \geq 0,\end{aligned}$$

*and the coefficient of* $X^{k_0}$ *in the finite power series*

$$X^{\epsilon\epsilon' l_p} \prod_{i=0}^{p-1} \frac{X^{l_i+1} - X^{-l_i-1}}{X - X^{-1}}$$

*does not vanish, where*

$$\begin{aligned}l_0 &:= h_0 - \max\{h_1, k_1\},\\ l_i &:= \min\{h_i, k_i\} - \max\{h_{i+1}, k_{i+1}\} \quad (1 \leq i \leq p-1),\\ l_p &:= \min\{h_p, k_p\}.\end{aligned} \tag{6.4}$$

*Moreover, the coefficient of* $X^{k_0}$ *is equal to the multiplicity of* $V'_{\Lambda'}$ *appearing in the irreducible decomposition.*

**Lemma 6.2** (Branching laws for $(SO(2p+3), SO(2) \times SO(2p+1))$, $p \geq 1)$**.** *Let* $\Lambda = h_0\varepsilon_0 + h_1\varepsilon_1 + \cdots + h_{p-1}\varepsilon_{p-1} + h_p\varepsilon_p \in D(SO(2p+3))$*, where* $h_0, h_1, \cdots, h_p$ *are integers satisfying* (6.2) *and* $\Lambda' = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_{p-1}\varepsilon_{p-1} + k_p\varepsilon_p \in D(SO(2) \times SO(2p+1))$*, where* $k_0, k_1, \cdots, k_p$ *are integers satisfying* (6.3)*. The irreducible decomposition of* $V_\Lambda$ *as an* $SO(2) \times SO(2p+1)$*-module contains an irreducible* $SO(2) \times SO(2p+1)$*-module* $V'_{\Lambda'}$ *if and only if*

$$\begin{aligned}&h_{i-1} \geq k_i \geq h_{i+1}, \quad (1 \leq i \leq p-1)\\ &h_{p-1} \geq k_p \geq 0,\end{aligned}$$

*and the coefficient of* $X^{k_0}$ *in the finite power series*

$$\left(\prod_{i=0}^{p-1} \frac{X^{l_i+1} - X^{-l_i-1}}{X - X^{-1}}\right) \frac{X^{l_p+\frac{1}{2}} - X^{-l_p-\frac{1}{2}}}{X^{\frac{1}{2}} - X^{-\frac{1}{2}}}$$

*does not vanish, where integers $l_0, l_1, \cdots, l_p$ are defined by* (6.4). *Moreover, the coefficient of $X^{k_0}$ is equal to the multiplicity of $V'_{\Lambda'}$ appearing in the irreducible decomposition.*

**6.3. Description of $D(K, K_0)$ and eigenvalue computations.**

For $m = 2p$ ($p \geq 2$) or $m = 2p+1$ ($p \geq 1$), each $\tilde{\Lambda} \in D(K) = D(SO(2) \times SO(m))$ can be expressed as

$$\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + \cdots + k_p\varepsilon_p,$$

where $k_0\varepsilon_0 \in D(SO(2))$, $\Lambda := k_1\varepsilon_1 + \cdots + k_p\varepsilon_p \in D(SO(m))$, and $k_0, k_1, \cdots, k_p \in \mathbf{Z}$ satisfying

$$\begin{aligned} &k_1 \geq k_2 \geq \cdots \geq k_{p-1} \geq |k_p| \quad \text{if } m = 2p, \\ &k_1 \geq k_2 \geq \cdots \geq k_{p-1} \geq k_p \geq 0 \quad \text{if } m = 2p+1. \end{aligned}$$

Then we have

$$\tilde{V}_{\tilde{\Lambda}} = U_{k_0\varepsilon_0} \otimes V_\Lambda.$$

Note that

$$\begin{aligned} D(K, K_0) &= D(SO(2) \times SO(m), \mathbf{Z}_2 \times SO(m-2)) \\ &\subset D(SO(2) \times SO(m), SO(m-2)), \\ D(K_1, K_0) &= D(SO(2) \times SO(2) \times SO(m-2), \mathbf{Z}_2 \times SO(m-2)) \\ &\subset D(SO(2) \times SO(2) \times SO(m-2), SO(m-2)). \end{aligned}$$

By applying Lemmas 6.1 and 6.2 to both cases $(SO(2p), SO(2) \times SO(2p-2))$ and $(SO(2p), SO(2) \times SO(2p-1))$, we can describe $D(K, K_0)$ as follows:

**Lemma 6.3.** *Assume that $p \geq 2$. Let $\tilde{\Lambda} \in D(K)$. Then an irreducible $K$-module $\tilde{V}_{\tilde{\Lambda}}$ with the highest weight $\tilde{\Lambda}$ contains an irreducible $K_1$-module $\tilde{V}'_{\tilde{\Lambda}'}$ with the highest weight $\tilde{\Lambda}' \in D(K_1)$ satisfying $(\tilde{V}'_{\tilde{\Lambda}'})_{K_0} \neq \{0\}$ if and only if*

$$\begin{aligned} \tilde{\Lambda} &= k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K), \\ \tilde{\Lambda}' &= k_0\varepsilon_0 + k'_1\varepsilon_1 \in D(K_1), \end{aligned}$$

*where $k_0, k_1, k_2, k'_1 \in \mathbf{Z}$, $k_1 \geq k_2 \geq 0$ satisfy the following conditions:*

(i) *The coefficient of $X^{k'_1}$ in the finite series expansion $\dfrac{X^{k_1-k_2+1} - X^{-(k_1-k_2+1)}}{X - X^{-1}}$ of $X$ does not vanish;*
(ii) *$k_0 + k'_1$ is even.*

*In particular, $-(k_1 - k_2) \leq k'_1 \leq (k_1 - k_2)$. Here the coefficient is equal to the multiplicity of $\tilde{V}'_{\tilde{\Lambda}'}$.*

**6.3.1. The case $m = 2p$ ($p \geq 2$).**

Suppose that $m = 2p$ ($p \geq 2$). For each

$$\begin{aligned}\tilde{\Lambda} &= k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0)\\ &= D(SO(2) \times SO(2p), \mathbf{Z}_2 \times SO(2p-2))\end{aligned}$$

with $\tilde{\Lambda}' = k_0\varepsilon_0 + k_1'\varepsilon_1 \in D(K_1, K_0) = D(SO(2) \times SO(2) \times SO(2p-2), \mathbf{Z}_2 \times SO(2p-2))$ as in Lemma 6.3, $-\mathcal{C}_{K/K_0}$ and $-\mathcal{C}_{K_1/K_0}$ have eigenvalues

$$\begin{aligned}-c_{\tilde{\Lambda}} &= k_0^2 + k_1^2 + k_2^2 + 2(p-1)k_1 + 2(p-2)k_2,\\ -c_{\tilde{\Lambda}'} &= \frac{1}{2}(k_0^2 + {k'}_1^2).\end{aligned}$$

Hence by the formula (6.1) the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}-c_L = &\ -c_{\tilde{\Lambda}} + \frac{1}{2}c_{\tilde{\Lambda}'}\\ = &\ k_0^2 + k_1^2 + k_2^2 + 2(p-1)k_1 + 2(p-2)k_2 - \frac{1}{2}(k_0^2 + {k'}_1^2).\end{aligned} \tag{6.5}$$

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0)$ by $\tilde{\Lambda} = (k_0, k_1, k_2)$.

For each $\tilde{\Lambda} = k_0\varepsilon_0 = (k_0, 0, 0) \in D(K, K_0)$, as $k_1' = 0$, $k_0 = k_0 + k_1'$ is even and $-c_L = \frac{1}{2}k_0^2$, we see that

$$-c_L \leq 2m - 2 = 4p - 2 \text{ if and only if } k_0^2 \leq 4(2p-1). \tag{6.6}$$

Since $\tilde{V}_{\tilde{\Lambda}} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C} \cong U_{k_0\varepsilon_0}$, we have

$$\rho_{k_0\varepsilon_0}(g)(v \otimes 1) = e^{\sqrt{-1}\frac{\pi}{2}k_0}\,(v \otimes 1).$$

Hence

$$(k_0, 0, 0) \in D(K, K_{[\mathfrak{a}]}) \text{ if and only if } k_0 \in 4\mathbf{Z}. \tag{6.7}$$

(i) The case $\mathcal{G}(N^6) \cong \frac{SO(2)\times SO(4)}{(\mathbf{Z}_2\times SO(2))\cdot \mathbf{Z}_4} \to Q_6(\mathbf{C})$ with $p = 2$.

**Lemma 6.4.** $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0)$ *has eigenvalue* $-c_L \leq 6$ *if and only if* $(k_0, k_1, k_2)$ *is one of*

$$\{0, (\pm 2, 0, 0), (\pm 1, 1, 0), (0, 1, 1), (\pm 2, 1, 1), (0, 2, 0), (0, 1, -1), (\pm 2, 1, -1)\}.$$

*Proof.* Since $-\mathcal{C}_L = -\frac{1}{2}\mathcal{C}_{K/K_0} - \frac{1}{2}\mathcal{C}_{K/K_1} \geq -\frac{1}{2}\mathcal{C}_{K/K_0}$, the condition $-c_L \geq 6$ implies that $-c_{\tilde{\Lambda}} = -c_{K/K_0} \leq 12$. Using the eigenvalue formula (6.5), we obtain the result. q.e.d.

Suppose that $\tilde{\Lambda} = (\pm 2, 0, 0)$. Then by (6.7) $\tilde{\Lambda} = (\pm 2, 0, 0) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (\pm 1, 1, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 4$ and $\tilde{V}_{\tilde{\Lambda}} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C}^4$, where $\Lambda = \varepsilon_1 \in D(K)$ corresponds to the matrix multiplication of $SO(4)$ on $\mathbf{C}^4$. It follows from the branching law (Lemma 6.1, $p = 2$) of $(SO(4), SO(2) \times SO(2))$ that $k_1' = \pm 1$. Hence $-c_L = \frac{1}{2}k_0^2 + \frac{5}{2}$. Note that

$U_{k_0\varepsilon_0} \otimes \mathbf{C}^4$ can be decomposed into irreducible $SO(2)\times SO(2)\times SO(2)$-modules as

$$U_{k_0\varepsilon_0} \otimes \mathbf{C}^4 = (U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})) \oplus (U_{k_0\varepsilon_0} \otimes (\{0\} \oplus \mathbf{C}^2)).$$

There is no nonzero fixed vector by $\mathbf{Z}_2 \times SO(2)$ in $U_{k_0\varepsilon_0} \otimes (\{0\} \oplus \mathbf{C}^2)$. Moreover, since

$$\begin{aligned}
&\rho_{k_0\varepsilon_0+\varepsilon_1}\begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & T \end{pmatrix}(v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \end{pmatrix}) \\
&= e^{\sqrt{-1}\pi k_0} v \otimes \begin{pmatrix} -w_1 \\ -w_2 \\ 0 \\ 0 \end{pmatrix} = e^{\sqrt{-1}\pi(k_0+1)} v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}$$

it follows that $(\tilde{V}_{\tilde{\Lambda}})_{K_0} = (\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2\times SO(2)} \neq \{0\}$ if and only if $k_0$ is odd, and then $(\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2\times SO(2)} = U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$. Let $k_0$ be odd. However, since

$$\rho_{k_0\varepsilon_0+\varepsilon_1}(g)(v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \end{pmatrix}) = e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes \begin{pmatrix} w_2 \\ w_1 \\ 0 \\ 0 \end{pmatrix},$$

$U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ has no nonzero fixed vector by $(\mathbf{Z}_2 \times SO(2)) \cdot \mathbf{Z}_4$, and hence $(k_0, 1, 0) \notin D(K, K_{[\mathfrak{a}]})$. In particular, $(\pm 1, 1, 0) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda}_1 = (k_0, 1, 1)$ and $\tilde{\Lambda}_2 = (k_0, 1, -1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}_1} = \dim \tilde{V}_{\tilde{\Lambda}_2} = 3$ and $\tilde{V}_{\tilde{\Lambda}_1} \oplus \tilde{V}_{\tilde{\Lambda}_2} \cong \mathbf{C} \otimes \wedge^2 \mathbf{C}^4$. It follows from the branching law (Lemma 6.1, p=2) $(SO(4), SO(2)\times SO(2))$ that

$$\tilde{V}_{\tilde{\Lambda}_1} = \tilde{V}'_{(k_0,1,1)} \oplus \tilde{V}'_{(k_0,-1,-1)} \oplus \tilde{V}'_{(k_0,0,0)},$$

where $(k_0, 0, 0) \in D(K_1, K_0)$. Thus $-c_L = \frac{1}{2}k_0^2 + 4$, which is equal to 4 when $k_0 = 0$ and 6 when $k_0 = \pm 2$.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbf{C}^4$. Then we have

$$\begin{aligned}
\tilde{V}_{\tilde{\Lambda}_1} &= \operatorname{span}\{e_1\wedge e_2, e_1\wedge e_3 - e_2\wedge e_4, e_1\wedge e_4 + e_2\wedge e_3\}, \\
\tilde{V}_{\tilde{\Lambda}_2} &= \operatorname{span}\{e_3\wedge e_4, e_1\wedge e_3 + e_2\wedge e_4, e_1\wedge e_4 - e_2\wedge e_3\}.
\end{aligned}$$

Since $e_1 \wedge e_2 \in \wedge^2\mathbf{C}^4$ is fixed by the representation of $SO(2)\times SO(2)$ with respect to the highest weight $\tilde{\Lambda}_1$,

$$(\tilde{V}_{\tilde{\Lambda}_1})_{K_0} = \operatorname{span}\{1 \otimes (e_1 \wedge e_2)\}.$$

Moreover,

$$\rho_{\tilde{\Lambda}_1}(g)(v \otimes (e_1\wedge e_2)) = e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes (e_2 \wedge e_1).$$

Hence $\tilde{\Lambda}_1 = (0,1,1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda}_1 = (\pm 2,1,1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}_1})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C}\{e_1 \wedge e_2\}$ for $k_0 = 2$ or $-2$, both of which give eigenvalue 6. Similarly, $\tilde{\Lambda}_2 = (0,1,-1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda}_2 = (\pm 2,1,-1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}_2})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C}\{e_3 \wedge e_4\}$ for $k_0 = 2$ or $-2$, both of which give eigenvalue 6.

Suppose that $\tilde{\Lambda} = (0,2,0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 9$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^4)$, where the corresponding representation of $SO(4)$ is just the adjoint representation on $\mathrm{S}^2_0(\mathbf{C}^4)$. It follows from the branching law of $(SO(4), SO(2) \times SO(2))$ that $k_1' = 0, \pm 2$. Thus $-c_L = 8 - \frac{1}{2}{k_1'}^2$. When $k_1' = \pm 2$, $-c_L = 6$; otherwise, $-c_L = 8 > 6$. On the other hand, $\mathrm{S}^2_0(\mathbf{C}^4)$ can be decomposed into the following $SO(2) \times SO(2)$-modules:

$$V_{2\varepsilon_1} \cong \mathrm{S}^2_0(\mathbf{C}^4) = \mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathrm{S}^2_0(\mathbf{C}^2) \oplus M(2,2;\mathbf{C}) \oplus \mathbf{C}\begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Thus $\mathrm{S}^2_0(\mathbf{C}^2) \oplus \mathbf{C}\begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}$ is fixed by $\{-I_2\} \times SO(2)$ and $\dim(\tilde{V}_{\tilde{\Lambda}})_{K_0} = 3$. Moreover,

$$\rho_{\tilde{\Lambda}}(g)(v \otimes \begin{pmatrix} a & b & \\ b & -a & \\ & & 0 \end{pmatrix}) = v \otimes \begin{pmatrix} -a & b & \\ b & a & \\ & & 0 \end{pmatrix},$$
$$\rho_{\tilde{\Lambda}}(g)(v \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}) = v \otimes \begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Hence

$$(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} = \mathbf{C} \otimes \mathbf{C}\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \oplus \mathbf{C} \otimes \mathbf{C}\begin{pmatrix} I_2 & \\ & -I_2 \end{pmatrix}.$$

Notice that the first summand lies in the $SO(2) \times SO(2) \times SO(2)$-module $V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1}$, which gives eigenvalue 6 and the second summand lies in the $SO(2) \times SO(2) \times SO(2)$-module with respect to weight $(0,0,0) \in D(K_1, K_0)$, which gives eigenvalue $8 > 6$. Therefore, $\tilde{\Lambda} = (0,2,0) \in D(K, K_{[\mathfrak{a}]})$ and the multiplicity corresponding to eigenvalue 6 is 1.

Now we know that $\mathcal{G}(N^6) \subset Q_6(\mathbf{C})$ is Hamiltonian stable. Since $\tilde{\Lambda} = (2,1,1)$, $(-2,1,1)$, $(2,1,-1)$, $(-2,1,-1)$, $(0,2,0) \in D(K, K_{[\mathfrak{a}]})$ give the smallest eigenvalue 6 with multiplicity 1 and

$$\begin{aligned}
&n(L^6)\\
=&\dim \tilde{V}_{(2,1,1)} + \dim \tilde{V}_{(-2,1,1)} + \dim \tilde{V}_{(2,1,-1)} + \dim \tilde{V}_{(-2,1,-1)} + \dim \tilde{V}_{(0,2,0)}\\
=&3+3+3+3+9 = 21 = \dim SO(8) - \dim(SO(2) \times SO(4)) = n_{hk}(L^6).
\end{aligned}$$

Hence we obtain that $\mathcal{G}(N^6) \subset Q_6(\mathbf{C})$ is strictly Hamiltonian stable.

(ii) The case $\mathcal{G}(N^{4p-2}) \cong \frac{SO(2)\times SO(2p)}{(\mathbf{Z}_2 \times SO(2p-2))\cdot \mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ with $p \geq 3$.

Suppose that $\tilde{\Lambda} = (k_0, 0, 0)$ and $k_0 \in 4\mathbf{Z} \setminus \{0\}$. Then $k_1' = 0$ and by (6.6) $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$. As $p \geq 3$, we have $16 < 20 \leq 4(2p-1)$. Hence by (6.7) we see that for every $k_0 \in 4\mathbf{Z} \setminus \{0\}$ such that $16 \leq k_0^2 < 4(2p-1)$ we have eigenvalue $-c_L = \frac{1}{2}k_0^2 < 4p-2$. Therefore, $\mathcal{G}(N^{4p-2}) \cong \frac{SO(2)\times SO(2p)}{(\mathbf{Z}_2\times SO(2p-2))\cdot\mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ is not Hamiltonian stable if $p \geq 3$.

**Theorem 6.1.**

$$L^{4p-2} = (SO(2) \times SO(2p))/(\mathbf{Z}_2 \times SO(2p-2))\mathbf{Z}_4 \quad (p \geq 2)$$

*is not Hamiltonian stable if and only if* $(m-2)-1 = 2p-3 \geq 3$. *If* $p = 2$, *then it is strictly Hamiltonian stable.*

Remark. The index $i(L^{4p-2})$ goes to $\infty$ as $p \to \infty$.

**6.3.2. The case $m = 2p+1$ $(p \geq 1)$.**

Assume that $m = 2p+1$ $(p \geq 2)$. For each

$$\begin{aligned}\tilde{\Lambda} &= k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0)\\ &= D(SO(2) \times SO(2p+1), \mathbf{Z}_2 \times SO(2p-1))\end{aligned}$$

with $\tilde{\Lambda}' = k_0\varepsilon_0 + k_1'\varepsilon_1 \in D(K_1, K_0) = D(SO(2) \times SO(2) \times SO(2p-1), \mathbf{Z}_2 \times SO(2p-1))$ as in Lemma 6.3, $-\mathcal{C}_{K/K_0}$ and $-\mathcal{C}_{K_1/K_0}$ have eigenvalues

$$\begin{aligned}-c_{\tilde{\Lambda}} &= k_0^2 + k_1^2 + k_2^2 + (2p-1)k_1 + (2p-3)k_2,\\ -c_{\tilde{\Lambda}'} &= -\frac{1}{2}(k_0^2 + {k'}_1^2).\end{aligned}$$

Hence by the formula (6.1) the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}-c_L = &\ -c_{\tilde{\Lambda}} + \frac{1}{2}c_{\tilde{\Lambda}'}\\ = &\ k_0^2 + k_1^2 + k_2^2 + (2p-1)k_1 + (2p-3)k_2 - \frac{1}{2}(k_0^2 + {k'}_1^2).\end{aligned} \tag{6.8}$$

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0)$ by $\tilde{\Lambda} = (k_0, k_1, k_2)$.

For each $\tilde{\Lambda} = k_0\varepsilon_0 = (k_0, 0, 0) \in D(K, K_0)$, as $k_1' = 0$, $k_0 = k_0 + k_1'$ is even and $-c_L = \frac{1}{2}k_0^2$, we see that

$$-c_L \leq 2m-2 = 4p \text{ if and only if } k_0^2 \leq 8p. \tag{6.9}$$

As $\tilde{V}_{\tilde{\Lambda}} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C} \cong U_{k_0\varepsilon_0}$, we have

$$\rho_{k_0\varepsilon_0}(g)(v \otimes 1) = e^{\sqrt{-1}\frac{\pi}{2}k_0}(v \otimes 1).$$

Hence

$$(k_0, 0, 0) \in D(K, K_{[\mathfrak{a}]}) \text{ if and only if } k_0 \in 4\mathbf{Z}. \tag{6.10}$$

(i) The case $\mathcal{G}(N^4) \cong \frac{SO(2)\times SO(3)}{\mathbf{Z}_2\cdot\mathbf{Z}_4} \to Q_4(\mathbf{C})$ with $p = 1$.

In this case, $K = SO(2)\times SO(3)$, $K_1 = SO(2)\times SO(2)$, and $K_0 = \mathbf{Z}_2$, where $\mathbf{Z}_2$ is generated by $\begin{pmatrix} -I_4 & 0 \\ 0 & 1 \end{pmatrix} \in U = SO(5)$. Let $V_{\tilde{\Lambda}}$ be an irreducible $SO(2)\times SO(3)$-module with the highest weight $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 \in D(K) = D(SO(2)\times SO(3))$, where $k_0, k_1 \in \mathbf{Z}$ and $k_1 \geq 0$. It follows from the branching law of $(SO(3), SO(2))$ that $V_{\tilde{\Lambda}}$ contains an irreducible $SO(2)\times SO(2)$-module $V_{\tilde{\Lambda}'}$ with the highest weight $\tilde{\Lambda}' = k_0\varepsilon_0 + k_1'\varepsilon_1 \in D(K_1) = D(SO(2)\times SO(2))$, where $k_1' \in \mathbf{Z}$, if and only if $|k_1'| \leq k_1$. Then we see that $\tilde{\Lambda}' \in D(SO(2)\times SO(2), \mathbf{Z}_2)$ if and only if $k_0 + k_1'$ is even. By the formula (6.1) the corresponding eigenvalue of the Casimir operator $-\mathcal{C}_L$ is

$$-c_L = k_0^2 + k_1^2 + k_1 - \frac{1}{2}(k_0^2 + {k'}_1^2) = \frac{1}{2}k_0^2 + k_1^2 + k_1 - \frac{1}{2}{k'}_1^2. \tag{6.11}$$

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 \in D(SO(2)\times SO(3), \mathbf{Z}_2)$ by $\tilde{\Lambda} = (k_0, k_1)$. Using the eigenvalue formula (6.11), we compute the following.

**Lemma 6.5.** *$\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 \in D(K, K_0)$ has eigenvalue $-c_L \leq 4$ if and only if $(k_0, k_1)$ is one of*

$$\Big\{ 0, (\pm 2, 0), (\pm 2, 1), (\pm 1, 1), (0, 1), (0, 2) \Big\}.$$

Suppose that $\tilde{\Lambda} = (\pm 2, 0)$. Notice that for any $v\otimes w \in \tilde{V}_{k_0\varepsilon_0} \cong \mathbf{C}\otimes\mathbf{C}$,

$$\rho_{k_0\varepsilon_0}(g)(v\otimes w) = e^{\sqrt{-1}k_0\frac{\pi}{2}} v\otimes w,$$

$\tilde{\Lambda} = k_0\varepsilon_0 \in D(K, K_{[\mathfrak{a}]})$ if and only if $k_0 \in 4\mathbf{Z}$. Hence $\tilde{\Lambda} = (\pm 2, 0) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 3$. The complex representation of $K = SO(2)\times SO(3)$ with the highest weight $\tilde{\Lambda}$ corresponds to

$$\tilde{V}_{\tilde{\Lambda}} = U_{k_0\varepsilon_0}\otimes V_{\varepsilon_1} \cong U_{k_0\varepsilon_0}\otimes \mathbf{C}^3 = (U_{k_0\varepsilon_0}\otimes\mathbf{C}^2)\oplus(U_{k_0\varepsilon_0}\otimes\mathbf{C}^1).$$

For each $v\otimes w \in U_{k_0\varepsilon_0}\otimes\mathbf{C}^3$ and $\mathrm{diag}(-I_2, -I_2, 1) \in K_0$, where $w = (w_1, w_2, w_2)^t \in \mathbf{C}^3$, the representation of $K_0$ is given by

$$\rho_{\tilde{\Lambda}}(\mathrm{diag}(-I_2, -I_2, 1))(v\otimes w) = e^{\sqrt{-1}k_0\pi} v\otimes(-w_1, -w_2, w_3)^t.$$

Then $(V_{\tilde{\Lambda}})_{K_0} = \mathbf{C}\otimes\mathbf{C}(0,0,w_3)^t \cong \mathbf{C}\otimes\mathbf{C}$ if $k_0$ is even and $(V_{\tilde{\Lambda}})_{K_0} = \mathbf{C}\otimes\mathbf{C}(w_1, w_2, 0)^t \cong \mathbf{C}\otimes\mathbf{C}^2$ if $k_0$ is odd. Moreover,

$$\rho_{\tilde{\Lambda}}(g)(v\otimes w) = e^{\sqrt{-1}k_0\frac{\pi}{2}} v\otimes \begin{pmatrix} w_2 \\ w_1 \\ -w_3 \end{pmatrix}.$$

Thus $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$ if and only if $k_0 \equiv 2 \bmod 4$ and its multiplicity is 1. In particular, $\tilde{\Lambda} = (0,1)$ or $(\pm 1, 1) \notin D(K, K_{[\mathfrak{a}]})$ and $\tilde{\Lambda} = (\pm 2, 1) \in D(K, K_{[\mathfrak{a}]})$. For $\tilde{\Lambda} = (\pm 2, 1)$, it follows from the branching laws of $(SO(3), SO(2))$ that $|k'_1| \leq k_1$ thus $k'_1 = 0$ such that $k_0 + k'_1$ is even. Hence $-c_L = 4$.

Suppose that $\tilde{\Lambda} = (0, 2)$. Then $\dim_{\mathbf{C}} \tilde{V}_{\tilde{\Lambda}} = 5$. It follows from the branching law of $(SO(3), SO(2))$ that $k'_1 = 0$ or $\pm 2$. If $k'_1 = \pm 2$, then $-c_L = 4$. If $k'_1 = 0$, then $-c_L = 6 > 4$. On the other hand, $\Lambda = 2\varepsilon_1 \in D(SO(3))$ corresponds to $V_\Lambda \cong \mathrm{S}^2_0(\mathbf{C}^3)$, and the representation of $SO(3)$ on $\mathrm{S}^2_0(\mathbf{C}^3)$ is just the complexified isotropy representation of a symmetric pair $(SU(3), SO(3))$. Thus $\mathrm{S}^2_0(\mathbf{C}^3)$ can be decomposed into irreducible $SO(2)$-modules as

$$\begin{aligned}
V_{2\varepsilon_1} &\cong \mathrm{S}^2_0(\mathbf{C}^3) \\
&= \mathrm{S}^2_0(\mathbf{C}^2) \oplus \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{pmatrix} \mid a, b \in \mathbf{C} \right\} \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \\
&= \mathbf{C} \begin{pmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & -1 \end{pmatrix} \\
&\quad \oplus \mathbf{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 1 & \sqrt{-1} & 0 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -\sqrt{-1} \\ 1 & -\sqrt{-1} & 0 \end{pmatrix} \oplus \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \\
&= V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1} \oplus V'_{\varepsilon_1} \oplus V'_{-\varepsilon_1} \oplus V'_0.
\end{aligned}$$

Using this expression, we can directly show that

$$\begin{aligned}
&(\tilde{V}_{\tilde{\Lambda}})_{K_0} \cong (\mathbf{C} \otimes \mathrm{S}^2_0(\mathbf{C}^2)) \oplus \left( \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \right) \\
&\text{and } (\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} \cong \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \oplus \left( \mathbf{C} \otimes \mathbf{C} \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \right).
\end{aligned}$$

Hence $\tilde{\Lambda} = (0, 2) \in D(K, K_{[\mathfrak{a}]})$ with multiplicity 2. Note that the first summand of $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}}$ lies in $\mathbf{C} \otimes (V'_{2\varepsilon_1} \oplus V'_{-2\varepsilon_1})$, which gives eigenvalue 4 with multiplicity 1 and the second summand of $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}}$ lies in $\mathbf{C} \otimes V'_0$, which gives eigenvalue $6 (> 4)$ with multiplicity 1.

Now we obtain that $\mathcal{G}(N^4) \subset Q_4(\mathbf{C})$ is Hamiltonian stable. Moreover, since

$$\begin{aligned}
n(L^4) &= \dim \tilde{V}_{(2,1)} + \dim \tilde{V}_{(-2,1)} + \dim \tilde{V}_{(0,2)} = 3 + 3 + 5 \\
&= 11 = \dim SO(6) - \dim(SO(2) \times SO(3)) = n_{hk}(L^4),
\end{aligned}$$

$L^4 = \mathcal{G}(N^4) \subset Q_4(\mathbf{C})$ is Hamiltonian rigid. Therefore, $\mathcal{G}(N^4) \subset Q_4(\mathbf{C})$ is strictly Hamiltonian stable.

(ii) The case $\mathcal{G}(N^8) \cong \frac{SO(2) \times SO(5)}{(\mathbf{Z}_2 \times SO(3)) \cdot \mathbf{Z}_4} \to Q_8(\mathbf{C})$ with $p = 2$.

Denote $\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0) = D(SO(2) \times SO(5), \mathbf{Z}_2 \times SO(3))$ by $\tilde{\Lambda} = (k_0, k_1, k_2)$. Let $\tilde{\Lambda}' = k_0\varepsilon_0 + k'_1\varepsilon_1 \in D(K_1, K_0) = D(SO(2) \times SO(2) \times SO(3), \mathbf{Z}_2 \times SO(3))$ as in Lemma 6.3. Then, using the eigenvalue formula (6.8), we compute the following.

**Lemma 6.6.** *$\tilde{\Lambda} = k_0\varepsilon_0 + k_1\varepsilon_1 + k_2\varepsilon_2 \in D(K, K_0)$ has eigenvalue $-c_L \leq 8$ if and only if $(k_0, k_1, k_2)$ is one of*

$$\{\, 0, (\pm 4, 0, 0), (\pm 1, 1, 0), (\pm 3, 1, 0), (0, 1, 1), (\pm 2, 1, 1), (0, 2, 0)\,\}.$$

Suppose that $\tilde{\Lambda} = (\pm 4, 0, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 1$. It follows from the branching law of $(SO(5), SO(2)\times SO(3))$ that $k_1' = 0$. Thus $-c_L = 8$. On the other hand, it follows from (6.10) that $\tilde{\Lambda} = (\pm 4, 0, 0) \in D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 5$ and $\tilde{V}_{\tilde{\Lambda}} \cong U_{k_0\varepsilon_0} \otimes \mathbf{C}^5$, where $\Lambda = \varepsilon_1 \in D(K)$ corresponds to the matrix multiplication of $SO(5)$ on $\mathbf{C}^5$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k_1' = \pm 1$. Hence $-c_L = \frac{1}{2}k_0^2 + \frac{7}{2}$. Notice that $U_{k_0\varepsilon_0} \otimes \mathbf{C}^5$ can be decomposed into the $SO(2) \times SO(3)$-modules

$$U_{k_0\varepsilon_0} \otimes \mathbf{C}^5 = (U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})) \oplus (U_{k_0\varepsilon_0} \otimes (\{0\} \oplus \mathbf{C}^3)),$$

where $U_{k_0}\varepsilon_0 \otimes (\{0\} \oplus \mathbf{C}^3)$ has no nonzero fixed vector by $\mathbf{Z}_2 \times SO(3)$. If $k_0$ is odd, then

$$\rho_{k_0\varepsilon_0+\varepsilon_1}\begin{pmatrix} -I_2 & & \\ & -I_2 & \\ & & T \end{pmatrix}(v\otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}) = e^{\sqrt{-1}\pi k_0} v\otimes \begin{pmatrix} -w_1 \\ -w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v\otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

that is, $(\tilde{V}_{\tilde{\Lambda}})_{\mathbf{Z}_2\otimes SO(3)} = U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ if $k_0$ is odd. But since

$$\rho_{k_0\varepsilon_0+\varepsilon_1}(g)(v \otimes \begin{pmatrix} w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}) = e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes \begin{pmatrix} w_2 \\ w_1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$U_{k_0\varepsilon_0} \otimes (\mathbf{C}^2 \oplus \{0\})$ has no nonzero fixed vector by $(\mathbf{Z}_2 \times SO(3)) \cdot \mathbf{Z}_4$, i.e., neither $(\pm 1, 1, 0)$ and $(\pm 3, 1, 0)$ is in $D(K, K_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (k_0, 1, 1)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 10$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C}\otimes\wedge^2\mathbf{C}^5$. It follows from the branching law of $(SO(5), SO(2) \times SO(3))$ that $k_1' = 0$. Thus $-c_L = \frac{1}{2}k_0^2 + 6$. On the other hand, since $e_1 \wedge e_2 \in \wedge^2\mathbf{C}^5$ is fixed by $SO(2) \times SO(3)$, $v \otimes (e_1 \wedge e_2) \in \mathbf{C} \otimes \wedge^2\mathbf{C}^5$ is fixed by $\mathbf{Z}_2 \times SO(3) \subset SO(2) \times SO(2) \times SO(3)$. Moreover,

$$\rho_{k_0\varepsilon_0+\varepsilon_1+\varepsilon_2}(g)(v \otimes (e_1 \wedge e_2)) = e^{\sqrt{-1}\frac{\pi}{2}k_0} v \otimes (e_2 \wedge e_1).$$

Hence $\tilde{\Lambda} = (0, 1, 1) \notin D(K, K_{[\mathfrak{a}]})$ but $\tilde{\Lambda} = (\pm 2, 1, 1) \in D(K, K_{[\mathfrak{a}]})$ and $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} \cong \mathbf{C}\otimes\mathbf{C}\{e_1\wedge e_2\}$ for $k_0 = 2$ or $-2$, both of which give eigenvalue 8.

Suppose that $\tilde{\Lambda} = (0, 2, 0)$. Then $\dim \tilde{V}_{\tilde{\Lambda}} = 14$ and $\tilde{V}_{\tilde{\Lambda}} \cong \mathbf{C} \otimes \mathrm{Sym}_0^2(\mathbf{C}^5)$, where $\mathrm{Sym}_0^2$ is the space of traceless symmetric matrices and the representation of $SO(5)$ with highest weight $2\varepsilon_1$ is just the adjoint representation on $\mathrm{Sym}^2_0(\mathbf{C}^5)$. It follows from the branching law of $(SO(5), SO(2)\times SO(3))$ that $k_1' = 0, \pm 2$. Thus $-c_L = 10 - \frac{1}{2}{k_1'}^2$. When $k_1' = \pm 2$, $-c_L = 8$; otherwise, $-c_L = 10 > 8$. On the other hand, $\mathrm{Sym}^2_0(\mathbf{C}^5)$ can be decomposed into the following $SO(2)\times SO(3)$-modules:

$$\begin{aligned} V_{2\varepsilon_1} &\cong \mathrm{Sym}^2_0(\mathbf{C}^5) \\ &= \mathrm{Sym}^2_0(\mathbf{C}^2) \oplus \mathrm{Sym}^2_0(\mathbf{C}^3) \oplus M(2,3;\mathbf{C}) \\ &\oplus \left\{ \begin{pmatrix} zI_2 & \\ 0 & wI_3 \end{pmatrix} \mid z, w \in \mathbf{C}, 2z + 3w = 0 \right\}. \end{aligned}$$

Thus $\mathrm{Sym}^2_0(\mathbf{C}^2)$ is fixed by $\{-I_2\}\times SO(3)$ and

$$(\tilde{V}_{\tilde{\Lambda}})_{K_0} \cong \mathbf{C}\otimes \mathrm{Sym}^2_0(\mathbf{C}^2) \oplus \mathbf{C}\otimes\mathbf{C}\begin{pmatrix} 3I_2 & \\ & -2I_3 \end{pmatrix}.$$

Moreover,

$$\rho_{2\varepsilon_1}(g)(v\otimes \begin{pmatrix} a & b & \\ b & -a & \\ & & 0 \end{pmatrix}) = v \otimes \begin{pmatrix} -a & b & \\ b & a & \\ & & 0 \end{pmatrix}.$$

Hence $(\tilde{V}_{\tilde{\Lambda}})_{K_{[\mathfrak{a}]}} = \mathbf{C}\otimes\mathbf{C}\cdot \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \oplus \mathbf{C}\otimes\mathbf{C}\begin{pmatrix} 3I_2 & \\ & -2I_3 \end{pmatrix}$. Therefore, $\tilde{\Lambda} = (0,2,0) \in D(K, K_{[\mathfrak{a}]})$. Notice the first summand lies in $\tilde{V}'_{(0,2,0)} \oplus \tilde{V}'_{(0,-2,0)}$, which gives eigenvalue 8, and the second summand lies in $\tilde{V}'_{(0,0,0)}$, which gives eigenvalue 10. Hence the multiplicity corresponding to eigenvalue 8 is 1.

Since $\tilde{\Lambda} = (4,0,0), (-4,0,0), (2,1,1), (-2,1,1), (0,2,0) \in D(K, K_{[\mathfrak{a}]})$ give the smallest eigenvalue 8 with multiplicity 1 and

$$\begin{aligned} n(L^8) &= \dim \tilde{V}_{(4,0,0)} + \dim \tilde{V}_{(-4,0,0)} + \dim \tilde{V}_{(2,1,1)} + \dim \tilde{V}_{(-2,1,1)} \\ &\quad + \dim \tilde{V}_{(0,2,0)} \\ &= 1 + 1 + 10 + 10 + 14 = 36 \\ &> 34 = \dim SO(10) - \dim SO(2)\times SO(5) = n_{hk}(L^8), \end{aligned}$$

$\mathcal{G}(N^8) \subset Q_8(\mathbf{C})$ is not Hamiltonian rigid. Therefore, $\mathcal{G}(N^8) \subset Q_8(\mathbf{C})$ is Hamiltonian stable but not strictly Hamiltonian stable.

(iii) The case $\mathcal{G}(N^{4p}) \cong \frac{SO(2)\times SO(2p+1)}{(\mathbf{Z}_2\times SO(2p-1))\cdot \mathbf{Z}_4} \to Q_{4p}(\mathbf{C})$ with $p \geq 3$.

Suppose that $\tilde{\Lambda} = (k_0, 0, 0)$ and $k_0 \in 4\mathbf{Z}\setminus\{0\}$. Then $k_1' = 0$ and by (6.9) $\tilde{\Lambda} \in D(K, K_{[\mathfrak{a}]})$. As $p \geq 3$, we have $16 < 24 \leq 8p$. Hence by (6.10) we see that for every $k_0 \in 4\mathbf{Z}\setminus\{0\}$ such that $16 \leq k_0^2 < 8p$ we have eigenvalue $-c_L = \frac{1}{2}k_0^2 < 4p$. Therefore, $\mathcal{G}(N^{4p}) \cong \frac{SO(2)\times SO(2p+1)}{(\mathbf{Z}_2\times SO(2p-1))\cdot \mathbf{Z}_4} \to Q_{4p-2}(\mathbf{C})$ is not Hamiltonian stable if $p \geq 3$.

Therefore, we obtain the following.

**Theorem 6.2.** *The Gauss image* $L^{4p} = \frac{SO(2)\times SO(2p+1)}{(\mathbf{Z}_2\times SO(2p-1))\mathbf{Z}_4} \to Q_{4p}(\mathbf{C})$ $(p \geq 1)$ *is not Hamiltonian stable if and only if* $(m-2)-1 = 2p-2 \geq 3$. *If* $p = 1$, *then it is strictly Hamiltonian stable. If* $p = 2$, *then it is Hamiltonian stable but not strictly Hamiltonian stable.*

Remark. The index $i(L^{4p})$ goes to $\infty$ as $p \to \infty$.

## 7. The case $(U,K) = (SU(m+2), S(U(2)\times U(m)))$ $(m \geq 2)$

In this case, $U = SU(m+2)$ and $K = S(U(2)\times U(m))$ with $m \geq 2$. Then $(U,K)$ is of $B_2$ type for $m = 2$ and $BC_2$ type for $m \geq 3$.

In this case, we use the formulation by the unitary group $U(m)$ rather than one by the special unitary groups $SU(m)$. It seems to work more successfully in our argument of applying the branching laws. Here we will also indicate the relations between both formulations. Let $\tilde{U} := U(m+2)$, $\tilde{K} := U(2)\times U(m)$, $\tilde{K}_2 := U(2)\times U(2)\times U(m-2)$, $\tilde{K}_1 := U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)$, and $\tilde{K}_0 := U(1)\times U(1)\times U(m-2)$. Then $\tilde{U} = C(\tilde{U})\cdot U$, $\tilde{K} = C(\tilde{U})\cdot K$, $\tilde{K}_2 = C(\tilde{U})\cdot K_2$, $\tilde{K}_1 = C(\tilde{U})\cdot K_1$, and $\tilde{K}_0 = C(\tilde{U})\cdot K_0$, where $C(\tilde{U})$ is the center of $\tilde{U}$.

Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ and $\tilde{\mathfrak{u}} = \tilde{\mathfrak{k}} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ and $\tilde{\mathfrak{u}}$ corresponding to $(U,K)$ and $(\tilde{U},\tilde{K})$, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, where

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & H_{12} \\ -\bar{H}_{12}^t & 0 \end{pmatrix} \mid H_{12} = \begin{pmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\}.$$

Then the centralizer $\tilde{K}_0$ of $\mathfrak{a}$ in $\tilde{K}$ is given as follows:

$$\tilde{K}_0 = \left\{ P = \begin{pmatrix} e^{is} & & & & \\ & e^{it} & & & \\ & & e^{is} & & \\ & & & e^{it} & \\ & & & & T \end{pmatrix} \mid T \in U(m-2) \right\}$$
$$\cong U(1)\times U(1)\times U(m-2).$$

Moreover,

$$\tilde{K}_{[\mathfrak{a}]} = \tilde{K}_0 \cup (Q\cdot\tilde{K}_0) \cup (Q^2\cdot\tilde{K}_0) \cup (Q^3\cdot\tilde{K}_0),$$

where

$$Q = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & -1 & \\ & & -1 & 0 & \\ & & & & I_{m-2} \end{pmatrix} \in \tilde{K}_2 \subset \tilde{K}.$$

Thus the deck transformation group of the covering map $\mathcal{G} : N^{8m-2} \to \mathcal{G}(N^{4m-2})$ $(m \geq 2)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \tilde{K}_{[\mathfrak{a}]}/\tilde{K}_0 \cong \mathbf{Z}_4$. Remark that we will use $P$ and $Q$ to denote the element in $\tilde{K}_0$ and the generator of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ throughout this section.

**7.1. Description of the Casimir operator.**

Define an inner product $\langle X, Y\rangle_{\mathfrak{u}} := -\mathrm{tr}XY$ for each $X, Y \in \mathfrak{u} = \mathfrak{su}(m+2)$ or for each $X, Y \in \tilde{\mathfrak{u}} = \mathfrak{u}(m+2)$ . The restricted root system $\Sigma(U, K)$ is of type $B_2$ for $m = 2$ and type $BC_2$ for $m \geq 3$. Then the square length of each restricted roots with respect to $\langle\ ,\ \rangle_{\mathfrak{u}}$, is given by

$$\|\gamma\|^2_{\mathfrak{u}} = \begin{cases} 1 \text{ or } 2, & m = 2, \\ \frac{1}{2}, 1 \text{ or } 2, & m \geq 3. \end{cases}$$

Hence the Casimir operator $\mathcal{C}_L$ of $L$ with respect to the induced metric from $g^{\mathrm{std}}_{Q_{4m-2}(\mathbf{C})}$ can be expressed as follows:

$$\mathcal{C}_L = \begin{cases} \mathcal{C}_{K/K_0} - \frac{1}{2}\,\mathcal{C}_{K_1/K_0}, & m = 2, \\ 2\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, & m \geq 3, \end{cases} \tag{7.1}$$

where $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_2/K_0}$, and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of $K/K_0$, $K_2/K_0$, and $K_1/K_0$ relative to $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}}$, $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_2}$ and $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_1}$, respectively.

**7.2. Descriptions of $D(\tilde{U})$, $D(U)$ and etc.**

$D(\tilde{U})$, $D(C(\tilde{U}))$, and $D(U)$ are described as follows:

$$\begin{aligned} D(\tilde{U}) = D(U(m+2)) = \Big\{&\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_{m+2} y_{m+2} \mid \tilde{p}_1, \ldots, \tilde{p}_{m+2} \in \mathbf{Z}, \\ &\tilde{p}_i - \tilde{p}_{i+1} \geq 0\ (i = 1, \ldots, m+1)\Big\}, \\ D(C(\tilde{U})) = D(C(U(m+2))) = \Big\{&\Lambda = p_0(y_1 + \cdots + y_{m+2}) \mid p_0 \in \frac{1}{m+2}\mathbf{Z}\Big\}, \\ D(U) = D(SU(m+2)) = \Big\{&\Lambda = p_1 y_1 + \cdots + p_{m+2} y_{m+2} \mid \sum_{i=1}^{m+2} p_i = 0, \\ &p_i - p_{m+2} \in \mathbf{Z}, p_i - p_{i+1} \geq 0\ (i = 1, \ldots, m+1)\Big\}. \end{aligned}$$

Each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(U(m+2))$ can be decomposed as $\tilde{\Lambda} = \Lambda^0 + \Lambda$, where

$$\Lambda^0 = \left(\frac{1}{m+2}\sum_{i=1}^{m+2} \tilde{p}_i\right)\left(\sum_{i=1}^{m+2} y_i\right) \in D(C(U(m+2)))$$

and

$$\Lambda = (\tilde{p}_1 - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{p}_i)y_1 + \cdots + (\tilde{p}_{m+2} - \frac{1}{m+2}\sum_{i=1}^{m+2}\tilde{p}_i)y_{m+2} \in D(SU(m+2)).$$

Note that this projection $D(\tilde{U}) \to D(U), \tilde{\Lambda} \mapsto \Lambda$ is surjective.

$$\begin{aligned} &D(\tilde{K}) = D(U(2) \times U(m)) \\ =&\{\tilde{\Lambda} = \tilde{q}_1 y_1 + \tilde{q}_2 y_2 + \tilde{q}_3 y_3 + \cdots + \tilde{q}_{m+2} y_{m+2} \mid \\ &\tilde{q}_i \in \mathbf{Z}\ (i = 1, \ldots, m+2), \tilde{q}_1 - \tilde{q}_2 \geq 0, \tilde{q}_i - \tilde{q}_{i+1} \geq 0\ (i = 3, \ldots, m+1)\}, \end{aligned}$$

$$
\begin{aligned}
&D(K) = D(S(U(2)\times U(m)))\\
&=\{\Lambda = q_1y_1+q_2y_2+q_3y_3+\cdots+q_{m+2}y_{m+2} \mid \sum_{i=1}^{m+2} q_i = 0, q_i - q_j \in \mathbf{Z}\\
&\quad (i,j=1,2,\ldots,m+2), q_1-q_2 \geq 0, q_i - q_{i+1}\\
&\quad \geq 0\ (i=3,4,\ldots,m+1)\},\\
&D(\tilde{K_2}) = D(U(2)\times U(2)\times U(m-2))\\
&=\{\tilde{\Lambda} = \tilde{q}_1y_1+\tilde{q}_2y_2+\tilde{q}_3y_3+\tilde{q}_4y_4+\tilde{q}_5y_5+\cdots+\tilde{q}_{m+2}y_{m+2} \mid\\
&\qquad \tilde{q}_i \in \mathbf{Z}\ (i=1,\ldots,m+2),\\
&\qquad \tilde{q}_1-\tilde{q}_2, \tilde{q}_3-\tilde{q}_4, \tilde{q}_i-\tilde{q}_{i+1} \geq 0\ (i=5,\ldots,m+1)\},\\
&D(K_2) = D(S(U(2)\times U(2)\times U(m-2)))\\
&=\{\Lambda = q_1y_1+q_2y_2+q_3y_3+q_4y_4+q_5y_5+\cdots+q_{m+2}y_{m+2} \mid \sum_{i=1}^{m+2} q_i = 0,\\
&\quad q_i-q_j \in \mathbf{Z}\ (i,j=1,2,\ldots,m+2), q_1-q_2, q_3-q_4, q_i-q_{i+1}\\
&\quad \geq 0\ (i=5,\ldots,m+1)\},\\
&D(\tilde{K_1}) = D(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2))\\
&=\{\tilde{\Lambda} = \tilde{q}_1y_1+\tilde{q}_2y_2+\tilde{q}_3y_3+\tilde{q}_4y_4+\tilde{q}_5y_5+\cdots+\tilde{q}_{m+2}y_{m+2} \mid\\
&\qquad \tilde{q}_i \in \mathbf{Z}\ (i=1,\ldots,m+2), \tilde{q}_i-\tilde{q}_{i+1} \geq 0\ (i=5,\ldots,m+1)\ \},\\
&D(K_1) = D(S(U(1)\times U(1)\times U(1)\times U(1)\times U(m-2)))\\
&=\Big\{\Lambda = q_1y_1+q_2y_2+q_3y_3+q_4y_4+q_5y_5+\cdots+q_{m+2}y_{m+2} \mid \sum_{i=1}^{m+2} q_i = 0,\\
&\quad q_i-q_j \in \mathbf{Z}\ (i,j=1,\ldots,m+2), q_i-q_{i+1} \geq 0\ (i=5,\ldots,m+1)\Big\},\\
&D(\tilde{K_0}) = D(U(1)\times U(1)\times U(m-2))\\
&=\{\tilde{\Lambda} = \tilde{q}_1y_1+\tilde{q}_2y_2+\tilde{q}_3y_3+\tilde{q}_4y_4+\tilde{q}_5y_5+\cdots+\tilde{q}_{m+2}y_{m+2} \mid\\
&\qquad \tilde{q}_3 = \tilde{q}_1 \in \frac{1}{2}\mathbf{Z}, \tilde{q}_4 = \tilde{q}_2 \in \frac{1}{2}\mathbf{Z}, \tilde{q}_i \in \mathbf{Z}\ (i=5,\ldots,m+2),\\
&\qquad \tilde{q}_i-\tilde{q}_{i+1} \geq 0\ (i=5,6,\ldots,m+1)\},\\
&D(K_0) = D(S(U(1)\times U(1)\times U(m-2)))\\
&=\{\Lambda = q_1y_1+q_2y_2+q_3y_3+q_4y_4+q_5y_5+\cdots+q_{m+2}y_{m+2} \mid\\
&\qquad \sum_{i=1}^{m+2} q_i = 0, q_i-q_j \in \mathbf{Z}\ (i,j=1,\ldots,m+2),\\
&\qquad q_3 = q_1, q_4 = q_2, q_i-q_{i+1} \geq 0\ (i=5,\ldots,m+1)\}.
\end{aligned}
$$

The natural maps $D(\tilde{K}) \longrightarrow D(K)$, $D(\tilde{K_2}) \longrightarrow D(K_2)$, $D(\tilde{K_1}) \longrightarrow D(K_1)$ and $D(\tilde{K_0}) \longrightarrow D(K_0)$ are also surjective.

**7.3. Branching laws of** $(U(m), U(2)\times U(m-2))$**.** The branching laws for $(SU(m), S(U(m)\times U(2)))$ given in [**29**] can be reformulated to the branching laws for $(U(m), U(2)\times U(m-2))$ as follows:

**Lemma 7.1** (Branching law of $(U(m), U(2)\times U(m-2))$)**.** *For each* $\tilde{\Lambda} = \tilde{p}_1 y_1 + \cdots + \tilde{p}_m y_m \in D(U(m))$*, an irreducible* $U(m)$*-module* $V_{\tilde{\Lambda}}$ *with the highest weight* $\tilde{\Lambda}$ *can be decomposed into the direct sum of irreducible* $U(2)\times U(m-2)$*-modules as follows:*

$$V_{\tilde{\Lambda}} = \bigoplus_{\tilde{\Lambda}'\in D(U(2)\times U(m-2))} V'_{\tilde{\Lambda}'}.$$

*Here* $V_{\tilde{\Lambda}}$ *contains an irreducible* $U(2)\times U(m-2)$*-module* $V'_{\tilde{\Lambda}'}$ *with the highest weight* $\tilde{\Lambda}' = \tilde{q}_1 y_1 + \cdots + \tilde{q}_m y_m \in D(U(2)\times U(m-2))$ *if and only if the following conditions are satisfied:*

(i) $\tilde{q}_1 - \tilde{p}_1 \in \mathbf{Z}$;

(ii) $\tilde{p}_{i-2} \geq \tilde{q}_i \geq \tilde{p}_i \quad (i = 3, \ldots, m)$;

(iii) *in the finite power series expansion in* $X$ *of* $\dfrac{\prod_{i=2}^{m}(X^{r_i+1} - X^{-(r_i+1)})}{(X - X^{-1})^{m-2}}$, *where*

$$\begin{aligned} r_2 &:= \tilde{p}_1 - \max(\tilde{q}_3, \tilde{p}_2), \\ r_i &:= \min(\tilde{q}_i, \tilde{p}_{i-1}) - \max(\tilde{q}_{i+1}, \tilde{p}_i), \quad (3 \leq i \leq m-1), \\ r_m &:= \min(\tilde{q}_m, \tilde{p}_{m-1}) - \tilde{p}_m, \end{aligned}$$

*the coefficient of* $X^{\tilde{q}_1 - \tilde{q}_2 + 1}$ *does not vanish. Moreover, the value of this coefficient is equal to the multiplicity of the irreducible* $U(2)\times U(m-2)$*-module* $V'_{\tilde{\Lambda}'}$.

**7.4. Branching law of** $(U(3), U(2)\times U(1))$**.** Now following Lemma 5.1 the branching law of $(U(3), U(2)\times U(1))$ is described as follows.

**Lemma 7.2.** *Let* $\tilde{V}_{\tilde{\Lambda}}$ *be an irreducible* $U(3)$*-module with the highest weight* $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 \in D(U(3))$*, where* $\tilde{p}_i \in \mathbf{Z}$ $(i = 1, 2, 3)$ *and* $\tilde{p}_1 \geq \tilde{p}_2 \geq \tilde{p}_3$*. Then* $\tilde{V}_{\tilde{\Lambda}}$ *can be decomposed into irreducible* $U(2)\times U(1)$*-modules as*

$$\tilde{V}_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3} = \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{p}_2 - \tilde{p}_3} \tilde{V}'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 - \beta) y_2 + (\tilde{p}_3 + \alpha + \beta) y_3}.$$

**7.5. Descriptions of** $D(\tilde{K}, \tilde{K}_0)$**,** $D(\tilde{K}_2, \tilde{K}_0)$**,** $D(\tilde{K}_1, \tilde{K}_0)$**.** Let

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(\tilde{K}) = D(U(2)\times U(m)),$$

where $\tilde{p}_1, \ldots, \tilde{p}_{m+2} \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \cdots \geq \tilde{p}_{m+2}$. Thus $\Lambda_\sigma = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 \in D(U(2))$, $\Lambda_\tau = \tilde{p}_3 y_3 + \cdots + \tilde{p}_{m+2} y_{m+2} \in D(U(m))$ and

$$\tilde{\rho}_{\tilde{\Lambda}} = \sigma \boxtimes \tau \in \mathcal{D}(\tilde{K}) = \mathcal{D}(U(2)\times U(m)),$$

where $\sigma \in \mathcal{D}(U(2))$, $\tau \in \mathcal{D}(U(m))$.

By Lemma 7.1, an irreducible $U(m)$-module $V_\tau$ with the highest weight $\Lambda_\tau$ can be decomposed into the direct sum of irreducible $U(2) \times U(m-2)$-modules as

$$V_\tau = \bigoplus V'_{\tilde{\Lambda}'_\tau},$$

where $\tilde{\Lambda}'_\tau = \sum_{i=3}^{m+2} \tilde{q}_i y_i \in D(U(2) \times U(m-2))$ with $\tilde{q}_3, \ldots, \tilde{q}_{m+2} \in \mathbf{Z}$, $\tilde{q}_i - \tilde{q}_{i+1} \geq 0$ $(i = 3, 5, \ldots, m+1)$. Note that setting $\Lambda_\varsigma := \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(U(2))$ and $\Lambda_\gamma := \tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2} \in D(U(m-2))$, $V_{\tilde{\Lambda}}$ is decomposed into the direct sum of irreducible $\tilde{K}_2$-modules as

$$V_{\tilde{\Lambda}} = \bigoplus_{\varsigma,\gamma} (V_\sigma \boxtimes V_\varsigma \boxtimes V_\gamma).$$

By the branching law of $(U(2), U(1) \times U(1))$ (see Lemma 5.1),

$$V_\sigma = V_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2} = \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2},$$

$$V_\varsigma = V_{\tilde{q}_3 y_3 + \tilde{q}_4 y_4} = \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4}.$$

Thus $V_{\tilde{\Lambda}}$ is decomposed into the direct sum of irreducible $\tilde{K}_1$-modules as

$$V_{\tilde{\Lambda}} = \bigoplus \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} (V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2} \boxtimes V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}).$$

Since as a $U(1) \times U(1)$-module

$$\begin{aligned} &V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2} \boxtimes V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4} \\ =&V''_{\frac{1}{2}(\tilde{p}_1 + \tilde{q}_3 - \alpha - \beta)(y_1 + y_3) + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_4 + \alpha + \beta)(y_2 + y_4)}, \end{aligned}$$

$V_{\tilde{\Lambda}}$ is decomposed into the direct sum of irreducible $\tilde{K}_0$-modules:

$$\begin{aligned} &V_{\tilde{\Lambda}} \\ =&\bigoplus_{\varsigma,\gamma} (V_{\tilde{p}_1 y_1 + \tilde{p}_2 y_2} \boxtimes V_{\tilde{q}_3 y_3 + \tilde{q}_4 y_4} \boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}) \\ =&\bigoplus_{\varsigma,\gamma} \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} (V'_{(\tilde{p}_1 - \alpha) y_1 + (\tilde{p}_2 + \alpha) y_2} \boxtimes V'_{(\tilde{q}_3 - \beta) y_3 + (\tilde{q}_4 + \beta) y_4} \\ &\boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}}) \\ =&\bigoplus_{\varsigma,\gamma} \bigoplus_{\alpha=0}^{\tilde{p}_1 - \tilde{p}_2} \bigoplus_{\beta=0}^{\tilde{q}_3 - \tilde{q}_4} V''_{\frac{1}{2}(\tilde{p}_1 + \tilde{q}_3 - \alpha - \beta)(y_1 + y_3) + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_4 + \alpha + \beta)(y_2 + y_4)} \\ &\boxtimes V_{\tilde{q}_5 y_5 + \cdots + \tilde{q}_{m+2} y_{m+2}} . \end{aligned}$$

Thus we obtain that $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ if and only if there exist $\alpha, \beta \in \mathbf{Z}$ with $0 \leq \alpha \leq \tilde{p}_1 - \tilde{p}_2$ and $0 \leq \beta \leq \tilde{q}_3 - \tilde{q}_4$ such that

$$V''_{\frac{1}{2}(\tilde{p}_1+\tilde{q}_3-\alpha-\beta)(y_1+y_3)+\frac{1}{2}(\tilde{p}_2+\tilde{q}_4+\alpha+\beta)(y_2+y_4)} \boxtimes V_{\tilde{q}_5 y_5+\cdots+\tilde{q}_{m+2}y_{m+2}}$$

is a trivial $\tilde{K}_0$-module, that is,

$$\begin{cases} \tilde{p}_1 + \tilde{q}_3 - \alpha - \beta = 0, \\ \tilde{p}_2 + \tilde{q}_4 + \alpha + \beta = 0, \\ \tilde{q}_5 = \cdots = \tilde{q}_{m+2} = 0. \end{cases}$$

Hence we have the following.

**Lemma 7.3.** $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ *if and only if*

$$\begin{aligned} &\tilde{p}_5 = \tilde{p}_6 = \cdots = \tilde{p}_m = 0, \\ &\tilde{p}_3 \geq \tilde{p}_4 \geq 0, \ \tilde{p}_{m+2} \leq \tilde{p}_{m+1} \leq 0, \\ &\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0. \end{aligned}$$

If $m \geq 4$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is expressed as

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_{m+1} y_{m+1} + \tilde{p}_{m+2} y_{m+2},$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4 \geq 0 \geq \tilde{p}_{m+1} \geq \tilde{p}_{m+2}$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0.$$

If $m = 3$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is expressed as

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5,$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4 \geq \tilde{p}_5$, $\tilde{p}_3 \geq 0$, $\tilde{p}_5 \leq 0$,

$$\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 = 0.$$

If $m = 2$, then each $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ is expressed as

$$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4,$$

where $\tilde{p}_i \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{p}_3 \geq \tilde{p}_4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 = 0$.

Correspondingly, each $\tilde{\Lambda}' \in D(\tilde{K}_2, \tilde{K}_0)$ is expressed as $\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4$, where $\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4 \in \mathbf{Z}$, $\tilde{p}_1 \geq \tilde{p}_2$, $\tilde{q}_3 \geq \tilde{q}_4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{q}_3 + \tilde{q}_4 = 0$; in other words, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_{m+1} + \tilde{p}_{m+2} = 0$ if $m \geq 4$, $\tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 + \tilde{p}_4 + \tilde{p}_5 = 0$ if $m = 3$. Each $\tilde{\Lambda}'' \in D(\tilde{K}_1, \tilde{K}_0)$ is expressed as $\tilde{\Lambda}'' = \tilde{q}'_1 y_1 + \tilde{q}'_2 y_2 + \tilde{q}'_3 y_3 + \tilde{q}'_4 y_4$, where $\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4 \in \mathbf{Z}$, $\tilde{q}'_1 + \tilde{q}'_3 = 0$, $\tilde{q}'_2 + \tilde{q}'_4 = 0$, $\tilde{q}'_1 = -\alpha + \tilde{p}_1$, $\tilde{q}'_2 = \alpha + \tilde{p}_2$ for some $\alpha = 0, \ldots, \tilde{p}_1 - \tilde{p}_2$, and $\tilde{q}'_3 = -\beta + \tilde{q}_3$, $\tilde{q}'_4 = \beta + \tilde{q}_4$ for some $\beta = 0, \ldots, \tilde{q}_3 - \tilde{q}_4$.

Moreover, the coefficient of $X^{\tilde{q}_3-\tilde{q}_4+1}$ in

$$\begin{aligned}&\frac{1}{X-X^{-1}}(X^{\tilde{p}_3-\tilde{p}_4+1}-X^{-(\tilde{p}_3-\tilde{p}_4+1)})\\&\quad(X^{\tilde{p}_{m+1}-\tilde{p}_{m+2}+1}-X^{-(\tilde{p}_{m+1}-\tilde{p}_{m+2}+1)})\\&=\sum_{i=0}^{\tilde{p}_3-\tilde{p}_4}(X^{(\tilde{p}_3-\tilde{p}_4)+(\tilde{p}_{m+1}-\tilde{p}_{m+2})-2i+1}-X^{(\tilde{p}_3-\tilde{p}_4)-(\tilde{p}_{m+1}-\tilde{p}_{m+2})-2i-1})\end{aligned}$$

is equal to the multiplicity of the $\tilde{K}_2$-module with the highest weight $\tilde{\Lambda}'=\Lambda_\sigma+\tilde{\Lambda}'_\tau=\tilde{p}_1y_1+\tilde{p}_2y_2+\tilde{q}_3y_3+\tilde{q}_4y_4\in D(\tilde{K}_2,\tilde{K}_0)$.

**7.6. Eigenvalue computation when** $m=2$**.** For each $\tilde{\Lambda}=\tilde{p}_1y_1+\tilde{p}_2y_2+\tilde{p}_3y_3+\tilde{p}_4y_4\in D(\tilde{K},\tilde{K}_0)$ and $\tilde{\Lambda}''=\tilde{q}'_1y_1+\tilde{q}'_2y_2+\tilde{q}'_3y_3+\tilde{q}'_4y_4\in D(\tilde{K}_1,\tilde{K}_0)$ defined as above, the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}-c_L=&-c_{\tilde{\Lambda}}+\frac{1}{2}c_{\tilde{\Lambda}''}\\=&\ \tilde{p}_1^2+\tilde{p}_2^2+\tilde{p}_3^2+\tilde{p}_4^2+(\tilde{p}_1-\tilde{p}_2)+(\tilde{p}_3-\tilde{p}_4)\\&-\frac{1}{2}((\tilde{q}'_1)^2+(\tilde{q}'_2)^2+(\tilde{q}'_3)^2+(\tilde{q}'_4)^2).\end{aligned}\tag{7.2}$$

Since

$$-\mathcal{C}_L=-\frac{1}{2}\mathcal{C}_{K/K_0}-\frac{1}{2}\mathcal{C}_{K/K_1}\geq-\frac{1}{2}\mathcal{C}_{K/K_0},$$

the first eigenvalue of $-\mathcal{C}_L$, $-c_L\leq n=6$ implies $-c_{\tilde{\Lambda}}\leq 12$. Notice that

$$-c_{\tilde{\Lambda}}=\tilde{p}_1^2+\tilde{p}_2^2+\tilde{p}_3^2+\tilde{p}_4^2+(\tilde{p}_1-\tilde{p}_2)+(\tilde{p}_3-\tilde{p}_4)\geq 2(\tilde{p}_2^2+\tilde{p}_4^2);$$

we know $\tilde{p}_2^2+\tilde{p}_4^2\leq 6$, which follows that the possible choice for $\tilde{p}_2$ and $\tilde{p}_4$ is $|\tilde{p}_2|=0,1$ or $2$ and $|\tilde{p}_4|=0,1$ or $2$. Taking into account $\sum_{i=1}^4\tilde{p}_i=0$ and using the eigenvalue formula (7.2), we obtain the following.

**Lemma 7.4.** *$\tilde{\Lambda}=\tilde{p}_1y_1+\tilde{p}_2y_2+\tilde{p}_3y_3+\tilde{p}_4y_4\in D(\tilde{K},\tilde{K}_0)$ has eigenvalue $-c_L\leq 6$ if and only if $(\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4)$ is one of*

$$\begin{aligned}\Big\{&(0,0,0,0),(1,1,-1,-1),(1,0,0,-1),(1,-1,0,0),(1,-1,1,-1),\\&(1,1,0,-2),(2,0,-1,-1),(0,-1,1,0),(0,0,1,-1),\\&(0,-2,1,1),(-1,-1,2,0),(-1,-1,1,1)\Big\}.\end{aligned}$$

Denote $\tilde{\Lambda}=\tilde{p}_1y_1+\tilde{p}_2y_2+\tilde{p}_3y_3+\tilde{p}_4y_4\in D(\tilde{K},\tilde{K}_0)$ by $\tilde{\Lambda}=(\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4)$.

Suppose that $\tilde{\Lambda}=(1,1,-1,-1)$. Then $\dim_{\mathbf{C}}V_{\tilde{\Lambda}}=1$. By the branching law of $(U(2),U(1)\times U(1))$, $(\tilde{q}'_1,\tilde{q}'_2,\tilde{q}'_3,\tilde{q}'_4)=(1,1,-1,-1)\in D(\tilde{K}_1,\tilde{K}_0)$. Then $-c_{\tilde{\Lambda}}=4$, $-c_{\tilde{\Lambda}''}=4$, $-c_L=-c_{\tilde{\Lambda}}+\frac{1}{2}c_{\tilde{\Lambda}''}=2<6$. On the other hand, $V_{\tilde{\Lambda}}=\mathbf{C}\boxtimes\mathbf{C}$, which is fixed by the $\rho_{\tilde{\Lambda}}|_{\tilde{K}_0}$-action. But for a generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$, $\rho_{\tilde{\Lambda}}(Q)=-\mathrm{Id}$ on $V_{\tilde{\Lambda}}$. Hence $\tilde{\Lambda}=(1,1,-1,-1)\notin D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda}=(-1,-1,1,1)\notin D(\tilde{K},\tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1, 0, 0, -1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 4$. It follows from the branching law of $(U(2), U(1) \times U(1))$ that $(\tilde{q}'_1, \tilde{q}'_2) = (1, 0) \oplus (0, 1)$ and $(\tilde{q}'_3, \tilde{q}'_4) = (0, -1)$ or $(-1, 0)$. Then $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1, 0, -1, 0)$ or $(0, 1, 0, -1) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence $-c_{\tilde{\Lambda}} = 4$, $-c_{\tilde{\Lambda}''} = 2$, $-c_L = -c_{\tilde{\Lambda}} + \frac{1}{2} c_{\tilde{\Lambda}''} = 3 < 6$.

Recall that the complete set of all inequivalent irreducible unitary representations of $SU(2)$ is given by

$$\mathcal{D}(SU(2)) = \{(V_m, \rho_m) \mid m \in \mathbf{Z}, m \geq 0\},$$

where $V_m$ denotes the complex vector space of complex homogeneous polynomials of degree $m$ with two variables $z_0, z_1$ and the representation $\rho_m$ of $SU(2)$ on $V_m$ is defined by $(\rho_m(g) f)(z_0, z_1) = f((z_0, z_1) g)$ for each $g \in SU(2)$. Set

$$v_k^{(m)}(z_0, z_1) := \frac{1}{\sqrt{k!(m-k)!}} z_0^{m-k} z_1^k \in V_m \quad (k = 0, 1, \ldots, m), \tag{7.3}$$

and define the standard Hermitian inner product of $V_m$ invariant under $\rho_m(SU(2))$ such that $\{v_0^{(m)}, \ldots, v_m^{(m)}\}$ is a unitary basis of $V_m$. Then

$$V_{\tilde{\Lambda}} = (W'_{\frac{1}{2}(y_1+y_2)} \otimes V_1) \boxtimes (W'_{-\frac{1}{2}(y_1+y_2)} \otimes V_1).$$

The representation of $\tilde{K}_0$ on $v_i^{(1)} \otimes v_j^{(1)} \in V_{\tilde{\Lambda}}$ $(i, j = 0, 1)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(P)(v_i^{(1)} \otimes v_j^{(1)}) \\
=&\left[\rho_1 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}\right](v_i^{(1)}) \\
&\otimes \left[\rho_1 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix}\right](v_j^{(1)}) \\
=&e^{\sqrt{-1}(s-t)[1-(i+j)]} v_i^{(1)} \otimes v_j^{(1)}.
\end{aligned}$$

Then $(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(1)} \otimes v_0^{(1)}, v_0^{(1)} \otimes v_1^{(1)}\}$. But for $\mathrm{diag}(1, 1, -1, -1) \in \tilde{K}_{[\mathfrak{a}]}$ and $i, j = 0, 1$, $\rho_{\tilde{\Lambda}}(\mathrm{diag}(1, 1, -1, -1))(v_i^{(1)} \otimes v_j^{(1)}) = -v_i^{(1)} \otimes v_j^{(1)}$. So $(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1, 0, 0, -1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda} = (0, -1, 1, 0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1, -1, 0, 0)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 3$. It follows from the branching law of $(U(2), U(1) \times U(1))$ that $(\tilde{q}'_1, \tilde{q}'_2) = (1, -1), (0, 0)$, or $(-1, 1)$ and $(\tilde{q}'_3, \tilde{q}'_4) = (0, 0)$. Then $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0, 0, 0, 0) \in D(\tilde{K}, \tilde{K}_0)$. Hence $-c_{\tilde{\Lambda}} = 4$, $-c_{\tilde{\Lambda}''} = 0$, $-c_L = -c_{\tilde{\Lambda}} + \frac{1}{2} c_{\tilde{\Lambda}''} = 4 < 6$. On the other hand, $V_{\tilde{\Lambda}} \cong V_2 \boxtimes \mathbf{C}$. The representation of $\tilde{K}_0$ on $v_i^{(2)} \otimes w \in V_{\tilde{\Lambda}}$ is given by

$$\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes w) = e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes w.$$

Then $(V_{\tilde\Lambda})_{\tilde K_0} = \mathrm{span}_{\mathbf C}\{v_1^{(2)}\otimes w\}$. But for the generator $Q\in\tilde K_{[\mathfrak a]}$,

$$\rho_{\tilde\Lambda}(Q)(v_1^{(2)}\otimes w) = -v_1^{(2)}\otimes w.$$

So $(V_{\tilde\Lambda})_{\tilde K_{[\mathfrak a]}} = \{0\}$ and $\tilde\Lambda = (1,-1,0,0)\notin D(\tilde K,\tilde K_{[\mathfrak a]})$. Similarly, $\tilde\Lambda = (0,0,1,-1)\notin D(\tilde K,\tilde K_{[\mathfrak a]})$.

Suppose that $\tilde\Lambda = (1,-1,1,-1)$. Then $\dim_{\mathbf C} V_{\tilde\Lambda} = 9$. It follows from the branching laws of $(U(2), U(1)\times U(1))$ that $(\tilde q_1', \tilde q_2') = (1,-1)$ or $(0,0)$ and $(\tilde q_3',\tilde q_4') = (1,-1)$ or $(0,0)$. Then $(\tilde q_1',\tilde q_2',\tilde q_3',\tilde q_4') = (1,-1,-1,1)$, $(-1,1,1,-1)$, or $(0,0,0,0)\in D(\tilde K,\tilde K_0)$. When $(\tilde q_1',\tilde q_2',\tilde q_3',\tilde q_4') = (0,0,0,0)$, $-c_{\tilde\Lambda} = 8$, $-c_{\tilde\Lambda''} = 0$, $-c_L = -c_{\tilde\Lambda} + \frac12 c_{\tilde\Lambda''} = 8 > 6$. When $(\tilde q_1',\tilde q_2',\tilde q_3',\tilde q_4') = (1,-1,-1,1)$ or $(-1,1,1,-1)$, $-c_{\tilde\Lambda} = 8$, $-c_{\tilde\Lambda''} = 4$, $-c_L = -c_{\tilde\Lambda} + \frac12 c_{\tilde\Lambda''} = 6$. On the other hand, $V_{\tilde\Lambda}\cong V_2\boxtimes V_2$. The representation of $\tilde K_0$ on $v_i^{(2)}\otimes v_j^{(2)}\in V_{\tilde\Lambda}$ $(i,j=0,1,2)$ is given by

$$\begin{aligned}
&\rho_{\tilde\Lambda}(P)(v_i^{(2)}\otimes v_j^{(2)})\\
=&\left[\rho_2\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_i^{(2)})\\
&\otimes\left[\rho_2\begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}}\end{pmatrix}\right](v_j^{(2)})\\
=&e^{\sqrt{-1}(s-t)[2-(i+j)]}v_i^{(2)}\otimes v_j^{(2)}.
\end{aligned}$$

Hence $(V_{\tilde\Lambda})_{\tilde K_0} = \mathrm{span}_{\mathbf C}\{v_0^{(2)}\otimes v_2^{(2)}, v_1^{(2)}\otimes v_1^{(2)}, v_2^{(2)}\otimes v_0^{(2)}\}$. Moreover, the action of the generator $Q$ of $\mathbf Z_4$ in $\tilde K_{[\mathfrak a]}$ on $v_i^{(2)}\otimes v_j^{(2)}$ is given by

$$\rho_{\tilde\Lambda}(Q)(v_i^{(2)}\otimes v_j^{(2)}) = (-1)^{3-i}v_{2-i}^{(2)}\otimes v_{2-j}^{(2)}.$$

Therefore, $(V_{\tilde\Lambda})_{\tilde K_{[\mathfrak a]}} = \mathrm{span}\{v_0^{(2)}\otimes v_2^{(2)} - v_2^{(2)}\otimes v_0^{(2)}, v_1^{(2)}\otimes v_1^{(2)}\}$ and $\tilde\Lambda = (1,-1,1,-1)\in D(\tilde K,\tilde K_{[\mathfrak a]})$. Note that the $\tilde K_{[\mathfrak a]}$-fixed vector $v_1^{(2)}\otimes v_1^{(2)}\in V_0'$, which corresponds eigenvalue 8, and the $\tilde K_{[\mathfrak a]}$-fixed vector $v_0^{(2)}\otimes v_2^{(2)} - v_2^{(2)}\otimes v_0^{(2)}\in V'_{y_1-y_2-y_3+y_4}\oplus V'_{-y_1+y_2+y_3-y_4}$, which gives eigenvalue 6.

Suppose that $\tilde\Lambda = (2,0,-1,-1)$. Then $\dim_{\mathbf C} V_{\tilde\Lambda} = 3$. It follows from the branching law of $(U(2), U(1)\times U(1))$ that $(\tilde q_1',\tilde q_2') = (2,0),(1,1)$, or $(0,2)$ and $(\tilde q_3',\tilde q_4') = (-1,-1)$. Then $(\tilde q_1',\tilde q_2',\tilde q_3',\tilde q_4') = (1,1,-1,-1)\in D(\tilde K,\tilde K_0)$. Hence $-c_{\tilde\Lambda} = 8$, $-c_{\tilde\Lambda''} = 4$, $-c_L = -c_{\tilde\Lambda} + \frac12 c_{\tilde\Lambda''} = 6$. On the other hand,

$$V_{\tilde\Lambda}\cong (V_2\otimes\mathbf C)\boxtimes\mathbf C.$$

The representation of $\tilde{K}_0$ on $v_i^{(2)} \otimes w \in V_{\tilde{\Lambda}}$ $(i = 0, 1, 2)$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes w) \\
&= e^{\sqrt{-1}(s+t)} \left[ \rho_2 \begin{pmatrix} e^{\frac{\sqrt{-1}(s-t)}{2}} & \\ & e^{-\frac{\sqrt{-1}(s-t)}{2}} \end{pmatrix} \right] (v_i^{(2)}) \otimes e^{-\sqrt{-1}(s+t)} w \\
&= e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes w.
\end{aligned}$$

Hence $(V_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes 1\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes w$ is given by $\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes 1) = (-1)^{1-i} v_{2-i}^{(2)} \otimes 1$. Therefore, $(V_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}\{v_1^{(2)} \otimes 1\}$ and $\tilde{\Lambda} = (2, 0, -1, -1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 6. Similarly, $\tilde{\Lambda} = (-1, -1, 2, 0)$, $(1, 1, 0, -2)$, $(0, -2, 1, 1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which give eigenvalue 6 and with multiplicity 1, respectively.

Moreover, we observe that

$$\begin{aligned}
n(L^6) =& \dim_{\mathbf{C}} V_{(2,0,-1,-1)} + \dim_{\mathbf{C}} V_{(-1,-1,2,0)} + \dim_{\mathbf{C}} V_{(1,1,0,-2)} \\
&+ \dim_{\mathbf{C}} V_{(0,-2,1,1)} + \dim_{\mathbf{C}} V_{(1,-1,1,-1)} = 3 + 3 + 3 + 3 + 9 \\
=& 21 = \dim SO(8) - \dim S(U(2) \times U(2)) = n_{hk}(L^6).
\end{aligned}$$

Therefore, we obtain that $L^6 = \mathcal{G}(\frac{S(U(2)\times U(2))}{S(U(1)\times U(1))}) \subset Q_6(\mathbf{C})$ is strictly Hamiltonian stable.

**7.7. Eigenvalue computation when** $m = 3$**.** For each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5 \in D(\tilde{K}, \tilde{K}_0)$, $\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(\tilde{K}_2, \tilde{K}_0)$, and $\tilde{\Lambda}'' = \tilde{q}_1' y_1 + \tilde{q}_2' y_2 + \tilde{q}_3' y_3 + \tilde{q}_4' y_4 \in D(\tilde{K}_1, \tilde{K}_0)$ given as in Subsection 7.5, the corresponding eigenvalue of $-\mathcal{C}_L$ is

$$\begin{aligned}
-c_L =& -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2} c_{\tilde{\Lambda}''} \\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + 2(\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_5^2) + (\tilde{p}_1 - \tilde{p}_2) + 4(\tilde{p}_3 - \tilde{p}_5) \\
&- (\tilde{q}_3^2 + \tilde{q}_4^2) - (\tilde{q}_3 - \tilde{q}_4) - \frac{1}{2}((\tilde{q}_1')^2 + (\tilde{q}_2')^2 + (\tilde{q}_3')^2 + (\tilde{q}_4')^2).
\end{aligned} \tag{7.4}$$

Since $-\mathcal{C}_L \geq -\frac{1}{2}\mathcal{C}_{K/K_0}$, the condition $-c_L \leq n = 10$ implies that $-c_{\tilde{\Lambda}} \leq 20$. Notice that

$$-c_{\tilde{\Lambda}} = \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_5^2 + (\tilde{p}_1 - \tilde{p}_2) + 2(\tilde{p}_3 - \tilde{p}_5) \geq 2\tilde{p}_2^2 + 3\tilde{p}_5^2,$$

and we have

$$\begin{cases} 2\tilde{p}_2^2 + 3\tilde{p}_5^2 \leq 20, \\ \tilde{p}_i \in \mathbf{Z}, \quad \sum_{i=1}^5 \tilde{p}_i^2 \leq 20, \\ \sum_{i=1}^5 \tilde{q}_i = 0, \quad \tilde{p}_1 \geq \tilde{p}_2, \quad \tilde{p}_3 \geq \tilde{p}_4 \geq \tilde{p}_5. \end{cases}$$

Then by the similar calculations we obtain the following.

**Lemma 7.5.** *$\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_5 y_5 \in D(\tilde{K}, \tilde{K}_0)$ has eigenvalue $-c_L \leq 10$ if and only if $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)$ is one of*

$$\begin{aligned}\big\{&(0,0,0,0,0), (1,-1,1,0,-1), (2,0,0,-1,-1), (0,-2,1,1,0),\\ &(1,1,0,0,-2), (-1,-1,2,0,0), (1,-1,0,0,0), (1,0,0,0,-1),\\ &(0,-1,1,0,0), (1,1,0,-1,-1), (-1,-1,1,1,0), (0,0,1,0,-1)\big\}.\end{aligned}$$

Suppose that $\tilde{\Lambda} = (1,-1,1,0,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 24$. It follows from Lemma 7.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,-1,0)$ or $(0,0,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,0)$, by the branching law of $(U(2), U(1)\times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (0,0,0,0,0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 16 > 10$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,-1,0)$, by the branching law of $(U(2), U(1)\times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (1,-1,-1,1,0)$, $(0,0,0,0,0)$, or $(-1,1,1,-1,0) \in D(\tilde{K}, \tilde{K}_0)$, respectively. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 10$, $12$, or $10$, respectively. On the other hand, now

$$\begin{aligned}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} &\subset (W_{y_1-y_2} \boxtimes W_{y_3-y_4} \boxtimes W_0) \oplus (W_{y_1-y_2} \boxtimes W_0 \boxtimes W_0)\\ &\cong (V_2 \boxtimes V_2 \boxtimes \mathbf{C}) \oplus (V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}),\end{aligned}$$

where the latter is a $\tilde{K}_2$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes v_j \otimes w \in V_2 \boxtimes V_2 \boxtimes \mathbf{C}$ $(i,j = 0,1,2)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v_j^{(2)} \otimes w)\\ &= \rho_{y_1-y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(v_i^{(2)}) \otimes \rho_{y_3-y_4}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(v_j^{(2)}) \otimes w\\ &= e^{\sqrt{-1}(s-t)(2-i-j)} v_i^{(2)} \otimes v_j^{(2)} \otimes w.\end{aligned}$$

The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(2)} \otimes v \otimes w \in V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v \otimes w) =& \rho_{y_1-y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t}\end{pmatrix}(v_i^{(2)}) \otimes v \otimes w\\ =& e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes v \otimes w.\end{aligned}$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_2^{(2)} \otimes v_0^{(2)} \otimes w, v_0^{(2)} \otimes v_2^{(2)} \otimes w, v_1^{(2)} \otimes v_1^{(2)} \otimes w, v_1^{(2)} \otimes v \otimes w\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes v_{2-i}^{(2)} \otimes w$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v_{2-i}^{(2)} \otimes w) &= \rho_2\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}(v_i^{(2)})\\ &\qquad \otimes \rho_2\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{pmatrix}(v_{2-i}^{(2)}) \otimes w\\ &= (-1)^{1-i} u_{2-i} \otimes v_i^{(2)} \otimes w,\end{aligned}$$

and the action on $v_i^{(2)} \otimes v \otimes w$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v \otimes w) &= \rho_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v_i^{(2)}) \otimes v \otimes w \\ &= (-1)^{2-i} v_{2-i}^{(2)} \otimes v \otimes w.\end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_2^{(2)} \otimes v_0^{(2)} \otimes w - v_0^{(2)} \otimes v_2^{(2)} \otimes w, v_1^{(2)} \otimes v_1^{(2)} \otimes w\}$ and $\tilde{\Lambda} = (1,-1,1,0,-1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Notice that the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_1^{(2)} \otimes v_1^{(2)} \otimes w \in V_{\tilde{\Lambda}''}$, which corresponds eigenvalue 12, where $\tilde{\Lambda}'' = 0$. And the $\tilde{K}_{[\mathfrak{a}]}$-fixed vector $v_2^{(2)} \otimes v_0^{(2)} \otimes w - v_0^{(2)} \otimes v_2^{(2)} \otimes w \in V_{\tilde{\Lambda}_1''} \oplus V_{\tilde{\Lambda}_2''}$, which gives eigenvalue 10, where $\tilde{\Lambda}_1'' = (1,-1,-1,1,0)$ and $\tilde{\Lambda}_2'' = (-1,1,1,-1,0)$.

Suppose that $\tilde{\Lambda} = (2,0,0,-1,-1)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 9$. It follows from the branching law of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-1,-1)$ or $(-1,-1,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (-1,-1,0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (1,1,-1,-1,0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2} c_{\tilde{\Lambda}''} = 10$. On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset (W_{2y_1} \boxtimes W_{-(y_3+y_4)} \boxtimes W_0) \cong V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C},$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(2)} \otimes v \otimes w \in V_2 \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned}&\rho_{\tilde{\Lambda}}(P)(v_i^{(2)} \otimes v \otimes w) \\ =&\rho_{2y_1} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v_i^{(2)}) \otimes \rho_{-y_3-y_4} \begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix} (v) \otimes w \\ =&e^{\sqrt{-1}(s-t)(1-i)} v_i^{(2)} \otimes v \otimes w.\end{aligned}$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes v \otimes w\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(2)} \otimes v \otimes w$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(Q)(v_i^{(2)} \otimes v \otimes w) &= \rho_{2y_1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v_i^{(2)}) \\ &\quad \otimes \rho_{-(y_3+y_4)} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (v) \otimes w \\ &= (-1)^{1+i}\, v_{2-i}^{(2)} \otimes v \otimes w.\end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_1^{(2)} \otimes v \otimes w\}$, where $\dim_{\mathbf{C}} (\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = 1$ and $\tilde{\Lambda} = (2,0,0,-1,-1) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10. Similarly, $\tilde{\Lambda} = (0,-2,1,1,0) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10 and with multiplicity 1 and dimension 9.

Suppose that $\tilde{\Lambda} = (1,1,0,0,-2)$. Then $\dim_{\mathbf{C}} V_{\tilde{\Lambda}} = 6$. It follows from Lemma 7.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,-2), (0,-1,-1)$, or $(0,-2,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-2,0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (1,1,-1,-1,0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 10$. On the other hand,

$$(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset W_0 \boxtimes W_{-2y_4} \boxtimes W_0 \cong \mathbf{C} \boxtimes V_2 \boxtimes \mathbf{C},$$

and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v_i^{(2)} \otimes w \in \mathbf{C} \boxtimes V_2 \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned} &\rho_{\tilde{\Lambda}}(P)(u \otimes v_i^{(2)} \otimes w) \\ =&\rho_{y_1+y_2}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(u) \otimes \rho_{-2y_4}\begin{pmatrix} e^{\sqrt{-1}s} & \\ & e^{\sqrt{-1}t} \end{pmatrix}(v_i^{(2)}) \otimes w \\ =&e^{\sqrt{-1}(s-t)(1-i)} u \otimes v_i^{(2)} \otimes w. \end{aligned}$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{u \otimes v_1^{(2)} \otimes w\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v_i^{(2)} \otimes w$ is given by

$$\begin{aligned} &\rho_{\tilde{\Lambda}}(Q)(u \otimes v_i^{(2)} \otimes w) \\ =&\rho_{y_1+y_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(u) \otimes \rho_{-2y_4}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}(v_i^{(2)}) \otimes w = u \otimes v_{2-i}^{(2)} \otimes w. \end{aligned}$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{u \otimes v_1^{(2)} \otimes w\}$, where $\dim_{\mathbf{C}}(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = 1$ and $\tilde{\Lambda} = (1,1,0,0,-2) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, which gives eigenvalue 10. Similarly, $\tilde{\Lambda} = (-1,-1,2,0,0) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$, gives eigenvalue 10 and has multiplicity 1 and dimension 6.

Suppose that $\tilde{\Lambda} = (1,-1,0,0,0)$. Then $(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,-1,0,0,0)$. It follows from the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) = (0,0,0,0,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 4 < 10$. On the other hand, $\tilde{V}_{\tilde{\Lambda}} = W_{y_1-y_2} \boxtimes W_0 \cong V_2 \boxtimes \mathbf{C}$, and the representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u_i \otimes w \in V_2 \boxtimes \mathbf{C}$ $(i = 0,1,2)$ is given by

$$\begin{aligned} \rho_{\tilde{\Lambda}}(P)(u_i \otimes w) &= \rho_2\begin{pmatrix} e^{\sqrt{-1}\frac{s-t}{2}} & \\ & e^{-\sqrt{-1}\frac{s-t}{2}} \end{pmatrix}(u_i) \otimes w \\ &= e^{\sqrt{-1}(s-t)(1-i)} u_i \otimes w. \end{aligned}$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{u_1 \otimes w\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u_1 \otimes w$ is given by $\rho_{\tilde{\Lambda}}(Q)(u_1 \otimes w) = -u_1 \otimes w$. Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1,-1,0,0,0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1,0,0,0,-1)$. It follows from Lemma 7.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,-1)$ or $(0,-1,0)$. When $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-1,0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4, \tilde{q}'_5) =$

$(1,0,-1,0,0)$ or $(0,1,0,-1,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence, $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 5,5 < 10$. On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset W_{y_1} \boxtimes W_{-y_4} \boxtimes W_0 \cong V_1 \boxtimes V_1 \boxtimes \mathbf{C}$, where the latter is the $\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(1)} \otimes v_j^{(1)} \otimes w \in V_1 \boxtimes V_1 \boxtimes \mathbf{C}$ $(i,j = 0,1)$ is given by

$$\rho_{\tilde{\Lambda}}(P)(v_i^{(1)} \otimes v_j^{(1)} \otimes w) = e^{\sqrt{-1}(s-t)(1-i-j)}\, v_i^{(1)} \otimes v_j^{(1)} \otimes w.$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_1^{(1)} \otimes v_0^{(1)} \otimes w, u_0 \otimes v_1^{(1)} \otimes w\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $v_i^{(1)} \otimes v_{1-i}^{(1)} \otimes w$ $(i = 0,1)$ is given by

$$\rho_{\tilde{\Lambda}}(Q)(v_i^{(1)} \otimes v_{1-i}^{(1)} \otimes w) = (-1)^{1-i}\, v_{1-i}^{(1)} \otimes v_i^{(1)} \otimes w.$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1,0,0,0,-1) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$. Similarly, $\tilde{\Lambda} = (0,-1,1,0,0) \notin D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$.

Suppose that $\tilde{\Lambda} = (1,1,0,-1,-1)$. It follows from Lemma 7.2 that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,-1,-1)$ or $(-1,-1,0)$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,1,-1,-1,0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (1,1,-1,-1) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 6 < 10$. On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset W_{y_1+y_2} \boxtimes W_{-y_3-y_4} \boxtimes W_0 \cong \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C}$, where the latter is the $\tilde{K}_2 = U(2) \times U(2) \times U(1)$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v \otimes w \in \mathbf{C} \boxtimes \mathbf{C} \boxtimes \mathbf{C}$ is given by

$$\rho_{\tilde{\Lambda}}(P)(u \otimes v \otimes w) = e^{\sqrt{-1}(s+t)} u \otimes e^{-\sqrt{-1}(s+t)} v \otimes w = u \otimes v \otimes w.$$

It follows that $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes 1 \otimes 1\}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v \otimes w$ is given by

$$\rho_{\tilde{\Lambda}}(Q)(u \otimes v \otimes w) = -u \otimes v \otimes w.$$

Therefore $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \{0\}$ and $\tilde{\Lambda} = (1,1,0,-1,-1) \notin D(\tilde{K}, \tilde{K}_0)$. Similarly, $\tilde{\Lambda} = (-1,-1,1,1,0) \notin D(\tilde{K}, \tilde{K}_0)$.

Suppose that $\tilde{\Lambda} = (0,0,1,0,-1)$. It follows from the branching law of $(U(3), U(2) \times U(1))$ that $(\tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (1,0,-1), (0,0,0), (1,-1,0)$ or $(0,-1,1)$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,0,0,0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching law of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0,0,0,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 12 > 10$. For the element $(\tilde{p}_1, \tilde{p}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5) = (0,0,1,-1,0)$ in $D(\tilde{K}_2, \tilde{K}_0)$, by the branching laws of $(U(2), U(1) \times U(1))$, $(\tilde{q}'_1, \tilde{q}'_2, \tilde{q}'_3, \tilde{q}'_4) = (0,0,0,0) \in D(\tilde{K}_1, \tilde{K}_0)$. Hence $-c_L = -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''} = 8 < 10$. On the other hand, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} \subset \tilde{V}'_{(0,0,0,0,0)} \oplus \tilde{V}'_{(0,0,1,-1,0)}$. We are concerned with only $\tilde{V}'_{(0,0,1,-1,0)}$ since it corresponds to the smaller eigenvalue 8. Note that $\tilde{V}'_{(0,0,1,-1,0)} = W_0 \boxtimes$

$W_{y_3-y_4} \boxtimes W_0 \cong \mathbf{C} \boxtimes V_2 \boxtimes \mathbf{C}$, which is a $\tilde{K}_2$-module. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $u \otimes v_i^{(2)} \otimes w \in \tilde{V}'_{(0,0,1,-1,0)}$ $(i = 0, 1, 2)$ is given by

$$\rho_{\tilde{\Lambda}}(P)(u \otimes v_i^{(2)} \otimes w) = e^{\sqrt{-1}(s-t)(1-i)} u \otimes v_i^{(2)} \otimes w.$$

Thus $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{1 \otimes v_1 \otimes 1\} \oplus \tilde{V}'_{(0,0,0,0,0)}$. Moreover, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ on $u \otimes v_1^{(2)} \otimes w$ is given by

$$\begin{aligned}
&\rho_{\tilde{\Lambda}}(Q)(u \otimes v_1^{(2)} \otimes w)\\
=&u \otimes \rho_2(\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix})v_1 \otimes w = -u \otimes v_1^{(2)} \otimes w.
\end{aligned}$$

Therefore, $1 \otimes v_1^{(2)} \otimes 1 \notin (\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}}$ and $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \tilde{V}'_{(0,0,0,0,0)}$, which gives a larger eigenvalue 10.

Moreover,

$$\begin{aligned}
n(L^{10}) =& \dim_{\mathbf{C}} V_{(1,-1,1,0,-1)} + \dim_{\mathbf{C}} V_{(2,0,0,-1,-1)} + \dim_{\mathbf{C}} V_{(0,-2,1,1,0)}\\
&+ \dim_{\mathbf{C}} V_{(1,1,0,0,-2)} + \dim_{\mathbf{C}} V_{(-1,-1,2,0,0)}\\
=&24 + 9 + 9 + 6 + 6 = 54\\
=&\dim SO(12) - \dim S(U(2) \times U(3)) = n_{hk}(L^{10}).
\end{aligned}$$

Therefore we obtain that $L^{10} = \mathcal{G}(\frac{S(U(2)\times U(3))}{S(U(1)\times U(1)\times U(1))}) \subset Q_{10}(\mathbf{C})$ is strictly Hamiltonian stable.

**7.8. Eigenvalue computation when** $m \geq 4$**.** For each $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{p}_3 y_3 + \tilde{p}_4 y_4 + \tilde{p}_{m+1} y_{m+1} + \tilde{p}_{m+2} y_{m+2} \in D(\tilde{K}, \tilde{K}_0)$, $\tilde{\Lambda}' = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 + \tilde{q}_3 y_3 + \tilde{q}_4 y_4 \in D(\tilde{K}_2, \tilde{K}_0)$, and $\tilde{\Lambda}'' = \tilde{q}'_1 y_1 + \tilde{q}'_2 y_2 + \tilde{q}'_3 y_3 + \tilde{q}'_4 y_4 \in D(\tilde{K}_1, \tilde{K}_0)$ given as in Section 7.5, the corresponding eigenvalue of $-\mathcal{C}_L$ is given by

$$\begin{aligned}
-c_L =& -2c_{\tilde{\Lambda}} + c_{\tilde{\Lambda}'} + \frac{1}{2}c_{\tilde{\Lambda}''}\\
=& \tilde{p}_1^2 + \tilde{p}_2^2 + 2(\tilde{p}_3^2 + \tilde{p}_4^2 + \tilde{p}_{m+1}^2 + \tilde{p}_{m+2}^2)\\
&+ (\tilde{p}_1 - \tilde{p}_2) + 2(m-1)(\tilde{p}_3 - \tilde{p}_{m+2}) + 2(m-3)(\tilde{p}_4 - \tilde{p}_{m+1})\\
&- (\tilde{q}_3^2 + \tilde{q}_4^2) - (\tilde{q}_3 - \tilde{q}_4) - \frac{1}{2}((\tilde{q}'_1)^2 + (\tilde{q}'_2)^2 + (\tilde{q}'_3)^2 + (\tilde{q}'_4)^2).
\end{aligned}$$

In case $\tilde{\Lambda} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_{m+1}, \tilde{p}_{m+2}) = (\tilde{p}_1, \tilde{p}_2, 0, 0, 0, 0) \in D(\tilde{K}, \tilde{K}_0)$, since $\tilde{p}_3 = \tilde{p}_4 = \tilde{p}_{m+1} = \tilde{p}_{m+2} = 0$, we have $\tilde{q}_3 = \tilde{q}_4 = \tilde{q}_5 = \cdots = \tilde{q}_{m+2} = 0$ and thus $\tilde{q}'_3 = \tilde{q}'_4 = 0$. Since $\tilde{p}_1 + \tilde{p}_2 = 0$, by the branching law of $(U(2), U(1) \times U(1))$ we have $\tilde{q}'_1 = -\alpha + \tilde{p}_1$, $\tilde{q}'_2 = \alpha + \tilde{p}_2 = \alpha - \tilde{p}_1 = -\tilde{q}'_1$ for some $\alpha = 0, 1 \ldots, \tilde{p}_1 - \tilde{p}_2 = 2\tilde{p}_1$. $\tilde{\Lambda} \in D(\tilde{K}, \tilde{K}_0)$ implies that $\tilde{q}'_1 = \tilde{q}'_2 = 0$ since $\tilde{q}'_1 + \tilde{q}'_3 = 0$ and $\tilde{q}'_2 + \tilde{q}'_4 = 0$. Then $-c_L = 2\tilde{p}_1(\tilde{p}_1 + 1)$.

Now $\tilde{\Lambda} = \tilde{p}_1 y_1 + \tilde{p}_2 y_2 = 2\tilde{p}_1 \frac{1}{2}(y_1 - y_2)$. Set $\ell := 2\tilde{p}_1$. Then $\tilde{V}_{\tilde{\Lambda}} \cong V_\ell \boxtimes \mathbf{C}$. The representation $\rho_{\tilde{\Lambda}}$ of $\tilde{K}_0$ on $v_i^{(\ell)} \otimes w \in \tilde{V}_{\tilde{\Lambda}}$ is given by

$$\begin{aligned}\rho_{\tilde{\Lambda}}(P)(v_i^{(\ell)} \otimes w) &= \left[\rho_\ell \begin{pmatrix} e^{\sqrt{-1}(s-t)/2} & 0 \\ 0 & e^{-\sqrt{-1}(s-t)/2} \end{pmatrix}\right] (v_i^{(\ell)}) \otimes w \\ &= e^{\frac{\sqrt{-1}(s-t)}{2}(\ell - 2i)}\, v_i^{(\ell)} \otimes w.\end{aligned}$$

Hence $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_0} = \mathrm{span}_{\mathbf{C}}\{v_{\tilde{p}_1}^{(\ell)} \otimes w\}$. On the other hand, the action of the generator $Q$ of $\mathbf{Z}_4$ in $\tilde{K}_{[\mathfrak{a}]}$ is given by

$$\rho_{\tilde{\Lambda}}(Q)(v_{\tilde{p}_1}^{(\ell)} \otimes w) = \left[\rho_\ell \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] (v_{\tilde{p}_1}^{(\ell)}) \otimes w = (-1)^{\tilde{p}_1}\, v_{\tilde{p}_1}^{(\ell)} \otimes w.$$

Therefore, $(\tilde{V}_{\tilde{\Lambda}})_{\tilde{K}_{[\mathfrak{a}]}} = \mathrm{span}_{\mathbf{C}}\{v_{\tilde{p}_1}^{(\ell)} \otimes w\}$ for $\tilde{p}_1$ is even. As $m \geq 4$, for every even number $\tilde{p}_1 \geq 2$ such that $12 \leq 2\tilde{p}_1(\tilde{p}_1 + 1) < 4m - 2$, $\tilde{\Lambda} = \tilde{p}_1(y_1 - y_2) \in D(\tilde{K}, \tilde{K}_{[\mathfrak{a}]})$ has eigenvalue $12 \leq -c_L = 2\tilde{p}_1(\tilde{p}_1 + 1) < 4m - 2$. This means that $L^{4m-2} \subset Q_{4m-2}(\mathbf{C})$ is NOT Hamiltonian stable for $m \geq 4$.

From these results we conclude the following.

**Theorem 7.1.** *The Gauss image* $L^{4m-2} = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))\cdot \mathbf{Z}_4} \subset Q_{4m-2}(\mathbf{C})$ $(m \geq 2)$ *is not Hamiltonian stable if and only if* $m \geq 4$. *If* $m = 2$ *or* $3$, *it is strictly Hamiltonian stable.*

Remark. The index $i(L^{4m-2})$ goes to $\infty$ as $m \to \infty$.

## 8. The case $(U, K) = (Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$

In this case, $K = Sp(2) \times Sp(m) \subset U = Sp(m+2)$, $(U, K)$ is of type $B_2$ for $m = 2$ and type $BC_2$ for $m \geq 3$. Let $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ be the canonical decomposition of $\mathfrak{u}$ and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, where

$$\begin{aligned}\mathfrak{u} =& \mathfrak{sp}(m+2) \\ =& \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mid A \in \mathfrak{u}(m+2), B \in M(m+2, \mathbf{C}), B^t = B \right\} \\ & \subset \mathfrak{u}(2m+4),\end{aligned}$$

$$\begin{aligned}\mathfrak{k} &= \mathfrak{sp}(2) + \mathfrak{sp}(m) \\ &= \left\{ \begin{pmatrix} A_{11} & 0 & B_{11} & 0 \\ 0 & A_{22} & 0 & B_{22} \\ -\bar{B}_{11} & 0 & \bar{A}_{11} & 0 \\ 0 & -\bar{B}_{22} & 0 & \bar{A}_{22} \end{pmatrix} \right. \\ & \qquad \mid A_{11} \in \mathfrak{u}(2), B_{11} \in M(2, \mathbf{C}), B_{11}^t = B_{11}, \\ & \qquad \left. A_{22} \in \mathfrak{u}(m), B_{22} \in M(m, \mathbf{C}), B_{22}^t = B_{22} \right\},\end{aligned}$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A_{12} & 0 & B_{12} \\ -\bar{A}_{12}^t & 0 & B_{12}^t & 0 \\ 0 & -\bar{B}_{12} & 0 & \bar{A}_{12} \\ -\bar{B}_{11}^t & 0 & -A_{12}^t & 0 \end{pmatrix} \right.$$

$$\left. \mid A_{12} \in M(2, m; \mathbf{C}), B_{12} \in M(2, m; \mathbf{C}) \right\},$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & H_{12} & 0 & 0 \\ -\bar{H}_{12}^t & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{H}_{12} \\ 0 & 0 & -H_{12}^t & 0 \end{pmatrix} \right.$$

$$\left. \mid H_{12} = \begin{pmatrix} \xi_1 & 0 & 0 & \cdots & 0 \\ 0 & \xi_2 & 0 & \cdots & 0 \end{pmatrix}, \xi_1, \xi_2 \in \mathbf{R} \right\}.$$

Then the centralizer $K_0$ of $\mathfrak{a}$ in $K$ is given as follows:

$$\begin{aligned} K_0 =& Sp(1) \times Sp(1) \times Sp(m-2) \\ =& \left\{ \begin{pmatrix} a_1 & 0 & & & & b_1 & 0 & & & \\ 0 & a_2 & & & & 0 & b_2 & & & \\ & & a_1 & 0 & & & & b_1 & 0 & \\ & & 0 & a_2 & & & & 0 & b_2 & \\ & & & & A_{11} & & & & & A_{12} \\ -\bar{b}_1 & 0 & & & & \bar{a}_1 & 0 & & & \\ 0 & -\bar{b}_2 & & & & 0 & \bar{a}_2 & & & \\ & & -\bar{b}_1 & 0 & & & & \bar{a}_1 & 0 & \\ & & 0 & -\bar{b}_2 & & & & 0 & \bar{a}_2 & \\ & & & & A_{21} & & & & & A_{22} \end{pmatrix} \mid \right. \\ & \begin{pmatrix} a_1 & b_1 \\ -\bar{b}_1 & \bar{a}_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ -\bar{b}_2 & \bar{a}_2 \end{pmatrix} \in Sp(1) = SU(2), \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\ & \left. \in Sp(m-2) \right\}. \end{aligned}$$

Moreover,

$$K_{[\mathfrak{a}]} = K_0 \cup (Q \cdot K_0) \cup (Q^2 \cdot K_0) \cup (Q^3 \cdot K_0),$$

where

$$D = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & I_{m-2} \end{pmatrix} \text{ and } Q := \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

Thus the deck transformation group of the covering map $\mathcal{G} : N^{8m-2} \to \mathcal{G}(N^{8m-2})$ $(m \geq 2)$ is equal to $K_{[\mathfrak{a}]}/K_0 \cong \mathbf{Z}_4$.

**8.1. Description of the Casimir operator.**

Denote $\langle X, Y\rangle_{\mathfrak{u}} := -\frac{1}{2}\mathrm{tr}XY$ for each $X, Y \in \mathfrak{sp}(m+2) \subset \mathfrak{u}(2m+4)$. Then the square length of each restricted root relative to the above inner product $\langle\ ,\ \rangle_{\mathfrak{u}}$, is given by

$$\|\gamma\|^2_{\mathfrak{u}} = \begin{cases} 1 \text{ or } 2, & m = 2, \\ \frac{1}{2}, 1 \text{ or } 2, & m \geq 3. \end{cases}$$

Hence the Casimir operator $\mathcal{C}_L$ of $L$, with respect to the induced metric from $g^{\mathrm{std}}_{Q_{8m-2}(\mathbf{C})}$ can be expressed as follows:

$$\mathcal{C}_L = \begin{cases} \mathcal{C}_{K/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, & m = 2, \\ 2\,\mathcal{C}_{K/K_0} - \mathcal{C}_{K_2/K_0} - \frac{1}{2}\mathcal{C}_{K_1/K_0}, & m \geq 3, \end{cases} \tag{8.1}$$

where $\mathcal{C}_{K/K_0}, \mathcal{C}_{K_2/K_0}$, and $\mathcal{C}_{K_1/K_0}$ denote the Casimir operator of $K/K_0$, $K_2/K_0$, and $K_1/K_0$ relative to $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}}$, $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_2}$, and $\langle\ ,\ \rangle_{\mathfrak{u}}|_{\mathfrak{k}_1}$, respectively.

**8.2. Descriptions of $D(Sp(m))$ and $D(Sp(2)\times Sp(m))$.**

Let $G = Sp(m)$ and $K = Sp(2) \times Sp(m-2)$ in this subsection. Their Lie algebras are $\mathfrak{g}$ and $\mathfrak{k}$, respectively.

$$\mathfrak{t} = \{\xi = \sqrt{-1}\mathrm{diag}(\xi_1, \ldots, \xi_m, -\xi_1, \ldots, -\xi_m) \mid \xi_1, \ldots, \xi_m \in \mathbf{R}\}$$

is a maximal abelian subalgebra in both $\mathfrak{g}$ and $\mathfrak{k}$. Let $y_i : \xi \mapsto \xi_i$ be a linear form on $\mathfrak{t}$. Then the fundamental root system of $\mathfrak{g}$ relative to $\mathfrak{t}$ is given by $\{\alpha_1 = y_1 - y_2, \ldots, \alpha_{m-1} = y_{m-1} - y_m, \alpha_m = 2y_m\}$, and the fundamental root system of $\mathfrak{k}$ relative to $\mathfrak{t}$ can be given by $\{\alpha' = y_1 - y_2, \alpha' = 2y_2, \alpha'_3 = y_3 - y_4, \ldots, \alpha'_{m-1} = y_{m-1} - y_m, \alpha'_m = 2y_m\}$. Thus each $\Lambda \in D(G)$ for $G = Sp(m)$ relative to $\mathfrak{t}$ is uniquely expressed as $\Lambda = p_1y_1 + \cdots + p_my_m$ with $p_1, \ldots, p_m \in \mathbf{Z}$ and $p_1 \geq p_2 \geq \cdots \geq p_m \geq 0$. And also each $\Lambda \in D(K)$ for $K = Sp(2) \times Sp(m-2)$ relative to $\mathfrak{t}$ is uniquely expressed as $\Lambda' = q_1y_1 + \cdots + q_my_m$ with $q_1, \ldots, q_m \in \mathbf{Z}$ and $q_1 \geq q_2 \geq 0, q_3 \geq \cdots \geq q_m \geq 0$.

**8.3. Branching law of $(Sp(2), Sp(1)\times Sp(1))$.**

**Lemma 8.1** (Branching law of $(Sp(2), Sp(1) \times Sp(1))$ [**23**, **49**]). *Let $V_\Lambda$ be an irreducible $Sp(2)$-module with the highest weight $\Lambda = p_1y_1 + p_2y_2 \in D(Sp(2))$, where $p_1, p_2 \in \mathbf{Z}$ and $p_1 \geq p_2 \geq 0$. Then $V_\Lambda$ contains an irreducible $Sp(1) \times Sp(1)$-module $V_{\Lambda'}$ with the highest weight $\Lambda' = q_1y_1 + q_2y_2 \in D(Sp(1) \times Sp(1))$, where $q_1, q_2 \in \mathbf{Z}$ and $q_1 \geq 0, q_2 \geq 0$, if and only if*

(i) *$p_1 \geq q_2 \geq 0$, and*

(ii) *in the finite power series expansion in $X$ of $\frac{\prod_{i=0}^{1}(X^{r_i+1} - X^{-(r_i+1)})}{X - X^{-1}}$, where $r_i (i = 0, 1)$ are defined as*

$$r_0 := p_1 - \max(p_2, q_2), \quad r_1 := \min(p_2, q_2),$$

*the coefficient of $X^{q_1+1}$ does not vanish.*

*Here that coefficient is equal to the multiplicity of a $Sp(1)\times Sp(1)$-module $V_{\Lambda'}$ in $V_\Lambda$.*

**8.4. Descriptions of $D(K,K_0)$ and $D(K_1,K_0)$ when $m=2$.**

For each $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4 \in D(K) = D(Sp(2)\times Sp(2))$ with $p_1,\dots,p_4 \in \mathbf{Z}$ and $p_1 \geq p_2 \geq 0$, $p_3 \geq p_4 \geq 0$, we know that $p_1y_1+p_2y_2 \in D(Sp(2))$, $p_3y_3+p_4y_4 \in D(Sp(2))$ and $V_\Lambda = W_{p_1y_1+p_2y_2} \boxtimes W_{p_3y_3+p_4y_4}$. By Lemma 8.1, $W_{p_1y_1+p_2y_2}$ and $W_{p_3y_3+p_4y_4}$ can be decomposed into irreducible $Sp(1)\times Sp(1)$-modules as

$$W_{p_1y_1+p_2y_2} = \bigoplus_{q_1,q_2} W'_{q_1y_1+q_2y_2}, \quad W_{p_3y_3+p_4y_4} = \bigoplus_{q_3,q_4} W'_{q_3y_3+q_4y_4},$$

where $q_1,q_2$ and $q_3,q_4$ vary as in Lemma 8.1. Thus we have a decomposition of $V_\Lambda$ into the direct sum of irreducible $Sp(1)\times Sp(1)\times Sp(1)\times Sp(1)$-modules:

$$V_\Lambda = \bigoplus_{q_1,q_2}\bigoplus_{q_3,q_4}(W'_{q_1y_1+q_2y_2}\boxtimes W'_{q_3y_3+q_4y_4}).$$

Further, by the Clebsch–Gordan formula it can be decomposed into the sum of irreducible $Sp(1)\times Sp(1)$-modules as

$$V_\Lambda = \bigoplus_{q_1,q_2}\bigoplus_{q_3,q_4}\left(\bigoplus_{i=1}^{q_3} U_{q_1+q_3-2i}\right)\boxtimes\left(\bigoplus_{j=0}^{q_4} U_{q_2+q_4-2j}\right).$$

Here we assume that $q_1 \geq q_3 \geq 0$ and $q_2 \geq q_4 \geq 0$. Hence we have the following.

**Lemma 8.2.** *$\Lambda \in D(K,K_0)$ if and only if there exist $i,j \in \mathbf{Z}$ with $0 \leq i \leq q_3$ and $0 \leq j \leq q_4$ such that $U_{q_1+q_3-2i}\boxtimes U_{q_2+q_4-2j}$ is a trivial $Sp(1)\times Sp(1)$-module. Then it must be that $(q_1,q_2)=(q_3,q_4)$.*

**8.5. Eigenvalue computation when $m=2$.** For $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4 \in D(K,K_0)$ and $\Lambda' = q_1y_1+q_2y_2+q_3y_3+q_4y_4 \in D(K_1,K_0)$ with $q_1=q_3$, $q_2=q_4$ as in Lemma 8.2, the corresponding eigenvalue of $-\mathcal{C}_L$ is

(8.2)

$$\begin{aligned} -c_L =& -c_\Lambda + \frac{1}{2}c_{\Lambda'} \\ =& \Big(\sum_{i=1}^4 p_i^2 + 4p_1 + 2p_2 + 4p_3 + 2p_4\Big) - \Big(q_1^2+q_2^2+2q_1+2q_2\Big). \end{aligned}$$

Denote $\Lambda = p_1y_1+p_2y_2+p_3y_3+p_4y_4 \in D(K,K_0)$ by $\Lambda=(p_1,p_2,p_3,p_4)$. Since $-\mathcal{C}_L \geq -\frac{1}{2}\mathcal{C}_{K/K_0}$, the eigenvalue of $-\mathcal{C}_L$, $-c_L \leq n = 14$ implies $-c_\Lambda \leq 28$. Notice that

$$-c_\Lambda = \sum_{i=1}^4 p_i^2 + 4p_1 + 2p_2 + 4p_3 + 2p_4 \geq 2(p_2^2+p_4^2) + 6(p_2+p_4),$$

we have

$$\begin{cases} p_2^2 + p_4^2 + 3(p_2 + p_4) \leq 14, \\ \tilde{p}_i \in \mathbf{Z}, \quad \sum_{i=1}^4 p_i^2 \leq 28, \\ p_1 \geq p_2 \geq 0, \quad p_3 \geq p_4 \geq 0. \end{cases}$$

Then by the similar calculations using the eigenvalue formula (8.2), we obtain the following.

**Lemma 8.3.** *$\Lambda = p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4 \in D(K, K_0)$ has eigenvalue $-c_L \leq 14$ if and only if $(p_1, p_2, p_3, p_4)$ is one of*

$$\{ (0,0,0,0), (1,1,0,0), (0,0,1,1), (1,0,1,0), (1,1,1,1), \\ (1,1,2,0), (2,0,1,1) \}.$$

Suppose that $\Lambda = (1,1,0,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 5$. It follows from Lemma 8.1 that $(q_1, q_2) = (0,0)$ or $(1,1)$ and $(q_3, q_4) = (0,0)$. Then $(q_1, q_2, q_3, q_4) = (0,0,0,0) \in D(K_1, K_0)$. Hence $-c_\Lambda = 8$, $-c_{\Lambda'} = 0$, $-c_L = -c_\Lambda + \frac{1}{2}c_{\Lambda'} = 8 < 14$. On the other hand, there is a double covering $\pi : Sp(2) \to SO(5)$, and $\pi(Sp(1) \times Sp(1)) = SO(4)$. Let $\lambda_5$ denote the standard representation of $SO(5)$, and let 1 denote the trivial representation of $SO(5)$. Then the complex representation of $K = Sp(2) \times Sp(2)$ with the highest weight $(1,1,0,0)$ is $(\lambda_5 \otimes 1) \otimes \mathbf{C}$ and $V_\Lambda = \mathbf{C}^5$. It is easy to see that $(V_\Lambda)_{K_0} = \mathbf{C}\mathbf{e}_1$, where $\mathbf{e}_1 = (1,0,0,0,0)^t \in \mathbf{C}^5$. However, for

$$a = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix} \in K_{[\mathfrak{a}]} \subset K, \quad a \notin K_0,$$

$\pi(a) = \mathrm{diag}(-1,1,-1,-1,-1) \notin SO(4)$ and $\pi(a)\mathbf{e}_1 = -\mathbf{e}_1 \neq \mathbf{e}_1$. Therefore, $(V_\Lambda)_{K_{[\mathfrak{a}]}} = \{0\}$ and $\Lambda = (1,1,0,0) \notin D(K, K_{[\mathfrak{a}]})$. Similarly, $\Lambda = (0,0,1,1) \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,0,1,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 16$. The irreducible representation with the highest weight $\Lambda$ is just the complexified isotropy representation $\mathrm{Ad}_{\mathfrak{p}}(K)^{\mathbf{C}}$. Hence, $\Lambda \notin D(K, K_{[\mathfrak{a}]})$.

Suppose that $\Lambda = (1,1,1,1)$. Then $\dim_{\mathbf{C}} V_\Lambda = 25$. By Lemma 8.1, $(q_1, q_2) = (1,1)$ or $(0,0)$ and $(q_3, q_4) = (1,1)$ or $(0,0)$. Then $(q_1, q_2, q_3, q_4) = (1,1,1,1)$ or $(0,0,0,0) \in D(K_1, K_0)$. If $(q_1, q_2, q_3, q_4) = (1,1,1,1)$, then $-c_L = 10 < 14$. If $(q_1, q_2, q_3, q_4) = (0,0,0,0)$, then $-c_L = 16 > 14$. On the other hand, $V_{(1,1,1,1)}$ is explicitly given as

$$V_{(1,1,1,1)} = \mathbf{C}^5 \boxtimes \mathbf{C}^5 \cong M(5, \mathbf{C}).$$

There are doubly covering homomorphisms

$$\begin{aligned}\pi : K = Sp(2)\times Sp(2) &\longrightarrow SO(5)\times SO(5),\\ \pi|_{K_1} : K_1 = Sp(1)\times Sp(1)\times Sp(1)\times Sp(1) &\longrightarrow SO(4)\times SO(4),\\ \pi|_{K_0} : K_0 = Sp(1)\times Sp(1) &\longrightarrow SO(4).\end{aligned}$$

The representation of $K$ on $V_\Lambda$ is realized as the action of $\pi(K) = SO(5)\times SO(5)$ on $M(5,\mathbf{C})$ in the following way: For each $(A,B)\in SO(5)\times SO(5)$, $X\in M(5,\mathbf{C})$ is mapped to $AXB^{-1}\in M(5,\mathbf{C})$. Then as a $K_1$-module,

$$\begin{aligned}M(5,\mathbf{C}) &= \left\{\begin{pmatrix}0&0\\ *&0\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0&*\\ 0&0\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}*&0\\ 0&0\end{pmatrix}\right\}\oplus\left\{\begin{pmatrix}0&0\\ 0&*\end{pmatrix}\right\}\\ &= W_{(1,1,0,0)}\oplus W_{(0,0,1,1)}\oplus W_{(0,0,0,0)}\oplus W_{(1,1,1,1)}.\end{aligned}$$

$K_0$ acts on $M(5,\mathbf{C})$ by the adjoint action as a diagonal subgroup of $K_1$. Hence

$$\begin{aligned}(M(5,\mathbf{C}))_{K_0} &= \left\{\begin{pmatrix}x&0\\ 0&yI_4\end{pmatrix}\mid x,y\in\mathbf{C}\right\},\\ (M(5,\mathbf{C}))_{K_{[\mathfrak{a}]}} &= \mathbf{C}\begin{pmatrix}1&0\\ 0&0\end{pmatrix} = W(0,0,0,0).\end{aligned}$$

Though $\Lambda = (1,1,1,1)\in D(K,K_{[\mathfrak{a}]})$, by the preceding computation (in case $(q_1,q_2,q_3,q_4) = (0,0,0,0)$), we see that a nonzero element in $(M(5,\mathbf{C}))_{K_{[\mathfrak{a}]}} = W(0,0,0,0)$ gives eigenvalue $-c_L = 16 > 14$.

Suppose that $\Lambda = (1,1,2,0)$. Then $\dim_{\mathbf{C}} V_\Lambda = 50$. It follows from Lemma 8.1 that $(q_1,q_2) = (1,1)$ or $(0,0)$ and $(q_3,q_4) = (0,2),(1,1)$, or $(2,0)$. Thus

$$\begin{aligned}V_\Lambda = &(W_{(1,1)}\boxtimes U_{(0,2)})\oplus(W_{(1,1)}\boxtimes U_{(1,1)})\oplus(W_{(1,1)}\boxtimes U_{(2,0)})\\ &\oplus(W_{(0,0)}\boxtimes U_{(0,2)})\oplus(W_{(0,0)}\boxtimes U_{(1,1)})\oplus(W_{(0,0)}\boxtimes U_{(2,0)}).\end{aligned}$$

Here only $(q_1,q_2,q_3,q_4) = (1,1,1,1)$ $(W_{(1,1)}\boxtimes U_{(1,1)})$ belongs to $D(K_1,K_0)$, and the corresponding eigenvalue is $-c_L = 14$. On the other hand, the representation of $K$ with highest weight $\Lambda = (1,1,2,0)$ is $\lambda_5\boxtimes \mathrm{Ad}^{\mathbf{C}}_{\mathfrak{sp}(2)}$. Set $\Lambda_1 = (p_1,p_2) = (1,1)\in D(Sp(2))$. Then

$$V_{\Lambda_1}\cong\mathbf{C}^5 = \mathbf{C}\mathbf{e}_1\oplus\mathrm{span}_{\mathbf{C}}\{\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\} = W_{(0,0)}\oplus W_{(1,1)}.$$

Using the quaternionic representation

$$\mathfrak{sp}(2) = \{X\in M(2,\mathbf{H})\mid X^*+X=0\},$$

we chose the following basis of $\mathfrak{sp}(2)$:

$$E_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 := \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, E_4 := \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},$$
$$E_5 := \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, E_6 := \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, E_7 := \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},$$
$$E_8 := \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, E_9 := \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, E_{10} := \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix},$$

where $\{i, j, k\}$ denote the unit pure quaternions.

Set $\Lambda_2 = (p_3, p_4) = (2, 0) \in D(Sp(2))$. Then

$$\begin{aligned} V_{\Lambda_2} &\cong \operatorname{span}_{\mathbf{C}}\{E_1, E_2, E_3, E_4\} \oplus \operatorname{span}_{\mathbf{C}}\{E_5, E_6, E_7\} \oplus \operatorname{span}_{\mathbf{C}}\{E_8, E_9, E_{10}\} \\ &= W_{(1,1)} \oplus W_{(2,0)} \oplus W_{(0,2)}. \end{aligned}$$

By a direct computation, we get that

$$\begin{aligned} (V_\Lambda)_{K_0} &= \operatorname{span}_{\mathbf{C}}\{\mathbf{e}_2 \otimes E_1 + \mathbf{e}_3 \otimes E_2 + \mathbf{e}_4 \otimes E_3 + \mathbf{e}_5 \otimes E_4\} \\ &= (V_\Lambda)_{K_{[\mathfrak{a}]}} \subset W_{(1,1)} \otimes U_{(1,1)}. \end{aligned}$$

Therefore, $\Lambda = (1, 1, 2, 0) \in D(K, K_{[\mathfrak{a}]})$, which gives eigenvalue 14 with multiplicity 1. Similarly, we can show that $\Lambda = (2, 0, 1, 1) \in D(K, K_{[\mathfrak{a}]})$, which gives eigenvalue 14 with multiplicity 1.

Moreover, we observe that

$$\begin{aligned} n(L^{14}) &= \dim_{\mathbf{C}} V_{(1,1,2,0)} + \dim_{\mathbf{C}} V_{(2,0,1,1)} = 100 \\ &= \dim SO(16) - \dim Sp(2) \times Sp(2) = n_{hk}(L^{14}). \end{aligned}$$

From these results we obtain that $L^{14} = \mathcal{G}(\frac{Sp(2)\times Sp(2)}{Sp(1)\times Sp(1)}) \subset Q_{14}(\mathbf{C})$ is strictly Hamiltonian stable.

**8.6. Eigenvalue computation when $m \geq 3$.** For each

$$\Lambda = p_1 y_1 + p_2 y_2 + p_3 y_3 + \cdots + p_{m+2} y_{m+2} \in D(K, K_0)$$

with $p_i \in \mathbf{Z}$, $p_1 \geq p_2$, $p_3 \geq p_4 \geq \cdots \geq p_{m+2} \geq 0$,

$$\Lambda' = q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4 + q_5 y_5 + \cdots + q_{m+2} y_{m+2} \in D(K_2, K_0),$$

with $q_i \in \mathbf{Z}$, $q_1 \geq q_2 \geq 0$, $q_3 \geq q_4 \geq 0$, $q_5 \geq \cdots \geq q_{m+2} \geq 0$, $q_1 = p_1$, $q_2 = p_2$, and

$$\Lambda'' = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4 + k_5 y_5 + \cdots + k_{m+2} y_{m+2} \in D(K_1, K_0)$$

with $k_i \in \mathbf{Z}$, $k_i \geq 0$ for $1 \leq i \leq 4$, $k_5 \geq k_6 \geq \cdots \geq k_{m+2} \geq 0$, $k_j = q_j$ for $5 \leq j \leq m+2$, the corresponding eigenvalue of $-\mathcal{C}_L$ is expressed as follows:

(8.3)

$$
\begin{aligned}
-c_L = & -2c_\Lambda + c_{\Lambda'} + \frac{1}{2}c_{\Lambda''} \\
= & \, 2\Big(\sum_{i=1}^{m+2} p_i^2 + 4p_1 + 2p_2 + 2mp_3 + (2m-2)p_4 + \cdots + 2p_{m+2}\Big) \\
& - \Big(\sum_{i=1}^{m+2} q_i^2 + 4q_1 + 2q_2 + 4q_3 + 2q_4 + (2m-4)q_5 + \cdots + 2q_{m+2}\Big) \\
& - \frac{1}{2}\Big(\sum_{i=1}^{m+2} k_i^2 + 2k_1 + 2k_2 + 2k_3 + 2k_4 + (2m-4)k_5 + \cdots + 2k_{m+2}\Big),
\end{aligned}
$$

where $q_i = k_i$ for $5 \leq i \leq m+2$, $p_1 = q_1$, $p_2 = q_2$, and $k_1 = k_3$, $k_2 = k_4$.

Suppose that $\Lambda = (p_1, p_2, \ldots, p_{m+2}) = (2,2,0,\ldots,0) \in D(K)$. Then by using the branching law of $(Sp(2), Sp(1) \times Sp(1))$, we see that $\Lambda \in D(K, K_0)$, $\Lambda' = (q_1, q_2, \ldots, q_{m+2}) = (2,2,0,\ldots,0) \in D(K_2, K_0)$ and $\Lambda'' = (k_1, k_2, \ldots, k_{m+2}) = (0,0,0,\ldots,0) \in D(K_1, K_0)$. Hence by (8.3) the corresponding eigenvalue is $-c_L = 20 < 8m-2$ for $m \geq 3$. On the other hand, the irreducible representation of $K$ with the highest weight $\Lambda = (2,2,0,\ldots,0)$ is a 14-dimensional representation $\rho_{\mathrm{Sym}_0^2(\mathbf{C}^5)} \boxtimes \mathrm{I}$ of $Sp(2) \times Sp(m)$, where $\rho_{\mathrm{Sym}_0^2(\mathbf{C}^5)}$ is the composition of the natural surjective homomorphism $Sp(2) \to SO(5)$ and the traceless symmetric product representation of $SO(5)$ on $\mathrm{Sym}_0^2(\mathbf{C}^5) := \{X \in M(5;\mathbf{C}) \mid X^t = X, \mathrm{tr}X = 0\}$. Here each $A \in SO(5)$ acts on $\mathrm{Sym}_0^2(\mathbf{C}^5)$ by $\mathrm{Sym}_0^2(\mathbf{C}^5) \ni X \mapsto AXA^t \in \mathrm{Sym}_0^2(\mathbf{C}^5)$. So

$$
\begin{aligned}
\mathrm{Sym}_0(\mathbf{C}^5) = & \, \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4}I_4 \end{pmatrix} \oplus \Big\{ \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix} \mid X' \in \mathrm{Sym}_0(\mathbf{C}^4) \Big\} \\
& \oplus \Big\{ \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix} \mid Z \in M(1,4;\mathbf{C}) \Big\} \\
= & \, \mathbf{C} \oplus \mathrm{Sym}_0(\mathbf{C}^4) \oplus \mathbf{C}^4
\end{aligned}
$$

and

$$
(\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)} = \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4}I_4 \end{pmatrix} \cong \mathbf{C}.
$$

Under the natural surjective homomorphism $Sp(2)(\subset SU(4)) \to SO(5)$, the element

$$
\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \in Sp(2)
$$

corresponds to $\mathrm{diag}(-1,1,-1,-1,-1) \in SO(5)$, denoted by $Q'$. By a direct computation, we know that $(\mathrm{Sym}_0(\mathbf{C}^5))_{Q' \cdot SO(4)} \cap (\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)} =$

$(\mathrm{Sym}_0(\mathbf{C}^5))_{SO(4)}$. Thus

$$(V_{\Lambda=(2,2,0,\dots,0)})_{K_0} = \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} I_4 \end{pmatrix} \boxtimes \mathbf{C}$$

and, moreover,

$$(V_{\Lambda=(2,2,0,\dots,0)})_{K_{[\mathfrak{a}]}} = \mathbf{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4} I_4 \end{pmatrix} \boxtimes \mathbf{C}.$$

This means that $\Lambda = (2,2,0,\dots,0) \in D(K, K_{[\mathfrak{a}]})$ has multiplicity 1, which corresponds to eigenvalue $20 < 8m-2$. Therefore, $L^{8m-2} \subset Q_{8m-2}(\mathbf{C})$ is not Hamiltonian stable.

From our results of this section we conclude the following.

**Theorem 8.1.** *The Gauss image* $L = \frac{Sp(2)\times Sp(m)}{(Sp(1)\times Sp(1)\times Sp(m-2))\cdot \mathbf{Z}_4} \subset Q_{8m-2}(\mathbf{C})$ $(m \geq 2)$ *is not Hamiltonian stable if and only if* $m \geq 3$. *If* $m = 2$*, it is strictly Hamiltonian stable.*

## References


[1] A. Amarzaya & Y. Ohnita, *Hamiltonian stability of certain minimal Lagrangian submanifolds in complex projective spaces*, Tohoku Math. J. (2) **55** (2003) 583–610, MR 2017227, Zbl 1062.53053.

[2] A. Amarzaya & Y. Ohnita, *Hamiltonian stability of parallel Lagrangian submanifolds embedded in complex space forms*, a preprint, http://www.sci.osaka-cu.ac.jp/ ohnita/paper/Amar-Ohnita08.pdf.

[3] T. Asoh, *Compact transformation groups on* $\mathbf{Z}_2$*-cohomology spheres with orbits of codimension* 1, Hiroshima Math. J. **11** (1981) 571–616, MR 0635040, Zbl 0515.57021.

[4] T. Asoh, *Supplement to "Compact transformation groups on* $\mathbf{Z}_2$*-cohomology spheres with orbits of codimension* 1*,"* Hiroshima Math. J. **13** (1983) 647–652, MR 0725969, Zbl 0568.57021.

[5] L. Bérard-Bergery & J.-P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibres*, Illinois J. Math. **26** (1982), no. 2, 181–200, MR 0650387, Zbl 0483.58021.

[6] A. Borel & F. Hirzebruch, *Characteristic classes and homogeneous spaces, I.* Amer. J. Math. **80** (1958), 458–538, MR 0102800, Zbl 0097.36401.

[7] N. Bourbaki, *Elements de mathematique. Fasc. XXXIV. Groupes et algebres de Lie*, Actualites Scientifiques et Industrielles, No. 1337, Hermann, Paris 1968, MR 0240238, Zbl 0186.33001.

[8] T. Bröcker & T. Dieck, *Representations of compact Lie groups*, GTM 98, Springer-Verlag, New York, 1985, MR 0781344, Zbl 0581.22009.

[9] I. Castro & F. Urbano, *Minimal Lagrangian surfaces in* $S^2 \times S^2$, Commun. Anal. Geom. **15** (2007), 217–248, MR 2344322, Zbl 1185.53063.

[10] T. Cecil, Q.-S. Chi & G.R. Jensen, *Isoparametric hypersurfaces with four principal curvatures*, Ann. of Math. **166** (2007), 1–76, MR 2342690, Zbl 1143.53058.

[11] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures, II*, Nagoya Math. J. **204** (2011), 1–18, MR 2863363, Zbl 1243.53094.

[12] Q.-S. Chi, *Isoparametric hypersurfaces with four principal curvatures, III*, J. Diff. Geom. **94** (2013), 469–504, Zbl 06190450.

[13] D. Ferus, H. Karcher & H.F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502, MR 0624227, Zbl 0443.53037.

[14] W. Fulton & J. Harris, *Representation Theory: A First Course*, GTM 129, Springer-Verlag, 1991, MR 1153249, Zbl 0744.22001.

[15] H. Gluck, F. Morgan & W. Ziller, *Calibrated geometries in Grassmann manifolds*, Comment. Math. Helv. **69** (1989), 256–268, MR 0997365, Zbl 0681.53039.

[16] M. Goto & F.D. Grosshans, *Semisimple Lie Algebras*, Lecture Notes in Pure and Applied Mathematics, Vol. 38. Marcel Dekker, Inc., New York/Basel, 1978, MR 0573070, Zbl 0391.17004.

[17] R. Harvey & H.B. Lawson, *Calibrated geometries*, Acta Mathematica **148** (1982), 47–157, MR 0666108, Zbl 0584.53021

[18] W.-Y. Hsiang & H.B. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. Diff. Geom. **5** (1971), 1–38, MR 0298593, Zbl 0219.53045.

[19] A. Ikeda & Y. Taniguchi, *Spectra and eigenforms of the Laplacians on $S^n$ and $P^n(\mathbf{C})$*, Osaka J. Math. **15** (1978), 515–546, MR 0510492, Zbl 0392.53033.

[20] S. Immervoll, *On the classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres*, Ann. of Math. (2) **168** (2008), 1011–1024, MR 2456889, Zbl 1176.53057.

[21] H. Iriyeh, T. Sakai & H. Tasaki, *Lagrangian Floer homology of a pair of real forms in Hermitian symmetric spaces of compact type*, arXiv:1108.0260v2, 2012.

[22] H. Li, H. Ma & G. Wei, *A class of minimal Lagrangian submanifolds in complex hyperquadrics*, Geom. Dedicata **158**, no.1 (2012), 137–148, MR 2922708, Zbl 1250.53057.

[23] J. Lepowsky, *Multiplicity formulas for certain semisimple Lie groups*, Bull. Amer. Math. Soc. **77** (1971), 601–605, MR 0301142, Zbl 0228.17003.

[24] T. Kimura & M.S. Tanaka, *Totally geodesic submanifolds in compact symmetric spaces of rank Two*, Tokyo J. Math. **31**, no.2 (2008), 421–447, MR 2477882, Zbl 1215.53046.

[25] H. Ma & Y. Ohnita, *On Lagrangian submanifolds in complex hyperquadrics and isoparametric hypersurfaces in spheres*, Math. Z. **261** (2009), 749–785, MR 2480757, Zbl 1165.53037.

[26] H. Ma & Y. Ohnita, *Differential geometry of Lagrangian submanifolds and Hamiltonian variational problems*, Contemporary Mathematics **542** (2011), 115–134. Proceedings of the Conference in honor of J. C. Wood's sixtieth birthday, edited by E. Loubeau and S. Montaldo, MR 2796644, Zbl 1222.53085.

[27] H. Ma & Y. Ohnita, *Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces II*, preprint.

[28] W.G. McKay & J. Patera, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Lecture Notes in Pure and Applied Mathematics, Vol. 69, Marcel Dekker, Inc., New York/Basel, 1981, MR 0604363, Zbl 0728.17005.

[29] J.-L. Milhorat, *Spectrum of the Dirac operator on $Gr_2(\mathbf{C}^{m+2})$*, J. Math. Phys. **39** (1) (1998), 594–609, MR 1489638, Zbl 0919.58062.

[30] R. Miyaoka, *Isoparametric hypersurfaces with* $(g,m) = (6,2)$, Ann. of Math. (2) **177** (2013), 53–110, MR 2999038, Zbl 1263.53049.

[31] H.F. Münzner, *Isoparametrische Hyperflächen in Sphären*, Math. Ann. **251** (1980), 57–71, MR 0583825, Zbl 0417.53030.

[32] H.F. Münzner, *Isoparametrische Hyperfläche in Sphären, II*, Math. Ann. **256** (1981), 215–232, MR 0620709, Zbl 0438.53050.

[33] Y.G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), 501–519, MR 1062973, Zbl 0721.53060.

[34] Y.G. Oh, *Tight Lagrangian submanifolds in* $\mathbf{C}P^n$, Math. Z. **207** (1991), 409–416, MR 1115173, Zbl 0713.53032.

[35] Y.G. Oh, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Z. **212** (1993), 175–192, MR 1202805, Zbl 0791.53050.

[36] Y.G. Oh, *Mean curvature vector and symplectic topology of Lagrangian submanifolds in Einstein-Kähler manifolds*, Math. Z. **216** (1994), 471–482, MR 1283083, Zbl 0810.53045.

[37] Y. Ohnita, *Geometry of Lagrangian Submanifolds and Isoparametric Hypersurfaces*, Proceedings of The Fourteenth International Workshop on Differential Geometry and Related Fields, 14 (2010), 43–67, edited by Y.-J. Suh, National Institute for Mathematical Sciences, the Korean Mathematical Society and Grassmann Research Group, MR 2757809, Zbl 1260.53104.

[38] H. Ono, *Minimal Lagrangian submanifolds in adjoint orbits and upper bounds on the first eigenvalue of the Laplacian*, J. Math. Soc. Japan **55** (2003), 243–254, MR 1939195, Zbl 1038.53073.

[39] H. Ono, *Minimality and Hamiltonian stability of Lagrangian submanifolds in adjoint orbits*, Tokyo J. Math. **26** (2003), 83–106, MR 1982001, Zbl 1056.53050.

[40] H. Ono, *Integral formula of Maslov index and its applications*, Japan J. Math. (N.S.) **30** (2004), 413–421, MR 2111556, Zbl 1073.53108.

[41] H. Ono, *Hamiltonian stability of Lagrangian tori in toric Kähler manifolds*, Ann. Glob. Anal. Geom. **37** (2007), 329–343, MR 2325219, Zbl 1122.53036.

[42] H. Ozeki & M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres I*, Tohoku Math. J. (2) **27** (1975), 515–559, MR 0454888, Zbl 0359.53011.

[43] H. Ozeki & M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres II*, Tohoku Math. J.(2) **28** (1976), 7–55, MR 0454889, Zbl 0359.53012.

[44] B. Palmer, *Hamiltonian minimality and Hamiltonian stability of Gauss maps*, Diff. Geom. and its Appl. **7** (1997), 51–58, MR 1441918, Zbl 0897.53042.

[45] R. Schoen & J. Wolfson, *Minimizing volume among Lagrangian submanifolds*, Differential Equations: La Pietra 1996, edited by Giaquinta, Shatah, and Varadhan, Proc. Symp. Pure Math **65** (1999), 181–199, MR 1662755, Zbl 1031.53112.

[46] R. Takagi & T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 469–481, MR 0334094, Zbl 0244.53042.

[47] M. Takeuchi, *Stability of certain minimal submanifolds of compact Hermitian symmetric spaces*, Tohoku Math. J. (2) **36** (1984), 293–314, MR 0742600, Zbl 0528.53047.

[48] M. Takeuchi, Modern theory of spherical functions, Translations of Mathematical Monographs, AMS, 1993, MR 1280269, Zbl 0830.43002.

[49] C. Tsukamoto, *Spectra of Laplace-Beltrami operators on $SO(n+2)/SO(2) \times SO(n)$ and $Sp(n+1)/Sp(1) \times Sp(n)$*, Osaka J. Math. **18** (1981), 407–426, MR 0628842, Zbl 0477.53048.

[50] F. Uchida, *An orthogonal transformation group of $(8k-1)$-sphere*, J. Diff. Geom. **15** (1980), 569–574, MR 0628345, Zbl 0472.57023.

[51] F. Urbano, *Hamiltonian stability and index of minimal Lagrangian surfaces of the complex projective plane*, Indiana Univ. Math. J. **56** (2007), 931–946, MR 2317551, Zbl 1122.53045.

[52] S. Yamaguchi, *Spectra of flag manifolds*, Mem. Fac. Sci. Kyushu Univ. **33** (1979), 95–112, MR 0530024, Zbl 0405.53025.

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