

LINKING, TWISTING, WRITHING, AND HELICITY ON THE 3-SPHERE AND IN HYPERBOLIC 3-SPACE

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Abstract

In the first paper of this series, “Electrodynamics and the Gauss Linking Integral on the 3-sphere and in Hyperbolic 3-space,” we developed a steady-state version of classical electrodynamics in these two spaces, including explicit formulas for the vector-valued Green’s operator, explicit formulas of Biot-Savart type for the magnetic field, and a corresponding Ampère’s Law contained in Maxwell’s equations, and then used these to obtain explicit integral formulas for the linking number of two disjoint closed curves.

In this second paper, we obtain integral formulas for twisting, writhing, and helicity, and prove that $\text{LINK} = \text{TWIST} + \text{WRITHE}$ on the 3-sphere and in hyperbolic 3-space. We then use these results to derive upper bounds for the helicity of vector fields and lower bounds for the first eigenvalue of the curl operator on subdomains of these two spaces.

An announcement of these results, and a hint of their proofs, can be found in the Math ArXiv, [math.GT/0406276](https://arxiv.org/abs/math/0406276), while an expanded version of the first paper, with full proofs, can be found at [math.GT/0510388](https://arxiv.org/abs/math/0510388).

The flow of this paper is indicated by the following list of sections. The first two are devoted to a summary of information from the preceding paper.

- 1) Linking integrals in R^3 , S^3 , and H^3 .
- 2) Magnetic fields in R^3 , S^3 , and H^3 .
- 3) Link, twist, and writhe in S^3 and H^3 .
- 4) Proof scheme for $\text{LINK} = \text{TWIST} + \text{WRITHE}$.
- 5) Some geometric formulas on S^3 .
- 6) Proof of $\text{LINK} = \text{TWIST} + \text{WRITHE}$ in S^3 .
- 7) Proof of $\text{LINK} = \text{TWIST} + \text{WRITHE}$ in H^3 .
- 8) Helicity of vector fields on S^3 and H^3 .
- 9) Upper bounds for helicity in R^3 , S^3 , and H^3 .
- 10) Hodge decomposition of vector fields.
- 11) Spectral geometry of the curl operator in R^3 , S^3 , and H^3 .

Received 9/17/2010.

The integral formulas in this paper contain vectors lying in different tangent spaces; in non-Euclidean settings these vectors must be moved to a common location to be combined.

In S^3 , regarded as the group of unit quaternions, equivalently as $SU(2)$, the differential $L_{\mathbf{y}\mathbf{x}^{-1}}$ of left translation by $\mathbf{y}\mathbf{x}^{-1}$ moves tangent vectors from \mathbf{x} to \mathbf{y} . In either S^3 or H^3 , parallel transport $P_{\mathbf{y}\mathbf{x}}$ along the geodesic segment from \mathbf{x} to \mathbf{y} also does this. As a result, we get three versions for each of the formulas that appear in the theorems below.

1. Linking integrals in R^3 , S^3 , and H^3

Let $K_1 = \{\mathbf{x}(s)\}$ and $K_2 = \{\mathbf{y}(t)\}$ be disjoint oriented smooth closed curves in either Euclidean 3-space R^3 , the unit 3-sphere S^3 , or hyperbolic 3-space H^3 , and let $\alpha(\mathbf{x}, \mathbf{y})$ denote the distance from \mathbf{x} to \mathbf{y} .

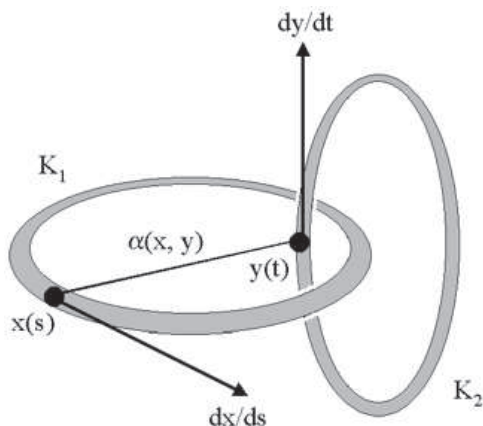


Figure 1. Two linked curves.

Carl Friedrich Gauss, in a half-page paper dated January 22, 1833, gave an integral formula for the linking number in Euclidean 3-space,

$$\text{Lk}(K_1, K_2) = \int_{K_1 \times K_2} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \frac{\mathbf{x} - \mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|^3} ds dt.$$

It will be convenient for us to write this as

$$\text{Lk}(K_1, K_2) = \int_{K_1 \times K_2} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y}) ds dt,$$

where $\varphi(\alpha) = 1/(4\pi\alpha)$, and where we use $\varphi(\mathbf{x}, \mathbf{y})$ as an abbreviation for $\varphi(\alpha(\mathbf{x}, \mathbf{y}))$. The subscript \mathbf{y} in the expression $\nabla_{\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y})$ tells us that the differentiation is with respect to the \mathbf{y} variable.

The following theorem from our first paper gives the corresponding linking integrals on the 3-sphere and in hyperbolic 3-space. Since the

location of the tangent vectors is now important, we note that the vector $\nabla_{\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y})$ is located at the point \mathbf{y} .

Theorem 1.1. Linking integrals in S^3 and H^3 .

(1) On S^3 in left-translation format:

$$\begin{aligned} \text{Lk}(K_1, K_2) &= \int_{K_1 \times K_2} L_{\mathbf{y}\mathbf{x}^{-1}} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y}) ds dt \\ &\quad - \frac{1}{4\pi^2} \int_{K_1 \times K_2} L_{\mathbf{y}\mathbf{x}^{-1}} \frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{y}}{dt} ds dt, \end{aligned}$$

where $\varphi(\alpha) = (\pi - \alpha) \cot \alpha / (4\pi^2)$.

(2) On S^3 in parallel transport format:

$$\text{Lk}(K_1, K_2) = \int_{K_1 \times K_2} P_{\mathbf{y}\mathbf{x}} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y}) ds dt,$$

where $\varphi(\alpha) = (\pi - \alpha) \csc \alpha / (4\pi^2)$.

(3) On H^3 in parallel transport format:

$$\text{Lk}(K_1, K_2) = \int_{K_1 \times K_2} P_{\mathbf{y}\mathbf{x}} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}}\varphi(\mathbf{x}, \mathbf{y}) ds dt,$$

where $\varphi(\alpha) = \text{csch } \alpha / (4\pi)$.

Greg Kuperberg (2008K) obtained, independently and by a totally different argument, an expression equivalent to formula (2) above.

The kernel functions used here have the following significance.

In Gauss's linking integral, the function $-\varphi(\alpha) = -1/(4\pi\alpha)$, where α is distance from a fixed point, is the fundamental solution of the Laplacian in R^3 ,

$$-\Delta\varphi = \delta.$$

Here δ is the Dirac δ -function.

In formula (1), the function $-\varphi(\alpha) = -(\pi - \alpha) \cot \alpha / (4\pi^2)$ is the fundamental solution of the Laplacian on S^3 ,

$$-\Delta\varphi = \delta - \frac{1}{2\pi^2}.$$

Since the volume of S^3 is $2\pi^2$, the right-hand side has average value zero.

In formula (2), the function $-\varphi(\alpha) = -(\pi - \alpha) \csc \alpha / (4\pi^2)$ is the fundamental solution of a shifted Laplacian on S^3 ,

$$-(\Delta\varphi - \varphi) = \delta.$$

In formula (3), the function $-\varphi(\alpha) = -\text{csch } \alpha / (4\pi)$ is the fundamental solution of a shifted Laplacian on H^3 ,

$$-(\Delta\varphi + \varphi) = \delta.$$

Our proof of the formula $\text{LINK} = \text{TWIST} + \text{WRITHE}$ will depend on the asymptotic properties of φ at its singularity. For example, in the case of S^3 in parallel transport format,

$$\varphi(\alpha) = \frac{1}{4\pi^2}(\pi - \alpha) \csc(\alpha) = \frac{1}{4\pi\alpha} - \frac{1}{4\pi^2} + \frac{1}{24\pi}\alpha - \frac{1}{24\pi^2}\alpha^2 + \alpha^3 f(\alpha),$$

where $f(\alpha)$ is bounded and smooth. Likewise

$$\begin{aligned} \varphi'(\alpha) &= -\frac{1}{4\pi\alpha^2} + \frac{1}{24\pi} - \frac{1}{12\pi^2}\alpha + \frac{7}{480\pi}\alpha^2 - \alpha^3 g(\alpha) \\ \text{and } \varphi''(\alpha) &= \frac{1}{2\pi\alpha^3} - \frac{1}{12\pi^2} + \frac{7}{240\pi}\alpha - \frac{7}{120\pi^2}\alpha^2 + \alpha^3 h(\alpha), \end{aligned}$$

where g and h are also bounded and smooth. Note that φ has no singularity at $\alpha = \pi$; in fact, φ is smooth and even around $\alpha = \pi$:

$$\varphi(\alpha) = \frac{1}{4\pi^2} + \frac{1}{24\pi^2}(\alpha - \pi)^2 + \frac{7}{1440\pi^2}(\alpha - \pi)^4 + \dots$$

near π . This implies that $\nabla_{\mathbf{y}}\varphi(\alpha(\mathbf{x}, \mathbf{y}))$ exists and is zero when \mathbf{y} is the antipodal point of x , even though $\nabla_{\mathbf{y}}\alpha$ is not defined there. Because of this, the functions $f(\alpha(\mathbf{x}, \mathbf{y}))$, $g(\alpha(\mathbf{x}, \mathbf{y}))$, and $h(\alpha(\mathbf{x}, \mathbf{y}))$ defined above are defined, smooth, and bounded for *all* \mathbf{x} and \mathbf{y} such that $\mathbf{x} \neq \mathbf{y}$.

Because we do not need so many terms of these expansions, we will simply write:

$$\varphi(\alpha) = \frac{1}{4\pi\alpha} + f(\alpha), \quad \varphi'(\alpha) = -\frac{1}{4\pi\alpha^2} + g(\alpha), \quad \varphi''(\alpha) = \frac{1}{2\pi\alpha^3} + h(\alpha),$$

where these new functions f , g , and h are bounded and smooth everywhere on S^3 .

Similar calculations show that, for $\varphi(\alpha) = \text{csch } \alpha/(4\pi)$ on H^3 , we again have

$$\varphi(\alpha) = \frac{1}{4\pi\alpha} + f(\alpha), \quad \varphi'(\alpha) = -\frac{1}{4\pi\alpha^2} + g(\alpha), \quad \varphi''(\alpha) = \frac{1}{2\pi\alpha^3} + h(\alpha),$$

where these latest functions f , g , and h are bounded and smooth everywhere on H^3 .

2. Magnetic fields in R^3 , S^3 , and H^3

In Euclidean 3-space R^3 , the classical convolution formula of Biot and Savart gives the magnetic field $\text{BS}(\mathbf{v})$ of a compactly supported current flow \mathbf{v} :

$$\text{BS}(\mathbf{v})(\mathbf{y}) = \int_{R^3} \mathbf{v}(\mathbf{x}) \times \frac{\mathbf{y} - \mathbf{x}}{4\pi|\mathbf{y} - \mathbf{x}|^3} d\mathbf{x}.$$

For simplicity, we write $d\mathbf{x}$ to mean $d\text{vol}_{\mathbf{x}}$.

The Biot-Savart formula can also be written as

$$\text{BS}(\mathbf{v})(\mathbf{y}) = \int_{R^3} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}}\varphi_0(\mathbf{x}, \mathbf{y}) d\mathbf{x},$$

where $\varphi_0(\alpha) = -1/(4\pi\alpha)$ is the fundamental solution of the Laplacian in R^3 .

In R^3 , if we start with a smooth, compactly supported current flow \mathbf{v} , then its magnetic field $\text{BS}(\mathbf{v})$ is a smooth vector field (although not in general compactly supported) which has the following properties:

- (1) It is divergence-free, $\nabla \cdot \text{BS}(\mathbf{v}) = 0$.
- (2) It satisfies Maxwell's equation

$$\nabla_{\mathbf{y}} \times \text{BS}(\mathbf{v})(\mathbf{y}) = \mathbf{v}(\mathbf{y}) + \nabla_{\mathbf{y}} \int_{R^3} \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \varphi_0(\mathbf{x}, \mathbf{y}) d\mathbf{x},$$

where φ_0 is the fundamental solution of the Laplacian in R^3 .

- (3) $\text{BS}(\mathbf{v})(\mathbf{y}) \rightarrow \mathbf{0}$ as $\mathbf{y} \rightarrow \infty$.

To see that (2) is one of Maxwell's equations, first integrate by parts to rewrite it as

$$\nabla_{\mathbf{y}} \times \text{BS}(\mathbf{v})(\mathbf{y}) = \mathbf{v}(\mathbf{y}) - \nabla_{\mathbf{y}} \int_{R^3} (\nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x})) \varphi_0(\mathbf{x}, \mathbf{y}) d\mathbf{x}.$$

If we think of the vector field $\mathbf{v}(\mathbf{x})$ as a steady current, then its negative divergence, $-\nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x})$, is the time rate of accumulation of charge at \mathbf{x} , and hence the integral

$$-\nabla_{\mathbf{y}} \int_{R^3} (\nabla_{\mathbf{x}} \cdot \mathbf{v}(\mathbf{x})) \varphi_0(\mathbf{x}, \mathbf{y}) d\mathbf{x}$$

is the time rate of increase of the electric field E at \mathbf{y} . Thus equation (2) is simply Maxwell's equation

$$\nabla \times B = \mathbf{v} + \frac{\partial E}{\partial t}.$$

In R^3 , S^3 , and H^3 , a linear operator satisfying conditions (1), (2), and (3) above will be referred to as a ***Biot-Savart operator***.

Remarks.

- To see that equation (2) above is Maxwell's equation, we integrated by parts, in spite of the fact that the kernel function $\varphi_0(\alpha)$ has a singularity at $\alpha = 0$. We leave it to the reader to check that the validity of this depends on the fact that the singularity of φ_0 is of order $1/\alpha$. We will use this throughout the paper, without further mention.
- Recall Ampère's Law: Given a divergence-free current flow, the circulation of the resulting magnetic field around a loop is equal to the flux of the current through any surface bounded by that loop. This is an immediate consequence of Maxwell's equation (2) above, since if the current flow \mathbf{v} is divergence-free, this equation says that $\nabla \times \text{BS}(\mathbf{v}) = \mathbf{v}$. Then Ampère's Law is just the curl theorem of vector calculus.

In particular, if the current flows along a wire loop, the circulation of the resulting magnetic field around a second loop disjoint from it is equal to the flux of the current through a cross-section of the wire loop, multiplied by the linking number of the two loops. Thus linking numbers are built into Ampère's Law, and once we have an explicit integral formula for the magnetic field due to a given current flow, we easily get an explicit integral formula for the linking number.

- In R^3 , conditions (1), (2), and (3) are easily seen to characterize the Biot-Savart operator, as follows. Since conditions (1) and (2) specify the divergence and the curl of $\text{BS}(\mathbf{v})$, the difference $\text{BS}_1(\mathbf{v}) - \text{BS}_2(\mathbf{v})$ between two candidates for the Biot-Savart operator would be divergence-free and curl-free. Since R^3 is simply connected, this difference would be the gradient of a harmonic function. Hence the components of this gradient must also be harmonic functions. Since they go to zero at infinity, they have to be identically zero. Thus $\text{BS}_1(\mathbf{v}) = \text{BS}_2(\mathbf{v})$.
- In S^3 , conditions (1) and (2) alone suffice to characterize the Biot-Savart operator, since there are no non-zero vector fields on S^3 which are simultaneously divergence-free and curl-free (i.e., there are no non-constant harmonic functions).
- In H^3 , it is not yet clear to us how to characterize the Biot-Savart operator. Even strengthening (3) to require that $\text{BS}(\mathbf{v})(\mathbf{y})$ go to zero exponentially fast at infinity is not quite enough. And in H^3 , unlike R^3 , the field $\text{BS}(\mathbf{v})$ is not in general of class L^2 .

The following theorem is from our first paper.

Theorem 2.1. *Biot-Savart integrals in S^3 and H^3 .* *Biot-Savart operators exist in S^3 and H^3 , and are given by the following formulas, in which \mathbf{v} is a smooth, compactly supported vector field:*

(1) *On S^3 , in left-translation format:*

$$\begin{aligned} \text{BS}(\mathbf{v})(\mathbf{y}) = & \int_{S^3} L_{\mathbf{y}\mathbf{x}^{-1}} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} - \frac{1}{4\pi^2} \int_{S^3} L_{\mathbf{y}\mathbf{x}^{-1}} \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \\ & + 2 \nabla_{\mathbf{y}} \int_{S^3} L_{\mathbf{y}\mathbf{x}^{-1}} \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathbf{y}} \varphi_1(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}, \end{aligned}$$

where $\varphi_0(\alpha) = -(\pi - \alpha) \cot \alpha / (4\pi^2)$ and $\varphi_1(\alpha) = -\alpha(2\pi - \alpha) / (16\pi^2)$.

(2) *On S^3 in parallel transport format:*

$$\text{BS}(\mathbf{v})(\mathbf{y}) = \int_{S^3} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) \, d\mathbf{x},$$

where $\varphi_0(\alpha) = -(\pi - \alpha) \csc \alpha / (4\pi^2)$.

(3) On H^3 in parallel transport format:

$$\text{BS}(\mathbf{v})(\mathbf{y}) = \int_{H^3} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) d\mathbf{x},$$

where $\varphi_0(\alpha) = -\text{csch } \alpha / (4\pi)$.

In formula (1), the function $\varphi_1(\alpha) = -\alpha(2\pi - \alpha)/(16\pi^2)$ satisfies the equation

$$\Delta \varphi_1 = \varphi_0 - [\varphi_0],$$

where $[\varphi_0]$ denotes the average value of φ_0 over S^3 . The other kernel functions already appeared in the linking integrals in Theorem 1.1.

In formula (3), the magnetic field $\text{BS}(\mathbf{v})(\mathbf{y})$ goes to zero at infinity like $e^{-\alpha}$, where α is the distance from \mathbf{y} to a fixed point in H^3 .

3. Link, twist, and writhe in S^3 and H^3

In a series of three papers (1959–1961), Georges Călugăreanu defined a real-valued invariant of a smooth *simple* closed curve in R^3 by allowing the two curves in Gauss’s linking integral to come together. In the limit, the points $\mathbf{x}(s)$ and $\mathbf{y}(t)$ now run along the same curve, and therefore can coincide, making Gauss’s integral seem improper because of the $|\mathbf{x} - \mathbf{y}|^3$ in the denominator. But Călugăreanu noted that in this case the numerator goes to zero even faster than the denominator, so that the whole integrand goes to zero as \mathbf{x} and \mathbf{y} come together, and the integral converges. In (1971), F. Brock Fuller called this invariant, which measures the extent to which the curve wraps and coils around itself, the “writhing number”:

$$\text{Wr}(K) = \int_{K \times K} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \frac{\mathbf{x} - \mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|^3} ds dt.$$

In those papers, Călugăreanu also discovered and proved the formula $\text{LINK} = \text{TWIST} + \text{WRITHE}$, in which LINK is the linking number of the two edges of a closed ribbon, TWIST measures the extent to which the ribbon twists around one of its edges, and WRITHE is the writhing number of that edge.

Călugăreanu proved this formula under the assumption that the simple closed curve K has nowhere-vanishing curvature, but the basic ideas for proving the formula without this assumption are already present in his papers. This can be seen in sections 6 and 7 of this paper, where the proofs we give in S^3 and H^3 follow Călugăreanu’s original proof in R^3 , but require no curvature restriction. Nevertheless, Călugăreanu’s formula without the curvature restriction was proved by James White (1969) in his thesis, using a totally different approach based on ideas of William Pohl (1968a, b).

Moving on to S^3 and H^3 , we follow Călugăreanu’s lead and replace the two closed curves K_1 and K_2 in the linking integrals of Theorem 1.1

with one simple closed curve. Again all the integrals converge, and we use them to extend the notion of writhing number to these spaces.

Definition of the writhing integrals in S^3 and H^3 .

(1) On S^3 in left-translation format:

$$\begin{aligned} \text{Wr}_L(K) := \int_{K \times K} L_{\mathbf{y}\mathbf{x}^{-1}} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x}, \mathbf{y}) ds dt \\ - \frac{1}{4\pi^2} \int_{K \times K} L_{\mathbf{y}\mathbf{x}^{-1}} \frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{y}}{dt} ds dt, \end{aligned}$$

where $\varphi(\alpha) = (\pi - \alpha) \cot \alpha / (4\pi^2)$.

(2) On S^3 in parallel transport format:

$$\text{Wr}_P(K) := \int_{K \times K} P_{\mathbf{y}\mathbf{x}} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x}, \mathbf{y}) ds dt,$$

where $\varphi(\alpha) = (\pi - \alpha) \csc \alpha / (4\pi^2)$.

(3) On H^3 in parallel transport format:

$$\text{Wr}_P(K) := \int_{K \times K} P_{\mathbf{y}\mathbf{x}} \frac{d\mathbf{x}}{ds} \times \frac{d\mathbf{y}}{dt} \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x}, \mathbf{y}) ds dt,$$

where $\varphi(\alpha) = \text{csch } \alpha / (4\pi)$.

The two versions of the writhing number on S^3 are *not* the same, and one can show that

$$\text{Wr}_L(K) = \text{Wr}_P(K) + \frac{\text{length of } K}{2\pi}.$$

The parallel transport version of writhe is more intuitively satisfying, since in this version the writhing number of a great circle is zero.

We turn next to the definition of “twist”.

Let K be a smooth simple closed curve in S^3 or H^3 , parametrized by arclength s . Let $\mathbf{x}(s)$ be a moving point along K , and $\mathbf{T}(s) = \mathbf{x}'(s)$ the unit tangent vector field.

Let $\mathbf{v}(s)$ be a unit normal vector field along K . Our intention is to define the (total) twist of \mathbf{v} along K by a formula similar to

$$\text{Tw}(\mathbf{v}) = \frac{1}{2\pi} \int_K \mathbf{T}(s) \times \mathbf{v}(s) \cdot \mathbf{v}'(s) ds,$$

the formula for twist in Euclidean 3-space.

But on S^3 there are two flavors of twist, according to whether $\mathbf{v}'(s)$ is calculated as a “left-invariant” derivative or as a covariant derivative. If we fall back into Euclidean mode and write

$$\mathbf{v}'(s) = \lim_{h \rightarrow 0} \frac{\mathbf{v}(s+h) - \mathbf{v}(s)}{h},$$

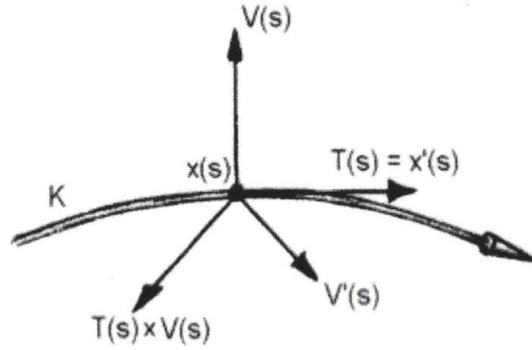


Figure 2. Vectors in the definition of twist.

then the vectors $\mathbf{v}(s+h)$ and $\mathbf{v}(s)$ lie in different tangent spaces, and we must move them together in order to subtract. If we use left-translation in the group S^3 to move $\mathbf{v}(s+h)$ back to the tangent space at $\mathbf{x}(s)$ which contains $\mathbf{v}(s)$, then the resulting limit is the *left-invariant derivative* $\mathbf{v}'_L(s)$. If we use parallel transport to move $\mathbf{v}(s+h)$ back, then the resulting limit is the *covariant derivative* $\mathbf{v}'_P(s)$.

The two flavors of twist on S^3 are then given by

$$\text{Tw}_L(\mathbf{v}) = \frac{1}{2\pi} \int_K \mathbf{T}(s) \times \mathbf{v}(s) \cdot \mathbf{v}'_L(s) ds \quad (1)$$

and

$$\text{Tw}_P(\mathbf{v}) = \frac{1}{2\pi} \int_K \mathbf{T}(s) \times \mathbf{v}(s) \cdot \mathbf{v}'_P(s) ds. \quad (2)$$

One can show that

$$\text{Tw}_L(\mathbf{v}) = \text{Tw}_P(\mathbf{v}) - \frac{\text{length of } K}{2\pi}.$$

Example. Let $K = \{(\cos s, \sin s, 0, 0) : s \in [0, 2\pi]\}$ be a great circle on S^3 , and along it the unit normal vector field $\mathbf{v}(s) = (0, 0, \cos s, \sin s)$. Then we have $\text{Tw}_L(\mathbf{v}) = 0$ and $\text{Tw}_P(\mathbf{v}) = 1$.

In hyperbolic 3-space H^3 , we have only the parallel transport version of twist,

$$\text{Tw}_P(\mathbf{v}) = \frac{1}{2\pi} \int_K \mathbf{T}(s) \times \mathbf{v}(s) \cdot \mathbf{v}'_P(s) ds. \quad (3)$$

Now consider in S^3 or H^3 a narrow ribbon of width ε obtained by starting with a simple closed curve $K = \{\mathbf{x}(s)\}$ and then exponentiating a unit normal vector field $\mathbf{v}(s)$ along K . One edge of this ribbon is the original curve K , and the other edge is the curve $K_\varepsilon = \{\mathbf{y}_\varepsilon(s)\}$, given explicitly (see section 5) by

$$\begin{aligned} \mathbf{y}_\varepsilon(s) &= \cos \varepsilon \mathbf{x}(s) + \sin \varepsilon \mathbf{v}(s) \quad \text{in } S^3; \\ \mathbf{y}_\varepsilon(s) &= \cosh \varepsilon \mathbf{x}(s) + \sinh \varepsilon \mathbf{v}(s) \quad \text{in } H^3. \end{aligned}$$

Since K is simple, the ribbon will be *embedded* in S^3 or H^3 provided ε is small enough.

The vector field $\mathbf{v}(s)$ then points “across” the ribbon.

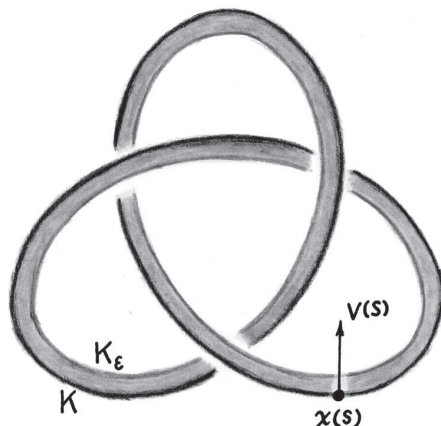


Figure 3. A ribbon, its generating curve, and its vector field.

Theorem 3.1. LINK = TWIST + WRITHE *in S^3 and H^3 .*

(1) *On S^3 in left-translation format:*

$$\text{Lk}(K, K_\varepsilon) = \text{Tw}_L(\mathbf{v}) + \text{Wr}_L(K).$$

(2) *On S^3 in parallel transport format:*

$$\text{Lk}(K, K_\varepsilon) = \text{Tw}_P(\mathbf{v}) + \text{Wr}_P(K).$$

(3) *On H^3 in parallel transport format:*

$$\text{Lk}(K, K_\varepsilon) = \text{Tw}_P(\mathbf{v}) + \text{Wr}_P(K).$$

We give an overview of the proof in the next section.

4. Proof scheme for LINK = TWIST + WRITHE

In spirit, our proof of Theorem 3.1 for ribbons in S^3 and H^3 follows Călugăreanu’s original proof in R^3 : we begin with the linking integrals given in Theorem 1.1 for the edges K and K_ε of our ribbon, let ε shrink to zero, and observe the behavior of the linking integrand.

The value of the linking integral is independent of ε for $\varepsilon > 0$ since the ribbon is embedded and since the linking number is invariant under homotopies which keep the two curves disjoint. But the linking integrand blows up as one approaches the diagonal of $K \times K$, and this is handled as follows.

Outside an appropriately chosen neighborhood of the diagonal, the linking integrand converges to the writhing integrand as $\varepsilon \rightarrow 0$, and its integral converges to the writhing number of the curve K . Inside this

neighborhood of the diagonal, the linking integrand blows up, but its integral converges to the total twist of the normal vector field \mathbf{v} along K .

The crucial thing, recognized by Călugăreanu, is that the width of the neighborhood of the diagonal in $K \times K$ must go to zero *much more slowly* than the width ε of the ribbon. In fact, we will choose the neighborhood of the diagonal to have width ε^p , where $0 < p < 1/3$.

To give a sense of this in action, we will outline here the proof of Theorem 3.1, part (2), dealing with $\text{LINK} = \text{TWIST} + \text{WRITHE}$ in parallel transport format on S^3 . The proofs for H^3 and for left-translation format on S^3 are essentially the same. In particular, in left-translation format, the integrand of the second integral in the expression for the linking number converges uniformly to the corresponding integrand for the writhing number.

Consider, in parallel transport format on S^3 , the linking integrand of K with K_ε ,

$$F_\varepsilon(s, t) = \frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}(s)\mathbf{y}_\varepsilon(t)} \left(\frac{d\mathbf{y}_\varepsilon}{dt} \times \nabla_{\mathbf{y}_\varepsilon(t)} \varphi(\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(t))) \right)$$

and the writhing integrand of K ,

$$F_0(s, t) = \frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}(s)\mathbf{x}(t)} \left(\frac{d\mathbf{x}}{dt} \times \nabla_{\mathbf{x}(t)} \varphi(\alpha(\mathbf{x}(s), \mathbf{x}(t))) \right),$$

where $\varphi(\alpha) = (\pi - \alpha) \csc \alpha / (4\pi^2)$.

Then the linking number of K and K_ε is given by

$$\text{Lk}(K, K_\varepsilon) = \iint_{0 \leq s, t \leq L} F_\varepsilon(s, t) ds dt,$$

and the writhing number of K is given by

$$\text{Wr}_P(K) = \iint_{0 \leq s, t \leq L} F_0(s, t) ds dt.$$

Since $\alpha(\mathbf{x}, \mathbf{y})$ is the distance between \mathbf{x} and \mathbf{y} , and since φ has a singularity just at $\alpha = 0$, the only difficulty in considering the convergence of the linking integral as $\varepsilon \rightarrow 0$ happens near the diagonal, where $s = t$.

To handle this, we first show that because the singularity of $\varphi''(\alpha)$ at $\alpha = 0$ is like $1/\alpha^3$, we have that for sufficiently small $\varepsilon > 0$,

$$|F_\varepsilon(s, t) - F_0(s, t)| \leq C\varepsilon^{1-3p},$$

provided that $|s - t| \geq \varepsilon^p$.

If $0 < p < 1/3$, then $1 - 3p > 0$, and hence $C\varepsilon^{1-3p} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore $F_\varepsilon(s, t)$ converges uniformly to $F_0(s, t)$ in the region where $|s - t| \geq \varepsilon^p$, and this region expands to the region $|s - t| > 0$ as $\varepsilon \rightarrow 0$. Since the writhing integrand $F_0(s, t)$ remains bounded even along the

diagonal, this shows that

$$\iint_{|s-t|>\varepsilon^p} F_\varepsilon(s, t) ds dt \xrightarrow{\varepsilon \rightarrow 0} \iint_{0 \leq s, t \leq L} F_0(s, t) ds dt = \text{Wr}_P(K),$$

that is, a *portion* of the linking integral converges to the *entire* writhing integral as $\varepsilon \rightarrow 0$. This is the content of Proposition 6.3 below.

The more delicate part of the argument is the integral near the diagonal. A careful analysis reveals that for $0 < p < 1$,

$$\lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon^p}^{t+\varepsilon^p} F_\varepsilon(s, t) ds = \frac{1}{2\pi} \mathbf{x}'(t) \times \mathbf{v}(t) \cdot \mathbf{v}'_P(t).$$

Hence

$$\iint_{|s-t|<\varepsilon^p} F_\varepsilon(s, t) ds dt \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_0^L \mathbf{x}'(t) \times \mathbf{v}(t) \cdot \mathbf{v}'_P(t) dt = \text{Tw}_P(\mathbf{v}).$$

That is, the remaining portion of the linking integral converges to the entire twisting integral. This is the content of Proposition 6.4. In this way, we see that

$$\text{Lk}(K, K_\varepsilon) = \text{Tw}_P(\mathbf{v}) + \text{Wr}_P(K).$$

5. Some geometric formulas on S^3

Before we can proceed with the details of the proof of the equation $\text{LINK} = \text{TWIST} + \text{WRITHE}$, we need to collect some basic geometric formulas on S^3 , which are treated in more detail in our (2008D) paper.

We consider $S^3 \subset R^4$ in the usual way, as the set

$$\{\mathbf{x} \in R^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\},$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the standard inner product on R^4 . Since the linking, twisting, and writhing integrands involve cross products of vectors, we remind the reader that if $\mathbf{x} \in S^3$, and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}S^3$, we define the cross product $\mathbf{v} \times \mathbf{w}$ to be the triple product $[\mathbf{x}, \mathbf{v}, \mathbf{w}]$, where

$$\mathbf{v} \times \mathbf{w} = [\mathbf{x}, \mathbf{v}, \mathbf{w}] = \det \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ v_0 & v_1 & v_2 & v_3 \\ w_0 & w_1 & w_2 & w_3 \\ \hat{\mathbf{x}}_0 & \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \end{bmatrix}.$$

In this formula, we view $\mathbf{x}, \mathbf{v}, \mathbf{w}$, and the result as vectors in R^4 , and $\{\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3\}$ is the canonical orthonormal basis of R^4 . From this, it is easy to see that if \mathbf{u} is also tangent to S^3 at \mathbf{x} , then the triple product $\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ is equal to the value of the 4-by-4 determinant whose rows are $\mathbf{x}, \mathbf{u}, \mathbf{v}$, and \mathbf{w} . We will use the notation $\|\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\|$ for this determinant.

Next, suppose \mathbf{v} is a *unit* vector in $T_{\mathbf{x}}S^3$. Then the unique unit-speed geodesic in S^3 through \mathbf{x} with initial tangent vector \mathbf{v} is given by

$$G(t) = \cos t \mathbf{x} + \sin t \mathbf{v}.$$

Because $\langle \mathbf{x}, \mathbf{v} \rangle = 0$, we have that $\langle \mathbf{x}, G(t) \rangle = \cos t$, and we can conclude in general that the geodesic distance $\alpha(\mathbf{x}, \mathbf{y})$ between two points \mathbf{x} and \mathbf{y} on S^3 is $\alpha(\mathbf{x}, \mathbf{y}) = \arccos \langle \mathbf{x}, \mathbf{y} \rangle$.

Moreover, if \mathbf{x} and \mathbf{y} are any distinct, non-antipodal points on S^3 , then the vector $\mathbf{v} = (\mathbf{y} - \cos \alpha \mathbf{x}) / \sin \alpha$ is a unit vector in $T_{\mathbf{x}}S^3$, and the geodesic it generates connects \mathbf{x} to \mathbf{y} . From this we deduce that

$$\nabla_{\mathbf{x}} \alpha(\mathbf{x}, \mathbf{y}) = \frac{\cos \alpha \mathbf{x} - \mathbf{y}}{\sin \alpha}$$

and

$$\nabla_{\mathbf{y}} \alpha(\mathbf{x}, \mathbf{y}) = \frac{\cos \alpha \mathbf{y} - \mathbf{x}}{\sin \alpha}.$$

We will also need the formula for parallel transport of a vector $\mathbf{v} \in T_{\mathbf{y}}S^3$ to $T_{\mathbf{x}}S^3$:

$$P_{\mathbf{xy}}(\mathbf{v}) = \mathbf{v} - \frac{\langle \mathbf{x}, \mathbf{v} \rangle}{1 + \langle \mathbf{x}, \mathbf{y} \rangle} (\mathbf{x} + \mathbf{y}).$$

Specifically, we need the observation that $P_{\mathbf{xy}}$ affects \mathbf{v} by adding a linear combination of \mathbf{x} and \mathbf{y} .

Using these formulas, we can make precise the definitions of $F_{\varepsilon}(s, t)$ and $F_0(s, t)$ and then derive equivalent expressions for them that will be useful in our proof of LINK = TWIST + WRITHE.

Proposition 5.1. *Let $\mathbf{x}(s)$ be a simple closed curve in S^3 , let $\mathbf{v}(s)$ be a unit vector in $T_{\mathbf{x}(s)}S^3$ which is perpendicular to $\mathbf{x}'(s)$, and define the curve $\mathbf{y}_{\varepsilon}(t)$ by setting $\mathbf{y}_{\varepsilon}(t) = \mathbf{x}(t) \cos \varepsilon + \mathbf{v}(t) \sin \varepsilon$ for each t . Then*

$$\begin{aligned} F_{\varepsilon}(s, t) &= \frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}(s)\mathbf{y}_{\varepsilon}(t)} \left(\frac{d\mathbf{y}_{\varepsilon}}{dt} \times \nabla_{\mathbf{y}_{\varepsilon}(t)} \varphi(\alpha(\mathbf{x}(s), \mathbf{y}_{\varepsilon}(t))) \right) \\ &= -\frac{\varphi'(\alpha_{\varepsilon})}{\sin \alpha_{\varepsilon}} \left\| \mathbf{y}_{\varepsilon}(t), \frac{d\mathbf{y}_{\varepsilon}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\| \end{aligned}$$

using the determinant notation given above and using the shorthand $\alpha_{\varepsilon}(s) = \alpha(\mathbf{x}(s), \mathbf{y}_{\varepsilon}(t))$ for the distance between $\mathbf{x}(s)$ and $\mathbf{y}_{\varepsilon}(t)$. Similarly, we have (for $s \neq t$)

$$\begin{aligned} F_0(s, t) &= \frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}(s)\mathbf{x}(t)} \left(\frac{d\mathbf{x}}{dt} \times \nabla_{\mathbf{x}(t)} \varphi(\alpha(\mathbf{x}(s), \mathbf{x}(t))) \right) \\ &= -\frac{\varphi'(\alpha_0)}{\sin \alpha_0} \left\| \mathbf{x}(t), \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\| \end{aligned}$$

where α_0 is the distance between $\mathbf{x}(s)$ and $\mathbf{x}(t)$.

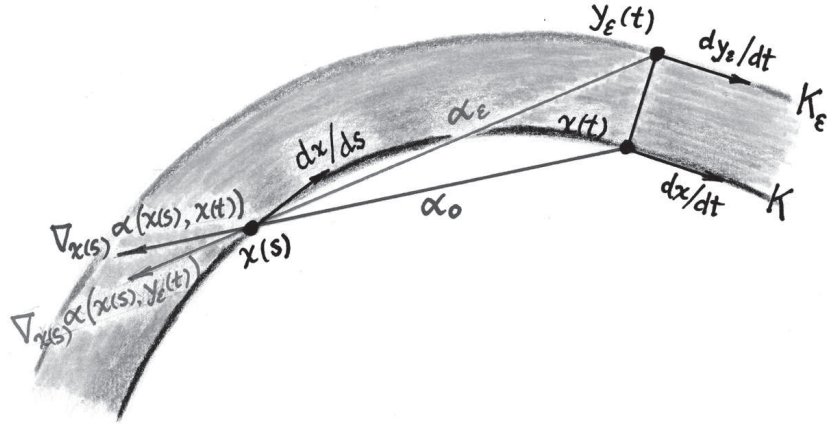


Figure 4. A ribbon, its generating curve, and relevant vectors.

Proof. Using the formulas given above for $\nabla_{\mathbf{y}_\varepsilon} \alpha$ and the cross product, we write

$$\begin{aligned} \frac{d\mathbf{y}_\varepsilon}{dt} \times \nabla_{\mathbf{y}_\varepsilon} \varphi(\alpha(\mathbf{x}, \mathbf{y}_\varepsilon(t))) &= \varphi'(\alpha_\varepsilon) \frac{d\mathbf{y}_\varepsilon}{dt} \times \frac{\cos \alpha_\varepsilon \mathbf{y}_\varepsilon - \mathbf{x}}{\sin \alpha_\varepsilon} \\ &= -\frac{\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \left[\mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt}, \mathbf{x}(s) \right]. \end{aligned}$$

Because the triple product is perpendicular to \mathbf{x} and \mathbf{y}_ε , this vector in R^4 is not changed by $P_{\mathbf{x}\mathbf{y}_\varepsilon}$. Therefore we can express

$$\begin{aligned} F_\varepsilon(s, t) &= \frac{d\mathbf{x}}{ds} \cdot \left(-\frac{\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \left[\mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt}, \mathbf{x}(s) \right] \right) \\ &= -\frac{\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \left\| \mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|. \end{aligned}$$

The proof for F_0 is identical.

q.e.d.

6. Proof of LINK = TWIST + WRITHE on S^3 in parallel transport format

In this section we prove the LINK = TWIST + WRITHE formula in parallel transport format for ribbons in the 3-sphere. As outlined above, the idea is to write the linking integral for the two edges of a ribbon of width ε , and then take its limit as $\varepsilon \rightarrow 0$. Of course the value of the linking integral stays constant, but the limit of the integral is not equal to the integral of the limit of the linking integrand. The latter limit is the writhe of the fixed edge of the ribbon, and the difference is the twist.

To avoid unnecessary complications, we assume all our curves and deformations of curves are smooth, so we are free to differentiate, commute derivatives, etc.

As indicated above, we begin with a smooth, simple closed curve K parametrized by arclength and given by $\mathbf{x}(s)$ for $0 \leq s \leq L = \text{length of } K$. We define our ribbon by letting $\mathbf{v}(s)$ be a *unit* vector, tangent to S^3 at $\mathbf{x}(s)$ and perpendicular to $\mathbf{T}(s) = \mathbf{x}'(s)$ for every s . The other edge of our ribbon of width ε will be at distance ε along the geodesic emanating from \mathbf{x} in the direction of \mathbf{v} , so it is $\mathbf{y}_\varepsilon(s) = \cos \varepsilon \mathbf{x}(s) + \sin \varepsilon \mathbf{v}(s)$. In general, s is not an arclength parameter for the curve $\mathbf{y}_\varepsilon(s)$.

The linking number of the two edges of the ribbon is:

$$\text{Lk}(\mathbf{x}, \mathbf{y}_\varepsilon) = \int_0^L \int_0^L F_\varepsilon(s, t) ds dt,$$

where $F_\varepsilon(s, t)$ is given by either of the expressions in Proposition 5.1.

The linking number is independent of ε , and so our strategy will be to take the limit of the linking integral of the edges of the ribbon of width ε as $\varepsilon \rightarrow 0$. We will examine the difference between the limit of the integral (the linking number) and the integral of the limit (the writhing number), and show that it is equal to the twist of the ribbon as defined earlier. Since the twist is defined by a single integral in contrast to the double integrals that define link and writhe, we'll use the following notation for the “halfway” integrations of the latter two quantities:

$$H_{Lk}(t; \varepsilon) = \int_0^L F_\varepsilon(s, t) ds$$

and

$$H_{Wr}(t) = \int_0^L F_0(s, t) ds.$$

Our objective will be to show that

$$\lim_{\varepsilon \rightarrow 0} H_{Lk}(t; \varepsilon) = H_{Wr}(t) + \text{“something”},$$

where the integral of “something” with respect to t will be the twist of the ribbon.

As we indicated above, the convergence of the linking integrand to the writhing integrand fails to be uniform only near the diagonal of $[0, L] \times [0, L]$, so we'll write

$$H_{Lk}(t) = \int_{|s-t| > \varepsilon^p} \cdots ds + \int_{|s-t| < \varepsilon^p} \cdots ds$$

where p is a number between 0 and 1 to be determined later. We will show that the first term converges to $H_{Wr}(t)$ and the second term will give us our “something”.

Before we can prove that the convergence is uniform away from the diagonal, we need the following preliminary lemma.

Lemma 6.1. *There is a constant $C > 0$ such that the spherical distance $\alpha(\mathbf{x}(s), \mathbf{x}(t)) > C|s - t|$, where we consider $|s - t|$ to be the “distance” on the circle with circumference L .*

Proof. This is true locally (i.e., for s near t) because \mathbf{x} is parametrized by arclength (as we will justify below), and globally by compactness.

To get the local estimate, we use Taylor's formula to write

$$\mathbf{x}(s) = \mathbf{x}(t) + (s-t)\mathbf{x}'(t) + \frac{(s-t)^2}{2}\mathbf{x}''(t) + (s-t)^3\mathbf{A}(s,t)$$

where $\mathbf{A}(s,t)$ is a bounded, smooth vector-valued function of s and t . Since $\mathbf{x}(s)$ lies on the sphere $S^3 \subset R^4$, we have $\langle \mathbf{x}(s), \mathbf{x}'(s) \rangle = 0$, and since \mathbf{x} is parametrized by arclength, we have $\langle \mathbf{x}'(s), \mathbf{x}'(s) \rangle = 1$. It follows that $\langle \mathbf{x}(s), \mathbf{x}''(s) \rangle = -1$, and hence

$$\cos \alpha(\mathbf{x}(s), \mathbf{x}(t)) = \langle \mathbf{x}(s), \mathbf{x}(t) \rangle = 1 - \frac{(s-t)^2}{2} + (s-t)^3 p_1(s,t),$$

where $p_1(s,t)$ is a bounded smooth scalar-valued function of s and t . In what follows, p_i will always stand for such a function without comment. Then clearly

$$\sin^2(\alpha) = 1 - \cos^2(\alpha) = (s-t)^2(1 + (s-t)p_2)$$

and using Taylor's theorem for $(1+z)^{1/2}$ and for $\arcsin(z)$ we conclude that

$$\alpha(\mathbf{x}(s), \mathbf{x}(t)) = |(s-t) + (s-t)^2 p_3|.$$

This is surely larger than $\frac{1}{2}|s-t|$ for $|s-t|$ sufficiently small, say for $|s-t| < \delta$. q.e.d.

Corollary 6.2. $\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(t)) > C'|s-t|$, with C' independent of ε , provided ε is small enough so that the ribbon never touches itself. When $|s-t| > \varepsilon^p$, this implies $\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(t)) > C'\varepsilon^p$.

Again, this is a combination of a local estimate and a global compactness argument.

Now we can begin to analyze the convergence of the linking integral. We start with the part away from the diagonal, which we expect to converge to the writhing integral.

Proposition 6.3. *If $0 < p < 1/3$, then*

$$\lim_{\varepsilon \rightarrow 0} \iint_{|s-t| > \varepsilon^p} F_\varepsilon(s,t) ds dt = \int_0^L \int_0^L F_0(s,t) ds dt = \text{Wr}(K).$$

In other words, the limit of the "away from the diagonal" part of $\text{Lk}(\mathbf{x}, \mathbf{y}_\varepsilon)$ is the integral of $H_{W_r}(\mathbf{x})$, which is the writhing number $\text{Wr}(K)$.

Proof. We need to analyze the difference

$$\begin{aligned} F_\varepsilon(s,t) - F_0(s,t) &= -\frac{\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \left\| \mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\| \\ &\quad + \frac{\varphi'(\alpha_0)}{\sin \alpha_0} \left\| \mathbf{x}(t), \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|, \end{aligned}$$

using the notation of Proposition 5.1.

Using properties of the determinant, we can rewrite the difference $F_\varepsilon - F_0$ as a sum as follows:

$$F_\varepsilon - F_0 = \frac{-\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \left\| \mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt} - \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\| \\ + \left\| \frac{\varphi'(\alpha_0)}{\sin \alpha_0} \mathbf{x}(t) - \frac{\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \mathbf{y}_\varepsilon(t), \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|.$$

We proceed to bound these two summands in terms of ε .

For the first summand of $F_\varepsilon - F_0$,

$$\frac{-\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \left\| \mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt} - \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|,$$

we begin with some easy preliminary observations: $\mathbf{x}(s)$ and $d\mathbf{x}/ds$ are unit vectors, and since $\mathbf{y}_\varepsilon(t) = \cos \varepsilon \mathbf{x}(t) + \sin \varepsilon \mathbf{v}(t)$, we have

$$\frac{d\mathbf{y}_\varepsilon}{dt} - \frac{d\mathbf{x}}{dt} = (\cos \varepsilon - 1) \frac{d\mathbf{x}}{dt} + \sin \varepsilon \frac{d\mathbf{v}}{dt},$$

and so can bound the second vector in the determinant as

$$\left| \frac{d\mathbf{y}_\varepsilon}{dt} - \frac{d\mathbf{x}}{dt} \right| \leq C\varepsilon,$$

where C depends on the maximum value of $|d\mathbf{v}/dt|$.

To handle the first vector in the determinant, we'll group the $1/\sin \alpha_\varepsilon$ with the $\mathbf{y}_\varepsilon(t)$, and then note that the determinant is unaffected if we subtract $(\cos \alpha_\varepsilon / \sin \alpha_\varepsilon) \mathbf{x}(s)$ from the first vector. In other words, the first summand of $F_\varepsilon - F_0$ is equal to

$$-\varphi'(\alpha_\varepsilon) \left\| \frac{1}{\sin \alpha_\varepsilon} \mathbf{y}_\varepsilon(t) - \frac{\cos \alpha_\varepsilon}{\sin \alpha_\varepsilon} \mathbf{x}(s), \frac{d\mathbf{y}_\varepsilon}{dt} - \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|.$$

And since $\cos \alpha_\varepsilon = \langle \mathbf{x}(s), \mathbf{y}_\varepsilon(t) \rangle$, the first vector in this latter determinant is a unit vector. Therefore, the entire determinant is bounded by $|\varphi'(\alpha_\varepsilon)|C\varepsilon$.

Finally, recall from section 1 that we can bound $|\varphi'(\alpha)|$ by a constant divided by α^2 , and since $\alpha_\varepsilon > \varepsilon^p$ by hypothesis, we conclude that the first summand of $F_\varepsilon - F_0$ is bounded by $M\varepsilon^{1-2p}$.

The second summand of $F_\varepsilon - F_0$ is the determinant

$$\left\| \frac{\varphi'(\alpha_0)}{\sin \alpha_0} \mathbf{x}(t) - \frac{\varphi'(\alpha_\varepsilon)}{\sin \alpha_\varepsilon} \mathbf{y}_\varepsilon(t), \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|,$$

and our job will be to handle its first vector, since the other three are all unit vectors.

Since the value of the determinant is unaffected if we replace its first vector with

$$\varphi'(\alpha_0) \left(\frac{1}{\sin \alpha_0} \mathbf{x}(t) - \frac{\cos \alpha_0}{\sin \alpha_0} \mathbf{x}(s) \right) - \varphi'(\alpha_\varepsilon) \left(\frac{1}{\sin \alpha_\varepsilon} \mathbf{y}_\varepsilon(t) - \frac{\cos \alpha_\varepsilon}{\sin \alpha_\varepsilon} \mathbf{x}(s) \right),$$

we will obtain a bound on the determinant by bounding this vector. Using the expressions for $\nabla\alpha$ derived in section 5, we write this as

$$\nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(t))) - \nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{x}(t))),$$

and then rewrite it as

$$\int_0^\varepsilon \frac{d}{d\sigma} \nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))) d\sigma,$$

where $\mathbf{y}_\sigma(t) = \cos \sigma \mathbf{x}(t) + \sin \sigma \mathbf{v}(t)$.

We now calculate and estimate:

$$\begin{aligned} \frac{d}{d\sigma} \nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))) &= \frac{d}{d\sigma} (\varphi'(\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))) \nabla_{\mathbf{x}(s)}\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))) \\ &= I + II, \end{aligned}$$

where

$$I = \varphi''(\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))) \frac{d\alpha}{d\sigma} \nabla_{\mathbf{x}(s)}\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))$$

and

$$II = \varphi'(\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t))) \frac{d}{d\sigma} \nabla_{\mathbf{x}(s)}\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t)).$$

To bound $|I|$, we recall that $|\nabla\alpha| = 1$, $|d\alpha/d\sigma| \leq 1$, and $\varphi''(\alpha) = 1/(2\pi\alpha^3) + \dots$. Moreover, since $\alpha > K'\varepsilon^p > K'\sigma^p$, we get

$$|I| < \frac{Q_1}{\sigma^{3p}}.$$

To bound $|II|$, we have to know more about

$$\begin{aligned} \frac{d}{d\sigma} \nabla_{\mathbf{x}(s)}\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t)) &= \frac{d}{d\sigma} \left(\frac{\cos \alpha_\sigma \mathbf{x} - \mathbf{y}_\sigma}{\sin \alpha_\sigma} \right) \\ &= -\frac{1}{\sin^2 \alpha_\sigma} \frac{d\alpha}{d\sigma} \mathbf{x} - \frac{1}{\sin \alpha_\sigma} \frac{d\mathbf{y}_\sigma}{d\sigma} + \frac{\cos \alpha_\sigma}{\sin^2 \alpha_\sigma} \frac{d\alpha}{d\sigma} \mathbf{y}_\sigma \\ &= -\frac{1}{\sin \alpha_\sigma} \frac{d\mathbf{y}_\sigma}{d\sigma} + \frac{1}{\sin \alpha_\sigma} \frac{d\alpha}{d\sigma} \left(\frac{\cos \alpha_\sigma \mathbf{y}_\sigma - \mathbf{x}}{\sin \alpha_\sigma} \right) \\ &= -\frac{1}{\sin \alpha_\sigma} \frac{d\mathbf{y}_\sigma}{d\sigma} + \frac{1}{\sin \alpha_\sigma} \frac{d\alpha}{d\sigma} \nabla_{\mathbf{y}_\sigma} \alpha. \end{aligned}$$

Once again, we'll use the facts that $|\nabla\alpha| = 1$, $|d\alpha/d\sigma| \leq 1$, and $|d\mathbf{y}_\sigma/d\sigma| = 1$ to conclude that

$$\left| \frac{d}{d\sigma} \nabla_{\mathbf{x}}\alpha(\mathbf{x}, \mathbf{y}_\sigma) \right| \leq \left| \frac{2}{\sin \alpha} \right| < \frac{C}{\alpha}.$$

Finally, we use that $\varphi'(\alpha) = -1/(4\pi\alpha^2) + \dots$, so that $|\varphi'(\alpha)| \leq C'/\alpha^2$, and the fact that $\alpha > K'\varepsilon^p > K'\sigma^p$ to conclude that

$$|II| \leq |\varphi'| \left| \frac{d}{d\sigma} \nabla\alpha \right| \leq \frac{Q_2}{\sigma^{3p}}.$$

Now we've estimated both terms into which we decomposed $(d/d\sigma)\nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{y}_\sigma(t)))$, so we can estimate its integral as σ goes from 0 to ε to obtain the result

$$\begin{aligned} \left| \nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(t))) - \nabla_{\mathbf{x}(s)}\varphi(\alpha(\mathbf{x}(s), \mathbf{x}(t))) \right| &\leq \int_0^\varepsilon \frac{Q_1 + Q_2}{\sigma^{3p}} d\sigma \\ &= Q\varepsilon^{1-3p}. \end{aligned}$$

So far, for $|s - t| > \varepsilon^p$, we have

$$|F_\varepsilon(s, t) - F_0(s, t)| \leq M\varepsilon^{1-2p} + Q\varepsilon^{1-3p}.$$

Therefore

$$\left| \int_{|s-t|>\varepsilon^p} F_\varepsilon(s, t) ds - \int_{|s-t|>\varepsilon^p} F_0(s, t) ds \right| \leq L(M\varepsilon^{1-2p} + Q\varepsilon^{1-3p}).$$

Since

$$\left| \int_{t-\varepsilon^p}^{t+\varepsilon^p} F_0(s, t) ds \right| \leq R\varepsilon^p,$$

we get

$$\left| \int_{|s-t|>\varepsilon^p} F_\varepsilon(s, t) ds - \int_0^L F_0(s, t) ds \right| \leq LM\varepsilon^{1-2p} + LQ\varepsilon^{1-3p} + R\varepsilon^p.$$

So if $0 < p < 1/3$, we can conclude that

$$\lim_{\varepsilon \rightarrow 0} \int_{|s-t|>\varepsilon^p} F_\varepsilon(s, t) ds = H_{Wr}(t)$$

uniformly in t , and so

$$\lim_{\varepsilon \rightarrow 0} \iint_{|s-t|>\varepsilon^p} F_\varepsilon(s, t) ds dt = \int_0^L \int_0^L F_0(s, t) ds dt = \text{Wr}(K).$$

This completes the proof of Proposition 6.3.

q.e.d.

Now we must analyze the part of the linking integral near the diagonal.

Proposition 6.4. *With \mathbf{x} , \mathbf{y}_ε and F_ε defined as above, for $0 < p < 1/3$,*

$$\lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon^p}^{t+\varepsilon^p} F_\varepsilon(s, t) ds = \frac{1}{2\pi} \mathbf{x}'(t) \times \mathbf{v}(t) \cdot \mathbf{v}'_P(t).$$

Proof. To begin, we apply Taylor's theorem to K and write:

$$\mathbf{x}(s) = \mathbf{x}(t) + (s-t)\mathbf{T}(t) + \frac{(s-t)^2}{2}\mathbf{x}''(t) + (s-t)^3\mathbf{A}_1(s, t),$$

where $\mathbf{T}(t) = \mathbf{x}'(t)$ is the (unit) tangent vector to K at $\mathbf{x}(t)$ and $\mathbf{A}_1(s, t)$ is a smooth, bounded (independent of ε) vector-valued function of s and t .

Because the link and writhe integrals (even the partial ones) are invariant under shifting the intervals of integration (i.e., adding different constants mod L to s and t), we may, without loss of generality, assume that $t = 0$. Then we can write:

$$\mathbf{x}(s) = \mathbf{x} + s\mathbf{T} + \frac{s^2}{2}\mathbf{x}'' + s^3\mathbf{A}_1(s)$$

where $\mathbf{x} = \mathbf{x}(0)$, where $\mathbf{T} = \mathbf{x}'(0)$ is the unit tangent vector to K at \mathbf{x} , where $\mathbf{x}'' = \mathbf{x}''(0)$, and where $\mathbf{A}_1(s)$ is a smooth, bounded (independent of ε and uniformly in t) vector-valued function of s . As in Proposition 5.1, we will write $\alpha_\varepsilon(s)$ for $\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(0))$ in what follows.

Similarly, we can write

$$\frac{d\mathbf{x}}{ds} = \mathbf{T} + s\mathbf{x}'' + s^2\mathbf{A}_2(s),$$

and we recall that the other edge of the ribbon and its derivative are given by

$$\mathbf{y}_\varepsilon(0) = \cos \varepsilon \mathbf{x} + \sin \varepsilon \mathbf{v}$$

and

$$\left. \frac{d\mathbf{y}_\varepsilon}{dt} \right|_{t=0} = \cos \varepsilon \mathbf{T} + \sin \varepsilon \mathbf{v}',$$

where $\mathbf{v} = \mathbf{v}(0)$ and $\mathbf{v}' = \mathbf{v}'(0)$. Because we are differentiating \mathbf{v} as though it were a vector field in R^4 , the derivative here coincides with the covariant derivative on S^3 , rather than the left-invariant derivative of section 4. Here and for the remainder of this section, until the statement of the theorem, we will omit the subscript in the notation \mathbf{v}'_P .

Using the notation of Proposition 5.1, we can express

$$F_\varepsilon(s, 0) = -\frac{\varphi'(\alpha_\varepsilon(s))}{\sin \alpha_\varepsilon(s)} \left\| \mathbf{y}_\varepsilon(0), \frac{d\mathbf{y}_\varepsilon}{dt}(0), \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|$$

as $-\varphi'(\alpha_\varepsilon(s))/\sin \alpha_\varepsilon(s)$ times the determinant

$$\left\| \begin{array}{c} \cos \varepsilon \mathbf{x} + \sin \varepsilon \mathbf{v}, \cos \varepsilon \mathbf{T} + \sin \varepsilon \mathbf{v}', \\ \mathbf{x} + s\mathbf{T} + \frac{s^2}{2}\mathbf{x}'' + s^3\mathbf{A}_1, \mathbf{T} + s\mathbf{x}'' + s^2\mathbf{A}_2 \end{array} \right\|.$$

We proceed to analyze the factor $\varphi'(\alpha_\varepsilon(s))/\sin \alpha_\varepsilon(s)$ in front of the determinant, and the following four terms, into which the determinant

can be expanded:

$$\begin{aligned}
I &= \left\| \cos \varepsilon \mathbf{x}, \cos \varepsilon \mathbf{T}, \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, s \mathbf{x}'' + s^2 \mathbf{A}_2 \right\| \\
II &= \left\| \cos \varepsilon \mathbf{x}, \sin \varepsilon \mathbf{v}', s \mathbf{T} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, \mathbf{T} + s \mathbf{x}'' + s^2 \mathbf{A}_2 \right\| \\
III &= \left\| \sin \varepsilon \mathbf{v}, \cos \varepsilon \mathbf{T}, \mathbf{x} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, s \mathbf{x}'' + s^2 \mathbf{A}_2 \right\| \\
IV &= \left\| \sin \varepsilon \mathbf{v}, \sin \varepsilon \mathbf{v}', \mathbf{x} + s \mathbf{T} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, \mathbf{T} + s \mathbf{x}'' + s^2 \mathbf{A}_2 \right\|.
\end{aligned}$$

First, we derive an expansion of $\varphi'(\alpha_\varepsilon(s))/\sin \alpha_\varepsilon(s)$ in powers of s and ε . To begin, since $\mathbf{x}(s)$ is a curve on S^3 and is parametrized by arclength, we have $\langle \mathbf{x}, \mathbf{x} \rangle = 1$, $\langle \mathbf{x}, \mathbf{T} \rangle = 0$, $\langle \mathbf{T}, \mathbf{T} \rangle = 1$, and $\langle \mathbf{x}, \mathbf{x}'' \rangle = -1$ (the last equation comes from differentiating $\langle \mathbf{x}, \mathbf{T} \rangle = 0$). Using these observations, we derive

$$\begin{aligned}
\cos \alpha_\varepsilon(s) &= \langle \mathbf{x}(s), \mathbf{y}_\varepsilon(0) \rangle \\
&= \langle \mathbf{x} + s \mathbf{T} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, \cos \varepsilon \mathbf{x} + \sin \varepsilon \mathbf{v} \rangle \\
&= \cos \varepsilon - \frac{s^2}{2} \cos \varepsilon + \frac{s^2}{2} \sin \varepsilon \langle \mathbf{x}'', \mathbf{v} \rangle + s^3 p_0 \\
&= 1 - \frac{\varepsilon^2 + s^2}{2} + s^3 p_1 + s^2 \varepsilon p_2 + s \varepsilon^2 p_3 + \varepsilon^3 p_4
\end{aligned}$$

where, as before, p_i stands for a function of s and ε that is bounded for all s and ε , and smooth except perhaps for $s = \varepsilon = 0$.

Since $\sin^2 \alpha = 1 - \cos^2 \alpha$, we can conclude that

$$\begin{aligned}
\sin^2 \alpha_\varepsilon(s) &= \varepsilon^2 + s^2 + s^3 p_5 + s^2 \varepsilon p_6 + s \varepsilon^2 p_7 + \varepsilon^3 p_8 \\
&= (\varepsilon^2 + s^2) \left(1 + \frac{s^3}{\varepsilon^2 + s^2} p_5 + \frac{s^2 \varepsilon}{\varepsilon^2 + s^2} p_6 + \frac{s \varepsilon^2}{\varepsilon^2 + s^2} p_7 + \frac{\varepsilon^3}{\varepsilon^2 + s^2} p_8 \right).
\end{aligned}$$

Using the Taylor series $\sqrt{1+z} = 1 + \frac{1}{2}z + \dots$ we can conclude that

$$\sin \alpha_\varepsilon(s) = (\varepsilon^2 + s^2)^{1/2} \left(1 + \frac{s^3}{\varepsilon^2 + s^2} p_9 + \frac{s^2 \varepsilon}{\varepsilon^2 + s^2} p_{10} + \frac{s \varepsilon^2}{\varepsilon^2 + s^2} p_{11} + \frac{\varepsilon^3}{\varepsilon^2 + s^2} p_{12} \right).$$

Using the Taylor series $\arcsin z = z + \dots$ we can conclude that

$$\alpha = (\varepsilon^2 + s^2)^{1/2} \left(1 + \frac{s^3}{\varepsilon^2 + s^2} p_{13} + \frac{s^2 \varepsilon}{\varepsilon^2 + s^2} p_{14} + \frac{s \varepsilon^2}{\varepsilon^2 + s^2} p_{15} + \frac{\varepsilon^3}{\varepsilon^2 + s^2} p_{16} \right).$$

We combine this with the expansion of $\varphi'(\alpha) = -1/(4\pi\alpha^2) +$ something bounded so that

$$\varphi'(\alpha_\varepsilon(s)) = -\frac{1}{4\pi(\varepsilon^2 + s^2)} + p_{17},$$

and finally conclude that

$$\begin{aligned} \frac{\varphi'(\alpha_\varepsilon(s))}{\sin \alpha_\varepsilon(s)} &= \left(\frac{-1}{4\pi(\varepsilon^2 + s^2)} + p_{17} \right) \left(\frac{1}{(\varepsilon^2 + s^2)^{1/2}} \right) (1 + sp_{18} + \varepsilon p_{19}) \\ &= \frac{-1}{4\pi(\varepsilon^2 + s^2)^{3/2}} (1 + sp_{20} + \varepsilon p_{21}). \end{aligned}$$

The utility of this expression for $\varphi'(\alpha_\varepsilon(s))/\sin \alpha_\varepsilon(s)$ will become apparent when we multiply it by the determinants, integrate from $-\varepsilon^p$ to ε^p , and then take the limit as $\varepsilon \rightarrow 0$. Because $(\varepsilon^2 + s^2)^{1/2}$ is larger than either s or ε , we can see that whenever $a + b \geq 3$, the product of $\varphi'(\alpha_\varepsilon(s))/\sin \alpha_\varepsilon(s)$ with $s^a \varepsilon^b$ will integrate to something comparable to ε^p , and the integral will go to zero as ε does.

Next, we will use the observation about $(\varepsilon^2 + s^2)^{1/2}$ from the preceding paragraph to deal with the four determinants. The first one,

$$I = \left\| \cos \varepsilon \mathbf{x}, \cos \varepsilon \mathbf{T}, \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, s\kappa \mathbf{N} + s^2 \mathbf{A}_2 \right\|,$$

clearly has a factor of s^3 , so it will not contribute to our limit. Similarly, the second one,

$$II = \left\| \cos \varepsilon \mathbf{x}, \sin \varepsilon \mathbf{v}', s\mathbf{T} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, \mathbf{T} + s\kappa \mathbf{N} + s^2 \mathbf{A}_2 \right\|,$$

has a factor of $s^2 \sin \varepsilon$ (since you can't use the \mathbf{T} from both the third and fourth rows), and so II doesn't contribute to our limit, either.

Using the expansion $\cos \varepsilon = 1 - \varepsilon^2/2 + \dots$, we can express the third determinant,

$$III = \left\| \sin \varepsilon \mathbf{v}, \cos \varepsilon \mathbf{T}, \mathbf{x} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, s\mathbf{x}'' + s^2 \mathbf{A}_2 \right\|,$$

as the sum of two terms:

$$s \sin \varepsilon \left\| \mathbf{v}, \mathbf{T}, \mathbf{x}, \mathbf{x}'' \right\| + s^2 \sin \varepsilon p_{22},$$

from which only the first term could contribute to our limit.

Finally, the fourth determinant,

$$IV = \left\| \sin \varepsilon \mathbf{v}, \sin \varepsilon \mathbf{v}', \mathbf{x} + s\mathbf{T} + \frac{s^2}{2} \mathbf{x}'' + s^3 \mathbf{A}_1, \mathbf{T} + s\mathbf{x}'' + s^2 \mathbf{A}_2 \right\|,$$

can be decomposed as

$$\sin^2 \varepsilon \left\| \mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{T} \right\| + s \sin^2 \varepsilon p_{23},$$

from which only the first term could contribute to our limit.

From our analysis so far, we conclude that

$$F_\varepsilon(s, 0) = \frac{1}{4\pi(\varepsilon^2 + s^2)^{3/2}} (s\varepsilon \left\| \mathbf{v}, \mathbf{T}, \mathbf{x}, \mathbf{x}'' \right\| + \varepsilon^2 \left\| \mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{T} \right\| + Z(\varepsilon, s)),$$

where

$$\lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon^p}^{\varepsilon^p} \frac{Z(\varepsilon, s)}{(\varepsilon^2 + s^2)^{3/2}} ds = 0.$$

We are now ready to calculate the limit of the integral:

$$\lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon^p}^{\varepsilon^p} F_\varepsilon(s, 0) ds.$$

From the formula for the integrand given above, this limit will equal the limit as $\varepsilon \rightarrow 0$ of

$$\|\mathbf{v}, \mathbf{T}, \mathbf{x}, \mathbf{x}''\| \int_{-\varepsilon^p}^{\varepsilon^p} \frac{s\varepsilon}{4\pi(\varepsilon^2 + s^2)^{3/2}} ds + \|\mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{T}\| \int_{-\varepsilon^p}^{\varepsilon^p} \frac{\varepsilon^2}{4\pi(\varepsilon^2 + s^2)^{3/2}} ds.$$

The integrand in the first of these integrals is odd, so the integral is always zero (and hence the limit of that term is zero). For the second term, we will need the fact that (for $p < 1$)

$$\lim_{\varepsilon \rightarrow 0} \int_{-\varepsilon^p}^{\varepsilon^p} \frac{\varepsilon^2}{(\varepsilon^2 + s^2)^{3/2}} ds = 2,$$

which one calculates using the substitution $x = s/\varepsilon$ and the fact that the anti-derivative of $1/(1+x^2)^{3/2}$ is $x/\sqrt{1+x^2}$.

We have thus reached our final conclusion, namely that

$$\lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon^p}^{t+\varepsilon^p} F_\varepsilon(s, t) ds = \frac{1}{2\pi} \|\mathbf{v}, \mathbf{v}', \mathbf{x}, \mathbf{T}\| = \frac{1}{2\pi} (\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}').$$

This completes the proof of Proposition 6.4.

q.e.d.

We can use Propositions 6.3 and 6.4 and a little arithmetic to start from the definitions of $H_{Lk}(t; \varepsilon)$ and $H_{Wr}(t)$ and deduce:

Proposition 6.5.

$$\lim_{\varepsilon \rightarrow 0} H_{Lk}(t; \varepsilon) = H_{Wr}(t) + \frac{1}{2\pi} (\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}').$$

We integrate the expression in Proposition 6.5 with respect to t from 0 to L to reach our final conclusion:

Theorem 6.6.

$$\text{Lk}(\mathbf{x}, \mathbf{y}) = \text{Tw}_P(\mathbf{x}, \mathbf{v}) + \text{Wr}_P(\mathbf{x}).$$

In other words, LINK = TWIST + WRITHE.

Example. The simplest example of two linked curves on S^3 is a pair of great circles from the same Hopf fibration. We verify Theorem 6.6 in this case. The curve $\mathbf{x}(s) = [\cos s, \sin s, 0, 0]$ is a great circle parametrized by arclength as s runs from 0 to $L = 2\pi$. We will take \mathbf{x} as one edge of our ribbon.

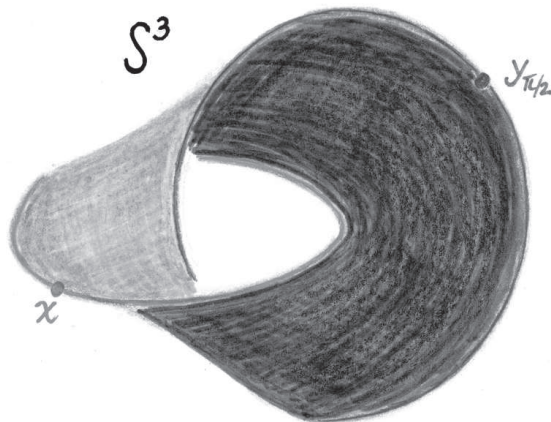


Figure 5. The ribbon in the example.

Let $\mathbf{v}(t) = [0, 0, \cos t, \sin t]$. Then \mathbf{v} is the restriction of a left-invariant vector field to the great circle, and we will take the other edge of our ribbon to be

$$\begin{aligned} \mathbf{y}_\varepsilon(t) &= \cos \varepsilon \mathbf{x}(t) + \sin \varepsilon \mathbf{v}(t) \\ &= [\cos \varepsilon \cos t, \cos \varepsilon \sin t, \sin \varepsilon \cos t, \sin \varepsilon \sin t]. \end{aligned}$$

If $\varepsilon = \pi/2$, then $\mathbf{y}_\varepsilon(t) = [0, 0, \cos t, \sin t]$, which is the “orthogonal” great circle to \mathbf{x} , and we compute the linking number of these two circles as follows. Since $\langle \mathbf{x}(s), \mathbf{y}_\varepsilon(t) \rangle = 0$ for all s and t , we have that $\alpha(\mathbf{x}(s), \mathbf{y}_\varepsilon(t)) = \pi/2$ for all s and t . Therefore, the linking integrand is given by

$$\begin{aligned} \frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}\mathbf{y}_\varepsilon} \left(\frac{d\mathbf{y}_\varepsilon}{dt} \times \nabla_{\mathbf{y}_\varepsilon} \varphi \right) &= \frac{-\varphi'(\alpha)}{\sin \alpha} \left\| \mathbf{y}_\varepsilon(t), \frac{d\mathbf{y}_\varepsilon}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\| \\ &= \frac{-\varphi'(\frac{\pi}{2})}{\sin \frac{\pi}{2}} \begin{vmatrix} 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \\ \cos s & \sin s & 0 & 0 \\ -\sin s & \cos s & 0 & 0 \end{vmatrix} \\ &= \frac{1}{4\pi^2} \end{aligned}$$

The integration takes place for $(s, t) \in [0, 2\pi] \times [0, 2\pi]$, so the formula for the linking number of \mathbf{x} and \mathbf{y}_ε yields 1, as expected.

To calculate the twist of our ribbon, we note that $\mathbf{T}(s) = \mathbf{x}'(s) = [-\sin s, \cos s, 0, 0]$, and $\mathbf{v}'(s) = [0, 0, -\sin s, \cos s]$. It is then easy to calculate that $\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}' = [\mathbf{x}, \mathbf{T}, \mathbf{v}, \mathbf{v}'] = 1$ for all s , which gives us that the twist of the ribbon is

$$\text{Tw}(\mathbf{x}, \mathbf{v}) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{T}(t) \times \mathbf{v}(t) \cdot \mathbf{v}'(t) dt = \frac{1}{2\pi} \int_0^{2\pi} 1 dt = 1.$$

To calculate the writhe of \mathbf{x} , we use the fact that \mathbf{x} is a geodesic, and so we have $P_{\mathbf{x}(s)\mathbf{x}(t)}\mathbf{T}(t) = \mathbf{T}(s)$. From this it is easy to conclude that

$$\frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}(s)\mathbf{x}(t)} \left(\frac{d\mathbf{x}}{dt} \times \nabla_{\mathbf{x}(t)}\varphi(\alpha(\mathbf{x}(s), \mathbf{x}(t))) \right) = 0$$

for all s and t . Therefore

$$\text{Wr}(\mathbf{x}) = 0.$$

Theorem 6.6 then reads

$$\text{Lk}(\mathbf{x}, \mathbf{y}_\varepsilon) = \text{Tw}(\mathbf{x}, \mathbf{v}) + \text{Wr}(\mathbf{x}) = 1 + 0 = 1$$

as it should.

7. Proof of LINK = TWIST + WRITHE in H^3

The proof of LINK = TWIST + WRITHE in H^3 is essentially a repetition of the parallel transport format proof in S^3 , except for various changes of sign and for replacing trigonometric functions with their corresponding hyperbolic ones. In this section, we highlight the places where differences occur.

As in the first paper in this series, we view $H^3 \subset R^{1,3}$, the four-dimensional Minkowski space endowed with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$$

so that

$$H^3 = \{\mathbf{x} \in R^4 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1 \text{ and } x_0 > 0\}.$$

We reserve the notation $\mathbf{v} \cdot \mathbf{w}$ for the induced inner product on H^3 ; namely for $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}H^3$, we define $\mathbf{v} \cdot \mathbf{w} = -\langle \mathbf{v}, \mathbf{w} \rangle$. Because the tangent vectors are spacelike, this inner product provides H^3 with a Riemannian metric which is complete and has constant curvature -1 .

If $\mathbf{x} \in H^3$, and $\mathbf{u}, \mathbf{v} \in T_{\mathbf{x}}H^3$, then we have

$$\mathbf{u} \times \mathbf{v} = \det \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \\ -\hat{\mathbf{x}}_0 & \hat{\mathbf{x}}_1 & \hat{\mathbf{x}}_2 & \hat{\mathbf{x}}_3 \end{vmatrix}.$$

Then for $\mathbf{w} \in T_{\mathbf{x}}H^3$, the triple product $\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \|\mathbf{x}, \mathbf{u}, \mathbf{v}, \mathbf{w}\|$.

For geodesics and the distance function, we will have

$$G(t) = \cosh t \mathbf{x} + \sinh t \mathbf{v}$$

for the unit-speed geodesic through \mathbf{x} in the direction of $\mathbf{v} \in T_{\mathbf{x}}H^3$, and the geodesic distance between \mathbf{x} and \mathbf{y} in H^3 will satisfy $\cosh \alpha = \langle \mathbf{x}, \mathbf{y} \rangle$. We have

$$\nabla_{\mathbf{y}}\alpha(\mathbf{x}, \mathbf{y}) = \frac{\cosh \alpha \mathbf{y} - \mathbf{x}}{\sinh \alpha}.$$

Except for the change in the inner product, the formula for parallel transport remains the same: the result of parallel transport in H^3 of \mathbf{v} from \mathbf{y} to \mathbf{x} is

$$P_{\mathbf{xy}}(\mathbf{v}) = \mathbf{v} - \frac{\langle \mathbf{x}, \mathbf{v} \rangle}{1 + \langle \mathbf{x}, \mathbf{y} \rangle} (\mathbf{x} + \mathbf{y}).$$

Armed with these changes, and with $\varphi(\alpha) = \operatorname{csch}(\alpha)/(4\pi)$, the proofs of Lemma 6.1, Corollary 6.2, Proposition 6.3 (where the biggest change is to have $\sinh \alpha_\varepsilon$ rather than $\sin \alpha_\varepsilon$ in the denominator), and Proposition 6.4 proceed in the hyperbolic space case essentially without change from the spherical case.

We are then led to the conclusion of Proposition 6.5,

$$\lim_{\varepsilon \rightarrow 0} H_{Lk}(t; \varepsilon) = H_{Wr}(t) + \frac{1}{2\pi} (\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}').$$

And once again, we define the writhe of the \mathbf{x} edge of our ribbon as

$$Wr(\mathbf{x}) = \int_0^L \int_0^L \frac{d\mathbf{x}}{ds} \cdot P_{\mathbf{x}(s)\mathbf{x}(t)} \left(\frac{d\mathbf{x}}{dt} \times \nabla_{\mathbf{x}(t)} \varphi(\alpha(\mathbf{x}(s), \mathbf{x}(t))) \right) ds dt,$$

and the twist of our ribbon as

$$Tw(\mathbf{x}, \mathbf{v}) = \frac{1}{2\pi} \int_0^L (\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}') dt.$$

Finally, we integrate the expressions from the hyperbolic version of Proposition 6.5 with respect to t from 0 to L to reach our final conclusion:

Theorem 7.1.

$$\operatorname{Lk}(\mathbf{x}, \mathbf{y}_\varepsilon) = \operatorname{Tw}(\mathbf{x}, \mathbf{v}) + \operatorname{Wr}(\mathbf{x}).$$

In other words, LINK = TWIST + WRITHE.

Example. A simple example of a ribbon in $H^3 \subset R^{1,3}$ has as one edge the circle

$$\mathbf{x}(s) = [\sqrt{2}, \cos s, \sin s, 0]$$

in $R^{1,3}$. The unit tangent vector to this curve is

$$\mathbf{T} = [0, -\sin s, \cos s, 0],$$

and we can choose the vector field

$$\mathbf{v}(s) = \frac{1}{\sqrt{1 - \frac{1}{2} \cos^2 s}} \left[\frac{\cos s}{\sqrt{2}}, \cos^2 s, \cos s \sin s, \sin s \right]$$

along \mathbf{x} . Clearly we have $\langle \mathbf{x}'(s), \mathbf{v}(s) \rangle = \langle \mathbf{T}(s), \mathbf{v}(s) \rangle = 0$ for all s , and $\langle \mathbf{v}(s), \mathbf{v}(s) \rangle = -1$, so \mathbf{v} is a unit vector perpendicular to T . We can make the ribbon by choosing the other edge to be the curve given by $\mathbf{y}_\varepsilon(s) = \cosh \varepsilon \mathbf{x}(s) + \sinh \varepsilon \mathbf{v}(s)$.

By looking at the projections of the \mathbf{x} and \mathbf{y}_ε curves into R^3 (ignoring the first coordinates), it's easy to see that these curves have linking

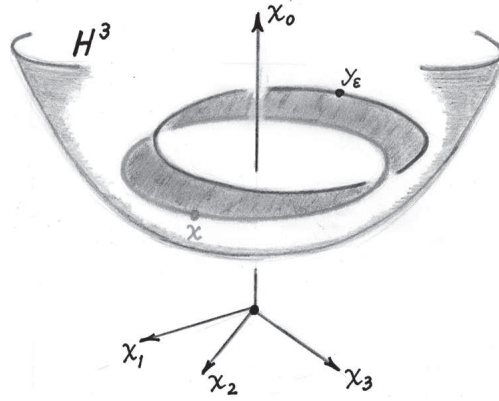


Figure 6. The ribbon in the example.

number -1 . The writhing integrand of the \mathbf{x} curve is easily seen to be zero, since the writhing integrand is given by

$$-\frac{\varphi'(\alpha)}{\sinh \alpha} \left\| \mathbf{x}(t), \frac{d\mathbf{x}}{dt}, \mathbf{x}(s), \frac{d\mathbf{x}}{ds} \right\|,$$

and the determinant is zero because the last component of each vector in the determinant is zero.

For the twist of the ribbon, we must calculate $\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}'$, which is given by the determinant

$$\left\| \mathbf{x}(s), \mathbf{v}(s), \frac{d\mathbf{v}}{ds}, \mathbf{T}(s) \right\|,$$

and calculating this determinant yields

$$\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}' = \frac{\sqrt{2}}{\cos^2 s - 2}.$$

So we can calculate that

$$\begin{aligned} \text{Tw}(\mathbf{x}, \mathbf{v}) &= \frac{1}{2\pi} \int_0^{2\pi} (\mathbf{T} \times \mathbf{v} \cdot \mathbf{v}') ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{2}}{\cos^2 s - 2} ds = -1. \end{aligned}$$

Theorem 7.1 then reads

$$\text{Lk}(\mathbf{x}, \mathbf{y}_\varepsilon) = \text{Tw}(\mathbf{x}, \mathbf{v}) + \text{Wr}(\mathbf{x}) = -1 + 0 = -1$$

as it should.

8. Helicity of vector fields on S^3 and H^3

Lodewijk Woltjer introduced in 1958 the notion of “helicity” of a vector field \mathbf{v} defined on a domain Ω in Euclidean 3-space,

$$H(\mathbf{v}) = \int_{\Omega \times \Omega} \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{4\pi|\mathbf{x} - \mathbf{y}|^3} d\mathbf{x} d\mathbf{y}, \quad (8.1)$$

as an invariant during ideal magnetohydrodynamic evolution of plasma fields. Keith Moffatt (1969M), recognizing that this quantity measures the extent to which the field lines of \mathbf{v} wrap and coil around one another, named it “helicity” and showed that Woltjer’s original formula could be written in the above form.

If \mathbf{v} is a smooth vector field on R^3 with compact support, then the above formula for its helicity can be written succinctly as

$$H(\mathbf{v}) = \int_{R^3} \text{BS}(\mathbf{v})(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) d\mathbf{y}, \quad (8.2)$$

where we recall that $\text{BS}(\mathbf{v})$ denotes the magnetic field due to the steady current flow \mathbf{v} .

This is how Woltjer originally presented his invariant, $\int \mathbf{A} \cdot \mathbf{B} d\mathbf{x}$, with the role of \mathbf{v} played by the magnetic field \mathbf{B} and the role of $\text{BS}(\mathbf{v})$ played by its vector potential \mathbf{A} .

We use (8.2) to define the helicity of a vector field \mathbf{v} on S^3 or H^3 , and then immediately obtain explicit integral formulas from Theorem 2.1.

Theorem 8.3. Helicity integrals in S^3 and H^3 .

(1) On S^3 , in left-translation format:

$$\begin{aligned} H(\mathbf{v}) &= \int_{S^3 \times S^3} L_{\mathbf{y}\mathbf{x}^{-1}} \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} \\ &\quad - \frac{1}{4\pi^2} \int_{S^3 \times S^3} L_{\mathbf{y}\mathbf{x}^{-1}} \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{y}) d\mathbf{x} d\mathbf{y} \\ &\quad + 2 \int_{S^3 \times S^3} \nabla_{\mathbf{y}} (L_{\mathbf{y}\mathbf{x}^{-1}} \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathbf{y}} \varphi_1(\mathbf{x}, \mathbf{y})) \cdot \mathbf{v}(\mathbf{y}) d\mathbf{x} d\mathbf{y}, \end{aligned}$$

where $\varphi(\alpha) = (\pi - \alpha) \cot \alpha / (4\pi^2)$ and $\varphi_1(\alpha) = -\alpha(2\pi - \alpha) / (16\pi^2)$.

(2) On S^3 in parallel transport format:

$$H(\mathbf{v}) = \int_{S^3 \times S^3} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y},$$

where $\varphi(\alpha) = (\pi - \alpha) \csc \alpha / (4\pi^2)$.

(3) On H^3 in parallel transport format:

$$H(\mathbf{v}) = \int_{H^3 \times H^3} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \varphi(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$$

where $\varphi(\alpha) = \text{csch } \alpha / (4\pi)$.

In formula (1), if \mathbf{v} is divergence-free, then the third integral in the definition of $H(\mathbf{v})$ vanishes, and this formula then resembles the linking formula (1) of Theorem 1.1. Formulas (2) and (3) already resemble the corresponding linking formulas of Theorem 1.1.

In formulas (1) and (2), if the smooth vector field \mathbf{v} on S^3 is divergence-free, then its helicity is the same as its *asymptotic* (or *mean*) *Hopf invariant*, as defined by Arnold (1974), and is invariant under the group of volume-preserving diffeomorphisms of S^3 .

In formula (3), we assume that \mathbf{v} has compact support in order to guarantee convergence of the integral.

9. Upper bounds for helicity in R^3 , S^3 , and H^3

Let Ω be a compact, smoothly bounded subdomain of R^3 , S^3 , or H^3 , and let \mathbf{v} be a smooth vector field defined on Ω . Thinking of \mathbf{v} as a current flow, its magnetic field $BS(\mathbf{v})$ is defined by the same formulas as in Theorem 2.1, except that the integration is carried out only over Ω .

For uniformity of approach, we ignore the left-translation format on S^3 and write

$$BS(\mathbf{v})(\mathbf{y}) = \int_{\Omega} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) \, d\mathbf{x}, \quad (9.1)$$

where

$$\text{in } R^3 \quad \text{we have } \varphi_0(\alpha) = -\frac{1}{4\pi\alpha} \quad \text{so } \Delta\varphi_0 = \delta;$$

$$\text{in } S^3 \quad \text{we have } \varphi_0(\alpha) = -\frac{1}{4\pi^2}(\pi - \alpha) \csc \alpha \quad \text{so } \Delta\varphi_0 - \varphi_0 = \delta;$$

$$\text{in } H^3 \quad \text{we have } \varphi_0(\alpha) = -\frac{1}{4\pi} \operatorname{csch} \alpha \quad \text{so } \Delta\varphi_0 + \varphi_0 = \delta.$$

The magnetic field $BS(\mathbf{v})$ is defined throughout the ambient space. It is continuous everywhere, but its first derivatives suffer a discontinuity as one crosses the boundary of Ω . This is a familiar situation from electrodynamics in Euclidean 3-space.

In what follows, we will restrict $BS(\mathbf{v})$ to Ω , and ignore its behavior outside this domain.

Let $VF(\Omega)$ denote the space of all smooth vector fields on Ω , with the L^2 inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \int_{\Omega} \mathbf{v} \cdot \mathbf{w} \, d \operatorname{vol},$$

and associated *energy* $\langle \mathbf{v}, \mathbf{v} \rangle$ and *norm* $|\mathbf{v}| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$.

We seek a bound for the energy or norm of the output magnetic field $BS(\mathbf{v})$ on Ω in terms of the input current flow \mathbf{v} . Or to put it

another way, we seek an upper bound for the L^2 -operator norm of the Biot-Savart operator,

$$\text{BS}: \text{VF}(\Omega) \rightarrow \text{VF}(\Omega),$$

in terms of the geometry of the underlying domain Ω .

As a consequence, we will determine an upper bound for the helicity $\mathbb{H}(\mathbf{v}) = \langle \text{BS}(\mathbf{v}), \mathbf{v} \rangle$ of the vector field \mathbf{v} in terms of its energy $\langle \mathbf{v}, \mathbf{v} \rangle$ and the geometry of Ω .

Theorem 9.2. *Let Ω be a compact, smoothly bounded subdomain of R^3 , S^3 , or H^3 and let $R = R(\Omega)$ be the radius of a ball in that space having the same volume as Ω . Let \mathbf{v} be a smooth vector field defined on Ω . Then*

$$|\text{BS}(\mathbf{v})| \leq N(R)|\mathbf{v}|,$$

where

- in R^3 we have $N(R) = R$
- in S^3 we have $N(R) = \frac{1}{\pi}(2(1 - \cos R) + (\pi - R) \sin R)$
- in H^3 we have $N(R) = \sinh R$.

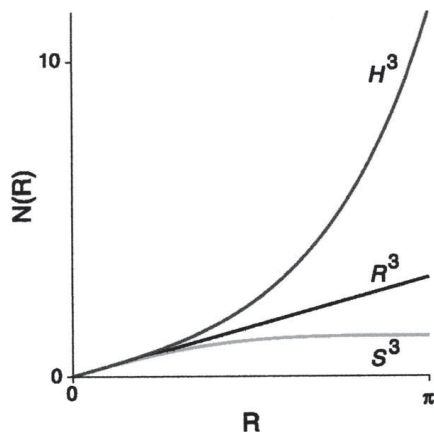


Figure 7. $N(R)$ for H^3 , R^3 , and S^3 .

It follows immediately that the helicity $\mathbb{H}(\mathbf{v}) = \langle \text{BS}(\mathbf{v}), \mathbf{v} \rangle$ is bounded by

$$|\mathbb{H}(\mathbf{v})| \leq N(R)|\mathbf{v}|^2.$$

In R^3 , the overestimate $N(R) = R$ for the norm of the Biot-Savart operator grows like the *cube root* of the volume $\frac{4}{3}\pi R^3$ of Ω .

By contrast, in H^3 the overestimate $N(R) = \sinh R$ for the norm of the Biot-Savart operator grows like the *square root* of the volume $2\pi(\sinh R \cosh R - R)$ of Ω .

Setting up for the proof of Theorem 9.2.

To begin, let $\psi(\alpha)$ be a real-valued function of the real variable $\alpha > 0$, where we think of α as distance from a fixed point (and on S^3 , we have the additional condition $0 < \alpha \leq \pi$). Assume ψ has the property that

$$N_\Omega(\psi) := \max_{\mathbf{y}} \int_{\mathbf{x} \in \Omega} |\psi(\mathbf{x}, \mathbf{y})| d\mathbf{x}$$

is finite, where as usual we write $\psi(\mathbf{x}, \mathbf{y})$ as an abbreviation for $\psi(\alpha(\mathbf{x}, \mathbf{y}))$. We note explicitly that the point \mathbf{y} need *not* be chosen in Ω .

Proposition 9.3. *Under the above circumstances, the operator*

$$T_\psi : \text{VF}(\Omega) \rightarrow \text{VF}(\Omega)$$

defined by

$$T_\psi(\mathbf{v})(\mathbf{y}) = \int_{\Omega} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \psi(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{x}$$

is a bounded operator with respect to the L^2 -norm, and

$$|T_\psi(\mathbf{v})| \leq N_\Omega(\psi) |\mathbf{v}|.$$

The proof of this proposition in the R^3 case can be found in our (2001) paper, Lemma 3 on pages 897 and 898. The argument there follows along the lines of the usual Young's inequality proof that convolution operators on spaces of scalar-valued functions are bounded; see Folland (1995) page 9, or Zimmer (1990Z) Proposition B.3 on page 10. The proof carries over to the S^3 and H^3 cases with virtually no changes.

We want to apply Proposition 9.3 to the Biot-Savart operator (9.1), which we write as

$$\text{BS}(\mathbf{v})(\mathbf{y}) = \int_{\Omega} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \varphi'_0(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} \alpha(\mathbf{x}, \mathbf{y}) d\mathbf{x} = T_{\varphi'_0}(\mathbf{v})(\mathbf{y}), \quad (9.4)$$

where

$$\begin{aligned} \text{in } R^3 \quad & \text{we have } \varphi'_0(\alpha) = \frac{1}{4\pi\alpha^2} \\ \text{in } S^3 \quad & \text{we have } \varphi'_0(\alpha) = \frac{1}{4\pi^2} (\csc \alpha + (\pi - \alpha) \csc \alpha \cot \alpha) \\ \text{in } H^3 \quad & \text{we have } \varphi'_0(\alpha) = \frac{1}{4\pi} \text{csch } \alpha \coth \alpha. \end{aligned}$$

Then by Proposition 9.3 we have

Proposition 9.5. $|\text{BS}(\mathbf{v})| \leq N_\Omega(\varphi'_0) |\mathbf{v}|.$

We turn next to estimating $N_\Omega(\varphi'_0)$.

Lemma 9.6. *If $\psi(\alpha)$ is a positive, decreasing function of α , then*

$$N_\Omega(\psi) = \max_{\mathbf{y}} \int_{\mathbf{x} \in \Omega} \psi(\mathbf{x}, \mathbf{y}) d\mathbf{x}$$

is maximized over all subdomains Ω of R^3 , S^3 , or H^3 having a given volume when Ω is a round ball and \mathbf{y} is its center.

We leave the proof of this, as well as that of the next elementary lemma, to the reader.

Lemma 9.7. *The functions*

$$\begin{aligned}\varphi'_0(\alpha) &= \frac{1}{4\pi\alpha^2} & \alpha \in (0, \infty) \\ \varphi'_0(\alpha) &= \frac{1}{4\pi^2}(\csc \alpha + (\pi - \alpha) \csc \alpha \cot \alpha) & \alpha \in (0, \pi] \\ \varphi'_0(\alpha) &= \frac{1}{4\pi} \operatorname{csch} \alpha \operatorname{coth} \alpha & \alpha \in (0, \infty)\end{aligned}$$

are decreasing functions of α on their respective domains.

In view of Lemmas 9.6 and 9.7, we next compute $N_\Omega(\varphi'_0)$, where Ω is a round ball of radius R in R^3 , S^3 , or H^3 , and $\varphi'_0(\alpha)$ is as given above. We use the shorthand $N(R) = N_\Omega(\varphi'_0)$.

Proposition 9.8. *Let Ω be a round ball of radius R in R^3 , S^3 , or H^3 . Then*

$$\begin{aligned}\text{in } R^3 \text{ we have} \quad N(R) &= R \\ \text{in } S^3 \text{ we have} \quad N(R) &= \frac{1}{\pi}(2(1 - \cos R) + (\pi - R) \sin R) \\ \text{in } H^3 \text{ we have} \quad N(R) &= \sinh R.\end{aligned}$$

Proof. We give the proof in S^3 and leave the other two cases to the reader.

$$\begin{aligned}N(R) &= \int_{\alpha=0}^R \varphi'_0(\alpha) 4\pi \sin^2 \alpha \, d\alpha \\ &= \int_{\alpha=0}^R \frac{1}{4\pi^2}(\csc \alpha + (\pi - \alpha) \csc \alpha \cot \alpha) 4\pi \sin^2 \alpha \, d\alpha \\ &= \frac{1}{\pi} \int_{\alpha=0}^R (\sin \alpha + (\pi - \alpha) \cos \alpha) \, d\alpha \\ &= \frac{1}{\pi}(2(1 - \cos R) + (\pi - R) \sin R).\end{aligned}$$

Remark. If we put $R = \pi$, then we get $N_{S^3}(\varphi'_0) = 4/\pi$, in which case

$$|\operatorname{BS}(\mathbf{v})| \leq N_{S^3}(\varphi'_0)|\mathbf{v}| = \frac{4}{\pi}|\mathbf{v}|,$$

for smooth vector fields \mathbf{v} defined on the entire 3-sphere.

We contrast this with the sharp estimate

$$|\operatorname{BS}(\mathbf{v})| \leq \frac{1}{2}|\mathbf{v}|,$$

with equality if and only if \mathbf{v} is a vector field of constant length tangent to a left or right Hopf fibration of S^3 . See our (2008D) paper for details.

Proof of Theorem 9.2. By Lemma 9.7, the functions $\varphi'_0(\alpha)$ in R^3 , S^3 , and H^3 are all positive and decreasing. Hence by Lemma 9.6, the quantity

$$N_\Omega(\varphi'_0) = \max_{\mathbf{y}} \int_{\mathbf{x} \in \Omega} \varphi'_0(\mathbf{x}, \mathbf{y}) d\mathbf{x}$$

is maximized over subdomains Ω having a given volume when Ω is a round ball and \mathbf{y} is its center. The values of $N_\Omega(\varphi'_0)$ in that case were calculated in Proposition 9.8, and inserting them into the estimate

$$|\text{BS}(\mathbf{v})| \leq N_\Omega(\varphi'_0)|\mathbf{v}|$$

of Proposition 9.5, we get Theorem 9.2.

For more information about energy bounds, we call the reader's attention to the papers of Moffatt (1990M) and Ricca (2008R).

10. Hodge decomposition of vector fields

In this section we collect, without proof, some information about the topology of compact subdomains in R^3 , S^3 , and H^3 , and about the structure of the space of vector fields on such domains. The reader will find the details in our (2002) paper.

Let Ω be a compact, smoothly bounded domain in R^3 , S^3 , or H^3 , and $\text{VF}(\Omega)$ the space of all smooth vector fields on Ω , with the L^2 inner product and associated energy and norm, as defined in the preceding section.

Let $\text{K}(\Omega) \subset \text{VF}(\Omega)$ denote the subspace consisting of vector fields which are divergence-free and tangent to the boundary of Ω ,

$$\text{K}(\Omega) = \{\mathbf{v} \in \text{VF}(\Omega) : \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0\},$$

where \mathbf{n} denotes the unit outward normal vector field along the boundary $\partial\Omega$ of Ω . These vector fields are just the incompressible fluid flows within a bounded domain, and in real life are naturally tangent to the boundary. In the traditional passage from geometric knot theory to fluid dynamics, a knot is modeled by such a flow within a tubular neighborhood of itself, and the flows are then called *fluid knots*, accounting for the “K” in the notation $\text{K}(\Omega)$.

Let $\text{G}(\Omega) \subset \text{VF}(\Omega)$ denote the subspace of *gradient fields*,

$$\text{G}(\Omega) = \{\mathbf{v} \in \text{VF}(\Omega) : \mathbf{v} = \nabla\varphi \text{ for some smooth function } \varphi : \Omega \rightarrow R\}.$$

Then we have an L^2 -orthogonal direct sum decomposition

$$\text{VF}(\Omega) = \text{K}(\Omega) \oplus \text{G}(\Omega). \tag{10.1}$$

The spaces $\text{VF}(\Omega)$, $\text{K}(\Omega)$, and $\text{G}(\Omega)$ are all infinite-dimensional.

Let $\text{HK}(\Omega) \subset \text{K}(\Omega)$ denote the subspace of vector fields which are not only divergence-free and tangent to the boundary, but also curl-free,

$$\text{HK}(\Omega) = \{\mathbf{v} \in \text{VF}(\Omega) : \nabla \cdot \mathbf{v} = 0, \nabla \times \mathbf{v} = \mathbf{0}, \mathbf{v} \cdot \mathbf{n} = 0\}.$$

We call the elements of $\text{HK}(\Omega)$ *harmonic knots*. The subspace $\text{HK}(\Omega)$ is finite-dimensional, and isomorphic to $H_1(\Omega)$, the one-dimensional homology of Ω with real coefficients.

The orthogonal decomposition (10.1), when further refined, yields the *Hodge decomposition* of $\text{VF}(\Omega)$; see our (2002) paper for details.

Let Ω^* denote the closure of the complement of Ω in R^3 , S^3 , or H^3 . Let g denote the *total genus* of $\partial\Omega$, that is, the sum of the genera of its components. Then, using real coefficients, $H_1(\partial\Omega)$ is a $2g$ -dimensional vector space, while $H_1(\Omega)$ and $H_1(\Omega^*)$ are each g -dimensional, and we have the direct sum decomposition

$$H_1(\partial\Omega) = \ker(H_1(\partial\Omega) \rightarrow H_1(\Omega)) + \ker(H_1(\partial\Omega) \rightarrow H_1(\Omega^*)), \quad (10.2)$$

where the above homomorphisms are induced by the inclusions $\partial\Omega \subset \Omega$ and $\partial\Omega \subset \Omega^*$.

Let a_1, a_2, \dots, a_g be a basis for $\ker(H_1(\partial\Omega) \rightarrow H_1(\Omega))$, and b_1, b_2, \dots, b_g a basis for $\ker(H_1(\partial\Omega) \rightarrow H_1(\Omega^*))$.

If $\mathbf{v} \in \text{HK}(\Omega)$, then, since \mathbf{v} is curl-free, its circulation

$$\text{Circ}(\mathbf{v}, \gamma) = \int_{\gamma} (\mathbf{v}(\mathbf{x}(t)) \cdot \frac{d\mathbf{x}}{dt} dt$$

about any curve γ in Ω depends only on the homology class of γ . So we can denote this circulation by $\text{Circ}(\mathbf{v}, [\gamma])$.

With this notation, the real numbers $\text{Circ}(\mathbf{v}, a_1), \dots, \text{Circ}(\mathbf{v}, a_g)$ are all zero, since the homology classes a_i on $\partial\Omega$ bound in Ω . By contrast,

(10.3) The real numbers $\text{Circ}(\mathbf{v}, b_1), \text{Circ}(\mathbf{v}, b_2), \dots, \text{Circ}(\mathbf{v}, b_g)$ are in general not zero, and in fact define an *isomorphism* of $\text{HK}(\Omega) \rightarrow R^g$.

11. Spectral geometry of the curl operator in R^3 , S^3 , and H^3

As before, let Ω be a compact, smoothly bounded subdomain of R^3 , S^3 , or H^3 , and $\text{VF}(\Omega)$ the infinite-dimensional space of smooth vector fields on Ω with the L^2 inner product.

Now we are interested in curl eigenfields on Ω , that is, vector fields \mathbf{v} on Ω which satisfy $\nabla \times \mathbf{v} = \lambda \mathbf{v}$ for $\lambda \neq 0$. In R^3 , these fields are used to model stable plasma flows; see our (1999) paper.

Curl eigenfields exist for every value of λ . For example, in R^3 , if

$$\mathbf{v} = \sin \lambda z \mathbf{i} + \cos \lambda z \mathbf{j},$$

then $\nabla \times \mathbf{v} = \lambda \mathbf{v}$.

We want to constrain the choice of vector fields \mathbf{v} by interior and boundary conditions which guarantee that the curl operator on $\text{VF}(\Omega)$

will have a discrete spectrum, while at the same time being reasonable for physical applications. Then we want to find a lower bound for the absolute values of the nonzero eigenvalues.

To begin, we will restrict our attention to the subspace $K(\Omega)$ of fluid knots, discussed in section 10. The vector fields in $K(\Omega)$ are divergence-free and tangent to the boundary of Ω . Since a curl eigenfield \mathbf{v} with nonzero eigenvalue λ is automatically divergence-free, the only real constraint here is that of tangency to the boundary.

Let $CK(\Omega) \subset K(\Omega)$ denote the subspace of vector fields whose curl lies in $K(\Omega)$. Any eigenfield of the curl operator in $K(\Omega)$ must lie in $CK(\Omega)$, so restricting our attention to $CK(\Omega)$ is no further constraint.

Lemma 11.1. *A vector field $\mathbf{v} \in K(\Omega)$ lies in the subspace $CK(\Omega)$ if and only if the circulation of \mathbf{v} around small loops on $\partial\Omega$ vanishes.*

Proof. The circulation of \mathbf{v} around a small loop on $\partial\Omega$ equals the flux of $\nabla \times \mathbf{v}$ through the small disk bounded by that loop. If this flux is zero for all such loops, then the normal component of $\nabla \times \mathbf{v}$ along $\partial\Omega$ must be zero, telling us that $\nabla \times \mathbf{v}$ is tangent to $\partial\Omega$, and hence that $\mathbf{v} \in CK(\Omega)$. q.e.d.

Remarks.

(1) If the circulation of \mathbf{v} vanishes around small loops on $\partial\Omega$, then it also vanishes around homologically trivial loops there.

(2) Any divergence-free vector field on Ω which vanishes on $\partial\Omega$ must lie in $CK(\Omega)$.

The kernel of the map $\text{curl}: CK(\Omega) \rightarrow K(\Omega)$ consists of vector fields on Ω which are divergence-free, curl-free, and tangent to the boundary. These are the harmonic knots $HK(\Omega)$ introduced in section 10.

Since we are interested in the spectral theory of the curl operator, we would like to know when $\text{curl}: CK(\Omega) \rightarrow K(\Omega)$ is self-adjoint with respect to the L^2 inner product; that is, when can we promise that

$$\langle \nabla \times \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \nabla \times \mathbf{w} \rangle$$

for vector fields \mathbf{v} and \mathbf{w} in $CK(\Omega)$?

Lemma 11.2. *Suppose that Ω is simply connected, or equivalently, that all the components of $\partial\Omega$ are 2-spheres. Then $\text{curl}: CK(\Omega) \rightarrow K(\Omega)$ is self-adjoint.*

Proof. Recall the formula from vector calculus,

$$\nabla \cdot (\mathbf{v} \times \mathbf{w}) = (\nabla \times \mathbf{v}) \cdot \mathbf{w} - \mathbf{v} \cdot (\nabla \times \mathbf{w}),$$

and integrate this over Ω to get

$$\begin{aligned} \int_{\Omega} \nabla \cdot (\mathbf{v} \times \mathbf{w}) \, d \text{vol} &= \int_{\Omega} (\nabla \times \mathbf{v}) \cdot \mathbf{w} \, d \text{vol} - \int_{\Omega} \mathbf{v} \cdot (\nabla \times \mathbf{w}) \, d \text{vol} \\ &= \langle \nabla \times \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}, \nabla \times \mathbf{w} \rangle. \end{aligned}$$

The left-hand side equals

$$\int_{\partial\Omega} (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{n} \, d \text{area},$$

and so the issue is to see when $\mathbf{v} \times \mathbf{w}$ has zero flux through $\partial\Omega$.

By Lemma 11.1, we know that $\mathbf{v} \in \text{CK}$ if and only if \mathbf{v} has zero circulation around small loops on $\partial\Omega$. Since Ω is simply connected, $\partial\Omega$ is a union of 2-spheres, and so \mathbf{v} must have zero circulation around *all* loops on $\partial\Omega$.

But this means that the restriction of \mathbf{v} to $\partial\Omega$ is a gradient field on that surface. So we write $\mathbf{v}|_{\partial\Omega} = \nabla f$, where $f: \partial\Omega \rightarrow R$ is some smooth function, and the gradient is the “surface gradient” on $\partial\Omega$.

Likewise, $\mathbf{w}|_{\partial\Omega} = \nabla g$ for some smooth function $g: \partial\Omega \rightarrow R$.

Now extend f and g to smooth functions F and G from $\Omega \rightarrow R$, and consider the vector fields ∇F and ∇G defined on Ω .

Because the cross product of two gradient fields is always divergence-free, that is,

$$\nabla \cdot (\nabla F \times \nabla G) = (\nabla \times \nabla F) \cdot \nabla G - \nabla F \cdot (\nabla \times \nabla G) = 0,$$

we have

$$\int_{\partial\Omega} (\nabla F \times \nabla G) \cdot \mathbf{n} \, d \text{area} = 0.$$

Since ∇f and ∇g are, respectively, the tangential components of ∇F and ∇G along $\partial\Omega$, we can write

$$\nabla F(\mathbf{x}) = \nabla f(\mathbf{x}) + a(\mathbf{x}) \mathbf{n}(\mathbf{x}) \quad \text{and} \quad \nabla G(\mathbf{x}) = \nabla g(\mathbf{x}) + b(\mathbf{x}) \mathbf{n}(\mathbf{x})$$

for $\mathbf{x} \in \partial\Omega$. From this we can see that

$$(\nabla F(\mathbf{x}) \times \nabla G(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) = (\nabla f(\mathbf{x}) \times \nabla g(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x})$$

along $\partial\Omega$, and hence

$$\begin{aligned} \int_{\partial\Omega} (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{n} \, d \text{area} &= \int_{\partial\Omega} (\nabla f \times \nabla g) \cdot \mathbf{n} \, d \text{area} \\ &= \int_{\partial\Omega} (\nabla F \times \nabla G) \cdot \mathbf{n} \, d \text{area} \\ &= 0. \end{aligned}$$

Thus $\langle \nabla \times \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \nabla \times \mathbf{w} \rangle$ for all \mathbf{v} and \mathbf{w} in $\text{CK}(\Omega)$, and so $\text{curl}: \text{CK}(\Omega) \rightarrow \text{K}(\Omega)$ is self-adjoint when Ω is simply connected, completing the proof of Lemma 11.2. q.e.d.

When Ω is not simply connected, the operator $\text{curl}: \text{CK}(\Omega) \rightarrow \text{K}(\Omega)$ is *not* self-adjoint. So we seek further sensible boundary conditions which will make this operator self-adjoint for any domain Ω .

To this end, let \mathbf{v} be a vector field in $\text{CK}(\Omega)$. By definition of $\text{CK}(\Omega)$, the circulation of \mathbf{v} around all small loops on $\partial\Omega$ vanishes. But then the circulation of \mathbf{v} around *any* loop on $\partial\Omega$ depends only on the homology class of that loop, giving us a linear map

$$\text{Circ}(\mathbf{v}): H_1(\partial\Omega) \rightarrow \mathbb{R},$$

from the one-dimensional real homology of $\partial\Omega$ to the reals.

To see where this is heading, let Ω^* denote the closure of the complement of Ω in R^3 , S^3 , or H^3 , as in section 10, and let g be the total genus of $\partial\Omega$.

Recall from section 10 the direct sum decomposition

$$H_1(\partial\Omega) = \ker(H_1(\partial\Omega) \rightarrow H_1(\Omega)) + \ker(H_1(\partial\Omega) \rightarrow H_1(\Omega^*)), \quad (10.2)$$

which splits a $2g$ -dimensional space into two g -dimensional summands.

Now let $\text{AK}(\Omega) \subset \text{CK}(\Omega) \subset \text{K}(\Omega)$ consist of all vector fields \mathbf{v} in $\text{CK}(\Omega)$ whose circulation vanishes around any loop on $\partial\Omega$ which bounds in Ω^* . The subspace $\text{AK}(\Omega)$ has codimension g in $\text{CK}(\Omega)$.

From (10.3), we get

$$\text{AK}(\Omega) \cap \text{HK}(\Omega) = \{0\}. \quad (11.3)$$

We call $\text{AK}(\Omega)$ the space of *Ampèrian knots* because, by Ampère's Law, the magnetic field due to a current running entirely within Ω will have zero circulation around all loops on $\partial\Omega$ which bound in Ω^* .

We intend to show that the operator $\text{curl}: \text{AK}(\Omega) \rightarrow \text{K}(\Omega)$ is self-adjoint, and proceed as follows.

Start with a vector field $\mathbf{v} \in \text{VF}(\Omega)$, and let $\text{BS}(\mathbf{v})$ be the corresponding magnetic field defined throughout R^3 , S^3 , or H^3 . Let the same symbol denote its restriction to Ω , so that we may consider the operator $\text{BS}: \text{VF}(\Omega) \rightarrow \text{VF}(\Omega)$. The magnetic field $\text{BS}(\mathbf{v})$ is always divergence-free, but in general is not tangent to the boundary of Ω .

Now define $\text{BS}'(\mathbf{v})$ to be the L^2 -orthogonal projection of $\text{BS}(\mathbf{v})$ into $\text{K}(\Omega)$. We are only going to apply BS' to vector fields \mathbf{v} already in $\text{K}(\Omega)$, so we regard this *modified Biot-Savart operator* as a map

$$\text{BS}': \text{K}(\Omega) \rightarrow \text{K}(\Omega).$$

We see from the orthogonal decomposition (10.1) that, for any vector field $\mathbf{v} \in \text{K}(\Omega)$, we have

$$\text{BS}(\mathbf{v}) = \text{BS}'(\mathbf{v}) + \text{the gradient component of } \text{BS}(\mathbf{v}). \quad (11.4)$$

Proposition 11.5. *The image of the map $\text{BS}': \text{K}(\Omega) \rightarrow \text{K}(\Omega)$ is the subspace $\text{AK}(\Omega)$ of Ampèrian knots.*

Proof. Let $\mathbf{v} \in \mathbf{K}(\Omega)$, so that \mathbf{v} is divergence-free and tangent to $\partial\Omega$. Then we have $\nabla \times \mathbf{BS}(\mathbf{v}) = \mathbf{v}$. This follows from Maxwell's equation for subdomains of R^3 by Proposition 1 of Cantarella, DeTurck, and Gluck (2001), for subdomains of S^3 by Proposition 3.1 of Parsley (2009), and similarly in H^3 .

By (11.4), we then also have $\nabla \times \mathbf{BS}'(\mathbf{v}) = \mathbf{v}$.

Now let γ be a loop on $\partial\Omega$ which bounds the surface Σ in Ω^* . Then the circulation of $\mathbf{BS}'(\mathbf{v})$ around γ equals the flux of $\nabla \times \mathbf{BS}'(\mathbf{v}) = \mathbf{v}$ through Σ , according to Ampère's Law. But the flux of \mathbf{v} through Σ is zero, since \mathbf{v} is confined to Ω . Thus $\mathbf{BS}'(\mathbf{v}) \in \mathbf{AK}(\Omega)$, and we have shown that

$$\mathbf{BS}'(\mathbf{K}(\Omega)) \subset \mathbf{AK}(\Omega).$$

To see the reverse inclusion, start with $\mathbf{w} \in \mathbf{AK}(\Omega)$, and let $\mathbf{v} = \nabla \times \mathbf{w} \in \mathbf{K}(\Omega)$. We claim that $\mathbf{w} = \mathbf{BS}'(\mathbf{v})$.

To see this, first note that $\nabla \times \mathbf{w} = \mathbf{v} = \nabla \times \mathbf{BS}'(\mathbf{v})$; hence $\mathbf{BS}'(\mathbf{v}) - \mathbf{w}$ is curl-free, and therefore lies in $\mathbf{HK}(\Omega)$.

Now we showed above that $\mathbf{BS}'(\mathbf{v}) \in \mathbf{AK}(\Omega)$, and we have $\mathbf{w} \in \mathbf{AK}(\Omega)$ by hypothesis, so $\mathbf{BS}'(\mathbf{v}) - \mathbf{w}$ also lies in $\mathbf{AK}(\Omega)$.

Therefore $\mathbf{BS}'(\mathbf{v}) - \mathbf{w}$ lies in $\mathbf{AK}(\Omega) \cap \mathbf{HK}(\Omega) = \{0\}$, according to (11.3), and so $\mathbf{BS}'(\mathbf{v}) = \mathbf{w}$. This shows that

$$\mathbf{AK}(\Omega) \subset \mathbf{BS}'(\mathbf{K}(\Omega)),$$

completing the proof.

q.e.d.

In the course of the proof, we actually showed a little more.

Corollary 11.6. *The maps*

$$\mathbf{BS}': \mathbf{K}(\Omega) \rightarrow \mathbf{AK}(\Omega) \quad \text{and} \quad \text{curl}: \mathbf{AK}(\Omega) \rightarrow \mathbf{K}(\Omega)$$

are inverses of one another.

Proposition 11.7. *The map $\text{curl}: \mathbf{AK}(\Omega) \rightarrow \mathbf{K}(\Omega)$ is self-adjoint.*

Proof. Self-adjointness of this curl map is equivalent to self-adjointness of its inverse $\mathbf{BS}': \mathbf{K}(\Omega) \rightarrow \mathbf{AK}(\Omega)$, and this in turn is a consequence of self-adjointness of the Biot-Savart operator $\mathbf{BS}: \mathbf{VF}(\Omega) \rightarrow \mathbf{VF}(\Omega)$, which can be seen directly from its defining formula as follows.

Suppose that Ω is a compact, smoothly bounded subdomain of R^3 , S^3 or H^3 , and that $\mathbf{v}, \mathbf{w} \in \text{VF}(\Omega)$. Then

$$\begin{aligned}
\langle \text{BS}(\mathbf{v}), \mathbf{w} \rangle &= \int_{\Omega} \text{BS}(\mathbf{v})(\mathbf{y}) \cdot \mathbf{w}(\mathbf{y}) \, d\mathbf{y} \\
&= \int_{\Omega} \left(\int_{\Omega} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \right) \cdot \mathbf{w}(\mathbf{y}) \, d\mathbf{y} \\
&= \int_{\Omega \times \Omega} P_{\mathbf{y}\mathbf{x}} \mathbf{v}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) \cdot \mathbf{w}(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \\
&= \int_{\Omega \times \Omega} \mathbf{v}(\mathbf{x}) \times (-\nabla_{\mathbf{x}} \varphi_0(\mathbf{x}, \mathbf{y})) \cdot P_{\mathbf{x}\mathbf{y}} \mathbf{w}(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \\
&= \int_{\Omega \times \Omega} P_{\mathbf{x}\mathbf{y}} \mathbf{w}(\mathbf{y}) \times \nabla_{\mathbf{x}} \varphi_0(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v}(\mathbf{x}) \, d\mathbf{x} \, d\mathbf{y} \\
&= \int_{\Omega \times \Omega} P_{\mathbf{y}\mathbf{x}} \mathbf{w}(\mathbf{x}) \times \nabla_{\mathbf{y}} \varphi_0(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \, d\mathbf{x} \, d\mathbf{y} \\
&= \langle \mathbf{v}, \text{BS}(\mathbf{w}) \rangle,
\end{aligned}$$

where we went from the third line above to the fourth by applying the parallel transport $P_{\mathbf{x}\mathbf{y}}$ to every term without changing the value of the integrand, from the fourth to the fifth by interchanging two terms and reversing the sign, from the fifth to the sixth by interchanging the variables \mathbf{x} and \mathbf{y} , and finally, comparing the sixth line to the third, moved on to the seventh line. This completes the proof of the proposition. q.e.d.

Finally, we come to the desired result.

Theorem 11.8. *Let Ω be a compact, smoothly bounded subdomain of R^3 , S^3 , or H^3 and let $R = R(\Omega)$ be the radius of a ball in that space having the same volume as Ω .*

Then $\text{curl}: \text{AK}(\Omega) \rightarrow \text{K}(\Omega)$ is a self-adjoint operator, and for each $\mathbf{v} \in \text{AK}(\Omega)$,

$$|\nabla \times \mathbf{v}| \geq \frac{|\mathbf{v}|}{N(R)},$$

where

$$\begin{aligned}
\text{in } R^3 \text{ we have } & N(R) = R \\
\text{in } S^3 \text{ we have } & N(R) = \frac{1}{\pi}(2(1 - \cos R) + (\pi - R) \sin R) \\
\text{in } H^3 \text{ we have } & N(R) = \sinh R.
\end{aligned}$$

In particular, if λ is any curl eigenvalue on $\text{AK}(\Omega)$, then $|\lambda| \geq \frac{1}{N(R)}$.

Proof. We already know from Proposition 11.7 that the map $\text{curl}: \text{AK}(\Omega) \rightarrow \text{K}(\Omega)$ is self-adjoint.

Let $\mathbf{v} \in K(\Omega)$. Then, since $BS'(\mathbf{v})$ is the orthogonal projection of $BS(\mathbf{v})$ back into the subspace $K(\Omega)$, we certainly have $|BS'(\mathbf{v})| \leq |BS(\mathbf{v})|$.

But $|BS(\mathbf{v})| \leq N(R)|\mathbf{v}|$ by Theorem 9.2, so that same bound applies to the modified Biot-Savart operator:

$$|BS'(\mathbf{v})| \leq N(R)|\mathbf{v}|.$$

Now suppose that $\mathbf{v} \in AK(\Omega)$. Then

$$\nabla \times \mathbf{v} \in K(\Omega) \quad \text{and} \quad BS'(\nabla \times \mathbf{v}) = \mathbf{v}.$$

Hence

$$|\mathbf{v}| = |BS'(\nabla \times \mathbf{v})| \leq N(R)|\nabla \times \mathbf{v}|,$$

and therefore

$$|\nabla \times \mathbf{v}| \geq \frac{1}{N(R)}|\mathbf{v}|,$$

completing the proof of Theorem 11.8.

q.e.d.

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