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CURVATURE ESTIMATES FOR STABLE MARGINALLY TRAPPED SURFACES

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Abstract

We derive local integral and sup- estimates for the curvature of stable marginally outer trapped surfaces in a sliced space-time. The estimates bound the shear of a marginally outer trapped surface in terms of the intrinsic and extrinsic curvature of a slice containing the surface. These estimates are well adapted to situations of physical interest, such as dynamical horizons.

1. Introduction

The celebrated regularity result for stable minimal surfaces, due to Schoen, Simon, and Yau [**SSY75**], gives a bound on the second fundamental form in terms of ambient curvature and area of the surface. The proof of the main result of [**SSY75**] makes use of the Simons formula [**Sim68**] for the Laplacian of the second fundamental form, together with the non-negativity of the second variation of area. In this paper we will prove a generalization of the regularity result of Schoen, Simon, and Yau to the natural analogue of stable minimal surfaces in the context of Lorentz geometry, stable marginally trapped surfaces. In this case, a generalization of the Simons formula holds for the null second fundamental form, and the appropriate notion of stability is that of stably outermost in the sense of [**AMS05**, **New87**]. A local area estimate for stable marginally trapped surfaces, a generalization of a result due to Pogorelov [**Pog81**], allows us to give a curvature bound independent of assumptions on the area of the surface. An interesting feature of our estimates is that they imply curvature bounds for stable minimal surfaces or surfaces of constant mean curvature that do not depend on bounds for the derivative of the ambient curvature.

Let Σ be a spacelike surface of co-dimension two in a $(3+1)$-dimensional Lorentz manifold L and let $l^{\pm}$ be the two independent future

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directed null sections of the normal bundle of Σ, with corresponding mean curvatures, or null expansions, $\theta^{\pm}$. Σ is called trapped if the future directed null rays starting at Σ converge, i.e., if $\theta^{\pm} < 0$. If L contains a trapped surface and satisfies certain causal conditions, then, if in addition, the null energy condition is satisfied, L is future causally incomplete [**Pen65**]. Let l^+ be the outgoing null normal. If L is an asymptotically flat spacetime this notion is well defined; otherwise the outgoing direction can be fixed by convention. We call Σ a marginally outer trapped surface (MOTS) if the outgoing lightrays are marginally converging, i.e., if $\theta^+ = 0$. No assumption is made on the ingoing null expansion θ^- of a MOTS. If Σ is contained in a time symmetric Cauchy surface, then $\theta^+ = 0$ if and only if Σ is minimal.

Marginally trapped surfaces are of central importance in general relativity, where they play the role of apparent horizons, or quasilocal black hole boundaries. The conjectured Penrose inequality, proved in the Riemannian case by Huisken and Ilmanen [**HI01**] and Bray [**Bra01**], may be formulated as an inequality relating the area of the outermost apparent horizon and the ADM mass. The technique of excising the interior of black holes using apparent horizons as excision boundaries plays a crucial role in current work in numerical relativity, where much of the focus is on modelling binary black hole collisions.

In spite of the importance of marginally trapped surfaces in the geometry of space-times, the extent of our knowledge of the regularity and existence of these objects is rather limited compared to the situation for minimal surfaces.

A smooth marginally outer trapped surface is stationary with respect to variations of area within its outgoing null cone, in view of the formula

$$\delta_{fl^+}\mu_\Sigma = f\theta^+\mu_\Sigma$$

where f is a function on Σ. The second variation of area at a MOTS in the direction l^+ is

$$\delta_{fl^+}\theta^+ = -(|\chi^+|^2 + G(l^+, l^+))f$$

where G denotes the Einstein tensor of L, and χ^+ is the second fundamental form of Σ with respect to l^+. For minimal surfaces in a Riemannian manifold, or maximal hypersurfaces in a Lorentz manifold, the second variation operator is an elliptic operator of second order. In contrast, the above equation shows that the second variation operator for area of a MOTS, with respect to variations in the null direction l^+, is an operator of order zero. Therefore, although MOTS can be characterized as stationary points of area, this point of view alone is not sufficient to yield a useful regularity result. In spite of this, as we shall see below, there is a natural generalization of the stability condition for minimal surfaces, as well as of the regularity result of Schoen, Simon, and Yau, to marginally outer trapped surfaces.

It is worth remarking at this point that if we consider variations of area of spacelike hypersurfaces in a Lorentz manifold, the stationary points are maximal surfaces. Maximal surfaces satisfy a quasilinear non-uniformly elliptic equation closely related to the minimal surface equation. Due to the fact that maximal hypersurfaces are spacelike, they are Lipschitz submanifolds. Moreover, in a space-time satisfying the timelike convergence condition, every maximal surface is stable. Hence, the regularity theory for maximal surfaces is of a different flavor than the regularity theory for minimal surfaces (cf. [**Bar84**]).

Assume that L is provided with a reference foliation consisting of spacelike hypersurfaces $\{M_t\}$, and that Σ is contained in one of the leaves M of this foliation. Let (g, K) be the induced metric and the second fundamental form of M with respect to the future directed timelike normal n. Further, let ν be the outward pointing normal of Σ in M and let A be the second fundamental form of Σ with respect to ν. After possibly changing normalization, $l^\pm = n \pm \nu$, we have

$$\theta^\pm = H \pm \operatorname{tr}_\Sigma K$$

where $H = \operatorname{tr} A$ is the mean curvature of Σ and $\operatorname{tr}_\Sigma K$ is the trace of the projection of K to Σ. Thus the condition for Σ to be a MOTS, $\theta^+ = 0$, is a prescribed mean curvature equation.

The condition that plays the role of stability for MOTS is the stably outermost condition (see [**AMS05**, **New87**]). Suppose Σ is contained in a spatial hypersurface M. Then Σ is stably locally outermost in M if there is an outward infinitesimal deformation of Σ, within M, which does not decrease θ^+. This condition, which is equivalent to the condition that Σ is stable in case M is time symmetric, turns out to be sufficient to apply the technique of [**SSY75**] to prove a bound on the second fundamental form A of Σ in M. In contrast to the situation for minimal surfaces the stability operator defined by the deformation of θ^+ is not self-adjoint. Nevertheless, it has a real principal eigenvalue with a corresponding principal eigenfunction which does not change sign.

The techniques of [**SSY75**] were first applied in the context of general relativity by Schoen and Yau [**SY81**], where existence and regularity for Jang's equation were proved. Jang's equation is an equation for a graph in $N = M \times \mathbf{R}$, and is of a form closely related to the equation $\theta^+ = 0$. Let u be a function on M, and let $\bar{K}$ be the pull-back to N of K along the projection $N \to M$. Jang's equation is the equation

$$\bar{g}^{ij}\left(\frac{D_iD_ju}{\sqrt{1+|Du|^2}} + \bar{K}_{ij}\right) = 0$$

where $\bar{g}^{ij} = g^{ij} - \frac{D_iuD_ju}{1+|Du|^2}$ is the induced metric on the graph $\bar{\Sigma}$ of u in N. Thus Jang's equation can be written as $\bar{\theta} = 0$ with

$$\bar{\theta} = \bar{H} + \operatorname{tr}_{\bar{\Sigma}} \bar{K},$$

where $\bar{H}$ is the mean curvature of $\bar{\Sigma}$ in N. This shows that Jang's equation $\bar{\theta} = 0$ is a close analog to the equation $\theta^+ = 0$ characterizing a MOTS. Solutions to Jang's equation satisfy a stability condition closely related to the stably outermost condition stated above. This is due to the fact that Jang's equation is translation invariant in the sense that if u solves Jang's equation, then also $u + c$ is a solution where c is a constant. Thus, in the sense of section 5, graphical solutions to Jang's equation are stable. This fact allows Schoen and Yau [**SY81**] to apply the technique of [**SSY75**] to prove regularity for solutions of Jang's equation. It is worth remarking that although the dominant energy condition is assumed to hold throughout [**SY81**], in fact the proof of the existence and regularity result for solutions of Jang's equation presented in [**SY81**] can be carried out without this assumption. In the present paper, the dominant energy condition is not used in the proof of our main regularity result (cf. Theorem 1.2 below).

It was proved by Galloway and Schoen [**GS06**], based on an argument for solutions of the Jang's equation in [**SY81**], that the stability of MOTS implies a "symmetrized" stability condition, which states that the spectrum of a certain self-adjoint operator analogous to the second variation operator for minimal surfaces is non-negative. The fact that stability in the sense of stably outermost implies this symmetrized version of stability was used in [**GS06**] to give conditions on the Yamabe type of stable marginal surfaces in general dimension. It turns out that this weaker symmetrized notion of stability is in fact sufficient for the curvature estimates proved here. The symmetrized notion of stability is also used in our local area estimates. However, since this notion has no direct interpretation in terms of the geometry of the ambient spacetime, we prefer to state our results in terms of the stably outermost condition.

Statement of results. The stability condition for MOTS which replaces the stability condition for minimal surfaces and which allows one to apply the technique of [**SSY75**] is the following.

Definition 1.1. Σ is stably outermost if there is a function $f \geq 0$ on Σ, $f \neq 0$ somewhere, such that $\delta_{f\nu}\theta^+ \geq 0$.

When there is no room for confusion we will refer to a stably outermost MOTS simply as a stable MOTS. This is analogous to the stability condition for a minimal surface $N \subset M$. The condition that there exists a function f on N with $f \geq 0$ and $f \neq 0$ somewhere, such that $\delta_{f\nu}H \geq 0$, is equivalent to the condition that N is stable.

The main result of this paper is the following theorem (cf. theorem 6.10, corollary 6.11 as well as theorem 7.1).

Theorem 1.2. *Suppose Σ is a stable MOTS in (M, g, K). Then the second fundamental form A satisfies the inequality*

$$\|A\|_\infty \le C(\|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, \mathrm{inj}(M,g)^{-1}) .$$

Here $\|\cdot\|_\infty$ denotes the sup-norm of the respective quantity, taken on Σ. As an application we prove a compactness result for MOTS (cf. theorem 8.1).

Theorem 1.3. *Let (g_n, K_n) be a sequence of initial data sets on a manifold M. Let (g, K) be another initial data set on M such that*

$$\begin{aligned} \|{}^M\mathrm{Rm}\|_\infty &\le C\,, \\ \|K\|_\infty + \|{}^M\nabla K\|_\infty &\le C, \\ \mathrm{inj}(M,g) &\ge C^{-1}\,, \end{aligned}$$

for some constant C. Assume that

$$\begin{aligned} g_n \to g \quad &\mathit{in} \quad C^2_{loc}(M,g) \ \mathit{and}, \\ K_n \to K \quad &\mathit{in} \quad C^1_{loc}(M,g). \end{aligned}$$

Furthermore, let $\Sigma_n \subset M$ be a sequence of immersed surfaces which are stable marginally outer trapped with respect to (g_n, K_n) and have an accumulation point in M. In addition, assume that the Σ_n have uniformly locally finite area, that is, for all $x \in M$ there exists $0 < r = r(x)$ and $a = a(x) < \infty$ such that

$$|\Sigma_n \cap B_{M_{t_n}}(x,r)| \le ar^2 \qquad \textit{uniformly in } n,$$

where $B_{M_{t_n}}(x,r)$ denotes the ball in M around x with radius r.

Then a subsequence of the Σ_n converges to a smooth immersed surface Σ locally in the sense of $C^{1,\alpha}$ graphs. Σ is a MOTS with respect to (g, K). If Σ is compact, then it is also stable.

Outline of the paper. In sections 2 and 3 we discuss the notation and preliminary results, as well as a Simons identity which holds for the shear of a MOTS. Section 4 introduces the linearization of the operator θ^+ acting on surfaces represented as graph over a MOTS. The stability conditions we use are discussed in section 5. The curvature estimates are derived in section 6 under the assumption of local area bounds. In section 7 we show how these bounds can be derived in terms of the ambient geometry. Finally section 8 uses the established curvature bounds to prove the compactness theorem.

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2. Preliminaries and notation

In this section we set up notation and recall some preliminaries from differential geometry. In the sequel we will consider two-dimensional spacelike submanifolds Σ of a four-dimensional manifold L. As a space-time manifold, L is equipped with a metric h of signature $(-,+,+,+)$. The inner product induced by h will frequently be denoted by $\langle\cdot,\cdot\rangle$. In addition, we will assume, that Σ is contained in a spacelike hypersurface M in L. The metric on M induced by h will be denoted by g, the metric on Σ by γ. We will denote the tangent bundles by TL, TM, and $T\Sigma$, and the space of smooth tangential vector fields along the respective manifolds by $\mathcal{X}(\Sigma)$, $\mathcal{X}(M)$, and $\mathcal{X}(L)$. Unless otherwise stated, we will assume that all manifolds and fields are smooth.

We denote by n the future directed unit timelike normal of M in L, which we will assume to be a well-defined vector field along M. The normal of Σ in M will be denoted by ν, which again is assumed to be a well-defined vector field along Σ.

The two directions n and ν span the normal bundle $\mathcal{N}\Sigma$ of Σ in L, and moreover, we can use them to define two canonical null directions, which also span this bundle, namely, $l^\pm := n \pm \nu$.

In addition to the metrics, h and its Levi-Civita connection ${}^L\nabla$ induce the second fundamental form K of M in L. It is the normal part of ${}^L\nabla$, in the sense that for all vector fields $X, Y \in \mathcal{X}(M)$,

$$ {}^L\nabla_X Y = {}^M\nabla_X Y + K(X,Y)n\,. \tag{1}$$

The second fundamental form of Σ in M will be denoted by A. For vector fields $X, Y \in \mathcal{X}(\Sigma)$ we have

$$ {}^M\nabla_X Y = {}^\Sigma\nabla_X Y - A(X,Y)\nu\,. \tag{2}$$

For vector fields $X, Y \in \mathcal{X}(\Sigma)$, the connection of L therefore splits according to

$$ {}^L\nabla_X Y = {}^\Sigma\nabla_X Y + K^\Sigma(X,Y)n - A(X,Y)\nu = {}^\Sigma\nabla_X Y - \mathrm{I\!I}(X,Y)\,, \tag{3}$$

where $\mathrm{I\!I}(X,Y) = A(X,Y)\nu - K^\Sigma(X,Y)n$ is the second fundamental form of Σ in L. Here K^Σ denotes the restriction of K to $T\Sigma$, the tangential space of Σ.

The trace of $\mathrm{I\!I}$ with respect to γ, which is a vector in the normal bundle of Σ, is called the mean curvature vector and is denoted by

$$ \mathcal{H} = \sum_i \mathrm{I\!I}(e_i, e_i)\,, \tag{4}$$

for an orthonormal basis e_1, e_2 of Σ. Since $\mathcal{H}$ is normal to Σ, it satisfies

$$ \mathcal{H} = H\nu - Pn \tag{5}$$

where $H = \gamma^{ij}A_{ij}$ is the trace of A and $P = \gamma^{ij}K^\Sigma_{ij}$ is the trace of K^Σ, with respect to γ. For completeness, we note that the norms of $\mathrm{I\!I}$ and

$\mathcal{H}$ are given by

$$|\mathrm{I\!I}|^2 = |A|^2 - |K^\Sigma|^2 \qquad \text{and} \tag{6}$$

$$|\mathcal{H}|^2 = H^2 - P^2\,. \tag{7}$$

Recall that since $\mathcal{H}$ and $\mathrm{I\!I}$ have values normal to Σ, the norms are taken with respect to h and are therefore not necessarily non-negative.

We use the following convention to represent the Riemann curvature tensor ${}^\Sigma\mathrm{Rm}$, the Ricci tensor ${}^\Sigma\mathrm{Rc}$, and the scalar curvature ${}^\Sigma\mathrm{Sc}$ of Σ. Here $X, Y, U, V \in \mathcal{X}(\Sigma)$ are vector fields.

$$\begin{aligned}
&{}^\Sigma\mathrm{Rm}(X,Y,U,V) = \big\langle {}^\Sigma\nabla_X {}^\Sigma\nabla_Y U - {}^\Sigma\nabla_Y {}^\Sigma\nabla_X U - {}^\Sigma\nabla_{[X,Y]}U, V\big\rangle\,,\\
&{}^\Sigma\mathrm{Rc}(X,Y) = \sum_i {}^\Sigma\mathrm{Rm}(X,e_i,e_i,Y)\,,\\
&{}^\Sigma\mathrm{Sc} = \sum_i {}^\Sigma\mathrm{Rc}(e_i,e_i)\,.
\end{aligned}$$

Analogous definitions hold for ${}^M\mathrm{Rm}$, ${}^M\mathrm{Rc}$, and ${}^M\mathrm{Sc}$ as well as ${}^L\mathrm{Rm}$, ${}^L\mathrm{Rc}$, and ${}^L\mathrm{Sc}$, with the exception that for ${}^L\mathrm{Rc}$ and ${}^L\mathrm{Sc}$ we take the trace with respect to the indefinite metric h.

We recall the Gauss and Codazzi equations of Σ in L, which relate the respective curvatures. The Riemann curvature tensors ${}^\Sigma\mathrm{Rm}$ and ${}^L\mathrm{Rm}$ of Σ and L, respectively, are related by the Gauss equation. For vector fields X, Y, U, V we have

$$\begin{aligned}
&{}^\Sigma\mathrm{Rm}(X,Y,U,V) = \\
&\qquad {}^L\mathrm{Rm}(X,Y,U,V) + \big\langle \mathrm{I\!I}(X,V), \mathrm{I\!I}(Y,U)\big\rangle - \big\langle \mathrm{I\!I}(X,U), \mathrm{I\!I}(Y,V)\big\rangle\,.
\end{aligned} \tag{8}$$

In two dimensions, all curvature information of Σ is contained in its scalar curvature, which we will denote by ${}^\Sigma\mathrm{Sc}$. The scalar curvature of L will be denoted by ${}^L\mathrm{Sc}$. The information of the Gauss equation above is fully contained in the following equation, which emerges from the above one by first taking the trace with respect to Y, U and then with respect to X, V:

$${}^\Sigma\mathrm{Sc} = {}^L\mathrm{Sc} + 2\,{}^L\mathrm{Rc}(n,n) - 2\,{}^L\mathrm{Rc}(\nu,\nu) - 2\,{}^L\mathrm{Rm}(\nu,n,n,\nu) + |\mathcal{H}|^2 - |\mathrm{I\!I}|^2\,. \tag{9}$$

The Codazzi equation, which relates ${}^L\mathrm{Rm}$ to $\mathrm{I\!I}$, has the following form:

$$\big\langle {}^L\nabla_X \mathrm{I\!I}(Y,Z), S\big\rangle = \big\langle \nabla_Y \mathrm{I\!I}(X,Z), S\big\rangle + {}^L\mathrm{Rm}(X,Y,S,Z) \tag{10}$$

for vector fields $X, Y, Z \in \mathcal{X}(\Sigma)$ and $S \in \Gamma(\mathcal{N}\Sigma)$.

There is also a version of the Gauss and Codazzi equations for the embedding of M in L. They relate the curvature ${}^L\mathrm{Rm}$ of L to the curvature ${}^{M}\mathrm{Rm}$ of M. For vector fields $X, Y, U, V \in \mathcal{X}(M)$ we have

$$\begin{aligned}&{}^{M}\mathrm{Rm}(X,Y,U,V)\\ &\quad = {}^{L}\mathrm{Rm}(X,Y,U,V) - K(Y,U)K(X,V) + K(X,U)K(Y,V)\,,\end{aligned} \tag{11}$$

$$ {}^{M}\nabla_X K(Y,U) - {}^{M}\nabla_Y K(X,U) = {}^{L}\mathrm{Rm}(X,Y,n,U)\,. \tag{12}$$

These equations also have a traced form, namely,

$$ {}^{M}\mathrm{Sc} = {}^{L}\mathrm{Sc} + 2\,{}^{L}\mathrm{Rc}(n,n) - (\operatorname{tr} K)^2 + |K|^2 \ \text{ and} \tag{13}$$

$$ {}^{M}\mathrm{div}K - {}^{M}\nabla \operatorname{tr} K = {}^{L}\mathrm{Rc}(\cdot, n)\,. \tag{14}$$

We now investigate the connection ${}^{N}\nabla$ on the normal bundle $\mathcal{N}\Sigma$ of Σ. Recall that for sections N of $\mathcal{N}\Sigma$ and $X \in \mathcal{X}(\Sigma)$, this connection is defined as

$$ {}^{N}\nabla_X N = \big({}^{L}\nabla_X N\big)^{\perp}\,,$$

where again $(\cdot)^{\perp}$ means taking the normal part. We have

$$ 0 = X(1) = X\big(\langle n, n\rangle\big) = 2\langle {}^{N}\nabla_X n, n\rangle\,,$$

and similarly $\langle {}^{N}\nabla_X \nu, \nu\rangle = 0$. Therefore the relevant component of ${}^{N}\nabla$ is

$$ \big\langle {}^{N}\nabla_X \nu, n\big\rangle = \big\langle {}^{L}\nabla_X \nu, n\big\rangle = -K(X,\nu)\,.$$

Recall that X is tangential to Σ. This lead us to define the 1-form S along Σ by the restriction of $K(\cdot,\nu)$ to $T\Sigma$.

$$ S(X) := K(X,\nu)\,. \tag{15}$$

Then, for an arbitrary section N of $\mathcal{N}\Sigma$ with $N = f\nu + gn$, we have

$$ {}^{N}\nabla_X N = X(f)\nu + X(g)n + S(X)\big(fn + g\nu\big)\,.$$

In particular

$$ {}^{N}\nabla_X l^{\pm} = \pm S(X) l^{\pm}\,. \tag{16}$$

We will later consider the decomposition of $\mathrm{I\!I}$ into its null components. For $X, Y \in \mathcal{X}(\Sigma)$ let

$$ \chi^{\pm}(X,Y) := \big\langle \mathrm{I\!I}(X,Y), l^{\pm}\big\rangle = K(X,Y) \pm A(X,Y)\,. \tag{17}$$

The traces of $\chi^{\pm}$ respectively will be called $\theta^{\pm}$:

$$ \theta^{\pm} = \langle \mathcal{H}, l^{\pm}\rangle = P \pm H\,. \tag{18}$$

The Codazzi equation (10) implies a Codazzi equation for $\chi^{\pm}$.

Lemma 2.1. *For vector fields $X, Y, Z \in \mathcal{X}(\Sigma)$ the following relation holds:*

$$\begin{aligned}\nabla_X \chi^{\pm}(Y,Z) &= \nabla_Y \chi^{\pm}(X,Z) + Q^{\pm}(X,Y,Z)\\ &\qquad \mp \chi^{\pm}(X,Z)S(Y) \pm \chi^{\pm}(Y,Z)S(X)\,.\end{aligned} \tag{19}$$

Here,

$$ Q^{\pm}(X,Y,Z) = {}^{L}\mathrm{Rm}(X,Y,l^{\pm},Z). \tag{20}$$

3. A Simons identity for $\chi^{\pm}$

We use the Codazzi equation we derived in the previous section to compute an identity for the Laplacian of $\chi^{\pm}$, which is very similar to the Simons identity for the second fundamental form of a hypersurface [**Sim68**, **SSY75**].

The Laplacian on the surface Σ is defined as the operator

$$^{\Sigma}\Delta = \gamma^{ij\,\Sigma}\nabla^2_{ij}\,.$$

In the sequel, we will drop the superscript on $^{\Sigma}\Delta$ and $^{\Sigma}\nabla$, since all tensors below will be defined only along Σ. We will switch to index notation, since this is convenient for the computations to follow. In this notation

$$T^{i_1\cdots i_p}_{j_1\cdots j_q}$$

denotes a (p,q)-tensor T as the collection of its components in an arbitrary basis $\{\partial_i\}_{i=1}^2$ for the tangent spaces. To make the subsequent computations easier, we will usually pick a basis of normal coordinate vectors. Also note that we use Latin indices ranging from 1 to 2 to denote components tangential to the surface Σ.

Recall, that the commutator of the connection is given by the Riemann curvature tensor, such that for a $(0,2)$-tensor T_{ij},

$$\nabla_k\nabla_l T_{ij} - \nabla_l\nabla_k T_{ij} = {}^{\Sigma}\mathrm{Rm}_{klmi}T_{mj} + {}^{\Sigma}\mathrm{Rm}_{klmj}T_{im}\,. \tag{21}$$

Note that we use the shorthand ${}^{\Sigma}\mathrm{Rm}_{klmj}T_{im} = {}^{\Sigma}\mathrm{Rm}_{klpj}T_{iq}\gamma^{pq}$ when there is no ambiguity. That is, we assume that we are in normal coordinates where $\gamma_{ij} = \gamma^{ij} = \delta_{ij}$. Also note that this fixes the sign convention for ${}^{\Sigma}\mathrm{Rm}_{ijkl}$ such that ${}^{\Sigma}\mathrm{Rc}_{ij} = {}^{\Sigma}\mathrm{Rm}_{ikkj}$ is positive on the round sphere.

Lemma 3.1. *The Laplacian of $\chi = \chi^+$ satisfies the following identity:*

$$\begin{aligned}\chi_{ij}\Delta\chi_{ij} &= \chi_{ij}\nabla_i\nabla_j\theta^+ + \chi_{ij}\big({}^{L}\mathrm{Rm}_{kilk}\chi_{lj} + {}^{L}\mathrm{Rm}_{kilj}\chi_{kl}\big)\\ &\quad + \chi_{ij}\nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big)\\ &\quad + \chi_{ij}\nabla_i\big(Q_{kjk} - \theta^+S_j + \chi_{jk}S_k\big)\\ &\quad - |\mathrm{I\!I}|^2|\chi|^2 + \theta^+\chi^+_{ij}\chi^+_{jk}\chi^+_{ki} - \theta^+\chi^+_{ij}\chi^+_{jk}K^{\Sigma}_{ki} - P\chi^+_{ij}\chi^+_{jk}\chi^+_{ki}\end{aligned}$$

where $P = \gamma^{ij}K^{\Sigma}_{ij}$ is the trace of K^{Σ}.

Proof. Recall that in coordinates the Codazzi equation (19) for χ_{ij} reads

$$\nabla_i\chi_{jk} = \nabla_j\chi_{ik} + Q_{ijk} - \chi_{ik}S_j + \chi_{jk}S_i\,. \tag{22}$$

Then compute, using (22) in the first and third step, and the commutator relation (21) in the second, to obtain

(23)
$$\begin{aligned}
\nabla_k\nabla_l\chi_{ij} &= \nabla_k\nabla_i\chi_{lj} + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big)\\
&= \nabla_i\nabla_k\chi_{lj} + {}^\Sigma\mathrm{Rm}_{kiml}\chi_{mj} + {}^\Sigma\mathrm{Rm}_{kimj}\chi_{lm}\\
&\quad + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big)\\
&= \nabla_i\nabla_j\chi_{kl} + {}^\Sigma\mathrm{Rm}_{kiml}\chi_{mj} + {}^\Sigma\mathrm{Rm}_{kimj}\chi_{lm}\\
&\quad + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big) + \nabla_i\big(Q_{kjl} - \chi_{kl}S_j + \chi_{jl}S_k\big)\,.
\end{aligned}$$

We will use the Gauss equation (8) to replace the ${}^\Sigma$Rm-terms by ^{L}Rm-terms. Observe that

$$\mathrm{I\!I}_{ij} = -\tfrac12\chi^+_{ij}l^- - \tfrac12\chi^-_{ij}l^+\,.$$

Plugging this into the Gauss equation (8) gives

$${}^\Sigma\mathrm{Rm}_{ijkl} = {}^L\mathrm{Rm}_{ijkl} + \tfrac12\big(\chi^+_{ik}\chi^-_{jl} + \chi^-_{ik}\chi^+_{jl} - \chi^+_{il}\chi^-_{jk} - \chi^-_{il}\chi^+_{jk}\big)\,.$$

Combining with (23), we infer that

$$\begin{aligned}
\nabla_k\nabla_l\chi_{ij} &= \nabla_i\nabla_j\chi_{kl} + {}^L\mathrm{Rm}_{kiml}\chi_{mj} + {}^L\mathrm{Rm}_{kimj}\chi_{lm}\\
&\quad + \tfrac12\big(\chi^+_{il}\chi^-_{km} + \chi^-_{il}\chi^+_{km} - \chi^+_{kl}\chi^-_{im} - \chi^-_{kl}\chi^+_{im}\big)\chi^+_{mj}\\
&\quad + \tfrac12\big(\chi^+_{km}\chi^-_{ij} + \chi^-_{km}\chi^+_{ij} - \chi^+_{kj}\chi^-_{im} - \chi^-_{kj}\chi^+_{im}\big)\chi^+_{lm}\\
&\quad + \nabla_k\big(Q_{lij} - \chi_{lj}S_i + \chi_{ij}S_l\big) + \nabla_i\big(Q_{kjl} - \chi_{kl}S_j + \chi_{jl}S_k\big)\,.
\end{aligned}$$

Taking the trace with respect to k, l yields

$$\begin{aligned}
\Delta\chi_{ij} &= \nabla_i\nabla_j\theta^+ + {}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl}\\
&\quad + \nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big) + \nabla_i\big(Q_{kjk} - \theta^+S_j + \chi_{jk}S_k\big)\\
&\quad + \frac12\big(\chi^-_{ij}|\chi^+|^2 + \langle\chi^+,\chi^-\rangle\chi^+_{ij} - \theta^+\chi^+_{jk}\chi^-_{ki} - \theta^-\chi^+_{jk}\chi^+_{ki}\big)\\
&\quad + \frac12\big(\chi^+_{jk}\chi^-_{kl}\chi^+_{li} - \chi^-_{jk}\chi^+_{kl}\chi^+_{li}\big).
\end{aligned}$$

We contract this equation with χ^+_{ij} and obtain

$$\begin{aligned}
\chi_{ij}\Delta\chi_{ij} &= \chi_{ij}\nabla_i\nabla_j\theta^+ + \chi_{ij}\big({}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl}\big)\\
&\quad + \chi_{ij}\nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big)\\
&\quad + \chi_{ij}\nabla_i\big(Q_{kjk} - \theta^+S_j + \chi_{jk}S_k\big)\\
&\quad + \langle\chi^+,\chi^-\rangle|\chi|^2 - \tfrac12\theta^+\chi^+_{ij}\chi^+_{jk}\chi^-_{ki} - \tfrac12\theta^-\chi^+_{ij}\chi^+_{jk}\chi^+_{ki}\,.
\end{aligned}$$

Now observe that $\chi^-_{ij} = 2K^\Sigma_{ij} - \chi^+_{ij}$ and $\theta^- = 2P - \theta^+$. Substituting this into the last two terms, together with $\langle\chi^+,\chi^-\rangle = -|\mathrm{I\!I}|^2$, we arrive at the identity we claimed. q.e.d.

4. The Deformation of θ^+

This section is concerned with the deformation of the operator θ^+, as defined in equation (18). We begin by considering an arbitrary, spacelike surface $\Sigma \subset L$. Assume that the normal bundle is spanned by the globally defined null vector fields $l^\pm$, such that $\langle l^+, l^- \rangle = -2$. We call such a frame a *normalized null frame*. As before, let $\theta^\pm := \langle \mathcal{H}, l^\pm \rangle$. We abbreviate $\chi = \chi^+$.

A variation of Σ is a differentiable map

$$F : \Sigma \times (-\varepsilon, \varepsilon) \to L : (x,t) \mapsto F(x,t)\,,$$

such that $F(\cdot, 0) = \mathrm{id}_\Sigma$ is the identity map on Σ. The vector field $\frac{\partial F}{\partial t}\big|_{t=0} = V$ is called variation vector field of F. We will only consider variations, with variation vector fields V of the form $V = \alpha l^+ + \beta l^-$.

Note that in this setting, as a normalized null frame is not uniquely defined by its properties, the notion of θ^+ depends on the frame chosen. The freedom we have here is the following. Assume $k^\pm$ is another normalized null frame for the normal bundle of Σ, that is, $h(k^\pm, k^\pm) = 0$ and $h(k^+, k^-) = -2$. Since the null cone at each point is unique, the directions of $k^\pm$ can be aligned with $l^\pm$. But their magnitudes can be different, so $k^+ = e^\omega l^+$ and $k^- = e^{-\omega} l^-$ with a function $\omega \in C^\infty(\Sigma)$.

Therefore, if we want to compute the deformation of θ^+, it will not only depend on the deformation of Σ, as encoded in the deformation vector V. It will also depend on the change of the frame, that is, on the change of the vector l^+, which is an additional degree of freedom.

To expose the nature of that freedom, observe that if $l^\pm(t)$ is a null frame on each $\Sigma_t := F(\Sigma, t)$, then $\frac{\partial l^\pm}{\partial t}\Big|_{t=0}$ is still normal to Σ. On the other hand,

$$0 = \frac{\partial}{\partial t}\Big|_{t=0} \langle l^+, l^+ \rangle = 2\Big\langle \frac{\partial l^+}{\partial t}\Big|_{t=0}, l^+ \Big\rangle \qquad \text{and}$$
$$0 = \frac{\partial}{\partial t}\Big|_{t=0} \langle l^+, l^- \rangle = \Big\langle \frac{\partial l^+}{\partial t}\Big|_{t=0}, l^- \Big\rangle + \Big\langle \frac{\partial l^-}{\partial t}\Big|_{t=0}, l^+ \Big\rangle$$

Therefore $\frac{\partial l^\pm}{\partial t}\Big|_{t=0} = w l^\pm$ for a function $w \in C^\infty(\Sigma)$. Thus the linearized change of the frame is described by the single function w, which we will call the *variation of the null frame*.

If we fix both of the quantities V and w, a straightforward (but lengthy) computation gives the deformation of θ^+.

Lemma 4.1. *Assume $F : \Sigma \times (-\varepsilon, \varepsilon) \to L$ is a variation of Σ with variation vector field $V = \alpha l^+ + \beta l^-$. Assume further that the variation of the null frame is w. Then the variation of θ^+ is given by*

$$\begin{aligned}\delta_{V,w}\theta^+ &= 2\Delta\beta - 4S(\nabla\beta) - \alpha\big(|\chi|^2 + {}^L\mathrm{Rc}(l^+, l^+)\big) + 2\theta^+ w \\ &\quad - \beta\big(2\,\mathrm{div}\, S - 2|S|^2 - |\mathrm{I\!I}|^2 + {}^L\mathrm{Rc}(l^+, l^-) - \tfrac{1}{2}{}^L\mathrm{Rm}(l^+, l^-, l^-, l^+)\big)\,.\end{aligned}$$

If we consider marginally trapped surfaces, then the term $\theta^+ w$ in the previous calculation vanishes, and we get expressions independent of the change in the frame. As a consequence, we state the following two corollaries, which also restrict the variations we take into account.

Corollary 4.2. *Assume Σ is a marginally trapped surface, that is, it satisfies the equation $\theta^+ = 0$. Then the deformation of θ^+ in direction of $-l^-$ is given by*

$$\delta_{-\beta l^-, w}\theta^+ = 2L_-\beta\,,$$

where the operator L_- is given by

$$L_-\beta = -\Delta\beta + 2S(\nabla\beta) + \beta\big(\operatorname{div} S - \tfrac{1}{2}|\mathrm{I\!I}|^2 - |S|^2 - \Psi_-\big)\,,$$

and $\Psi_- = \frac{1}{4}{}^L\mathrm{Rm}(l^+, l^-, l^-, l^+) - \frac{1}{2}{}^L\mathrm{Rc}(l^+, l^-)$.

If we assume that $\Sigma \subset M$, where M is a three-dimensional spacelike surface, then Σ can be deformed in the direction of ν, the normal of Σ in M. The deformation of θ^+ then turns out to be the following.

Corollary 4.3. *Assume Σ is a marginally trapped surface; then the deformation of θ^+ in the spatial direction of $\nu := \frac{1}{2}(l^+ - l^-)$ is given by*

$$\delta_{f\nu, w} = L_M f\,,$$

where the operator L_M is given by

$$L_M f = -\Delta f + 2S(\nabla f) + f\big(\operatorname{div} S - |\chi|^2 + \langle K^\Sigma, \chi\rangle - |S|^2 - \Psi_M\big)\,,$$

and $\Psi_M = \frac{1}{4}{}^L\mathrm{Rm}(l^+, l^-, l^-, l^+) + {}^L\mathrm{Rc}(\nu, l^+)$.

Remark 4.4. 1) Using the Gauss equation (9), we can rewrite the expression for L_M as follows:

$$L_M f = -\Delta f + 2S(\nabla f) + f\big(\operatorname{div} S - \tfrac{1}{2}|\chi|^2 - |S|^2 + \tfrac{1}{2}{}^\Sigma\mathrm{Sc} - \tilde\Psi_M\big)\,. \tag{24}$$

Here $\tilde\Psi_M = G(n, l^+)$ where $G = {}^L\mathrm{Rc} - \frac{1}{2}{}^L\mathrm{Sc}\,h$ denotes the Einstein tensor of h.

Note that in view of the Gauss and Codazzi equations of the embedding $M \hookrightarrow L$, equations (13) and (14), the term $\tilde\Psi_M$ can be rewritten as

$$\begin{aligned}\tilde\Psi_M &= \tfrac{1}{2}\big({}^M\mathrm{Sc} + (\operatorname{tr} K)^2 - |K|^2\big) - \langle {}^M\mathrm{div}K - {}^M\nabla \operatorname{tr} K, \nu\rangle \\ &=: \mu + J(\nu)\,,\end{aligned} \tag{25}$$

where $J = {}^M\mathrm{div}K - {}^M\nabla \operatorname{tr} K$ is the projection of $G(n,\cdot)$ to M and $\mu = \frac{1}{2}({}^M\mathrm{Sc} + (\operatorname{tr} K)^2 - |K|^2) = G(n,n)$. The dominant energy condition is equivalent to $|J| \leq \mu$. Thus, if the dominant energy condition holds, $\tilde\Psi_M$ turns out to be non-negative.

2) The same procedure gives that we can write L_- as

$$L_- f = -\Delta f + 2S(\nabla f) + f\big(\operatorname{div} S - |S|^2 + \tfrac{1}{2}{}^\Sigma\mathrm{Sc} - \tilde\Psi_-\big)\,. \tag{26}$$

with $\tilde{\Psi}_- = G(l^+, l^-)$. Note that $\tilde{\Psi}_-$ is non-negative if the dominant energy condition holds. However, this representation does not contain a term $|\chi|^2$, which does not allow us to get estimates on $\sup|\chi|^2$. However, in the case of strict L_- stability there is a sheet M such that the surface is L_M-stable. We can then apply the subsequent results to get the estimates of theorem 1.2 in this case.

5. Stability of marginally outer trapped surfaces

As before, consider a four-dimensional space time L^4, with a three-dimensional spacelike slice M^3. As in the previous sections, the future directed unit normal to M in L will be denoted by n. In M consider a two-dimensional surface Σ, such that there exists a global unit normal vector field ν of Σ in M. The vector fields n and ν span the normal bundle of Σ in L and give rise to two canonical null vectors $l^\pm = n \pm \nu$. Again we use the shorthand $\chi = \chi^+$.

In this section we introduce two notions of stability for a marginally trapped surface. These are related to variations of the surface in different directions. The first definition is equivalent to definition 2 in [**AMS05**]. There a *stably outermost marginally outer trapped surface*, is defined as surface, on which the principal eigenvalue of L_M is positive. We recall from definition 1.1 that an *L_M-stable MOTS* is defined as follows.

Definition 5.1. A two-dimensional surface $\Sigma \subset M \subset L$ is called a *L_M-stable marginally outer trapped surface* if:

1) Σ is marginally trapped with respect to l^+ , that is $\theta^+ = 0$.
2) There exists a function $f \geq 0, f \not\equiv 0$ such that $L_M f \geq 0$. Here L_M is the operator from corollary 4.3.

Remark 5.2. 1) Although L_M is not formally self-adjoint, the eigenvalue of L_M with the smallest real part is real and non-negative (cf. [**AMS05**, Lemma 1]). This definition is equivalent to saying that the principal eigenvalue of L_M is non-negative. This is seen as follows:

Let λ be the principal eigenvalue L_M. Then, since λ is real, the L^2-adjoint L_M^* of L_M has the same principal eigenvalue and a corresponding eigenfunction $g > 0$. Pick $f \geq 0$ as in the definition of L_M-stability, i.e., $L_M f \geq 0$. Then compute

$$\lambda \int_\Sigma fg \,\mathrm{d}\mu = \int_\Sigma f L_M^* g \,\mathrm{d}\mu = \int_\Sigma L_M f g \,\mathrm{d}\mu \,.$$

As $f \geq 0$, $f \not\equiv 0$, $g > 0$, and $L_M f \geq 0$, this implies $\lambda \geq 0$.

The eigenfunction ψ of L_M with respect to the principal eigenvalue does not change sign. Therefore it can be chosen positive,

$\psi > 0$. Thus, the definition in fact is equivalent to the existence of $\psi > 0$ such that $L_M\psi = \lambda\psi \geq 0$. We will use this fact frequently in the subsequent sections. Note that L_M-stability is equivalent to the notion of a *stably outermost MOTS* in [**AMS05**, definition 2].

2) The conditions from the above definition are satisfied in the following situation. Let $\Sigma = \partial\Omega$ be the boundary of the domain Ω and satisfy $\theta^+ = 0$. Furthermore assume that there is a neighborhood U of Σ such that the exterior part $U \setminus \Omega$ does not contain any trapped surface, i.e., a surface with $\theta^+ < 0$. Then Σ is stable. Assume not. Then the principal eigenvalue would be negative and the corresponding eigenfunction ψ would satisfy $L_M\psi < 0$, $\psi > 0$. This would imply the existence of trapped surfaces outside of Σ, since the variation of Σ in direction $\psi\nu$ would decrease θ^+.

Note that the condition $\theta^+ = 0$ does not depend on the choice of the particular frame. Therefore, to say that a surface is marginally trapped, we do not need any additional information. In contrast, the notion of stability required here does depend on the frame, since clearly there is no distinct selection of ν when only Σ—and not M—is specified.

To address this issue, we introduce the second notion of stability of marginally outer trapped surfaces, namely with reference to the direction $-l^-$. This definition is more in spirit of Newman [**New87**] and recent interest in the so called dynamical horizons [**AK03**, **AG05**].

Definition 5.3. A two-dimensional surface $\Sigma \subset M \subset L$ is called a *L_--stable marginally outer trapped surface (L_--stable MOTS)* if:

1) Σ is marginally trapped with respect to l^+ , that is $\theta^+ = 0$.
2) There exists a function $f \geq 0, f \not\equiv 0$ such that $L_- f \geq 0$. Here L_- is the operator from corollary 4.2.

Remark 5.4. It turns out that this notion of stability does not depend on the choice of the null frame. This is due to the natural transformation law of the stability operator L_- when changing the frame according to $\tilde{l}^+ = fl^+$ and $\tilde{l}^- = f^{-1}l^-$. Then the operator $\tilde{L}_-$ with respect to this frame satisfies $f^{-1}\tilde{L}(f\beta) = L\beta$ for all functions $\beta \in C^\infty(\Sigma)$, as it is expected from the facts that $\tilde{\theta}^+ = f\theta^+$ and $-\beta l^- = -\beta f\tilde{l}^-$.

Remark 5.5. 1) Remark 5.2 is also valid here, in particular the definition implies that there exists a function $\psi > 0$ with $L_-\psi \geq 0$.

2) Technically speaking, the equation for a marginally trapped surface prescribes the mean curvature H of Σ in M to equal minus the value of a function $P : TM \to \mathbf{R} : (p, v) \mapsto trK - K_{ij}\nu^i\nu^j$, namely, $H(p) = -P(p, \nu)$ for all $p \in \Sigma$. This is a degenerate quasilinear elliptic equation for the position of the surface. These equations do not allow estimates for second derivatives without any additional information. This is where the two stability conditions come into

play. They give the additional piece of information needed in the estimates as in the case for stable minimal surfaces.

The two notions of stability above imply the positivity of certain symmetric differential operators as it was noticed in [**GS06**] for the operator L_M. However, the inequality there is not quite sufficient for our purposes; it needs some further rearrangement. This is the content of the following Lemmas. Basically lemmas 5.6 and 5.7 are the only way how stability is used in the subsequent estimates. Thus one could use these, in particular equation (27), to define a notion of symmetrized stability for MOTS.

Lemma 5.6. *If Σ is a stable MOTS, then for all $\varepsilon > 0$ and for all $\eta \in C_c^\infty(\Sigma)$ the following inequality holds:*

$$\int_\Sigma \eta^2|\chi|^2 \,\mathrm{d}\mu \leq (1+\varepsilon)\int_\Sigma |\nabla\eta|^2 + \eta^2\big(\tfrac{1+\varepsilon}{4\varepsilon}|K^\Sigma|^2 - \Psi_M\big)\,\mathrm{d}\mu.$$

Proof. Take f as in the definition of a stable MOTS. From remark 5.2 we can assume $f > 0$. Then $f^{-1}L_M f \geq 0$. Multiply this relation by η^2, integrate, and expand L_M as in corollary 4.3. This yields

$$0 \leq \int_\Sigma \eta^2\big(-f^{-1}\Delta f + 2f^{-1}S(\nabla f) + \operatorname{div} S - |\chi|^2 + \langle K^\Sigma, \chi\rangle - |S|^2 - \Psi_M\big)\,\mathrm{d}\mu\,.$$

By sorting terms, and partially integrating the Laplacian and the divergence term, we obtain

$$\begin{aligned}\int_\Sigma \eta^2|\chi|^2 &+ \eta^2\big(f^{-2}|\nabla f|^2 - 2f^{-1}S(\nabla f) + |S|^2\big)\,\mathrm{d}\mu\\ &\leq \int_\Sigma 2\eta\langle\nabla\eta, f^{-1}\nabla f - S\rangle + \eta^2|\chi|\,|K^\Sigma| - \eta^2\Psi_M\,\mathrm{d}\mu\,.\end{aligned}$$

By the Schwarz inequality

$$\int_\Sigma 2\eta\langle\nabla\eta, f^{-1}\nabla f - S\rangle\,\mathrm{d}\mu \leq \int_\Sigma |\nabla\eta|^2 + \eta^2|f^{-1}\nabla f - S|^2\,\mathrm{d}\mu,$$

and for any $\varepsilon > 0$

$$\int_\Sigma \eta^2|K^\Sigma|\,|\chi|\,\mathrm{d}\mu \leq (4\varepsilon)^{-1}\int_\Sigma \eta^2|K^\Sigma|^2\,\mathrm{d}\mu + \varepsilon\int_\Sigma \eta^2|\chi|^2\,.$$

Cancelling the terms $\int_\Sigma \eta^2|f^{-1}\nabla f - S|^2\,\mathrm{d}\mu$ and $\varepsilon\int_\Sigma \eta^2|\chi|^2\,\mathrm{d}\mu$ on both sides and redefining ε, we conclude the claimed inequality. q.e.d.

The following lemma is based on the original computation of [**GS06**].

Lemma 5.7. *If Σ is a stable MOTS, then for all $1 > \varepsilon > 0$ there exist constants c and $C(\varepsilon^{-1})$ such that for all $\eta \in C_c^\infty(\Sigma)$ the following inequality holds:*

$$\int_\Sigma \eta^2|\chi|^2\,\mathrm{d}\mu \leq (1+\varepsilon)\int_\Sigma |\nabla\eta|^2 + \eta^2\big(c|{}^M\mathrm{Rc}| - \tilde\Psi_M + C(\varepsilon^{-1})|K^\Sigma|^2\big)\,\mathrm{d}\mu.$$

Proof. We proceed as in the computation of lemma 5.6, but with the alternative representation (24) for L_M. As in [**GS06**], we get

$$\int_\Sigma \eta^2|\chi|^2 \,\mathrm{d}\mu \le \int_\Sigma 2|\nabla\eta|^2 + \eta^2\big({}^\Sigma\mathrm{Sc} - 2\tilde\Psi_M\big)\,\mathrm{d}\mu. \tag{27}$$

We use Gauss' equation to replace ${}^\Sigma\mathrm{Sc}$ in the following way:

$${}^\Sigma\mathrm{Sc} = {}^M\mathrm{Sc} - 2{}^M\mathrm{Rc}(\nu,\nu) + H^2 - |A|^2,$$

where A is the second fundamental form of $\Sigma \subset M$ and H is the mean curvature. We can move the $|A|^2$ term to the left-hand side. Then $H^2 = P^2$ by $\theta^+ = 0$ and thus $H^2 \le 4|K^\Sigma|^2$. The remaining terms are controlled by $3|{}^M\mathrm{Rc}|$. Inserting this, we find that

$$\int_\Sigma \eta^2(|\chi|^2 + |A|^2)\,\mathrm{d}\mu \le \int_\Sigma 2|\nabla\eta|^2 + \eta^2\big(3|{}^M\mathrm{Rc}| + 4|K^\Sigma|^2 - \tilde\Psi_M\big)\,\mathrm{d}\mu.$$

Now fix $\varepsilon > 0$. Since $\chi = A + K^\Sigma$ we can estimate

$$\begin{aligned} 2|\chi|^2 &\le (1+\varepsilon)|\chi|^2 + (1-\varepsilon)(|A|^2 + 2\langle A, K^\Sigma\rangle + |K^\Sigma|^2) \\ &\le (1+\varepsilon)(|\chi|^2 + |A|^2) + ((2\varepsilon)^{-1} - 2\varepsilon)|K^\Sigma|^2. \end{aligned}$$

Inserting this into the above inequality we find the claimed inequality. q.e.d.

A similar, but fundamentally different, inequality holds in the case of L_--stability. The fundamental difference is that the gradient term on the right has a factor of a little more than two, instead of a little more than one, as with the L_M-stability. In view of lemma 5.9 this factor of two is not at all surprising. This factor is the reason why the procedure in section 6 does not work to give curvature estimates for L_--stable surfaces.

Lemma 5.8. *If Σ is a L_--stable MOTS, then for all $\varepsilon > 0$ the following inequality holds:*

$$\int_\Sigma |\chi|^2\eta^2 \,\mathrm{d}\mu \le 2(1+\varepsilon)\int_\Sigma |\nabla\eta|^2 + \eta^2\big((2\varepsilon)^{-1}|K^\Sigma|^2 - \Psi_-\big)\,\mathrm{d}\mu.$$

We conclude with the remark that L_M-stability implies L_--stability.

Lemma 5.9. *Let (L,h) satisfy the null energy condition, i.e., assume that for all null vectors l we have that ${}^L\mathrm{Rc}(l,l) \ge 0$. Then if Σ is an L_M-stable MOTS, then it is also L_--stable.*

Proof. We use the notation from section 4, where we introduced the linearization of θ^+. For any function f compute

$$\begin{aligned} L_M f - L_- f &= \delta_{f\nu,w}\theta^+ - \tfrac{1}{2}\delta_{fl^-,w}\theta^+ = \tfrac{1}{2}\delta_{fl^+,w}\theta^+ \\ &= -\tfrac{1}{2} f\big(|\chi|^2 + {}^L\mathrm{Rc}(l^+,l^+)\big)\,. \end{aligned}$$

If $f > 0$, then by the null energy condition, the right-hand side is non-positive. If in addition $L_M f \geq 0$, as in the definition of L_M-stability, then this implies that

$$L_- f \geq L_M f \geq 0\,.$$

Hence Σ is also L_M stable. q.e.d.

6. A priori estimates

In this section we derive the actual estimates for stable outermost marginally trapped surfaces. All but the most basic estimates hold only for L_M-stable surfaces, as defined in section 5. This is due to the factor of two appearing in front of the gradient term in lemma 5.8, which does not allow us to carefully balance the Simons inequality and the stability inequalities.

Throughout this section we will make the assumption that the surfaces in question have *locally uniformly finite area*.

Definition 6.1. If there exists $r > 0$ and $a < \infty$, such that for all $x \in \Sigma$

$$|B_\Sigma(x,r)| \leq a, \tag{28}$$

then we say that Σ has (r,a)-locally uniformly finite area.

Here $B_\Sigma(x,r)$ denotes the the ball of radius r around x in Σ. In the sequel we will denote $B_\Sigma(x,r)$ by $B(x,r)$. The estimates below work in exactly the same way if intrinsic balls are replaced by extrinsic balls. Later, we will derive such bounds for stable MOTS in terms of the ambient geometry.

We first begin with the observation that the stability of a MOTS gives a local L^2-estimate for the shear tensor $\chi = \chi^+$.

In the sequel, for a tensor T, we denote by $\|T\|_\infty = \sup_\Sigma |T|$. That is, ∞-norms are taken on Σ only. Constant are always denoted by C; if we want to clarify the dependence of the constants, we denote by $C(a,b,\ldots)$ a constant that depends on the quantities $a, b\ldots$ in such a way that C deteriorates as $a + b + \ldots \to \infty$.

Lemma 6.2. *Suppose Σ is an L_M-stable MOTS with (r,a)-locally uniformly finite area. Then for all $x \in \Sigma$*

$$\int_{B(x,r/2)} |\chi|^2 \,\mathrm{d}\mu \leq C(r^{-1}, a, \|K^\Sigma\|_\infty, \|\Psi_M\|_\infty)\,.$$

Alternatively, the constant can be chosen to depend on $\|\tilde\Psi_M\|_\infty$ and $\|{}^M\mathrm{Rc}\|_\infty$ instead of $\|\Psi_M\|_\infty$.

Proof. The desired bound is easily derived from lemma 5.6 or 5.7. We will restrict ourselves to the proof of the first statement. To this end, fix $\varepsilon = \frac{1}{2}$, $x \in \Sigma$, and choose a cut-off function $\eta \geq 0$ such that $\eta \equiv 1$ on

$B(x,r/2)$, $\eta = 0$ on $\partial B(x,r)$, and $|\nabla\eta| \leq 4r^{-1}$. The left-hand side of the equation in lemma 5.6 is then is an upper bound for the left-hand side in the claim, whereas the right-hand side can be estimated by the claimed quantities. q.e.d.

This estimate can also be derived from L_--stability:

Lemma 6.3. *Suppose Σ is an L_--stable MOTS with (r,a)-locally uniformly finite area. Then for all $x \in \Sigma$*

$$\int_{B(x,r/2)} |\chi|^2 \,\mathrm{d}\mu \leq C(r^{-1}, a, \|K^\Sigma\|_\infty, \|\Psi_-\|_\infty)\,.$$

Proposition 6.4. *Let Σ be an L_M-stable MOTS. For any $\varepsilon > 0$, any $p \geq 2$, and any function η we have the estimate*

$$\begin{aligned}\int_\Sigma \eta^2|\chi|^{p+2}\,\mathrm{d}\mu \leq \tfrac{p^2}{4}(1+\varepsilon)\int_\Sigma \eta^2|\chi|^{p-2}\big|\nabla|\chi|\big|^2\,\mathrm{d}\mu \\ + C(\varepsilon^{-1}, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty)\int_\Sigma \big(\eta^2 + |\nabla\eta|^2\big)|\chi|^p\,\mathrm{d}\mu.\end{aligned}$$

Alternatively, we can make the constant on the right-hand side to be of the form $C(\varepsilon^{-1}, \|\tilde\Psi_M\|_\infty, \|K^\Sigma\|_\infty, \|{}^M\mathrm{Rc}\|_\infty))$.

This proposition also holds for L_--stable surfaces with appropriate modifications of the dependencies of C, and a general factor of two on the right-hand side. This factor of two is the reason why an argument like the subsequent one fails to give curvature estimates for L_--stable MOTS.

Proof. We will restrict to the proof of the first statement, since the other is proved similarly. From lemma 5.6 we find that for any $\delta > 0$ there is $C(\delta^{-1})$ such that for all functions ϕ, we have

$$\int_\Sigma \phi^2|\chi|^2\,\mathrm{d}\mu \leq (1+\delta)\int_\Sigma |\nabla\phi|^2 + \phi^2\big(C(\delta^{-1})|K^\Sigma|^2 - \Psi_M\big)\,\mathrm{d}\mu.$$

We substitute ϕ by $\eta|\chi|^{p/2}$. To this end compute

$$\nabla(\eta|\chi|^{p/2}) = \nabla\eta|\chi|^{p/2} + \eta\,\tfrac{p}{2}|\chi|^{p/2-1}\nabla|\chi|.$$

For any $\delta > 0$ we thus can estimate

$$|\nabla(\eta|\chi|^{p/2})|^2 \leq (1-\delta)\frac{p^2}{4}|\chi|^{p-2}\eta^2\big|\nabla|\chi|\big|^2 + C(\delta^{-1})|\nabla\eta|^2|\chi|^p.$$

Inserting this estimate into the first inequality, we find

$$\begin{aligned}\int_\Sigma \eta^2|\chi|^{p+2}\,\mathrm{d}\mu \leq (1+\delta)^2\,\tfrac{p^2}{4}\int_\Sigma \eta^2|\chi|^{p-2}\big|\nabla|\chi|\big|^2 \\ + \int_\Sigma C(\delta^{-1})|\nabla\eta|^2 + \eta^2|\chi|^p\big(C(\delta^{-1})|K^\Sigma|^2 - \Psi_M\big)\,\mathrm{d}\mu.\end{aligned}$$

Adjusting δ yields the claimed estimate. q.e.d.

We now aim for an estimate on the gradient term on the right-hand side of the estimate in proposition 6.4. The main tool will be the Simons identity from section 3. To avoid that the estimated depend on derivatives of curvature, we use similar techniques as in [**Met07**].

Proposition 6.5. *Let Σ be an L_M-stable MOTS. Then there exists $p_0 > 2$ such that for $2 \leq p \leq p_0$ and all functions η we have the estimate*

$$\begin{aligned}\int_\Sigma \eta^2 |\chi|^{p-2} \big|\nabla|\chi|\big|^2 \,\mathrm{d}\mu &\\ \leq C(p, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty)&\\ \cdot \int_\Sigma (\eta^2 + |\nabla\eta|^2)|\chi|^p + \eta^2|\chi|^{p-2}.&\end{aligned}$$

Alternatively, as before, we can replace the dependence of the constant on $\|\Psi_M\|_\infty$ by a dependence on $\|\tilde{\Psi}_M\|_\infty$ and $\|{}^M\mathrm{Rc}\|_\infty$.

Before we can start the proof of the proposition, we state the following lemma. It states an improved Kato's inequality similar to [**SY81**]. A general reference for such inequalities is [**CGH00**].

Lemma 6.6. *On a surface Σ with $\theta^+ = 0$ we have the estimate*

$$|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2 \geq \tfrac{1}{33}\big(\big|\nabla|\chi|\big|^2 + |\nabla\chi|^2\big) - c\left(|Q|^2 + |S|^2|\chi|^2\right).$$

Here c is a purely numerical constant.

Proof. The proof goes along the lines of a similar argument in Schoen and Yau in [**SY81**, p. 237], but for the sake of completeness, we include a sketch of it here.

In the following computation we do not use the Einstein summation convention and work in a local orthonormal frame for $T\Sigma$. Let $T := |\nabla\chi|^2 - \big|\nabla|\chi|\big|^2$. We compute

$$\begin{aligned}|\chi|^2 T &= |\chi|^2|\nabla\chi|^2 - \tfrac{1}{4}\big|\nabla|\chi|^2\big|^2\\ &= \sum_{i,j,k,l,m} (\chi_{ij}\nabla_k\chi_{lm})^2 - \sum_k \Big(\sum_{ij} \chi_{ij}\nabla_k\chi_{ij}\Big)^2\\ &= \tfrac{1}{2}\sum_{i,j,k,l,m}\Big(\chi_{ij}\nabla_k\chi_{lm} - \chi_{lm}\nabla_k\chi_{ij}\Big)^2.\end{aligned}$$

In the last term consider only summands with $i = k$ and $j = m$. This gives

$$\begin{aligned}|\chi|^2 T &\geq \tfrac{1}{2}\sum_{i,j,l}\Big(\chi_{ij}\nabla_i\chi_{jl} - \chi_{jl}\nabla_i\chi_{ij}\Big)^2\\ &\geq \tfrac{1}{8}\sum_l \Big(\sum_{i,j}\chi_{ij}\nabla_i\chi_{jl} - \chi_{jl}\nabla_i\chi_{ij}\Big)^2.\end{aligned}$$

Use the Codazzi equation (22) to swap indices in the gradient terms. We arrive at

$$|\chi|^2 T \geq \tfrac{1}{8}\sum_l \Big(\sum_{i,j}\big(\chi_{ij}\nabla_l\chi_{ij} + \chi_{ij}Q_{ilj} - \chi_{lj}Q_{iji}\big) + \sum_i \big(\theta S_i\chi_{il} - \chi_{il}\nabla_i\theta\big) - |\chi|^2 S_l\Big)^2.$$

By the fact that $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$, this implies

$$\begin{aligned}
|\chi|^2 T &\geq \tfrac{1}{16}\sum_l\Big(\sum_{i,j}\chi_{ij}\nabla_l\chi_{ij}\Big)^2 \\
&\quad - \tfrac{1}{8}\sum_l\Big(\sum_{i,j}\big(\chi_{ij}Q_{ilj} - \chi_{lj}Q_{iji}\big) + \sum_i \chi_{il}(\theta S_i - \nabla_i\theta) - |\chi|^2 S_l\Big)^2 \\
&\geq \tfrac{1}{16}|\chi|^2\big|\nabla|\chi|\big|^2 - c\,|\chi|^2\big(|Q|^2 + |S|^2|\chi|^2\big).
\end{aligned}$$

Dividing by $|\chi|^2$, we get

$$|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2 \geq \tfrac{1}{16}\big|\nabla|\chi|\big|^2 - c\big(|Q|^2 + |S|^2|\chi|^2\big).$$

Adding $\frac{1}{32}\big(|\nabla\chi|^2 - \big|\nabla|\chi|\big|^2\big)$ to both sides of this inequality and multiplying by $\frac{32}{33}$ yields the desired estimate. q.e.d.

Proof of proposition 6.5. We will restrict to the proof of the first statement, since the other is proved similarly. Compute

$$\Delta|\chi|^2 = 2|\chi|\Delta|\chi| + 2\big|\nabla|\chi|\big|^2.$$

On the other hand,

$$\Delta|\chi|^2 = 2\chi_{ij}\Delta\chi_{ij} + 2|\nabla\chi|^2.$$

Subtracting these equations yields

$$|\chi|\Delta|\chi| = \chi_{ij}\Delta\chi_{ij} + |\nabla\chi|^2 - \big|\nabla|\chi|\big|^2.$$

In the case $\theta^+ = 0$, the Simons identity from lemma 3.1 gives

$$\begin{aligned}
\chi_{ij}\Delta\chi_{ij} &= \chi_{ij}\big({}^L\mathrm{Rm}_{kilk}\chi_{lj} + {}^L\mathrm{Rm}_{kilj}\chi_{kl}\big) - |\mathrm{I\!I}|^2|\chi|^2 - P\chi_{ij}\chi_{jk}\chi_{ki} \\
&\quad + \chi_{ij}\nabla_k\big(Q_{kij} - \chi_{kj}S_i + \chi_{ij}S_k\big) + \chi_{ij}\nabla_i\big(Q_{kjk} + \chi_{jk}S_k\big).
\end{aligned}$$

Note that $\chi_{ij}\chi_{jk}\chi_{ki} = \mathrm{tr}(\chi^3)$, and the trace of a 2×2 matrix A satisfies the relation $\mathrm{tr}\,A^3 = \mathrm{tr}\,A(\mathrm{tr}\,A^2 - \det A)$. Since χ is traceless, this term vanishes. In addition, $|\mathrm{I\!I}|^2 = \langle\chi,\chi^-\rangle = |\chi|^2 - 2\langle K^\Sigma,\chi\rangle$.

As we are not interested in the particular form of some terms, to simplify notation, we introduce the $*$-notation. For two tensors T_1 and T_2, the expression $T_1 * T_2$ denotes linear combinations of contractions of $T_1 \otimes T_2$.

To remember that in the above equation we need to evaluate ${}^{L}\mathrm{Rm}$ only on vectors tangential to Σ, we use the projection of ${}^{L}\mathrm{Rm}$ to $T\Sigma$ and denote this by ${}^{L}\mathrm{Rm}^{\Sigma}$. Then the above equations combine to

$$\begin{aligned}&-|\chi|\Delta|\chi|+|\nabla\chi|^2-\big|\nabla|\chi|\big|^2\\&\quad=|\chi|^4+|\chi|^2*\chi*K^{\Sigma}+\chi*\chi*{}^{L}\mathrm{Rm}^{\Sigma}+\chi*\nabla\big(Q+\chi*S\big)\,.\end{aligned}\tag{29}$$

Multiply this equation by $\eta^2|\chi|^{p-2}$ and integrate. This yields

$$\begin{aligned}&\int_\Sigma -\eta^2|\chi|^{p-1}\Delta|\chi|+\eta^2|\chi|^{p-2}\big(|\nabla\chi|^2-\big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu\\&=\int_\Sigma \eta^2\big(|\chi|^{p+2}+|\chi|^p\chi*K^{\Sigma}+|\chi|^{p-2}\chi*\chi*{}^{L}\mathrm{Rm}^{\Sigma}\\&\qquad+|\chi|^{p-2}\chi*\nabla(Q+\chi*S)\big)\,\mathrm{d}\mu.\end{aligned}$$

Next, do a partial integration on the term including the Laplacian and on the last term on the second line. We find that

$$\begin{aligned}&\int_\Sigma \eta^2(p-1)|\chi|^{p-2}\big|\nabla|\chi|\big|^2+\eta^2|\chi|^{p-2}\big(|\nabla\chi|^2-\big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu\\&\le\int_\Sigma \eta^2|\chi|^{p+2}\,\mathrm{d}\mu\\&\quad+c\int_\Sigma \eta|\chi|^{p-1}|\nabla\eta|\,|\nabla|\chi||+\eta|\chi|^{p-1}|\nabla\eta|(|Q|+|\chi||S|)\,\mathrm{d}\mu\\&\quad+c\int_\Sigma \eta^2\big\{|\chi|^{p+1}|K^{\Sigma}|+|\chi|^p|{}^{L}\mathrm{Rm}^{\Sigma}|\\&\qquad\quad+|\chi|^{p-2}\big(|\nabla\chi|+\big|\nabla|\chi|\big|\big)(|Q|+|\chi||S|)\big\}\,\mathrm{d}\mu\,.\end{aligned}\tag{30}$$

Here c is a purely numerical constant. For any $\varepsilon>0$, we can estimate

$$c\int_\Sigma \eta^2|\chi|^{p+1}|K^{\Sigma}|\,\mathrm{d}\mu\le\varepsilon\int_\Sigma \eta^2|\chi|^{p+2}\,\mathrm{d}\mu+C(\varepsilon^{-1})\int_\Sigma \eta^2|\chi|^p|K^{\Sigma}|^2\,\mathrm{d}\mu$$

as well as

$$\begin{aligned}&c\int_\Sigma \eta^2|\chi|^{p-2}\big(|\nabla\chi|+\big|\nabla|\chi|\big|\big)(|Q|+|\chi||S|)\,\mathrm{d}\mu\\&\quad\le\varepsilon\int_\Sigma \eta^2|\chi|^{p-2}\big(|\nabla\chi|^2+\big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu\\&\qquad+C(\varepsilon^{-1})\int_\Sigma \eta^2\big(|\chi|^p|S|^2+|\chi|^{p-2}|Q|^2\big)\,\mathrm{d}\mu\,.\end{aligned}$$

In addition we estimate

$$\begin{aligned}&c\int_\Sigma \eta|\chi|^{p-1}|\nabla\eta|\,|\nabla|\chi||\,\mathrm{d}\mu\\&\quad\le\varepsilon\int_\Sigma \eta^2\big|\nabla|\chi|\big|^2|\chi|^{p-2}+C(\varepsilon^{-1})\int_\Sigma |\nabla\eta|^2|\chi|^p\,\mathrm{d}\mu,\end{aligned}$$

and

$$\int_\Sigma \eta|\chi|^{p-1}|\nabla\eta|(|Q|+|\chi||S|)\,\mathrm{d}\mu \\ \leq \int_\Sigma |\nabla\eta|^2|\chi|^p + \eta^2\big(|\chi|^{p-2}|Q|^2+|\chi|^p|S|^2\big)\,\mathrm{d}\mu.$$

Inserting these estimates into the estimate (30) gives

(31)

$$\begin{aligned}&\int_\Sigma \eta^2(p-1)|\chi|^{p-2}\big|\nabla|\chi|\big|^2 + \eta^2|\chi|^{p-2}\big(|\nabla\chi|^2-\big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu\\ &\leq \int_\Sigma (1+\varepsilon)\eta^2|\chi|^{p+2} + 2\varepsilon\eta^2|\chi|^{p-2}\big(|\nabla\chi|^2+\big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu\\ &\quad + C(\varepsilon^{-1})\int_\Sigma |\nabla\eta|^2|\chi|^p\,\mathrm{d}\mu\\ &\quad + C(\varepsilon^{-1},\|K^\Sigma\|_\infty,\|{}^L\mathrm{Rm}^\Sigma\|_\infty,\|S\|_\infty,\|Q\|_\infty)\int_\Sigma \eta^2\big(|\chi|^p+|\chi|^{p-2}\big)\,\mathrm{d}\mu.\end{aligned}$$

We apply lemma 6.6 to estimate the second term on the left-hand side from below by $\frac{1}{33}\int_\Sigma \eta^2\big(\big|\nabla|\chi|\big|^2+|\nabla\chi|^2\big)\,\mathrm{d}\mu$. In addition, use proposition 6.4 to estimate the first term on the right-hand side. This yields

$$\begin{aligned}&\int_\Sigma \eta^2(p-1)|\chi|^{p-2}\big|\nabla|\chi|\big|^2 + (\tfrac{1}{33}-2\varepsilon)\eta^2|\chi|^{p-2}\big(|\nabla\chi|^2+\big|\nabla|\chi|\big|^2\big)\,\mathrm{d}\mu\\ &\quad\leq \tfrac{p^2}{4}(1+\varepsilon)^2\int_\Sigma \eta^2|\chi|^{p-2}\big|\nabla|\chi|\big|^2\,\mathrm{d}\mu\\ &\qquad + C(\varepsilon^{-1},\|K^\Sigma\|_\infty,\|Q\|_\infty,\|{}^L\mathrm{Rm}^\Sigma\|_\infty,\|S\|_\infty)\\ &\qquad\qquad \cdot\int_\Sigma \eta^2\big(|\chi|^p+|\chi|^{p-2}\big)+|\nabla\eta|^2|\chi|^p\,\mathrm{d}\mu\,.\end{aligned}$$

Choose $p_0 > 2$ close enough to 2 and ε small enough, such that for $2 < p < p_0$ the gradient term on the right-hand side can be absorbed on the left-hand side. This gives the desired estimate. q.e.d.

Combining propositions 6.4 and 6.5 and the initial L^2-estimate in lemma 6.2 gives the following L^p estimates for $|\chi|$.

Theorem 6.7. *There exists $p_0 > 2$ such that for all $2 \leq p \leq p_0$ and all L_M-stable MOTS Σ which have (r,a)-locally uniformly finite area, the shear χ satisfies for all $x \in \Sigma$ the estimates*

$$\int_{B(x,r/8)} |\chi|^{p+2}\,\mathrm{d}\mu \leq C, \tag{32}$$

$$\int_{B(x,r/8)} |\chi|^{p-2}\big|\nabla|\chi|\big|^2\mathrm{d}\mu \leq C, \tag{33}$$

and

$$\int_{B(x,r/8)} |\nabla\chi|^2 \,\mathrm{d}\mu \le C\,. \tag{34}$$

Here $C = C(r^{-1}, a, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty)$, *or alternatively, we can replace the dependence on* $\|\Psi_M\|_\infty$ *by* $\|\tilde\Psi_M\|_\infty$ *and* $\|{}^M\mathrm{Rc}\|_\infty$.

Proof. Fix $x \in \Sigma$ and choose η to be a cut-off function with $\eta \equiv 1$ in $B(x,r/4)$ and $\eta \equiv 0$ outside of $B(x,r/2)$, such that $0 \le \eta \le 1$ and $|\nabla\eta| \le 4r^{-1}$. Plugging this into the estimates of 6.5 and 6.4 for $p=2$, in view of the local area bound, and the local L^2-estimate from lemma 6.2, yields L^4-estimates for $|\chi|$ in $B(x,r/4)$.

We pick p_0 a little smaller than the value allowed by Proposition 6.5. Then for any $2 < p \le p_0$, proceed as before, but now choose a cut-off function $\bar\eta$ with $\bar\eta \equiv 1$ in $B(x,r/8)$ and $\bar\eta \equiv 0$ outside of $B(x,r/4)$, such that $0 \le \bar\eta \le 1$ and $|\nabla\bar\eta| \le 8r^{-1}$. The resulting L^p and L^{p-2}-norms of $|\chi|$ on the right-hand side can now be estimated by combinations of the L^4-norm of $|\chi|$ and the local area bound.

To see the last estimate, note that in the proof of proposition 6.5, by appropriately choosing ε, we can retain a small portion of the term $\int_\Sigma \eta^2|\chi|^{p-2}|\nabla\chi|^2\,\mathrm{d}\mu$ on the right-hand side. q.e.d.

For the next step—the derivation of sup-bounds on χ—we use the generalization by Hoffman and Spruck [**HS74**] of the Michael-Simon-Sobolev inequality [**MS73**] in the following form.

Lemma 6.8. *For* (M,g) *exist constants* c_0^S, c_1^S, *such that for all hypersurfaces* $\Sigma \subset M$ *and all functions* $f \in C^\infty(\Sigma)$ *with* $|\mathrm{supp} f| \le c_0^S$ *the following estimate holds:*

$$\left(\int_\Sigma |f|^2\,\mathrm{d}\mu\right)^{1/2} \le c_1^S \int_\Sigma |\nabla f| + |fH|\,\mathrm{d}\mu\,.$$

Here H *is the mean curvature of* Σ *and the constants* c_0^S, c_1^S *depend only on a lower bound for the injectivity radius and an upper bound for the curvature of* (M,g).

Remark 6.9. Replacing f by f^p in the above inequality and using Hölders inequality gives that for all $1 < p < \infty$ and all f with $|\mathrm{supp} f| \le c_0^S$,

$$\left(\int_\Sigma f^p\,\mathrm{d}\mu\right)^{2/p} \le c_p^S |\mathrm{supp} f|^{2/p} \int_\Sigma |\nabla f|^2 + |Hf|^2\,\mathrm{d}\mu\,.$$

The constant c_p^S only depends on c_1^S and p.

Theorem 6.10. *Let Σ be an L_M-stable MOTS with (r,a)-locally finite area. Then the shear χ satisfies the estimate*

$$\sup_\Sigma |\chi| \leq C.$$

The constant C depends only on r^{-1}, a, $\|\Psi_M\|_\infty$, $\|K^\Sigma\|_\infty$, $\|Q\|_\infty$, $\|{}^L\mathrm{Rm}^\Sigma\|_\infty$, $\|S\|_\infty$, $\|{}^M\mathrm{Rm}\|_\infty$, and $\mathrm{inj}(M,g)^{-1}$.

Alternatively, the dependence on $\|\Psi_M\|_\infty$ can be replaced by $\|\tilde\Psi_M\|_\infty$.

Proof. We will restrict to the proof of the first statement, since the others are proved similarly.

We will proceed in a Stampacchia iteration. Let η be a cut-off function with $\eta\equiv 1$ on $B(x,r/16)$ and $\eta\equiv 0$ outside $B(x,r/8)$ such that $0\leq\eta\leq 1$ and $|\nabla\eta|\leq 16$. Let $u:=|\chi|$ and for $k\geq 0$ set $u_k := \max\{\eta u - k, 0\}$. In addition, set $A(k) := \operatorname{supp} u_k$. Then clearly $A(k)\subset B(x,r/8)$, such that on $A(k)$ the estimates from theorem 6.7 hold.

The L^2-bound for $|\chi|$ from lemma 6.2 implies that

$$\begin{aligned} k^2|A(k)| &\leq \int_{A(k)} \eta^2 u^2 \,\mathrm{d}\mu \leq \int_{B(x,r/8)} \eta^2 u^2 \,\mathrm{d}\mu \\ &\leq C(r^{-1}, a, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty)\,. \end{aligned}$$

Therefore there exists $k_0 = k_0(|\Sigma|, \|\Psi_M\|_\infty, \|K^\Sigma\|_\infty, c_0) < \infty$, such that $|A(k)|\leq c_0$ for all $k\geq k_0$. Here we want c_0^S to be the constant from lemma 6.8, to be able to apply the estimate from there for all functions with support in $A(k)$, with $k\geq k_0$.

To proceed, let $q>2$. Multiply the Simons identity, in the form (29) from the proof of proposition 6.5, by u_k^q and integrate. This yields

$$\begin{aligned} &\int_{A(k)} -u_k^q u\Delta u + u_k^q(|\nabla\chi|^2 - |\nabla u|^2)\,\mathrm{d}\mu \\ &\quad\leq c\int_{A(k)} u_k^q u^4 + |K| u_k^q u^3 + |{}^L\mathrm{Rm}^\Sigma| u_k^q u^2 + u_k^q \chi * \nabla\big(Q + \chi * S\big)\,\mathrm{d}\mu\,. \end{aligned}$$

Here c is a purely numerical constant. Partially integrate the Laplacian on the right hand side and the last term on the left-hand side. This yields

$$\begin{aligned} &\int_{A(k)} q\eta u u_k^{q-1}|\nabla u|^2 + q u^2 u^{q-1}\langle\nabla\eta,\nabla u\rangle + u_k^q|\nabla\chi|^2\,\mathrm{d}\mu \\ &\quad\leq c\int_{A(k)} u_k^q u^4 + |K| u_k^q u^3 + |{}^L\mathrm{Rm}^\Sigma| u_k^q u^2 \\ &\qquad\qquad + \big(u_k^q|\nabla\chi| + u_k^{q-1}u|\nabla u_k|\big)\big(|Q| + u|S|\big)\,\mathrm{d}\mu. \end{aligned}$$

Note that the term $\int q\eta u u_k^{q-1}|\nabla u|^2\,\mathrm{d}\mu$ on the left-hand side controls $\int u_k^q|\nabla u|^2\,\mathrm{d}\mu$. But before we use this estimate, we absorb the gradient terms into the first term on the left-hand side using $|\nabla u_k| \leq \eta \nabla u + C(r^{-1})u$. Consider for example the term containing $|\nabla\chi|^2$:

$$\begin{aligned} c\int_{A(k)} &u_k^q|\nabla\chi|(|Q| + u|S|)\,\mathrm{d}\mu \\ &\leq \int_{A(k)} u_k^q|\nabla\chi|^2\,\mathrm{d}\mu + c\int_{A(k)} u_k^q|Q|^2 + u_k^q u^2|S|^2\,\mathrm{d}\mu. \end{aligned}$$

The other terms which contain $|\nabla u|$ can be treated similarly, such that the resulting terms can be absorbed on the left. This yields an estimate of the form

(35)
$$\begin{aligned} &\int_{A(k)} u_k^q|\nabla u|^2\,\mathrm{d}\mu \\ &\quad\leq C(q, \|K^\Sigma\|_\infty, \|Q\|_\infty, \|{}^L\mathrm{Rm}^\Sigma\|_\infty, \|S\|_\infty)\int_{A(k)} u_k^{q-2}u^6 + u_k^{q-2}u^2\,\mathrm{d}\mu. \end{aligned}$$

Note that we used that $u_k \leq u$ and $u \leq u^2 + 1$ here to get rid of the extra terms. We begin estimating the terms on the right-hand side of (35) using lemma 6.8. Rewrite and estimate the first term as follows:

$$\begin{aligned} &\int_{A(k)} u_k^{q-2}u^6\,\mathrm{d}\mu \\ &= \int_{A(k)} (u_k u^{6/q-2})^{q-2}\,\mathrm{d}\mu \\ &\leq |A(k)|\Big(\tilde{c}^S_{q-2}\int_{A(k)} \big|\nabla(u_k u^{6/q-2})\big|^2 + |Hu_k u^{6/q-2}|^2\,\mathrm{d}\mu\Big)^{q-2/2}. \end{aligned} \tag{36}$$

To estimate the first term on the right-hand side, we use $u_k/u \leq 1$ to compute on $A(k)$ that

$$\begin{aligned} \big|\nabla(u_k u^{6/q-2})\big| &\leq u^{6/q-2}|\nabla u_k| + \tfrac{6}{q-2}u^{6/q-2}|\nabla u|\tfrac{u_k}{u} \\ &\leq c(q, r^{-1})(u^{6/q-2}|\nabla u| + u^{\frac{6}{q-2}+1}). \end{aligned}$$

Observe that if q is large enough, namely, such that $2 + \frac{12}{q-2} < p_0$ and $\frac{12}{q-2} + 2 \leq 2 + p_0$, then theorem 6.7 yields that

$$\int_{A(k)} \big|\nabla(u_k u^{6/q-2})\big|^2\,\mathrm{d}\mu \leq C(q)\int_{A(k)} u^{12/q-2}|\nabla u|^2 + u^{\frac{12}{q-2}+2}\,\mathrm{d}\mu \leq C(q).$$

Here, and for the remainder of the proof, $C(q)$ denotes a constant that depends on q and, in addition to that, on all the quantities the constant in the statement of this theorem depends on.

To address the second term in (36), recall that since $0 = \theta^+ = H + P$, we have $\|H\|_\infty = \|P\|_\infty \leq 2\|K^\Sigma\|_\infty$. Therefore

$$\int_{A(k)} H^2 u_k^2 u^{12/q-2}\,\mathrm{d}\mu \leq 4\|K^\Sigma\|_\infty^2 \int_\Sigma u^{\frac{12}{q-2}+2}\,\mathrm{d}\mu \leq C(q)\,,$$

where the last estimate also follows from theorem 6.7 if q is large enough. Summarizing these steps, we have

$$\int_{A(k)} u_k^{q-2} u^6\,\mathrm{d}\mu \leq C(q)|A(k)|\,.$$

A similar procedure for the remaining terms in (35) finally yields the estimate

$$\int_{A(k)} u_k^q |\nabla u|^2\,\mathrm{d}\mu \leq C(q)|A(k)|\,, \tag{37}$$

provided $q > q_0$ is large enough. Fix such a $q > q_0$ and let $f = u_k^{1+q/2}$. Since

$$|\nabla f|^2 \leq C(q, r^{-1})(u_k^q |\nabla u|^2 + u_k^q u^2),$$

equation (37) and the above estimates imply that

$$\int_{A((k)} |\nabla f|^2\,\mathrm{d}\mu \leq C(q)|A(k)|.$$

The Hoffman-Spruck-Sobolev inequality from lemma 6.8, combined with theorem 6.7, furthermore yields

$$\begin{aligned}\int_{A(k)} f^2\,\mathrm{d}\mu = \int_{A(k)} u_k^{q+2}\,\mathrm{d}\mu &\leq C(q)|A(k)| \left(\int_{A(k)} |\nabla u|^2 + Hu^2\,\mathrm{d}\mu\right)^{\frac{q+2}{2}} \\ &\leq C(q)|A(k)|.\end{aligned}$$

Thus one further application of lemma 6.8 yields

$$\int_{A(k)} u_k^{q+2}\,\mathrm{d}\mu = \int_{A(k)} f^2\,\mathrm{d}\mu \leq C(q)|A(k)|^2\,.$$

Consider $h > k \geq k_0$. Then on $A(h)$ we have that $u_k \geq h - k$ and therefore we derive the following iteration inequality:

$$|h-k|^{q+2}|A(h)| \leq \int_{A(h)} u_k^{q+2}\,\mathrm{d}\mu \leq \int_{A(k)} u_k^{q+2}\,\mathrm{d}\mu \leq C(q)|A(k)|^2\,.$$

Then [**Sta66**, lemma 4.1] implies that $|A(k_0 + d)| = 0$ for

$$d^{q+2} \leq C(q)|A(k_0)| \leq C(q)a.$$

In view of the definition of $A(k) = \operatorname{supp}\max\{\eta u - k, 0\}$, this yields in particular $|\chi|(x)| \leq C(q)$. As x was arbitrary, the claim follows. q.e.d.

Corollary 6.11. *Let $\Sigma \subset M$ be an L_M-stable MOTS with (r,a)-locally uniformly finite area. Then Σ satisfies the following estimates:*

$$\sup_\Sigma |\chi| \leq C(a, r^{-1}, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, (\mathrm{inj}(M,g))^{-1})$$

In addition, for all $x \in \Sigma$ we have an L^2-gradient estimate of the form

$$\int_{B(x,r/8)} |\nabla\chi|^2 \,\mathrm{d}\mu \leq C(a, r^{-1}, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)\,.$$

Proof. The statement to prove is that the constants only depend on the stated quantities. This is due to the following reasons.

First, we use the above estimates using the representation of constants containing $\|\tilde\Psi_M\|_\infty$ an $\|{}^M\mathrm{Rm}\|_\infty$. As we have seen in remark 1, we can estimate

$$|\tilde\Psi_M| \leq c(|K|^2 + |\nabla K| + |{}^M\mathrm{Rm}|),$$

where c is a numerical constant. Second, since for all $X, Y, Z \in \mathcal{X}(\Sigma)$,

$$Q(X,Y,Z) = {}^L\mathrm{Rm}(X,Y,n,Z) + {}^L\mathrm{Rm}(X,Y,\nu,Z),$$

we can use the Gauss and Codazzi equations of the embedding $M \hookrightarrow L$ to estimate

$$|Q| + |{}^L\mathrm{Rm}^\Sigma| \leq c(|K|^2 + |\nabla K| + |{}^M\mathrm{Rm}|).$$

Third, obviously

$$|K^\Sigma|^2 + |S|^2 \leq |K|^2\,.$$

Thus we see that all quantities are controlled by $\|K\|_\infty$, $\|\nabla K\|_\infty$, and $\|{}^M\mathrm{Rm}\|_\infty$, where the ∞-norms are computed on Σ. Note that the dependency on $\mathrm{inj}(M)$ comes from the fact that the constants c_0^S and c_1^S in the Hoffman-Spruck-Sobolev inequality only depend on $\|{}^M\mathrm{Rm}\|_\infty$ and $\mathrm{inj}(M)^{-1}$. q.e.d.

We conclude with an estimate for the principal eigenfunction to L_M or L_-.

Theorem 6.12. *Let Σ be an L_M-stable MOTS. Let $\lambda \geq 0$ be the principal eigenvalue of L_M and $f > 0$ its corresponding eigenfunction. They satisfy the estimates*

$$\lambda|\Sigma| + \tfrac{1}{2}\int_\Sigma f^{-2}|\nabla f|^2 \,\mathrm{d}\mu \leq 4\pi + \int_\Sigma |S|^2 \,\mathrm{d}\mu - \int_\Sigma \tilde\Psi_M \,\mathrm{d}\mu$$

and

$$\begin{aligned}\int |\nabla^2 f|^2 \,\mathrm{d}\mu &\\ \leq \lambda^2 \int_\Sigma f^2 \,\mathrm{d}\mu &\\ + C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, \mathrm{inj}(M,g)^{-1}) \int_\Sigma f^2 + |\nabla f|^2 \,\mathrm{d}\mu.&\end{aligned}$$

The same estimates hold for L_--stable MOTS when f and λ are the principal eigenfunction and eigenvalue of L_- instead; then $\tilde\Psi_M$ has to be replaced by $\tilde\Psi_-$ in the first estimate.

Proof. The first estimate follows from a computation similar to the proof of lemma 5.6.

The second estimate then follows from the first by using the identity

$$\int_\Sigma |\nabla^2 f|^2 \,\mathrm{d}\mu = \int_\Sigma (\Delta f)^2 + {}^\Sigma\mathrm{Rc}(\nabla f, \nabla f)\,\mathrm{d}\mu\,.$$

To estimate the terms on the right-hand side, note that

$$-\Delta f = \lambda f - 2S(\nabla f) - f(\operatorname{div} S - \tfrac12|\chi|^2 - |S|^2 + \tfrac12{}^\Sigma\mathrm{Sc} - \tilde\Psi_M)$$

and as Σ is two-dimensional

$${}^\Sigma\mathrm{Rc}(\nabla f, \nabla f) = \tfrac12{}^\Sigma\mathrm{Sc}|\nabla f|^2\,.$$

In view of the Gauss equation for $\Sigma \subset M$ and the bounds for χ, we find the claimed estimate. q.e.d.

Corollary 6.13. *If Σ is an L_M-stable MOTS, then the principal eigenfunction $f > 0$ to L_M which is normalized such that $\|f\|_\infty = 1$ satisfies the estimate*

$$\begin{aligned}\int_\Sigma & f^2 + |\nabla f|^2 + |\nabla^2 f|^2 \,\mathrm{d}\mu \\ &\leq C(|\Sigma|, |\Sigma|^{-1}, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, \mathrm{inj}(M,g)^{-1}).\end{aligned}$$

The same estimate holds for L_--stable MOTS, when f is the principal eigenfunction to L_- instead, and the constant depends on $|\Sigma|$, $|\Sigma|^{-1}$, $\|K\|_\infty$, $\|{}^L\mathrm{Rm}\|_\infty$, and $\mathrm{inj}(M,g)^{-1})$.

Proof. Since $\|f\|_\infty = 1$, we have $\int_\Sigma f^2\,\mathrm{d}\mu \leq |\Sigma|$. Then since $f^{-2} \geq 1$, the first estimate from the previous theorem implies

$$\int_\Sigma |\nabla f|^2 \leq C(|\Sigma|, \|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)\,.$$

Since $\lambda^2 \int_\Sigma f^2 \leq \lambda^2|\Sigma| \leq C|\Sigma|^{-1}$ the above estimates combined with the previous theorem imply the claim. q.e.d.

Remark 6.14. A local version of these estimates can also be derived from local area bounds, like the curvature estimates before. In the subsequent application, however, we will not use this more general form.

7. Local area bounds

This section is devoted to derive the area bounds needed for the curvature estimates in the previous section.

The following theorem is analogous to Pogorelov's estimate for stable minimal surfaces [**Pog81**]. We will modify the proof of Colding and Minicozzi given in [**CM02**].

Theorem 7.1. *Let (M, g) be a Riemannian 3-manifold with bounded curvature $\|{}^M\mathrm{Rm}\|_\infty \le C$.*

Let $\Sigma \subset M$ be an immersed surface with bounded mean curvature $\|H\|_\infty \le C$. Assume that there exist $\alpha > 0$ and a constant $Z \ge 0$ such that for all functions $\eta \in C_c^\infty(\Sigma)$ we have

$$-\int_\Sigma {}^\Sigma\mathrm{Sc}\,\eta^2\,\mathrm{d}\mu \le \int_\Sigma (2-\alpha)|\nabla\eta|^2 + Z\eta^2\,\mathrm{d}\mu. \tag{38}$$

Then there exists $r_0 = r_0(\alpha^{-1}, Z, \|H\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)$ such that for all $r < r_0$, the area $|\Sigma \cap B_\Sigma(x, r)|$ is bounded by

$$|\Sigma \cap B_\Sigma(x,r)| \le \tfrac{4\pi}{\alpha} r^2.$$

Proof. Fix an arbitrary $x \in \Sigma$. By the Gauss equation for Σ we know that

$${}^\Sigma\mathrm{Sc} = {}^M\mathrm{Sc} - 2{}^M\mathrm{Rc}(\nu,\nu) + H^2 - |A|^2 \le C(\|H\|_\infty, \|{}^M\mathrm{Rm}\|_\infty).$$

Hence by the Rauch comparison theorems (cf. [**CE75**, section 1.10]), there is a radius $0 < r_1 = r_1(\|H\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)$ such that Σ has no conjugate points in $B_\Sigma(x, r_1)$. Hence the pull-back γ of the metric of Σ to the disc $D_{r_1} := B(0, r_1)$ in $T_x\Sigma$ is regular and satisfies (38).

Denote $D_s = B(0, s)$ the disk of radius $0 \le s \le r_1$ in D_{r_1} and $\Gamma_s = \partial D_s$ the boundary. Note that D_s is a topological disk and Γ_s is a single circle. Furthermore, the area of D_s with respect to γ is bigger than $|B_\Sigma(x, s)|$, for $s < r_1$.

In the stability inequality, we set $\eta = \eta(s) = \max\{1 - \frac{s}{r_0}, 0\}$, where $0 < r \le r_1$ will be chosen below. Denote by $K(s) = \int_{D_s} \mathrm{Scal}\,\mathrm{d}\mu$. Then $K'(s) = \int_{\Gamma_s} \mathrm{Scal}\,\mathrm{d}l$, where $\mathrm{d}l$ is the line element of Γ_s induced by γ. Hence, by the co-area formula and partial integration,

$$-\int_{D_r} \mathrm{Scal}\,\eta^2\,\mathrm{d}\mu = -\int_0^r K'(s)\eta^2(s)\,\mathrm{d}s = \int_0^r K(s)(\eta^2(s))'\,\mathrm{d}s.$$

Let $l(s) = \mathrm{length}(\Gamma_s)$. By the formula for the variation of arc length and the Gauss-Bonnet formula we find $l'(s) = 2\pi - K(s)$, as Γ_s is a circle, on which the geodesic curvature integrates to 2π. Thus we compute, using $(\eta^2(s))' = -\frac{2}{r}(1 - \frac{s}{r})$, $(\eta^2(s))'' = \frac{2}{r^2}$ and the co-area formula

$$\begin{aligned}
-\int_{D_r} \mathrm{Scal}\,\eta^2\,\mathrm{d}\mu &= 2\pi \int_0^r (\eta^2(s))'\,\mathrm{d}s + \int_0^r l(s)(\eta^2(s))''\,\mathrm{d}s \\
&= -\frac{2\pi}{r}\int_0^r \big(1 - \frac{s}{r}\big)\,\mathrm{d}s + \frac{2}{r^2}\int_0^r l(s)\,\mathrm{d}s = -2\pi + \frac{2}{r^2}|D_r|.
\end{aligned}$$

Furthermore, compute

$$\int_{D_r} |\nabla\eta|^2\,\mathrm{d}\mu = \frac{1}{r^2}\int_0^r l(s)\,\mathrm{d}s = \frac{1}{r^2}|D_r|,$$

and estimate

$$\int_{D_r} Z\eta^2 \,\mathrm{d}\mu \le Z|D_r|.$$

Hence equation (38) implies that

$$-2\pi + \tfrac{2}{r^2}\,|D_r| \le \tfrac{2-\alpha}{r^2}\,|D_r| + Z|D_r|.$$

Thus

$$\alpha|D_r| \le 2\pi r^2 + Zr^2|D_r|.$$

Choose $r^2 = \min\{\frac{\alpha}{2Z}, r_1^2\}$ and absorb the error term. This yields the claim. q.e.d.

The above theorem yields the local area bound for stable MOTS needed for the curvature estimates.

Corollary 7.2. *Let Σ be an L_M-stable MOTS. Then there exists $r_0 > 0$ depending only on $\|K\|_\infty$, $\|\nabla K\|_\infty$, and $\|{}^M\mathrm{Rc}\|_\infty$, such that for every $x \in \Sigma$ and $r < r_0$*

$$|\Sigma \cap B_\Sigma(x,r)| \le 6\pi r^2.$$

Proof. To see that theorem 7.1 is applicable on an L_M-stable MOTS, recall equation (27). As $|\chi|^2 \ge \frac{1}{2}|A|^2 - 4|K^\Sigma|^2$ and by the Gauss equation

$$|A|^2 = {}^M\mathrm{Sc} - {}^\Sigma\mathrm{Sc} - 2\,{}^M\mathrm{Rc}(\nu,\nu) + H^2,$$

we find from (27), taking the scalar curvature term to the left, that

$$-\frac{3}{2}\int_\Sigma {}^\Sigma\mathrm{Sc}\eta^2 \,\mathrm{d}\mu \le \int_\Sigma 2|\nabla\eta|^2 + Z\eta^2 \,\mathrm{d}\mu$$

where $Z = Z(\|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rc}\|_\infty)$. As on a MOTS $\|H\|_\infty = \|P\|_\infty \le 2\|K^\Sigma\|_\infty$, theorem 7.1 gives the desired bounds. q.e.d.

Theorem 6.10 and corollary 6.11 imply the following estimates.

Corollary 7.3. *Let Σ be an L_M-stable MOTS; then*

$$\|\chi\|_\infty \le C\big(\|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty, (\mathrm{inj}(M,g))^{-1}\big).$$

Furthermore, there exists $0 < \bar r = \bar r(\|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)$ such that for all $x \in \Sigma$

$$\int_{B(x,\bar r)} |\nabla\chi|^2 \,\mathrm{d}\mu \le C(\|K\|_\infty, \|\nabla K\|_\infty, \|{}^M\mathrm{Rm}\|_\infty)\,.$$

8. Applications

The main application of the curvature estimates proved in this paper is the following compactness property of stable MOTS.

Theorem 8.1. *Let (g_n, K_n) be a sequence of initial data sets on a manifold M. Let (g, K) be another initial dataset on M such that*

$$\|{}^M\mathrm{Rm}\|_\infty \leq C\,,$$
$$\|K\|_\infty + \|{}^M\nabla K\|_\infty \leq C,$$
$$\mathrm{inj}(M, g) \geq C^{-1}\,,$$

for some constant C. Assume that

$$g_n \to g \quad \text{in} \quad C^2_{loc}(M, g) \text{ and}$$
$$K_n \to K \quad \text{in} \quad C^1_{loc}(M, g).$$

Furthermore, let $\Sigma_n \subset M$ be a sequence of immersed surfaces which are stable marginally outer trapped with respect to (g_n, K_n) and have an accumulation point in M. In addition, assume that the Σ_n have uniformly locally finite area, that is, for all $x \in M$ there exists $0 < r = r(x)$ and $a = a(x) < \infty$ such that

$$|\Sigma_n \cap B_{M_{t_n}}(x, r)| \leq ar^2 \qquad \text{uniformly in } n, \tag{39}$$

where $B_{M_{t_n}}(x, r)$ denotes the ball in M around x with radius r.

Then a subsequence of the Σ_n converges to a smooth immersed surface Σ locally in the sense of $C^{1,\alpha}$ graphs. Σ is a MOTS with respect to (g, K). If Σ is compact, then it is also stable.

Proof. By the estimates in corollary 6.11, the above assumptions, even without (39), are sufficient to imply that the shears χ_n, and thus the second fundamental forms A_n, of the Σ_n, with respect to the metric g_n are uniformly bounded

$$|A_n| \leq C.$$

As g_n is eventually C^2-close to g, this bound translates to a bound for the second fundamental forms $\tilde{A}_n$ of Σ_n with respect to the metric g.

In the sequel $B_M(x, s)$ denotes an extrinsic ball in (M, g). As the geometry of the (M, g_n) is uniformly bounded, the uniform curvature bound implies in particular, that there exists a radius s such that for every $x \in \Sigma_n$ the connected component of $\Sigma_n \cap B_M(x, s)$ containing x can be written as graph of a function u^x_n over $T_x\Sigma_n$, where the function u^x_n is uniformly bounded in C^2. Without loss of generality, we can assume that $s < r$, where r is from equation (39).

Now let $x \in M$ be arbitrary. From the previous fact we conclude that each connected component of $\Sigma_n \cap B_M(x, s)$, which intersects $\Sigma_n \cap B_M(x, s/2)$, contains a uniform amount of area. In view of the local area bound (39), we conclude that there are only finitely many such components. Furthermore, the maximal number of those components is uniform in n.

Hence, in each ball $B_M(x, s/2)$, we can extract a convergent subsequence of the Σ_n, such that $\Sigma_n \cap B_M(x, s)$ converges in $C^{1,\alpha}$ to a smooth surface Σ^x. As the M_{t_n} can be covered by countably many such balls, a diagonal argument yields a convergent subsequence of the Σ_n and a limit surface Σ, which is immersed (cf. remark 8.2). Note that since the Σ_n have an accumulation point in (M, g), the limit Σ is non-empty.

Furthermore, $C^{1,\alpha}$ convergence yields that Σ is $C^{1,\alpha}$ and satisfies a weak version of the equation $\theta^+ = 0$. In view of standard regularity theory for prescribed mean curvature equations, we find that Σ is in fact smooth (cf. [**GT01**]).

If Σ is compact, we can cover Σ with finitely many balls $B(x, s/8)$. As before we know that locally the Σ_n converge to Σ in $C^{1,\alpha}$. Since we also have local $W^{1,2}$-bounds on χ we can furthermore assume that $\Sigma_n \to \Sigma$ in $W^{2,p}$ for a fixed, large p, which will be selected below.

From this we can conclude that the metrics of the Σ_n converge to the metric of Σ in $C^\alpha \cap W^{1,p}$. We can pull back the metrics of the Σ_n to Σ and call them γ_n. The metric on Σ will be denoted by γ. Then define the operators L_n as the pull-backs of the operator L_M on Σ_n to Σ. Let f_n be the principal eigenfunctions of L_n with eigenvalues λ_n and normalize such that $\|f_n\|_\infty = 1$. Since the area of the Σ_n is eventually bounded below by half of the area of Σ, theorem 6.12 implies that $0 \leq \lambda_n \leq C$, where $C = C(\bar{C}, \|K\|_\infty, \|\nabla K\|_\infty, \|^M\mathrm{Rm}\|_\infty)$. Thus we can assume that the λ_n converge to some λ with $0 \leq \lambda \leq C$.

By corollary 6.13, the $W^{2,2}$-norm of the f_n taken with respect to the metrics γ_n is uniformly bounded. Recall that the difference of the Hessian of f with respect to γ^n and γ is of the form

$$\big(\nabla^2_{\gamma_n} - \nabla^2_\gamma\big) f = -\big(\Gamma_{\gamma_n} - \Gamma_\gamma\big) * df$$

where Γ_γ and Γ_{γ_n} denote the connection coefficients of γ and γ_n. Furthermore, ∇f is bounded in any L^p and by $W^{1,p}$ convergence of the metrics $\Gamma_{\gamma_n} - \Gamma_\gamma \to 0$ in L^p. Thus we find that also $\|f_n\|_{W^{2,2}} \leq C$, where the norm is taken with respect to the metric γ on Σ. Hence we can assume that $f_n \to f$ in $W^{1,p}$. The Sobolev embedding $W^{1,p} \hookrightarrow C^0$ implies that $f \geq 0$, and $\|f\|_\infty = 1$, so $f \not\equiv 0$.

The next step is to take the equation $L_n f_n = \lambda_n f_n$ to the limit. Since $f_n \to f$ only in $W^{1,p}$, we have to use the weak version of this equation, namely, that for all $\phi \in C^\infty(\Sigma)$,

$$\int_\Sigma \gamma_n^{ij}(df_n)_i d\phi_j + B_n^i (df_n)_i \phi + C_n f \phi \,\mathrm{d}\mu = \lambda_n \int_\Sigma f_n \phi \,\mathrm{d}\mu\,,$$

where B_n and C_n are the coefficients of the operator L_n. By the $W^{2,p}$-convergence of the surfaces, we find that γ_n converges to γ in $W^{1,p}$, and B_n^i and C_n converge in L^p to the coefficients B^i and C of L_M on Σ.

Thus, since f_n converges in $W^{1,p}$ to f, we can choose p large enough to infer that the limit of the above integrals converges to the corresponding integral on Σ, that is f satisfies

$$\int_\Sigma \langle \nabla f, \nabla \phi \rangle + \langle B, \nabla f \rangle \phi + C f \phi \, \mathrm{d}\mu = \lambda \int_\Sigma f \phi \, \mathrm{d}\mu \, .$$

Thus f is a weak eigenfunction of L_M on Σ. Elliptic regularity implies that f is smooth and satisfies $L_M f = \lambda f$. Since $\lambda \geq 0$ and $f \geq 0$, $f \not\equiv 0$, we conclude that Σ is stable. q.e.d.

Remark 8.2. If the limit surface is not compact, then it still follows that it is "symmetrized" stable in the sense that inequality (27) holds for all test functions η with compact support.

Remark 8.3. 1) If the surfaces Σ_n are embedded, one would assume that the limit Σ is embedded as well. However this is not necessarily the case. This is due to the fact that at a point p where Σ touches itself, the equation $\theta^+ = 0$ is satisfied with respect to the outward normal. At p the normals corresponding to these two sheets point into opposite directions at p. If we flip one of the normals, to make them point into the same direction to apply the maximum principle in a graphical situation, the equation for one sheet will remain $\theta^+ = 0$, but for the other it will change to $\theta^- = 0$. Hence, one cannot compare the two sheets.

However, if Σ also satisfies $\theta^- \leq 0$, then the maximum principle implies that the set S of touching points is open. By continuity S is closed, and $S \neq \Sigma$, as the Σ_n are embedded and have bounded curvature. Hence $S = \emptyset$ and Σ is embedded.

2) If the assumption of uniformly locally finite area does not hold, but the surfaces Σ_n are embedded, the limit still exists in the sense of laminations. Here, the limit is a lamination, for which the leaves are "symmetrized" stable MOTS (cf. remark 8.2). Convergence is in the sense of laminations in the class C^α, with convergence of the leaves in $C^{1,\alpha}$, for any $0 < \alpha < 1$. For a proof of this statement, we refer to [**CM04**, appendix B]; in the reference this is stated for minimal laminations, but the modification to the MOTS case is straightforward.

For the sake of completeness we state the definition of a *lamination*. A *lamination* $\mathcal{L} \subset M$ is a closed set, which is a disjoint union of complete connected smooth surfaces, called *leaves*. Furthermore, there are coordinate charts for M, $\psi : V \subset M \to \mathbf{R}^3$, V a neighborhood of some point $x \in M$, such that the image of each leaf L of $\mathcal{L}$ is contained in a set of the form $\mathbf{R}^2 \times t$, where $t \in I$, and I is a closed subset of $\mathbf{R}$. A sequence of laminations $\mathcal{L}_n$ is said to converge to a lamination $\mathcal{L}$ if the coordinate charts converge and $\mathcal{L}$ is the set of accumulation points of the $\mathcal{L}_n$.

3) There are examples of compact three dimensional manifolds which contain sequences of compact stable minimal surfaces of fixed genus and unbounded area [**CM99**, **Dea03**]. Thus assumption (39) does not follow immediately from standard theory, even when the surfaces Σ_n are confined to a compact region.

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