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A REGULARITY AND COMPACTNESS THEORY FOR IMMERSED STABLE MINIMAL HYPERSURFACES OF MULTIPLICITY AT MOST 2

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Abstract

We prove that a stable minimal hypersurface of an open ball which is immersed away from a closed (singular) set of finite co-dimension 2 Hausdorff measure and weakly close to a multiplicity 2 hyperplane must in the interior be the graph over the hyperplane of a 2-valued function satisfying a local $C^{1,\alpha}$ estimate. This regularity is optimal under our hypotheses. As a consequence, we also establish compactness of the class of stable minimal hypersurfaces of an open ball which have volume density ratios uniformly bounded by $3-\delta$ for any fixed $\delta \in (0,1)$ and interior singular sets of vanishing co-dimension 2 Hausdorff measure.

Contents

1. Introduction

Our goal in this paper is to study the local structure of immersed, possibly branched, stable minimal hypersurfaces of the $(n+1)$-dimensional Euclidean space for arbitrary $n \geq 2$. Assuming the singular set of such

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a hypersurface has locally finite $(n-2)$-dimensional Hausdorff measure, we here develop a regularity theory that is applicable near those points of the hypersurface where the volume density is less than 3. Our definition of the singular set is such that it consists only of "genuine" singularities, which include possible branch points. Thus, the points of self-intersection, where the hypersurface is immersed, are considered regular. (See Section 2 for the precise definition of the singular set.)

In particular, we obtain a description of the asymptotic behavior of the hypersurface near any of its multiplicity 2 branch points; i.e., points at which the hypersurface has a multiplicity 2 hyperplane as one of its tangent cones while it fails to decompose as the union of two regular minimal submanifolds in any neighborhood of the point. Our main regularity result is the following:

Theorem 1.1. *For each $\delta \in (0,1)$, there exists a number $\epsilon \in (0,1)$, depending only on n and δ, such that the following is true. If M is an orientable immersed stable minimal hypersurface of $B_2^{n+1}(0)$ with $0 \in \overline{M}$, $\mathcal{H}^{n-2}(\text{sing}\, M) < \infty$, $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 3 - \delta$ and $\int_{M\cap(B_1(0)\times\mathbf{R})} |x^{n+1}|^2 \leq \epsilon$, then $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R}) = \text{graph}\, u$ where u is either a single valued or a 2-valued $C^{1,\alpha}$ function on $B_{1/2}(0)$ satisfying*

$$\|u\|_{C^{1,\alpha}(B_{1/2}(0))} \leq C \left(\int_{M\cap(B_1(0)\times\mathbf{R})} |x^{n+1}|^2 \right)^{1/2}.$$

Here the constants C and $\alpha \in (0,1)$ depend only on n and δ.

See Section 2 for the definition of the $C^{1,\alpha}$ "norm" of u when u is a 2-valued function.

This theorem in particular says that if an n-dimensional stable minimal hypersurface with a singular set of locally finite $(n-2)$-dimensional Hausdorff measure has a multiplicity 2 plane as one of its tangent cones at some point, then it is the unique tangent cone to the hypersurface at that point. The theorem rules out, for example, the possibility of having a sequence of "necks" connecting two sheets and accumulating at a branch point.

A direct consequence of the above theorem is the following decomposition theorem in case $\mathcal{H}^{n-2}(\text{sing}\, M) = 0$.

Theorem 1.2. *For each $\delta \in (0,1)$, there exists a number $\epsilon \in (0,1)$, depending only on n and δ, such that the following is true. If M is an orientable immersed stable minimal hypersurface of $B_2^{n+1}(0)$ with $0 \in \overline{M}$, $\mathcal{H}^{n-2}(\text{sing}\, M) = 0$, $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 3 - \delta$ and $\int_{M\cap(B_1(0)\times\mathbf{R})} |x^{n+1}|^2 \leq \epsilon$, then either $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R}) = \text{graph}\, u^0$ or $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R}) =$*

$\operatorname{graph} u^1 \cup \operatorname{graph} u^2$ *where* $u^i : B_{1/2}(0) \to \mathbf{R}$ *are* $C^{1,\alpha}$ *functions satisfying*

$$\|u^i\|_{C^{1,\alpha}(B_{1/2}(0))} \leq C \left(\int_{M \cap (B_1(0) \times \mathbf{R})} |x^{n+1}|^2 \right)^{1/2}$$

for $i = 0, 1, 2$. *Here the constants* C *and* $\alpha \in (0,1)$ *depend only on* n *and* δ.

Theorem 1.2 implies that if V is a varifold arising as the weak limit of a sequence of stable minimal hypersurfaces having singular sets of $(n-2)$-dimensional Hausdorff measure zero, then near every point where V has a tangent cone equal to the multiplicity 2 varifold associated with a hyperplane, the support of V decomposes as the union of two minimal graphs. In particular, classical branching (of multiplicity 2) cannot occur in the weak limit of a sequence of smooth, stable minimal hypersurfaces.

Based on Theorem 1.2 and the standard dimension reducing principle of Federer, we obtained the following compactness result:

Theorem 1.3. *Let* $\delta \in (0,1)$. *There exists* $\sigma = \sigma(n, \delta) \in (0, 1/2)$ *such that the following is true. Suppose* M_k *is a sequence of orientable stable minimal hypersurface immersed in* $B_2^{n+1}(0)$ *with* $0 \in \overline{M}_k$, $\mathcal{H}^{n-2}(\operatorname{sing} M_k \cap B_\sigma^{n+1}(0)) = 0$ *for each* k *and* $\limsup_{k\to\infty} \frac{\mathcal{H}^n(M_k)}{\omega_n 2^n} \leq 3 - \delta$. *Then there exists a stationary varifold* V *of* $B_2^{n+1}(0)$ *and a closed subset* S *of* $\operatorname{spt}\|V\| \cap B_\sigma^{n+1}(0)$ *with* $S = \emptyset$ *if* $2 \leq n \leq 6$, S *discrete if* $n = 7$ *and* $\mathcal{H}^{n-7+\gamma}(S) = 0$ *for every* $\gamma > 0$ *if* $n \geq 8$ *such that after passing to a subsequence, which we again denote* $\{k\}$, $M_k \to V$ *as varifolds and* $(\operatorname{spt}\|V\| \setminus S) \cap B_\sigma^{n+1}(0)$ *is an orientable immersed, smooth, stable minimal hypersurface of* $B_\sigma^{n+1}(0)$.

In low dimensions, the "smallness of excess" hypothesis of Theorem 1.1 can be dropped provided we assume that the mass ratio is sufficiently close to 2. Precisely, we have the following:

Theorem 1.4. *There exist fixed constants* $\epsilon \in (0,1)$, $C \in (0,\infty)$ *and* $\alpha \in (0,1)$ *such that the following holds. If* $2 \leq n \leq 6$, M *is an orientable immersed stable minimal hypersurface of* $B_2^{n+1}(0)$ *with* $0 \in \overline{M}$, $\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty$, $\Theta_M(0) \geq 2$ *and* $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 2 + \epsilon$, *then either there exists a hyperplane* P *of* $\mathbf{R}^{n+1}$ *such that* $\overline{M} \cap B_1^{n+1}(0) = \operatorname{graph} u$ *where* u *is either a single valued or a* 2*-valued* $C^{1,\alpha}(P \cap B_1^{n+1}(0); P^\perp)$ *function with*

$$\|u\|_{C^{1,\alpha}(B_1^{n+1}(0) \cap P)} \leq C \left(\int_{M \cap B_2^{n+1}(0)} \operatorname{dist}^2(x, P) \right)^{1/2}$$

or there exists a pair of transversely intersecting hyperplanes $P^{(1)}, P^{(2)}$ *of* $\mathbf{R}^{n+1}$ *such that* $\overline{M} \cap B_1^{n+1}(0) = \operatorname{graph} u^{(1)} \cup \operatorname{graph} u^{(2)}$, *where* $u^{(i)} \in C^{1,\alpha}\left(P^{(i)} \cap (B_1^{n+1}(0)); P^{(i)\,\perp}\right)$ *with*

$$\|u^{(i)}\|_{C^{1,\,\alpha}(P^{(i)} \cap (B_1^{n+1}(0))} \leq C \left(\int_{M \cap B_2^{n+1}(0)} \operatorname{dist}^2 \left(x, P^{(1)} \cup P^{(2)} \right) \right)^{1/2}$$

for $i = 1, 2$.

Finally, we have the following decomposition theorem for the singular set of a branched stable minimal hypersurface of the type considered in this paper.

Theorem 1.5. *There exist* $\epsilon = \epsilon(n) \in (0,1)$ *and* $\sigma = \sigma(n) \in (0,1)$ *such that the following holds. If* V *belongs to the varifold closure of orientable immersed stable minimal hypersurfaces* M *of* $B_2^{n+1}(0)$ *with* $0 \in \overline{M}$, $\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty$ *and* $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 2 + \epsilon$ *then*

$$\operatorname{sing} V \cap B_\sigma^{n+1}(0) = B \cup S$$

where

(a) B *is the set of branch points of* V *in* $B_\sigma^{n+1}(0)$; *thus* B *consists of those points of* $\operatorname{sing} V \cap B_\sigma^{n+1}(0)$ *where* V *has a (unique) multiplicity 2 tangent plane. Either* $B = \emptyset$ *or* $\mathcal{H}^{n-2}(B) > 0$.

(b) S *is a relatively closed subset of* $\operatorname{spt} \|V\| \cap B_\sigma^{n+1}(0)$ *with* $S \cap B = \emptyset$, $S = \emptyset$ *if* $2 \leq n \leq 6$, S *a finite set if* $n = 7$ *and* $\mathcal{H}^{n-7+\gamma}(S) = 0$ *for each* $\gamma > 0$ *if* $n \geq 8$.

The proofs of the above theorems will appear in Sections 7 and 8 of the paper. Other consequences of Theorems 1.2 and 1.3, which include a pointwise curvature estimate and a Bernstein type theorem in dimensions ≤ 6, will appear in Section 9.

In case the mass bound is $2 - \delta$ (instead of $3 - \delta$) for some $\delta \in (0,1)$, Theorem 1.1 (with the conclusion that $M \cap (B_{1/2}(0) \times \mathbf{R})$ is the graph of a single valued function) follows from (otherwise much more general) interior regularity theorem of W.K. Allard [**All72**], [**Sim83**]. In case the stable hypersurface is *embedded*, the theorem (under the weaker hypothesis of arbitrary mass bound and with the stronger conclusion as in Theorem 1.2 with a finite number of functions $u_1 < u_2 < \ldots < u_k$ in place of u_1, u_2, with k bounded in terms of the mass bound) is due to R. Schoen and L. Simon [**SS81**]. The Schoen-Simon theorem in fact plays an essential role in the present work.

The main ingredient in the proof of Theorem 1.1 is a height excess decay lemma (Lemma 6.3), where we show that under the hypotheses of the theorem, the height excess of the hypersurface M at a smaller scale, measured relative to a suitable new pair of hyperplanes (a transverse pair of hyperplanes or a multiplicity 2 hyperplane) improves by a fixed factor. The theorem follows by iteratively applying the lemma. At a key stage of the proof of the excess decay lemma, we use a type of harmonic approximation, where we show that whenever the L^2-height excess of the hypersurface relative to a hyperplane is small in a cylinder, the hypersurface in a smaller cylinder is well approximated by the graph of a certain type of "2-valued harmonic" function. F. J. Almgren Jr. [**Alm83**] used a somewhat different class of multi-valued harmonic functions in his work on area minimizing currents of arbitrary dimension and codimension, where harmonic meant Dirichlet energy minimizing. We are working with the weaker assumption of stability, so our two-valued harmonic functions do not satisfy this minimizing property. However, the codimension 1 setting we are working in gives them a lot more structure, and we are able to obtain (in Theorem 5.1) sufficiently detailed, geometric information about them.

A feature of our excess decay lemma perhaps worth pointing out here is that it gives, at every scale, decay of the excess of the stable minimal hypersurface at one of several (three in fact) possible, fixed smaller scales. The reason why excess improvement is exhibited at one of several possible scales in contrast to the more familiar scenario where the improvement is always seen at a single fixed, smaller scale is partly geometric and partly technical. The geometric part of the explanation is that the way an immersed hypersurface satisfying the hypotheses of the theorem (in particular, the mass bound $3 - \delta$ which guarantees that it is "two sheeted") looks as one goes down in scale (fixing a base point) may vary between different possibilities; namely, at any given scale, it may either look like a pair of distinct, more or less parallel planes (i.e., the hypersurface is embedded) or it may look like a pair of transversely intersecting planes (i.e., the hypersurface is embedded away from a small tubular neighborhood around the axis of a transverse pair of hyperplanes) or it may have many self-intersections distributed more or less evenly. Different techniques for these different cases are employed in obtaining excess improvement. The technical part of the reason for the three scales is not having at our disposal, a priori, a single decay estimate, valid uniformly at all points of the domain away from the boundary and for all scales less than a fixed scale, for the aforementioned approximating 2-valued harmonic functions (which arise as blow-ups of sequences of hypersurfaces satisfying the hypotheses of the theorem and converging to multiplicity 2 hyperplanes). Rather, what we obtain (in Theorem 5.1) is an asymptotic description which gives two alternatives depending on whether the blow-up itself has a non-empty interior branch set or not. The presence of two such alternatives for the asymptotics of this "linear problem" means that, at the stage where knowledge of the asymptotics of the linear problem becomes necessary (which is precisely when we are confronted with the picture where the minimal hypersurface has many self-intersections distributed approximately evenly), the excess improvement we get for the hypersurface is, correspondingly, at one of two different smaller scales.

We use methods and results due to L. Simon [**Sim93**]; R. Hardt and L. Simon [**HS79**]; R. Schoen and L. Simon [**SS81**]; F. J. Almgren Jr. [**Alm83**] and the author [**Wic04a**] at a number of crucial points in the present work. The present work in fact should be viewed as a generalization of the results of [**Wic04a**]. To prove that a stable minimal hypersurface, when it is weakly close to a multiplicity 2 hyperplane, is well approximated by the graph of a 2-valued harmonic function of the type aforementioned, we utilize a blow-up argument where we blow up sequences of hypersurfaces off affine hyperplanes. This blow up procedure is based on the approximate graphical decomposition of the hypersurfaces as in [**SS81**], and is carried out as described in [**Wic04a**], after making modifications to and replacement of some of the arguments of [**Wic04a**]. The main difference in the present context, as far as this blowing up step is concerned, is that we here allow the hypersurfaces to be singular unlike in [**Wic04a**] where they were assumed to be smooth. Consequently, in particular, we here need a different argument to establish continuity of the blow-ups. (See Proposition 3.10.)

A major part of this paper is devoted to analyzing the nature of these 2-valued approximating functions. Theorem 5.1 is the key result in this respect, where we establish crucial decay estimates for the two-valued harmonic functions. Our approach in analyzing these functions has been to use geometric arguments, aimed at proving excess decay estimates for the graphs of the functions. To investigate the local regularity properties of these functions at points where their graphs blow up to transversely intersecting pairs of hyperplanes, and also to prove global decay estimates when the base point is a branch point of the function, we use variants of powerful techniques developed by Simon [**Sim93**] and Hardt and Simon [**HS79**]. In particular, a crucial ingredient is an estimate for the radial derivatives of the blow-up (Lemma 3.8) due to Hardt and Simon [**HS79**].

An important technical tool used in the analysis of the 2-valued harmonic functions is the monotonicity of a frequency function, an idea used first in a geometric setting by F.J. Almgren Jr. [**Alm83**]. We here make use of the frequency function directly associated with the two-valued function, as well as the one associated with the single valued function obtained by taking the difference between the two values of the two-valued function. Either frequency function, for any given center point, is monotonically non-decreasing as a function of the radius. Thus, in particular, we may classify the points of the domain of the two-valued function according to the values assumed by the limit of the frequency function associated with the difference function. In a classical setting, e.g., if the function were single valued and harmonic, this limit is equal to the vanishing order of the function at the point in question. In our setting, it conveys analogous information, which may be regarded as the order of contact between the "two sheets" of the graph of the 2-valued function, (although admittedly at a branch point one does not have a useful notion of two sheets) and it reveals the local geometric picture of the graph; i.e., whether the graph locally consists of two disjoint harmonic disks, or of two self intersecting harmonic disks, or whether it is branched. Furthermore, the rate of decay of the graph of the two valued function to its (unique) multiplicity 2 tangent plane at a branch point has a fixed lower bound independent of the function. Said differently, there exists a fixed frequency gap, depending only on n and δ (δ as in Theorem 1.1), implying that the order of contact at a branch point cannot be arbitrarily close to 1.

The existence of a rather rich class of stable branched minimal immersions of the type studied in this paper has recently been established in [**SW**].

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2. Notation and preliminaries

We shall adopt the following notation, conventions and definitions throughout the paper.

$\mathbf{R}^{n+1}$ denotes the $(n+1)$-dimensional Euclidean space and $(x^1, \ldots, x^{n+1})$ denotes a general point in $\mathbf{R}^{n+1}$.

$B_\rho^{n+1}(X)$ denotes the open ball in $\mathbf{R}^{n+1}$ with radius ρ and center X. We identify $\mathbf{R}^n$ with the hyperplane $\{(x^1, x^2, \ldots, x^{n+1}) \in \mathbf{R}^{n+1} : x^{n+1} = 0\}$ and for $X \in \mathbf{R}^n$ let $B_\rho(X)$ denote the open ball in $\mathbf{R}^n$ with radius ρ and center X.

ω_n denotes the volume of a ball in $\mathbf{R}^n$ with radius 1.

For compact sets $S, T \subseteq \mathbf{R}^{n+1}$, $\mathrm{d}_{\mathcal{H}}(S,T)$ denotes the Hausdorff distance between S and T.

$\mathcal{H}^n(S)$ denotes the n-dimensional Hausdorff measure of the set S.

For $Y \in \mathbf{R}^{n+1}$ and $\rho > 0$, $\eta_{Y,\rho} : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ is the map defined by $\eta_{Y,\rho}(X) = \frac{X-Y}{\rho}$.

M denotes a properly immersed, smooth hypersurface of $B_2^{n+1}(0)$. Thus M is a subset of $B_2^{n+1}(0)$ such that for each $X \in M$, there exists a number $\sigma > 0$ such that $M \cap \overline{B}_\sigma^{n+1}(X)$ is the union of a finite number of, possibly intersecting, smooth, connected, compact, embedded n-dimensional submanifolds with boundary contained in in $\partial B_\sigma^{n+1}(X)$.

Let M be a properly immersed smooth hypersurface of $B_2^{n+1}(0)$ so that $\mathcal{H}^n(M \cap K) < \infty$ for each compact subset $K \subset B_2^{n+1})(0)$. M is said to be *minimal* (or *stationary*) if it has zero first variation of volume with respect to deformations by arbitrary C^1 vector fields of $B_2^{n+1}(0)$ with compact support. Minimality of M is equivalent to the condition that

$$\int_M \operatorname{div}_M \Phi \, d\mathcal{H}^n = 0 \tag{2.1}$$

for every C^1 vector field $\Phi = (\Phi^1, \Phi^2, \ldots, \Phi^{n+1}) : B_2^{n+1}(0) \to \mathbf{R}^{n+1}$ with compact support in $B_2^{n+1}(0)$. (See [**Sim83**], Chapter 2.) Here $\operatorname{div}_M \Phi$ is the tangential divergence of Φ with respect to M. Thus, $\operatorname{div}_M \Phi = \sum_{j=1}^{n+1} e_j \cdot \nabla^M \Phi^j$ where ∇^M denotes the gradient operator on M and $\{e_j\}_{j=1}^{n+1}$ is the standard basis of $\mathbf{R}^{n+1}$.

Let M be oriented. If M is minimal, we say M is *stable* if the stability inequality

$$\int_M |A|^2 \zeta^2 \leq \int_M |\nabla^M \zeta|^2 \tag{2.2}$$

holds for every C^1 function ζ with compact support in M. Here A denotes the second fundamental form of M and $|A|$ its length. Note that in case M is embedded, stability of M is equivalent to M having non-negative second variation of volume with respect to deformations by compactly supported C^1 vector fields of $B_2^{n+1}(0)$ which on M are normal to M ([**Sim83**], Chapter 2). If M is immersed, (2.2) says that M has non-negative second variation of volume with respect to deformations determined by moving each (locally defined) smooth embedded piece M' of M (so $M = M' \cup (M \setminus M')$ with M' a smooth embedded hypersurface) with initial velocity at each point $x \in M'$ equal to $\zeta(x)\nu'(x)$, where ν' is the unit normal to M'.

For a smooth hypersurface M of $B_2^{n+1}(0)$, we say a point $X \in \overline{M} \cap B_2^{n+1}(0)$ is (an interior) *regular point* of M if there exists a number $\sigma > 0$ such that $\overline{M} \cap \overline{B}_\sigma^{n+1}(X)$ is the union of finitely many smooth, compact, connected, embedded submanifolds with boundary contained in $\partial B_\sigma^{n+1}(X)$. We shall redefine M so that if $X \in \overline{M}$ is a regular point of M, then $X \in M$. The (interior) *singular set* of M is then defined by

$$\operatorname{sing} M = \big(\overline{M} \setminus M\big) \cap B_2^{n+1}(0).$$

$\mathcal{I}_b$ denotes the family of stable minimal hypersurfaces M of $B_2^{n+1}(0)$ satisfying $\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty$. (The subscript b in $\mathcal{I}_b$ indicates that the members M of $\mathcal{I}_b$ are allowed to carry *branch point* singularities; i.e., points $Z \in \operatorname{sing} M$ such that a hyperplane (with multiplicity > 1) occurs as a tangent cone to (the varifold associated with) M at Z.)

For a stationary, rectifiable n-varifold V of some open subset U of $\mathbf{R}^{n+1}$ and a point $X \in U$, $\Theta\,(\|V\|, X)$ denotes the n-dimensional *density* at X of the weight measure $\|V\|$ on U associated with V. We refer the reader to [**Sim83**], Chapters 4 and 8 for an exposition of the theory of rectifiable varifolds.

For a Radon measure μ on U, $\operatorname{spt}\mu$ denotes the support of μ.

If L is an affine hyperplane of $\mathbf{R}^{n+1}$, $\pi_L : \mathbf{R}^{n+1} \to L$ denotes the orthogonal projection of $\mathbf{R}^{n+1}$ onto L. We shall abbreviate $\pi_{\mathbf{R}^n\times\{0\}}$ as π.

Unless stated otherwise, all constants c, C depend only on n and δ, where δ is as in Theorems 1.1-1.3.

A *pair of affine hyperplanes* means the union of two not necessarily distinct affine hyperplanes of $\mathbf{R}^{n+1}$, neither of which is perpendicular to $\mathbf{R}^n\times\{0\}$. If $P = P_1 \cup P_2$ is a pair of affine hyperplanes, with P_1, P_2 affine hyperplanes, we use the notation $p^+ = \max\{l_1, l_2\}$ and $p^- = \min\{l_1, l_2\}$ where, for $i = 1, 2$, $l_i : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ is the affine function with $\operatorname{graph} l_i = P_i$, and we set $P^+ = \operatorname{graph} p^+$ and $P^- = \operatorname{graph} p^-$. For such a pair of affine hyperplanes P, $\angle P$ denotes the angle $\theta \in [0, \pi)$ between P_1 and P_2 given by $\cos\theta = \nu_1 \cdot \nu_2$ where, for $i = 1, 2$, $\nu_i = \frac{(-Dl_i, 1)}{\sqrt{1+|Dl_i|^2}}$.

By a *pair of hyperplanes* we mean a pair of affine hyperplanes $P = P_1 \cup P_2$ where P_1 and P_2 are hyperplanes (so that $0 \in P_1 \cap P_2$).

We now briefly explain the basic facts about 2-valued functions needed in this paper. For a detailed treatment of multi-valued functions, we refer the reader to [**Alm83**].

Let k be an integer ≥ 1. ($k = 1$ and $k = n$ are the only cases needed in this paper.) Denote by $\mathbf{Q}_2(\mathbf{R}^k)$ the set of unordered pairs of elements of $\mathbf{R}^k$. Define a metric $\mathcal{G}$ on $\mathbf{Q}_2(\mathbf{R}^k)$ by

$$\begin{aligned}&\mathcal{G}(\{v_1, v_2\}, \{w_1, w_2\})\\ &\qquad = \min\left\{\sqrt{|v_1 - w_1|^2 + |v_2 - w_2|^2}, \sqrt{|v_1 - w_2|^2 + |v_2 - w_1|^2}\right\}.\end{aligned}$$

Let Ω be a bounded open subset of $\mathbf{R}^n$. If $u : \Omega \to \mathbf{Q}_2(\mathbf{R}^k)$, we say u is a 2-valued function on Ω with values in $\mathbf{Q}_2(\mathbf{R}^k)$. A 2-valued function $u : \Omega \to \mathbf{Q}_2(\mathbf{R}^k)$ is continuous if it is continuous with respect to the $\mathcal{G}$ metric.

We say that a 2-valued function $u : \Omega \to \mathbf{Q}_2(\mathbf{R}^k)$ is differentiable (or affinely approximable) at a point $a \in \Omega$ if there exist two affine functions l_1^a, $l_2^a : \mathbf{R}^n \to \mathbf{R}^k$ such that

$$u(a) = Au(a)(a) \quad \text{and}$$

$$\lim_{x\to a} \frac{\mathcal{G}(u(x), Au(a)(x))}{|x-a|} = 0$$

where $Au(a)$ is the 2-valued function defined by $Au(a)(x) = \{l_1^a(x), l_2^a(x)\}$ for all $x \in \mathbf{R}^n$. It follows that if $Au(a)$ exists, it is unique, and that if u is differentiable at $a \in \Omega$ then it is continuous at a. In this case, we let $Du(a)$ denote the 2-valued gradient $\{Dl_1^a, Dl_2^a\}$ of u at a.

We say that u is differentiable in Ω if u is differentiable at a for every $a \in \Omega$.

Suppose u is differentiable in Ω and $\alpha \in (0,1)$. We say that u is uniformly $C^{1,\alpha}$ in Ω, and write $u \in C^{1,\alpha}(\overline{\Omega})$, provided

$$[u]_{1,\alpha;\Omega} \equiv \sup_{x_1, x_2 \in \Omega(0), x_1 \neq x_2} \frac{\mathcal{G}\left(\{Dl_1^{x_1}, Dl_2^{x_1}\}, \{Dl_1^{x_2}, Dl_2^{x_2}\}\right)}{|x_1 - x_2|^\alpha}$$

is finite. If $u \in C^{1,\alpha}(\overline{\Omega})$, we define

$$\|u\|_{C^{1,\alpha}(\Omega)} = \sup_{x\in\Omega} \mathcal{G}(u(x), \{0,0\}) + \sup_{x\in\Omega} \mathcal{G}(\{Dl_1^x, Dl_2^x\}, \{0,0\}) + [u]_{1,\alpha;\Omega}.$$

3. Blowing up off affine hyperplanes

For $M \in \mathcal{I}_b$, $\rho \in (0, 3/2]$ and P a pair of affine hyperplanes, define the height excess $E_M(\rho, P)$ of M relative to P at scale ρ by

$$E_M^2(\rho, P) = \rho^{-n-2} \int_{M\cap(B_\rho(0)\times\mathbf{R})} \mathrm{dist}^2\,(X, P). \tag{3.1}$$

In case L is a single affine hyperplane, we write

$$\hat{E}_M(\rho, L) = E_M(\rho, L). \tag{3.2}$$

Let $\delta \in (0,1)$ be a fixed number, $\{M_k\} \subset \mathcal{I}_b$ a sequence of hypersurfaces such that

$$\overline{M}_k \cap \pi^{-1}(0) \neq \emptyset, \tag{3.3}$$

$$\frac{\mathcal{H}^n(M_k \cap (B_2^{n+1}(0)))}{\omega_n 2^n} \leq 3 - \delta \quad \text{and} \tag{3.4}$$

$$\hat{E}_k = \hat{E}_{M_k}(3/2, L_k) \searrow 0 \tag{3.5}$$

for some sequence $\{L_k\}$ of affine hyperplanes of $\mathbf{R}^{n+1}$ converging to $\mathbf{R}^n \times \{0\}$. Note that by a standard argument using the first variation formula (2.1) (see e.g., proof of inequality (4.18) of [**Wic04a**]), we then have that for each $\sigma \in (0, 3/2)$ the estimate

$$(E^T_{M_k}(\sigma, L_k))^2 \leq \frac{C\,\sigma^{-n}}{(3/2 - \sigma)^2}\,\hat{E}_k^2 \tag{3.6}$$

where $C = C(n)$ and, for a hypersurface $M \in \mathcal{I}_b$ and an affine hyperplane L, $E^T_M(\sigma, L) \equiv \sqrt{\sigma^{-n}\int_{M\cap(B_\sigma(0)\times\mathbf{R})} 1 - (\nu\cdot\nu^L)^2}$ is the *tilt excess* of M relative to L at scale σ. Here ν and ν^L are the unit normals to M and L respectively.

We need to blow up the sequence of hypersurfaces $\{M_k\}$ off the sequence of affine hyperplanes $\{L_k\}$. This is carried out essentially as in [**Wic04a**]. For convenience, we choose here to blow up by the height excess $\hat{E}_k$ rather than by the tilt excess $E^T_{M_k}(1, L_k)$ which was used in

[**Wic04a**]. This is possible in view of (3.6). Note also that in [**Wic04a**], it is assumed that for each k, (i) $\text{sing}\, M_k = \emptyset$ and (ii) M_k approximates a cone having a singularity at the origin. Here we weaken hypothesis (i) to $\mathcal{H}^{n-2}(\text{sing}\, M_k) < \infty$ and drop the assumption (ii) altogether. The blow up argument of Sections 3 and 4 of [**Wic04a**] can, however, be repeated with some changes to accommodate the weaker hypotheses. We justify this assertion as follows:

(1) The conclusion of Lemma 3.2 of [**Wic04a**] holds without change under the present hypotheses. That is to say, if $M \in \mathcal{I}_b$ satisfies (3.4) and $M \cap (B_{3/2}(0) \times \mathbf{R}) \subset B^{n+1}_{15/8}(0)$, then for each bounded, locally Lipschitz function φ on M with $\varphi \equiv 0$ in a neighborhood of $M \cap (\partial B_{3/2}(0) \times \mathbf{R})$, we have that for any constant unit vector ν_0,

$$\int_{M\cap(B_{3/2}(0)\times\mathbf{R})} |A|^2\varphi^2 \leq C \int_{M\cap(B_{3/2}(0)\times\mathbf{R})} \left(1-(\nu\cdot\nu_0)^2\right)|\nabla^M\varphi|^2 \tag{3.7}$$

where A denotes the second fundamental form of M, $|A|$ the length of A and $C = C(n)$. This estimate was first proved by R. Schoen in [**SR77**], and later used by R. Schoen and L. Simon in [**SS81**] (Lemma 1 of [**SS81**]) under the hypothesis that $\mathcal{H}^{n-2}(\text{sing}\, M) = 0$. We here use an argument of H. Federer and W. Ziemer [**FZ72**] (see also [**EG99**]) to justify our claim that the estimate in fact continues to hold under the weaker hypothesis $\mathcal{H}^{n-2}(\text{sing}\, M) < \infty$.

First note that by the argument of the proof of Lemma 1 of [**SS81**], the estimate (3.7) holds if φ is locally Lipschitz with compact support in $M \cap (B_{3/2}(0) \times \mathbf{R})$. The issue is to argue that it holds for bounded, locally Lipschitz φ vanishing near $M \cap (\partial B_{3/2}(0) \times \mathbf{R})$ under the assumption $\mathcal{H}^{n-2}(\text{sing}\, M) < \infty$. Let $\tau \in (0, 1/8)$ be arbitrary. Since $\text{sing}\, M \cap (\overline{B}_{3/2}(0) \times \mathbf{R})$ is compact, for each $i = 1, 2, \ldots$ there exists a finite number N^i and balls $B^{n+1}_{r_j^{(i)}}(Z_j^{(i)})$, $j = 1, \ldots, N^{(i)}$ with $Z_j^{(i)} \in \text{sing}\, M \cap (\overline{B}_{3/2}(0) \times \mathbf{R})$ such that $\text{sing}\, M \cap (\overline{B}_{3/2}(0) \times \mathbf{R}) \subset \cup_{j=1}^{N^{(i)}} B^{n+1}_{r_j^{(i)}}(Z_j^{(i)})$, $\sum_{j=1}^{N^{(i)}} \omega_{n-2}(r_j^{(i)})^{n-2} \leq K \equiv 1 + 2^{n-2}\mathcal{H}^{n-2}(\text{sing}\, M \cap (B_{3/2}(0) \times \mathbf{R}))$ and $r_j^{(i)} \leq \tau^{(i)}$. Here $\tau^{(1)} = \tau$ and $\tau^{(i)} = \frac{1}{4}\text{dist}\,(\text{sing}\, M \cap (\overline{B}_{3/2}(0) \times \mathbf{R}), \mathbf{R}^{n+1} \setminus U^{(i-1)})$ for $i = 2, 3, \ldots$, where $U^{(i)} = \cup_{j=1}^{N^{(i)}} B^{n+1}_{r_j^{(i)}}(Z_j^{(i)})$. For each $i = 1, 2, \ldots$ and each $j \in \{1, \ldots, N^{(i)}\}$, let $\psi_j^{(i)}$ be a C^1 function on M such that $\psi_j^{(i)} \equiv 0$ on $B^{n+1}_{r_j^{(i)}}(Z_j^{(i)}) \cap M$, $\psi_j^{(i)} \equiv 1$ on $M \setminus B^{n+1}_{2r_j^{(i)}}(Z_j^{(i)})$, $0 \leq \psi_j^{(i)} \leq 1$ everywhere and $|\nabla \psi_j^{(i)}| \leq 2(r_j^{(i)})^{-1}$. Let $\zeta^{(i)} =$

$\min\{\psi_1^{(i)}, \ldots, \psi_{N^{(i)}}^{(i)}\}$. Then $\mathrm{spt}\,|\nabla\,\zeta^{(i)}| \subset M \cap (V^{(i)} \setminus V^{(i+1)})$ where $V^{(i)} = \cup_{j=1}^{N^{(i)}} B^{n+1}_{2r_j^{(i)}}(Z_j^{(i)})$ and $\int_M |\nabla\,\zeta^{(i)}|^2 \leq cK$, $c = c(n)$. The last inequality follows from the monotonicity formula in view of (3.4) and the assumption $M \cap (B_{3/2}(0) \times \mathbf{R}) \subset B^{n+1}_{15/8}(0)$. Finally, for $\ell = 1, 2, \ldots$, let

$$\beta_\ell = \frac{1}{S_\ell}\sum_{i=1}^{\ell}\frac{\zeta^{(i)}}{i} \tag{3.8}$$

where $S_\ell = \sum_{i=1}^{\ell} i^{-1}$. Then, since $\mathrm{spt}\,\nabla\,\zeta^{(i)}$, $i = 1, 2, \ldots$ are disjoint, we have that

$$\begin{aligned} \int_{M\cap(B_{3/2}(0)\times\mathbf{R})} |\nabla\,\beta_\ell|^2 &= \frac{1}{S_\ell^2}\sum_{i=1}^{\ell}\int_{M\cap(B_{3/2}(0)\times\mathbf{R})} i^{-2}|\nabla\,\zeta^{(i)}|^2 \\ &\leq \frac{cK}{S_\ell^2}\sum_{i=1}^{\ell} i^{-2}. \end{aligned} \tag{3.9}$$

Now, if φ is a bounded, locally Lipschitz function vanishing in a neighborhood of $M \cap (\partial\,B_{3/2}(0) \times \mathbf{R})$, then for each ℓ, $\beta_\ell\varphi$ is a locally Lipschitz function with compact support in $M \cap (B_{3/2}(0) \times \mathbf{R})$ and hence (3.7) holds with $\beta_\ell\varphi$ in place of φ. Thus

$$\begin{aligned} \int_{M\cap(B_{3/2}(0)\times\mathbf{R})} |A|^2\beta_\ell^2\varphi^2 \;&\leq\; C\int_{M\cap(B_{3/2}(0)\times\mathbf{R})} \left(1-(\nu\cdot\nu_0)^2\right)\beta_\ell^2|\nabla\,\varphi|^2 \\ &\quad + C\sup\varphi^2\int_{M\cap(B_{3/2}(0)\times\mathbf{R})} |\nabla\,\beta_\ell|^2. \end{aligned} \tag{3.10}$$

Since $\beta_\ell \leq 1$ and $\beta_\ell \equiv 1$ on $M_\tau \cap (B_{3/2}(0) \times \mathbf{R})$ where $M_\tau = M \setminus \{X : \mathrm{dist}\,(X, \mathrm{sing}\,M) \leq 2\tau\}$, we conclude from (3.9) and (3.10) that

$$\begin{aligned} &\int_{M_\tau\cap(B_{3/2}(0)\times\mathbf{R})} |A|^2\varphi^2 \\ &\leq C\int_{M\cap(B_{3/2}(0)\times\mathbf{R})} \left(1-(\nu\cdot\nu_0)^2\right)|\nabla\,\varphi|^2 + \frac{C\,K}{S_\ell^2}\sum_{i=1}^{\ell} i^{-2}. \end{aligned}$$

Letting first $\ell \to \infty$ and then $\tau \to 0$ in this, we conclude (3.7).

Remark. Note that the validity of (3.7) under the hypothesis $\mathcal{H}^{n-2}\,(\mathrm{sing}\,M) < \infty$, as justified above, shows that Schoen-Simon regularity theory [**SS81**] for embedded stable minimal hypersurfaces M holds under the hypothesis $\mathcal{H}^{n-2}\,(\mathrm{sing}\,M) < \infty$.

(2) Lemma 3.3 of [**Wic04a**] (which is essentially the same as Lemma 2 of [**SS81**]) holds and gives a good approximate graphical decomposition of M_k *relative to the affine hyperplane* L_k, provided we make the minor modification noted in item (3) below, which is necessary due to the presence of a singular set. Note that since $L_k \to \mathbf{R}^n \times \{0\}$, there exists a sequence of rigid motions q_k of $\mathbf{R}^{n+1}$ with $q_k \to$ *identity* such that $q_k(a_k) = \{0\}$ and $q_k\, L_k \equiv \mathbf{R}^n \times \{0\}$, where a_k is the nearest point of L_k to $0 \in \mathbf{R}^{n+1}$. Then, by essentially the same arguments as in [**SS81**], Section 3 (as detailed in [**Wic04a**], Section 3), for each given $\sigma \in (0, 3/2)$ and each sufficiently large k (depending on σ), there exists a "good set" $\widetilde{\Omega}_k = \widetilde{\Omega}_k(\sigma) \subset L_k \cap q_k^{-1}\,(B_\sigma(0) \times \{0\})$ (which corresponds to Ω_k of [**Wic04a**], Lemma 3.3), and two Lipschitz functions $\widetilde{u}_k^{\pm} : \widetilde{\Omega}_k \to \mathbf{R}$ with Lipschitz constants ≤ 1 (analogous to $u_k^{\pm}$ of [**Wic04a**], Section 3), such that $\operatorname{graph} \widetilde{u}_k^{+} \nu^{L_k} \cup \operatorname{graph} \widetilde{u}_k^{-} \nu^{L_k} \subseteq M_k \cap q_k^{-1}(B_\sigma(0) \times \mathbf{R})$ and

(3.11)
$$\mathcal{H}^n\left((M_k \setminus (\operatorname{graph} \widetilde{u}_k^{+} \nu^{L_k} \cup \operatorname{graph} \widetilde{u}_k^{-} \nu^{L_k})) \cap q_k^{-1}(B_\sigma(0) \times \mathbf{R})\right) \leq C_\sigma (\hat{E}_k)^{2+\mu}$$

where ν^{L_k} denotes the upward pointing unit normal to L_k, μ is a fixed constant depending only on n, and C_σ is a constant depending only on n and σ. Here $\operatorname{graph} \widetilde{u}_k^{\pm} \nu^{L_k} \equiv \{x + \widetilde{u}_k^{\pm}(x) \nu^{L_k} : x \in \widetilde{\Omega}_k\}$.

In the present paper, we shall use the notation $G_k^{\pm} = G_k^{\pm}(\sigma) = \operatorname{graph} \widetilde{u}_k^{\pm} \nu^{L_k}$, $\Omega_k = q_k\, \widetilde{\Omega}_k$ and $u_k^{\pm}(x) = \widetilde{u}_k^{\pm} \circ q_k^{-1}(x)$ for $x \in \Omega_k$.

(3) In Lemma 3.3 of [**Wic04a**], the definition of Γ_k needs to be modified to $\Gamma_k = \pi_{L_k}\{X \in M_k \cap q_k^{-1}\,(B_\sigma(0) \times \mathbf{R}) : g_k(X) = \theta_k\} \cup \pi_{L_k}(\operatorname{sing} M_k)$. (cf. [**SS81**].) Here g_k and θ_k are as in [**Wic04a**], Section 3, and π_{L_k} is the orthogonal projection of $\mathbf{R}^{n+1}$ onto L_k. The conclusions of Lemma 3.3 (with notational changes as indicated in item (2) above) hold with this modification and with $\hat{E}_k$ in place of ϵ_k (where by definition $\epsilon_k =$ *tilt* excess in [**Wic04a**]).

(4) We may construct locally Lipschitz cut-off functions $\widetilde{\varphi}_k^0$, $\widetilde{\psi}_k^{(\eta)}$ and $\widetilde{\overline{\psi}}_k$ analogous, respectively, to the cut-off functions φ_k^0, $\psi_k^{(\eta)}$ and $\overline{\psi}_k$ of [**Wic04a**], Section 3. The domains of these cut-off functions are $q_k^{-1}\,(B_\sigma(0) \times \{0\}) \setminus \pi_{L_k}(\operatorname{sing} M_k)$, $M_k \setminus \pi_{L_k}^{-1}(\pi_{L_k}(\operatorname{sing} M_k))$ and $q_k^{-1}(B_\sigma(0) \times \{0\}) \setminus \pi_{L_k}(\operatorname{sing} M_k)$ respectively, and they take values in $\mathbf{R}$. We then define $\overline{\psi}_k : B_\sigma(0) \times \{0\} \setminus q_k(\pi_{L_k} \operatorname{sing} M_k) \to \mathbf{R}$ by setting $\overline{\psi}_k(x) = \widetilde{\overline{\psi}}_k \circ q_k^{-1}(x)$. Note that

$$\mathcal{H}^n\left(B_\sigma(0) \setminus \{x : \overline{\psi}_k(x) = 1\}\right) \leq C_\sigma (\hat{E}_k)^{2+\mu}. \tag{3.12}$$

(See the estimate (3.26) of [**Wic04a**], Section 3.)

(5) We cannot assume Lemma 3.4 of [**Wic04a**] in the present context because it depends on M_k being free of singularities. (Specifically, the inequality (3.7) of [**Wic04a**] assumes that $\operatorname{sing} M_k = \emptyset$.) Notice that in [**Wic04a**], Lemma 3.4 was used precisely at two places; namely,

(a) to establish the estimate (3.28) of [**Wic04a**] which bounds the square of the L^2 norm of $|D\overline{\psi}_k|$ from above by a constant times $(E^T_{M_k})^{2+\mu}$, where $\overline{\psi}_k$ is the cut-off function described in item (4) above and $\mu = \mu(n) > 0$ is as in Lemma 3.5 of [**Wic04a**], and

(b) in the proof of the pointwise gradient estimate for the blow-up (i.e., Lemma 4.9 of [**Wic04a**]).

The modifications necessary for (a) above are minor. In fact, it suffices to have the estimate

$$\int_{B_\sigma(0)} |D\overline{\psi}_k|^2 \leq c\,\hat{E}_k^2, \tag{3.13}$$

$c = c(\sigma)$, and this weaker estimate follows easily from (3.7) and (3.6) in view of the fact that $|D\overline{\psi}_k|$ is pointwise bounded from above by a constant times the length of the second fundamental form of M_k. (See [**Wic04a**], Section 3.) That this weaker estimate suffices follows from the fact that $|Du_k^\pm|$ are bounded, that $u_k^\pm \to 0$ pointwise a.e. and that $Du_k^\pm \to 0$ in L^2.

As for (b) above, we shall give an argument in Lemma 3.10 which, under our present (weaker) hypotheses, in fact shows only that the blow-ups are continuous and satisfy a uniform cone condition at points where they are single valued. This suffices for proving asymptotic decay estimates for the blow-ups later in Section 5.

(6) Parts (a), (b), (f) and (g) of Lemma 4.6 of [**Wic04a**] hold (of course with the functions now having domain $B_\sigma(0)$). Thus, letting $v_k^\pm = \frac{u_k^\pm}{\hat{E}_k}$, there exist functions $v^\pm \in W^{1,2}_{\text{loc}}(B_{3/2}(0))$ —the blow-up of $\{M_k\}$ off $\{L_k\}$—with $v^+ \geq v^-$ such that, after passing to a subsequence of $\{k\}$ which we continue to label $\{k\}$, we have

$$\overline{\psi}_k v_k^\pm \to v^\pm \tag{3.14}$$

in $W^{1,2}(B_\sigma(0))$ for each $\sigma < 3/2$. Note that unlike in [**Wic04a**] (where each M_k was assumed to approximate a cone arbitrarily closely), $v^\pm$ here need not be homogeneous of degree 1. Note also that it is easy to see that $\overline{\psi}_k v_k^\pm \to v^\pm$ in $L^2(B_\sigma(0))$ and weakly in $W^{1,2}(B_\sigma(0))$ for each $\sigma < 3/2$ since it follows directly from the definition of $\hat{E}_k$ that $\overline{\psi}_k v_k^\pm$ are uniformly bounded in $L^2(B_\sigma(0))$, and from the estimates (3.6) and (3.13) that $D(\overline{\psi}_k v_k^\pm)$ are uniformly bounded in $L^2(B_\sigma(0))$. The proof that the convergence is strong in $W^{1,2}(B_\sigma(0))$ requires only some minor modification of the argument of [**Wic04a**] used to prove the same assertion (i.e., parts (f) and (g) of Lemma 4.6, [**Wic04a**].) See item (8) below.

(7) $h \equiv \frac{1}{2}(v^+ + v^-)$ is harmonic in $B_{3/2}(0)$. The proof of this is as in part (e) of Lemma 4.6, [**Wic04a**].

(8) The necessary modifications to the argument of parts (f) and (g) of Lemma 4.6, [**Wic04a**] to show that the convergence in (3.14) is strong in $W^{1,2}(B_\sigma(0))$ for each $\sigma < 3/2$ are as follows: The energy estimate (4.6) of [**Wic04a**] must be replaced by

$$\int_{B_\sigma(0)\cap\{|\overline{\psi}_k(v_k^+-h)|\leq\epsilon\}} |D(\overline{\psi}_k(v_k^+ - h))|^2 + \int_{B_\sigma(0)\cap\{|\overline{\psi}_k(v_k^--h)|\leq\epsilon\}} |D(\overline{\psi}_k(v_k^- - h))|^2 \leq c\epsilon,$$

$c = c(\sigma)$, and, consequently, the estimate (4.41) of [**Wic04a**] becomes

$$\int_{B_\sigma(0)\cap\{|v^+-h|\leq\epsilon\}} |D(v^+ - h)|^2 + \int_{B_\sigma(0)\cap\{|v^--h|\leq\epsilon\}} |D(v^- - h)|^2 \leq c\epsilon,$$

$c = c(\sigma)$. To prove the former estimate, define $\widetilde{h}(x', x^{n+1}) = h(x')$ and repeat the argument of the proof of estimate (4.6) of [**Wic04a**] after replacing $\widetilde{\zeta}$ in the first variation identity (4.1) of [**Wic04a**] simply with $F_\delta(x^{n+1} - \hat{E}_k\widetilde{h})\widetilde{\zeta}^2$ (rather than with $F_\delta(x^{n+1})\widetilde{\zeta}^2$ which was used in [**Wic04a**]; here notation is as in [**Wic04a**]), and use the estimate (3.6) above.

The only other change necessary in the proof of strong convergence is that the function V_k^ϵ (see paragraph preceding estimate (4.34) of [**Wic04a**]) must now be defined to be $V_k^\epsilon = \overline{\psi}_k(\gamma_\epsilon(v_k^+ - h)D(v_k^+ - h) + \gamma_\epsilon(v_k^- - h)D(v_k^- - h))$. Of course then subsequent estimates involving V_k^ϵ need to be modified accordingly in an obvious way.

Remark. Note that the hypothesis (3.4) allows the possibility that M_k are "single sheeted," in which case the blow up would be a single valued, harmonic function v. Theorem 5.1, which describes asymptotics of the blow-up, holds trivially in this case. Our analysis throughout the paper, however, contains this as a special case.

Definition. Let $\mathcal{F}_\delta$ denote the family of ordered pairs of functions $v = (v^+, v^-) \in W^{1,2}_{\text{loc}}(B_{3/2}(0); \mathbf{R}^2)$ arising as blow-ups of sequences of stable minimal hypersurfaces in the manner described above. Precisely, each $(v^+, v^-) \in \mathcal{F}_\delta$ is the blow-up, as in (3.14), of a sequence $\{M_k\} \subset \mathcal{I}_b$ satisfying (3.3), (3.4) and (3.5) for some sequence of hyperplanes L_k converging to $\mathbf{R}^n \times \{0\}$.

Lemma 3.1.
For each $\sigma \in (0,3/2)$, $\mathcal{F}_\delta$ is a compact subset of $W^{1,2}(B_\sigma(0)\,;\,\mathbf{R}^2)$.

Proof. The lemma follows directly from the "diagonal process". Specifically, fix $\sigma \in (0,3/2)$ and let $\{(v_i^+, v_i^-)\}$ be a sequence of functions in $\mathcal{F}_\delta$. Then for each i, there exists a sequence of hypersurfaces $\{M_k^i\} \subset \mathcal{I}_b$ with $\overline{M}_k \cap \pi^{-1}(0) \neq \emptyset$, $\frac{\mathcal{H}^n\,(M_k^i \cap B_2^{n+1}(0))}{\omega_n 2^n} \leq 3-\delta$ and a sequence of affine hyperplanes L_k^i of $\mathbf{R}^{n+1}$ converging to $\mathbf{R}^n \times \{0\}$ as $k \to \infty$ such that $\hat{E}_k^i \equiv \hat{E}_{M_k^i}(3/2, L_k^i) \to 0$ and (v_i^+, v_i^-) is the blow-up of $\{M_k^i\}$ by $\hat{E}_k^i$. Thus, for each i,

$$\frac{\overline{\psi}_{i,\,k} u_{i,\,k}^{\pm}}{\hat{E}_k^i} \to v_i^{\pm} \tag{3.15}$$

in $W^{1,2}(B_\sigma(0))$. (The notation here is as in items (2) and (6) of the discussion at the beginning of this section.) Now choose a diagonal sequence $\{M_{k(i)}^i\}$, $k(1) < k(2) < k(3) < \ldots$ such that $\mathrm{dist}_{\mathcal{H}}\,(L_{k(i)}^i \cap (B_1(0) \times \mathbf{R}), B_1(0)) < 2^{-i}$, $\hat{E}_{k(i)}^i < 2^{-i}$, and $\|\frac{\overline{\psi}_{i,\,k(i)} u_{i,\,k(i)}^{\pm}}{E_{k(i)}^i} - v_i^{\pm}\|_{W^{1,2}(B_\sigma(0))} \leq 2^{-i}$ for each i. (This is possible by the convergence (3.15)). Let $(v^+, v^-) \in W_{\mathrm{loc}}^{1,2}(B_{3/2}(0); \mathbf{R}^2)$ be the blow-up of $\{M_{k(i)}^i\}$ by $\hat{E}_{k(i)}^i$. i.e., for a subsequence $\{i'\}$ of $\{i\}$, (v^+, v^-) is, for each $\tau < 3/2$, the $W^{1,2}(B_\tau(0);\mathbf{R}^2)$ limit of the blow-up sequence $\left\{\left(\frac{\overline{\psi}_{i',k(i')} u_{i',k(i')}^{+}}{\hat{E}_{k(i')}^{i'}}, \frac{\overline{\psi}_{i',\,k(i')} u_{i',k(i')}^{-}}{\hat{E}_{k(i')}^{i'}}\right)\right\}$. Then, by the definition of $\mathcal{F}_\delta$, $(v^+, v^-) \in \mathcal{F}_\delta$, and it is easily seen using the triangle inequality that $v_{i'}^{\pm} \to v^{\pm}$ in $W^{1,2}(B_\sigma(0))$. q.e.d.

Lemma 3.2. *Let $z \in B_{3/2}(0)$ and $\sigma \in (0, 3/2 - |z|)$. If for all sufficiently large k, $M_k \cap (B_\sigma(z) \times \mathbf{R})$ are embedded, then $v^+|_{B_\sigma(z)}$ and $v^-|_{B_\sigma(z)}$ are individually (a. e. equal to) harmonic functions on $B_\sigma(z)$.*

Proof. Under the hypotheses of the lemma, we have that for all sufficiently large k, $u_k^+ > u_k^-$ in $\Omega_k \cap B_\sigma(z)$ and that $u_k^{\pm}$ are (smooth) solutions of the minimal surface equation:

$$\sum_{j=1}^n D_j\left(\frac{D_j u_k^{\pm}}{\sqrt{1+|Du_k^{\pm}|^2}}\right) = 0 \tag{3.16}$$

in $\Omega_k \cap B_\sigma(z)$. Let ζ be an arbitrary C^1 function with compact support in $B_\sigma(z)$. Multiplying (3.16) by $\overline{\psi}_k \zeta$ and integrating over $B_\sigma(z)$, we have (since $\mathrm{spt}\,\overline{\psi}_k \subset \Omega_k$)

$$\int_{B_\sigma(z)} \frac{Du_k^{\pm} \cdot D(\overline{\psi}_k \zeta)}{\sqrt{1+|Du_k^{\pm}|^2}} = 0,$$

which can be written as

$$\int_{B_\sigma(z)} \frac{D(\overline{\psi}_k u_k^\pm)\cdot D\zeta}{\sqrt{1+|Du_k^\pm|^2}} = -\int_{B_\sigma(z)} \frac{\zeta Du_k^\pm \cdot D\overline{\psi}_k}{\sqrt{1+|Du_k^\pm|^2}} + \int_{B_\sigma(z)} \frac{u_k^\pm D\overline{\psi}_k\cdot D\zeta}{\sqrt{1+|Du_k^\pm|^2}}.$$

Dividing this by $\hat{E}_k$ and passing to the limit as $k\to\infty$, we conclude using the Cauchy-Schwarz inequality and (3.13) that $\int_{B_\sigma(z)} Dv^\pm\cdot D\zeta = 0$ as required. q.e.d.

Any $v = (v^+, v^-) \in \mathcal{F}_\delta$ satisfies the properties listed in Propositions 3.3—3.11 below. Given $v\in\mathcal{F}_\delta$, here and subsequently we use the following notation:

$$h = \frac{v^+ + v^-}{2}, \qquad w = \frac{v^+ - v^-}{2}.$$

Proposition 3.3.

(1) *h is harmonic in $B_{3/2}(0)$.*

(2) $\int_{B_{3/2}(0)} (v^+)^2 + (v^-)^2 \leq \left(\frac{3}{2}\right)^{n+2}$.

(3) $\int(|Dv^+|^2 + |Dv^-|^2)\zeta = -\int((v^+-y)Dv^+ + (v^- - y)Dv^-)\cdot D\zeta$ *for each* $y\in\mathbf{R}$ *and* $\zeta \in C^1_c(B_{3/2}(0))$ *and hence (by replacing* ζ *with* ζ^2 *in this and using the Cauchy-Schwarz inequality on the right hand side)* $\int(|Dv^+|^2 + |Dv^-|^2)\zeta^2 \leq 4\int((v^+-y)^2 + (v^- - y)^2)|D\zeta|^2$ *for each* $y\in\mathbf{R}$, $\zeta\in C^1_c(B_{3/2}(0))$.

(4) $\int_{B_\sigma(z)}(|Dv^+|^2 + |Dv^-|^2) = \int_{\partial B_\sigma(z)} (v^+ - y)\frac{\partial v^+}{\partial R} + (v^- - y)\frac{\partial v^-}{\partial R}$ *for each* $z\in B_{3/2}(0)$ *and almost every* $\sigma\in(0, 3/2-|z|)$.

(5) $\sum_{i,j=1}^n \int_{B_\sigma(z)} \left((|Dv^+|^2 + |Dv^-|^2)\delta_{ij} - 2D_iv^+D_jv^+ - 2D_iv^-D_jv^-\right) D_i\zeta^j$ $= 0$ *for each ball* $B_\sigma(z)$ *with* $\overline{B}_\sigma(z)\subset B_{3/2}(0)$ *and each vector field* $\zeta = (\zeta^1, \zeta^2 \ldots, \zeta^n)$ *with* $\zeta^j \in C^1_c(B_\sigma(z))$ *for* $j = 1, 2, 3, \ldots, n$.

Proof. Part (2) is a direct consequence of the definition (3.2) and the estimate (3.11). The proofs of parts (1), (3), (4) and (5) are contained in [**Wic04a**], Section 4; part (1) follows from the identity (4.30) of [**Wic04a**]; parts (3), (4) and (5) follow from exactly the arguments of Lemma 4.7, part (i); Lemma 4.7, part (ii) and Lemma 4.8 of [**Wic04a**] respectively. q.e.d.

Definition. Let $v\in\mathcal{F}_\delta$, $z\in B_{3/2}(0)$ and $y\in\mathbf{R}$. Define the frequency function $N_{v,z,y}(\cdot)$ by

$$N_{v,z,y}(\rho) = \frac{\rho\int_{B_\rho(z)}|Dv|^2}{\int_{\partial B_\rho(z)}(v^+-y)^2 + (v^- - y)^2} \tag{3.17}$$

provided $\rho\in(0, 3/2-|z|)$ and $\int_{\partial B_\rho(z)}(v^+-y)^2 + (v^- - y)^2 \neq 0$.

Whenever $z \in B_{3/2}(0)$ is a Lebesgue point of both v^+ and v^-, and $v^+(z) = v^-(z) = y$ (as will be the case in most of our applications of the frequency function), we shall let $\mathcal{N}_{v,z}(\rho) = \mathcal{N}_{v,z,y}(\rho)$.

Proposition 3.4. *Suppose $v \in \mathcal{F}_\delta$, $0 \leq \rho_1 < \rho_2$, $\overline{B}_{\rho_2}(z) \subseteq B_{3/2}(0)$, $y \in \mathbf{R}$ and $\int_{\partial B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2 \neq 0$ for all $\rho \in (\rho_1, \rho_2)$. Then $N_{v,z,y}(\cdot)$ is monotonically non-decreasing in (ρ_1, ρ_2).*

Proof. The argument is the same as in the proof of Lemma 5.13, [**Wic04a**]. We reproduce it here for the reader's convenience. Note first that the identity of Proposition 3.3, part (4) implies that

$$\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(z)} |Dv|^2\right) = 2\rho^{2-n}\int_{\partial B_\rho(z)} \left|\frac{\partial v}{\partial R}\right|^2 \tag{3.18}$$

for almost all $\rho \in (0, 3/2 - |z|)$, where $\frac{\partial v}{\partial R}(x) = Dv(x) \cdot \frac{x-z}{|x-z|}$ is the radial derivative. This follows by taking $(x^j - z^j)\,\zeta_l$ in place of ζ^j in the identity of Proposition 3.3, part (4) and letting $l \to \infty$, where ζ_l is a sequence of $C_c^\infty(B_\rho(z))$ functions converging to the characteristic function of the ball $B_\rho(z)$. (We omit the details here. This is exactly the argument used to derive the standard monotonicity formula for stationary harmonic maps, and can be found e.g., in [**Sim96**], Chapter 2.) Note also that by Proposition 3.3, part (4),

$$\int_{B_\rho(z)} |Dv|^2 = \frac{1}{2}\int_{\partial B_\rho(z)} \frac{\partial}{\partial R}\left((v^+ - y)^2 + (v^- - y)^2\right) \tag{3.19}$$

for a. e. $\rho \in (0, 3/2 - |z|)$.

Now by a change of variables in the denominator of (3.17), we have that

$$N_{v,z,y}(\rho) = \frac{\rho^{2-n}\int_{B_\rho(z)} |Dv|^2}{\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2}$$

where $\hat{v}^\pm_{z,\rho,y}(\omega) = v^\pm(z + \rho\,\omega) - y$. Using this and the identities (3.18), (3.19), we have that for a.e. $\rho \in (\rho_1, \rho_2)$,

(3.20)

$$\begin{aligned}
&\frac{d}{d\rho}\,N_{v,z,y}(\rho)\\
&= \frac{\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(z)} |Dv|^2\right)}{\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2} - \frac{\rho^{2-n}\int_{B_\rho(z)} |Dv|^2\,\frac{d}{d\rho}\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2}{\left(\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2\right)^2}\\
&= \frac{2\rho^{2-n}\int_{\partial B_\rho(z)} \left|\frac{\partial v}{\partial R}\right|^2}{\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2} -\\
&\quad - \frac{\frac{\rho^{2-n}}{2}\int_{\partial B_\rho(z)} \frac{\partial}{\partial R}\left((v^+ - y)^2 + (v^- - y)^2\right)\frac{d}{d\rho}\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2}{\left(\int_{\mathbf{S}^{n-1}} (\hat{v}^+_{z,\rho,y})^2 + (\hat{v}^-_{z,\rho,y})^2\right)^2}
\end{aligned}$$

$$= \frac{2\rho^{-1}\left(\int_{\mathbf{S}^{n-1}}(\hat{v}^+_{z,\rho,y})^2+(\hat{v}^-_{z,\rho,y})^2 \int_{\mathbf{S}^{n-1}}\left|\frac{\partial \hat{v}_{z,\rho,y}}{\partial R}\right|^2-\left(\int_{\mathbf{S}^{n-1}} \hat{v}^+_{z,\rho,y}\frac{\partial \hat{v}^+_{z,\rho,y}}{\partial R}+\hat{v}^-_{z,\rho,y}\frac{\partial \hat{v}^-_{z,\rho,y}}{\partial R}\right)^2\right)}{\left(\int_{\mathbf{S}^{n-1}}(\hat{v}^+_{z,\rho,y})^2+(\hat{v}^-_{z,\rho,y})^2\right)^2}$$

$$\geq 0.$$

The inequality above follows from the Cauchy-Schwarz inequality. This completes the proof. q.e.d.

Remark. By the definition (3.17) of frequency function and the identity (3.19), it follows that for $z \in B_{3/2}(0)$ and a.e. $\rho \in (0, 3/2 - |z|)$,

$$N_{v,z,y}(\rho) = \frac{\rho \int_{\partial B_\rho(z)} \frac{\partial}{\partial R}((v^+ - y)^2 + (v^- - y)^2)}{2\int_{\partial B_\rho(z)}(v^+ - y)^2 + (v^- - y)^2} \tag{3.21}$$

whenever $N_{v,z,y}(\rho)$ is defined.

Lemma 3.5. *Let $v = (v^+, v^-) \in \mathcal{F}_\delta$, $z \in B_{3/2}(0)$ and suppose that $\int_{\partial B_{\sigma_0}(z)}(v^+ - y)^2 + (v^- - y)^2 > 0$ for some $\sigma_0 \in (0, \mathrm{dist}\,(z, 3/2 - |z|)$. Then*

(a) *$\int_{\partial B_\rho(z)}(v^+ - y)^2 + (v^- - y)^2 > 0$ for all $\rho \in (0, 3/2 - |z|)$ and hence $N_{v,z,y}(\rho)$ is defined for all $\rho \in (0, 3/2 - |z|)$.*

(b) *For each $\rho \in (0, 3/2 - |z|)$ and each $\theta \in (0, 1]$,*

$$\frac{\mathcal{E}^2_{z,\,y,\,\theta\rho}}{\mathcal{E}^2_{z,\,y,\,\rho}} \geq \theta^{2(N_{v,z,y}(\rho)-1)}$$

where $\mathcal{E}_{z,\,y,\,\rho} = \left(\rho^{-n-2}\int_{B_\rho(z)}(v^+ - y)^2 + (v^- - y)^2\right)^{1/2}$.

Proof. Since $\int_{\partial B_\sigma(z)}(v^+ - y)^2 + (v^- - y)^2$ is (absolutely) continuous as a function of σ and $\int_{\partial B_{\sigma_0}(z)}(v^+ - y)^2 + (v^- - y)^2 > 0$ by hypothesis, there exist $\sigma_1 \in (0, 3/2 - |z|)$ with $\sigma_1 < \sigma_0$ such that $\int_{\partial B_\sigma(z)}(v^+ - y)^2 + (v^- - y)^2 > 0$ for all $\sigma \in (\sigma_1, \sigma_0]$. Hence the frequency function $N_{v,z,y}(\sigma)$ is well defined for all $\sigma \in (\sigma_1, \sigma_0]$, and by the monotonicity of $N_{v,z,y}(\sigma)$ and the identity (3.21), we have that for all $\sigma \in (\sigma_1, \sigma_0]$,

$$N_{v,z,y}(\sigma) = \frac{\sigma \frac{d}{d\sigma}\int_{\mathbf{S}^{n-1}}(v^{(z,\sigma)\,+} - y)^2 + (v^{(z,\sigma)\,-} - y)^2}{2\int_{\mathbf{S}^{n-1}}(v^{(z,\sigma)\,+} - y)^2 + (v^{(z,\sigma)\,-} - y)^2} \leq N_{v,z,y}(\sigma_0) = N_0$$

where $v^{(z,\sigma)\,\pm}(\omega) = v^\pm(z + \sigma\,\omega)$. This is equivalent to

$$\frac{d}{d\sigma}\log\left(\frac{\sigma^{1-n}\int_{\partial B_\sigma(z)}(v^+ - y)^2 + (v^- - y)^2}{\sigma^{2N_0}}\right) \leq 0$$

and integrating this differential inequality with respect to σ from σ_1 to σ_0, we have that

$$\frac{\int_{\partial B_{\sigma_0}(z)}(v^+ - y)^2 + (v^- - y)^2}{\sigma_0^{2N_0+n-1}} \leq \frac{\int_{\partial B_{\sigma_1}(z)}(v^+ - y)^2 + (v^- - y)^2}{\sigma_1^{2N_0+n-1}}.$$

This readily implies that $\int_{\partial B_{\sigma_1}(z)} (v^+ - y)^2 + (v^- - y)^2 > 0$. Thus $\int_{\partial B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2 > 0$ for all $\rho \in (0, \sigma_0]$. Since by Proposition 3.3, part (2), the function $(v^+ - y)^2 + (v^- - y)^2$ is weakly subharmonic in $B_{3/2}(0)$, it follows from the maximum principle that $\int_{\partial B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2 > 0$ for all $\rho \in (\sigma_0, 3/2 - |z|)$. Thus part (a) of the lemma holds.

To prove part (b), fix $\rho \in (0, 3/2 - |z|)$. Using part (a) and arguing as above, we have that

$$\frac{d}{d\sigma} \log \left(\frac{\sigma^{1-n} \int_{\partial B_\sigma(z)} (v^+ - y)^2 + (v^- - y)^2}{\sigma^{2N}} \right) \leq 0$$

for all $\sigma \in (0, \rho)$, where $N = N_{v,z,y}(\rho)$, and by integrating this from σ_1 to σ_2, we obtain that for every $\sigma_1, \sigma_2 \in (0, 3/2 - |z|)$ with $0 < \sigma_1 < \sigma_2 \leq \rho$,

(3.22)
$$\frac{\int_{\partial B_{\sigma_1}(z)} (v^+ - y)^2 + (v^- - y)^2}{\sigma_1^{2N+n-1}} \leq \frac{\int_{\partial B_{\sigma_2}(z)} (v^+ - y)^2 + (v^- - y)^2}{\sigma_2^{2N+n-1}}.$$

Hold σ_1 fixed with $0 < \sigma_1 < \theta\rho$, multiply inequality (3.22) by σ_2^{2N+n-1} and integrate with respect to σ_2 from $\theta\rho$ to ρ to obtain, for each $\sigma_1 \in (0, \theta\rho)$, that

$$\begin{aligned} \text{(3.23)} \quad & \int_{B_\rho(z) \setminus B_{\theta\rho}(z)} (v^+ - y)^2 + (v^- - y)^2 \\ & \leq \frac{1}{2N+n} \left(\rho^{2N+n} - (\theta\rho)^{2N+n} \right) \frac{\int_{\partial B_{\sigma_1}(z)} (v^+ - y)^2 + (v^- - y)^2}{\sigma_1^{2N+n-1}}. \end{aligned}$$

Now multiply both sides of (3.23) by σ_1^{2N+n-1} and integrate with respect to σ_1 from 0 to $\theta\rho$. This gives

$$\begin{aligned} & (\theta\rho)^{2N+n} \int_{B_\rho(z) \setminus B_{\theta\rho}(z)} (v^+ - y)^2 + (v^- - y)^2 \\ & \leq \left(\rho^{2N+n} - (\theta\rho)^{2N+n} \right) \int_{B_{\theta\rho}(z)} (v^+ - y)^2 + (v^- - y)^2 \end{aligned}$$

which, upon rearrangement of terms, gives the desired estimate. q.e.d.

Definition. For $v \in \mathcal{F}_\delta$, $z \in B_{3/2}(0)$ and $y \in \mathbf{R}$ with $\int_{\partial B_{\sigma_0}(z)} (v^+ - y)^2 + (v^- - y)^2 > 0$ for some $\sigma_0 \in (0, 3/2 - |z|)$, define $\mathcal{N}_{v,y}(z) = \lim_{\rho \downarrow 0} N_{v,z,y}(\rho)$. Note that $N_{v,z,y}(\rho)$ is well defined for all $\rho \in (0, 3/2 - |z|)$ and this limit exists by Lemmas 3.5 and 3.4 above.

Whenever $z \in B_{3/2}(0)$ is a Lebesgue point of both v^+ and v^-, and $v^+(z) = v^-(z) = y$, we shall let $\mathcal{N}_v(z) = \mathcal{N}_{v,y}(z)$.

Lemma 3.6.
Let $v \in \mathcal{F}_\delta$ and $z \in B_{3/2}(0)$. Suppose that $\int_{\partial B_{\sigma_0}(z)} (v^+ - y)^2 + (v^- - y)^2 > 0$ for some $\sigma_0 \in (0, 3/2 - |z|)$. Then $N_{v,z,y}(\rho)$ is constant for $\rho \in (0, 3/2 - |z|)$ (with value $\mathcal{N}_{v,y}(z)$) if and only if $\sqrt{(v^+ - y)^2 + (v^- - y)^2}$ is homogeneous of degree $\mathcal{N}_{v,y}(z)$ from the point z in $B_{3/2-|z|}(z)$; i.e., if and only if

$$
\begin{aligned}
&(v^+(z + \rho\,\omega) - y)^2 + (v^-(z + \rho\,\omega) - y)^2 \\
&= \left(\frac{\rho}{\rho'}\right)^{2\mathcal{N}_{v,y}(z)} ((v^+(z + \rho'\,\omega) - y)^2 + (v^-(z + \rho'\,\omega) - y)^2)
\end{aligned}
$$

for each ρ, $\rho' \in (0, 3/2 - |z|)$ and $\omega \in \mathbf{S}^{n-1}$.

Proof. Note first that $N_{v,z,y}(\rho)$ is well defined for $\rho \in (0, 3/2 - |z|)$ by Lemma 3.5, part (a). If $\sqrt{(v^+ - y)^2 + (v^- - y)^2}$ is homogeneous of some degree α from z in $B_{3/2-|z|}(z)$, it is easy to see using the identity (3.21) that $N_{v,z,y}(\rho) = \alpha (= \mathcal{N}_{v,y}(z))$ for $\rho \in (0, 3/2 - |z|)$. Conversely, suppose $N_{v,z,y}(\rho)$ is constant in the interval $(0, 3/2 - |z|)$. Then by (3.20),

$$\frac{\partial}{\partial R}\,\hat{v}^{\pm}_{z,\rho,y}(\omega) = \alpha \hat{v}^{\pm}_{z,\rho,y}(\omega)$$

for some constant $\alpha(\rho)$, almost all $\rho \in (0, 3/2 - |z|)$ and almost all $\omega \in \mathbf{S}^{n-1}$, where $\hat{v}^{\pm}_{z,\rho,y}(\omega) = v^{\pm}(z + \rho\omega) - y$. (This just follows from the condition under which equality holds in Cauchy-Schwarz inequality.) This is equivalent to the differential identities

$$\rho \frac{d}{d\rho}\,(v^{\pm}(z + \rho\,\omega) - y) = \alpha(\rho)(v^{\pm}(z + \rho\,\omega) - y) \tag{3.24}$$

which imply $\alpha(\rho) \int_{\mathbf{S}^{n-1}} (v^+(z + \rho\omega) - y)^2 + (v^-(z + \rho\omega) - y)^2 d\omega = \left(\frac{\rho}{2}\right) \frac{d}{d\rho} \int_{\mathbf{S}^{n-1}} (v^+(z + \rho\omega) - y)^2 + (v^-(z + \rho\omega) - y)^2 d\omega$, so that $\alpha(\rho) = N_{v,z,y}(\rho)$ by (3.21). Since $N_{v,z,y}(\rho)$ is constant by hypothesis, it follows that $\alpha(\rho) = \alpha$ for some constant α, so that by (3.24), $(v^+(z + \rho\,\omega) - y)^2 + (v^-(z + \rho\,\omega) - y)^2 = \left(\frac{\rho}{\rho'}\right)^{2\alpha} ((v^+(z + \rho'\,\omega) - y)^2 + (v^-(z + \rho\,\omega) - y)^2)$ for ρ, $\rho' \in (0, 3/2 - |z|)$ and $\omega \in \mathbf{S}^{n-1}$. It then follows from (3.21) that $\alpha = \mathcal{N}_{v,y}(z)$. q.e.d.

The estimate in Lemma 3.8 below, essentially due to Hardt and Simon [**HS79**], will play a very important role first in our proof of continuity of functions in $\mathcal{F}_\delta$ (Lemma 3.10 below) and later in establishing crucial asymptotic decay properties (Theorem 5.1) of these functions. In the proof of this estimate, we shall need the following:

Lemma 3.7. *Let $\sigma \in (0, 3/2)$. There exist $\epsilon = \epsilon(n, \sigma) \in (0, 1)$ and $C = C(n, \sigma) \in (0, \infty)$ such that if $M \in \mathcal{I}_b$, $M \cap \pi^{-1}(0) \neq \emptyset$, $\frac{\mathcal{H}^n\,(M \cap B_2^{n+1}(0))}{\omega_n 2^n} \leq 3 - \delta$, L is an affine hyperplane of $\mathbf{R}^{n+1}$ with* dist $(L \cap$

$(B_1(0) \times \mathbf{R}), B_1(0)) < \epsilon$ *and* $\hat{E} = \hat{E}_M(3/2, L) < \epsilon$, *then for each* $Z \in M \cap (B_\sigma(0) \times \mathbf{R})$ *with* $\Theta_M(Z) \geq 2$ *we have that*

$$\operatorname{dist}(Z, L) \leq C\,\hat{E}.$$

Proof. By translating, scaling and rotating, we may assume without loss of generality that $L = \mathbf{R}^n \times \{0\}$. Let Z be as in the statement of the lemma and write $Z = (z', z^{n+1})$. Set $\sigma_0 = 3/2 - \sigma$. The monotonicity formula for M ([**Sim83**], Section 17) says that

$$\int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} \frac{(\nu \cdot (X - Z))^2}{|X - Z|^{n+2}} = \frac{\mathcal{H}^n\left(M \cap B^{n+1}_{\sigma_0/2}(Z)\right)}{\omega_n(\sigma_0/2)^n} - \Theta_M(Z) \tag{3.25}$$

where ν denotes the unit normal to M. Writing $\nu = (\nu', \nu^{n+1})$ where $\nu^{n+1} = \nu \cdot e^{n+1}$, we have that

(3.26)
$$\begin{aligned}
&\int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} \frac{(\nu \cdot (X - Z))^2}{|X - Z|^{n+2}} \\
&\geq (\sigma_0/2)^{-n-2} \int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} (\nu' \cdot (x' - z') + \nu^{n+1}(x^{n+1} - z^{n+1}))^2 \\
&\geq \frac{1}{2}(\sigma_0/2)^{-n-2} \int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} (\nu^{n+1})^2 |x^{n+1} - z^{n+1}|^2 - \\
&\quad - (\sigma_0/2)^{-n} \int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} 1 - (\nu^{n+1})^2 \\
&\geq \frac{1}{2}(\sigma_0/2)^{-n-2} \int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} (\nu^{n+1})^2 |x^{n+1} - z^{n+1}|^2 - \\
&\quad - c\,\sigma_0^{-n-2} \int_{M \cap (B_{\sigma_0}(z') \times \mathbf{R})} |x^{n+1}|^2 \\
&\geq \frac{1}{2}(\sigma_0/2)^{-n-2} \int_{M \cap B^{n+1}_{\sigma_0/2}(Z)} (\nu^{n+1})^2 |x^{n+1} - z^{n+1}|^2 - c\,\sigma_0^{-n-2} \hat{E}^2
\end{aligned}$$

where $c = c(n)$, and for the second of the inequalities in the above, we have used $(a+b)^2 \geq a^2/2 - b^2$ with $a = \nu^{n+1}(x^{n+1} - z^{n+1})$, $b = \nu' \cdot (x' - z')$ and the fact that $|\nu'|^2 = 1 - (\nu^{n+1})^2$, and the third inequality is standard and is analogous to (3.6).

On the other hand, provided $\epsilon = \epsilon(n,\sigma)$ is sufficiently small, we have that

$$
\begin{aligned}
(3.27)\qquad & \frac{\mathcal{H}^n\left(M \cap B^{n+1}_{\sigma_0/2}(Z)\right)}{\omega_n(\sigma_0/2)^n} - \Theta_M(Z) \\
&\leq \frac{\mathcal{H}^n\left(M \cap B^{n+1}_{\sigma_0/2}(Z)\right)}{\omega_n(\sigma_0/2)^n} - 2 \\
&\leq C\sigma_0^{-n}\int_{\Omega\cap B_{\sigma_0/2}(z')} \sqrt{1+|Du^+|^2} - 1 \\
&\quad + C\sigma_0^{-n}\int_{\Omega\cap B_{\sigma_0/2}(z')} \sqrt{1+|Du^-|^2} - 1 + C\sigma_0^{-n}\hat{E}^{2+\mu} \\
&= C\sigma_0^{-n}\int_{\Omega\cap B_{\sigma_0/2}(z')} \frac{|Du^+|^2}{1+\sqrt{1+|Du^+|^2}} \\
&\quad + C\sigma_0^{-n}\int_{\Omega\cap B_{\sigma_0/2}(z')} \frac{|Du^-|^2}{1+\sqrt{1+|Du^-|^2}} + C\sigma_0^{-n}\hat{E}^{2+\mu} \\
&\leq C\sigma_0^{-n}\int_{M\cap(B_{\sigma_0/2}(z')\times\mathbf{R})} 1-(\nu^{n+1})^2 + C\sigma_0^{-n}\hat{E}^2 \\
&\leq C\sigma_0^{-n-2}\int_{M\cap(B_{\sigma_0}(z')\times\mathbf{R})} |x^{n+1}|^2 + C\sigma_0^{-n}\hat{E}^2 \\
&\leq C\sigma_0^{-n-2}\hat{E}^2
\end{aligned}
$$

where $C = C(n)$ and Ω, u^+ and u^- correspond, respectively, to Ω_k, u_k^+ and u_k^- of item (2) of the discussion (with M in place of M_k and $L_k \equiv \mathbf{R}^n \times \{0\}$) at the beginning of Section 3. Note that we have used the estimate (3.11) here.

Combining the estimates (3.26) and (3.27), we have

$$
(3.28)\qquad \int_{M\cap B^{n+1}_{\sigma_0/2}(Z)} (\nu^{n+1})^2|x^{n+1} - z^{n+1}|^2 \leq C\hat{E}^2
$$

where $C = C(n)$, which implies by the triangle inequality that $|z^{n+1}|^2\int_{M\cap B^{n+1}_{\sigma_0/2}(Z)}(\nu^{n+1})^2 \leq C\hat{E}^2$. But if $\epsilon = \epsilon(n,\sigma)$ is sufficiently small, then $\int_{M\cap B^{n+1}_{\sigma_0/2}(Z)}(\nu^{n+1})^2 \geq c\sigma_0^n$, $c = c(n)$, and hence

$$
|z^{n+1}|^2 \leq C\sigma_0^{-n}\hat{E}^2
$$

where $C = C(n)$. This is the required estimate. q.e.d.

Lemma 3.8. *Suppose $v = (v^+, v^-) \in \mathcal{F}_\delta$. If $z \in B_{3/2}(0)$ is a Lebesgue point of both v^+ and v^-, $v^+(z) = v^-(z)$ and $v^+ \not\equiv v^-$ (as L^2 functions) in any ball centered at z, then for each $\rho \in (0, 3/2 - |z|)$,* *we have that*

$$\int_{B_{\rho/2}(z)} R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^+ - y}{R}\right)\right)^2 + R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^- - y}{R}\right)\right)^2$$
$$\leq C\rho^{-n-2}\int_{B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2$$

where $y = v^+(z) = v^-(z)$. *Here* $C = C(n)$.

Proof. Suppose the hypotheses of the lemma are satisfied for some $z \in B_{3/2}(0)$. Let $\{M_k\} \subset \mathcal{I}_b$ be a sequence of hypersurfaces whose blow-up is v. First we claim that for each $\tau \in (0, 3/2 - |z|)$, there exist infinitely many k such that $M_k \cap (B_\tau(z) \times \mathbf{R})$ contains a point Z_k with $\Theta_{M_k}(Z_k) \geq 2$. For if not, $M_k \cap (B_\tau(z) \times \mathbf{R})$ would be embedded for some $\tau \in (0, 3/2 - |z|)$ and all sufficiently large k, and hence, by Lemma 3.2, v^+ and v^- would both be individually harmonic in $B_\tau(z)$. Since $v^+ \geq v^-$ and $v^+(z) = v^-(z)$, we would then have by the maximum principle that $v^+ \equiv v^-$ in $B_\tau(z)$, contradicting one of the hypotheses of the lemma. Hence the claim must be true.

Now take an arbitrary sequence of numbers $\tau_j \searrow 0$ and apply this claim with τ_j in place of τ. This gives a subsequence of $\{k\}$, which we continue to denote $\{k\}$, such that $M_k \cap (B_{3/2}(0) \times \mathbf{R})$ contains a point $Z_k = (Z_k', Z_k^{n+1})$ with $\Theta_{M_k}(Z_k) \geq 2$, satisfying $Z_k' \to z$. By the monotonicity identity for minimal submanifolds ([**Sim83**], Section 17), we have that, for $\rho \in (0, 3/2 - |z|)$,

$$\begin{aligned}\int_{M_k \cap B^{n+1}_{\rho/2}(Z_k)} \frac{((X - Z_k)\cdot \nu_k)^2}{|X - Z_k|^{n+2}} &= \frac{\mathcal{H}^n(M_k \cap B^{n+1}_{\rho/2}(Z_k))}{\omega_n(\rho/2)^n} - \Theta_{M_k}(Z_k)\\ &\leq \frac{\mathcal{H}^n(M_k \cap B^{n+1}_{\rho/2}(Z_k))}{\omega_n(\rho/2)^n} - 2.\end{aligned}$$

Estimating as in (3.27), we have

$$\begin{aligned}&\frac{\mathcal{H}^n(M_k \cap B^{n+1}_{\rho/2}(Z_k))}{\omega_n(\rho/2)^n} - 2\\ &= \frac{\mathcal{H}^n((G_k^+ \cup G_k^-) \cap B^{n+1}_{\rho/2}(Z_k))}{\omega_n(\rho/2)^n} - 2\\ &\quad + \frac{\mathcal{H}^n((M_k \setminus G_k) \cap B^{n+1}_{\rho/2}(Z_k))}{\omega_n(\rho/2)^n}\\ &\leq \frac{1}{\omega_n(\rho/2)^n}\int_{B_{\rho/2}(Z_k')}\left(\sqrt{1 + |D(\overline{\psi}_k u_k^+)|^2} - 1\right)\\ &\quad + \frac{1}{\omega_n(\rho/2)^n}\int_{B_{\rho/2}(Z_k')}\left(\sqrt{1 + |D(\overline{\psi}_k u_k^-)|^2} - 1\right) + \frac{C\hat{E}_k^{2+\mu}}{\omega_n(\rho/2)^n}\end{aligned}$$

$$= \frac{1}{\omega_n(\rho/2)^n} \int_{B_{\rho/2}(Z'_k)} \frac{|D(\overline{\psi}_k u_k^+)|^2}{\sqrt{1+|D(\overline{\psi}_k u_k^+)|^2}+1} +$$

$$+ \frac{1}{\omega_n(\rho/2)^n} \int_{B_{\rho/2}(Z'_k)} \frac{|D(\overline{\psi}_k u_k^-)|^2}{\sqrt{1+|D(\overline{\psi}_k u_k^-)|^2}+1} + \frac{C\hat{E}_k^{2+\mu}}{\omega_n(\rho/2)^n},$$

which implies that

$$\limsup_{k\to\infty} \frac{1}{\hat{E}_k^2} \left(\frac{\mathcal{H}^n(M_k \cap B_{\rho/2}(Z_k))}{\omega_n(\rho/2)^n} - 2 \right) \leq \frac{1}{2\omega_n(\rho/2)^n} \int_{B_{\rho/2}(z)} |Dv^+|^2 + |Dv^-|^2. \tag{3.29}$$

On the other hand,

$$\int_{M_k \cap B^{n+1}_{\rho/2}(Z_k)} \frac{((X-Z_k)\cdot\nu_k)^2}{|X-Z_k|^{n+2}}$$

$$\geq \int_{G_k^+ \cap B^{n+1}_{\rho/2}(Z_k)} \frac{((X-Z_k)\cdot\nu_k)^2}{|X-Z_k|^{n+2}} + \int_{G_k^- \cap B^{n+1}_{\rho/2}(Z_k)} \frac{((X-Z_k)\cdot\nu_k)^2}{|X-Z_k|^{n+2}}$$

$$\geq \int_{B^n_{\rho/2}(Z'_k)} \frac{\left(-(X'-Z'_k)\cdot D(\overline{\psi}_k u_k^+) + (\overline{\psi}_k u_k^+ - Z_k^{n+1})\right)^2}{\left((\overline{\psi}_k u_k^+ - Z_k^{n+1})^2 + |X'-Z'_k|^2\right)^{\frac{n+2}{2}}}$$

$$+ \int_{B^n_{\rho/2}(Z'_k)} \frac{\left(-(X'-Z'_k)\cdot D(\overline{\psi}_k u_k^-) + (\overline{\psi}_k u_k^- - Z_k^{n+1})\right)^2}{\left((\overline{\psi}_k u_k^- - Z_k^{n+1})^2 + |X'-Z'_k|^2\right)^{\frac{n+2}{2}}}.$$

This implies by Fatou's lemma and (3.29) that

$$\begin{aligned} C\rho^{-n} \int_{B_{\rho/2}(z)} |Dv^+|^2 + |Dv^-|^2 &\geq \liminf_{k\to\infty} \frac{1}{\hat{E}_k^2} \int_{M_k\cap B_{\rho/2}(Z_k)} \frac{((X-Z_k)\cdot\nu_k)^2}{|X-Z_k|^{n+2}} \\ &\geq \int_{B^n_{\rho/2}(z)} \frac{((v^+-y)-(X'-z)\cdot Dv^+)^2}{|X'-z|^{n+2}} \\ &\quad + \int_{B^n_{\rho/2}(z)} \frac{((v^--y)-(X'-z)\cdot Dv^-)^2}{|X'-z|^{n+2}} \\ &= \int_{B_{\rho/2}(z)} R^{2-n} \left(\frac{\partial}{\partial R}\left(\frac{v^+-y}{R}\right)\right)^2 \\ &\quad + \int_{B_{\rho/2}(z)} R^{2-n} \left(\frac{\partial}{\partial R}\left(\frac{v^+-y}{R}\right)\right)^2 \end{aligned} \tag{3.30}$$

where $C = C(n)$ and $y = \lim_{k\to\infty} \frac{Z_k^{n+1}}{\hat{E}_k}$, possibly after passing to a subsequence of $\{k\}$. Note that the existence of the limit $y \in \mathbf{R}$ follows from Lemma 3.7. The required estimate follows by combining the inequalities (3.29) and (3.30), and using Proposition 3.3, part (2). Observe that the estimate automatically implies that $y = v^+(z) = v^-(z)$, for if not, the integral on the left hand side would not be finite. q.e.d.

Remark. Note that the proof of the preceding lemma shows the following: If $v = (v^+, v^-) \in \mathcal{F}_\delta$, $z \in B_{3/2}(0)$ is a Lebesgue point of both v^+ and v^-, $v^+(z) = v^-(z) = y$, $v^+ \not\equiv v^-$ (as L^2 functions) in any ball centered at z, and if $\{M_k\}$ is a sequence of hypersurfaces in $\mathcal{I}_b$ whose blow-up is v, then there exist a subsequence $\{k_j\}$ of $\{k\}$ and points $Z_{k_j} = (Z'_{k_j}, Z_{k_j}^{n+1}) \in M_{k_j} \cap (B_{3/2}(0) \times \mathbf{R})$ such that $\Theta_{M_{k_j}}(Z_{k_j}) \geq 2$ and $\lim_{j\to\infty} \left(Z'_{k_j}, \frac{Z_{k_j}^{n+1}}{\hat{E}_{k_j}}\right) = (z, y)$.

Lemma 3.9. *Let* $(v^+, v^-) \in \mathcal{F}_\delta$ *and* $\{M_k\}$ *be a sequence of hypersurfaces in* $\mathcal{I}_b$ *whose blow-up is* (v^+, v^-). *If* $z \in B_{3/2}(0)$ *is a Lebesgue point of both* v^+ *and* v^-, *and if* $v^+(z) > v^-(z)$, *then there exists* $\beta > 0$ *such that* $M_k \cap (B_\beta(z) \times \mathbf{R})$ *are embedded for all sufficiently large* k, *and hence* v^+ *and* v^- *are individually harmonic in* $B_\beta(z)$.

Proof. Suppose that $z \in B_{3/2}(0)$ is a Lebesgue point of both v^+ and v^-, $v^+(z) > v^-(z)$ but for no $\beta > 0$, $M_k \cap (B_\beta(z) \times \mathbf{R})$ are embedded for all sufficiently large k. Then, taking $\beta = 1/j$, we can find a subsequence $\{k_j\}$ of $\{k\}$ such that there exists $Z_{k_j} = (Z'_{k_j}, Z_{k_j}^{n+1}) \in M_{k_j} \cap (B_{1/j}(z) \times \mathbf{R})$ with $\Theta_{M_{k_j}}(Z_{k_j}) \geq 2$. In particular, $Z'_{k_j} \to z$. By the argument of Lemma 3.8 above, we then have that

$$\int_{B_\rho(z)} R^{2-n} \left(\frac{\partial}{\partial R}\left(\frac{v^+ - y}{R}\right)\right)^2 + R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^- - y}{R}\right)\right)^2 < \infty \tag{3.31}$$

for any $\rho \in (0, 3/2 - |z|)$ and some $y \in \mathbf{R}$, implying that $v^+(z) = v^-(z)\,(= y)$. This contradiction shows that there exists $\beta > 0$ such that $M_k \cap (B_\beta(z) \times \mathbf{R})$ are embedded for all sufficiently large k. It then follows from Lemma 3.2 that v^+ and v^- are individually harmonic in $B_\beta(z)$. The lemma is thus proved. q.e.d.

Remark. Note that the proof of the above lemma shows the following: If for some $\beta \in (0,1)$ there is no $z \in \overline{B}_\beta(0)$ such that $v^+(z) = v^-(z)$, then $M_k \cap (B_\beta(0) \times \mathbf{R})$ are embedded for all sufficiently large k.

In the next lemma and subsequently, we shall use the following notation: for any $v = (v^+, v^-) \in \mathcal{F}_\delta$ and any $\rho \in (0, 3/2)$,

$$\widetilde{v}_\rho = (\widetilde{v}^+_\rho, \widetilde{v}^-_\rho) \equiv \left(\frac{v^+_\rho}{\mathcal{E}_\rho}, \frac{v^-_\rho}{\mathcal{E}_\rho}\right) \tag{3.32}$$

where $v_\rho^\pm(x) = \frac{v^\pm(\frac{2\rho x}{3})}{\frac{2\rho}{3}}$ and $\mathcal{E}_\rho^2 = \rho^{-n-2}\int_{B_\rho(0)}(v^+)^2 + (v^-)^2$. More generally, if $v = (v^+, v^-) \in \mathcal{F}_\delta$, $z \in B_{3/2}(0)$ and $y \in \mathbf{R}$, we let, for $\rho \in (0, 3/2 - |z|)$,

$$\widetilde{v}_{z,\rho,y} = (\widetilde{v}^+_{z,\rho,y}, \widetilde{v}^-_{z,\rho,y}) \equiv \left(\frac{v^+_{z,\rho,y}}{\mathcal{E}_{z,\rho,y}}, \frac{v^-_{z,\rho,y}}{\mathcal{E}_{z,\rho,y}}\right) \tag{3.33}$$

where $v^\pm_{z,\rho,y}(x) = \frac{v^\pm(z+\frac{2\rho x}{3})-y}{\frac{2\rho}{3}}$ and $\mathcal{E}^2_{z,\rho,y} = \rho^{-n-2}\int_{B_\rho(z)}(v^+ - y)^2 + (v^- - y)^2$.

Note that if $v \in \mathcal{F}_\delta$, $z \in B_{3/2}(0)$, $\rho \in (0, 3/2 - |z|)$ and $y \in \mathbf{R}$, then $\widetilde{v}_{z,\rho,y} \in \mathcal{F}_\delta$. In fact, if v is the blow-up (in the sense of Section 3) of the sequence of hypersurfaces $\{M_k\} \subset \mathcal{I}_b$ off the sequence $\{L_k\}$ of affine hyperplanes converging to $\mathbf{R}^n \times \{0\}$, then $\widetilde{v}_{z,\rho,y}$ is the blow-up of the sequence $\widetilde{M_k} \equiv \eta_{(z,\hat{E}_k y), \frac{2}{3}\rho}\, M_k$ off the sequence $\widetilde{L}_k \equiv \left(\frac{2}{3}\rho\right)^{-1}\left(L_k + \hat{E}_k y \nu^{L_k} - (z, \hat{E}_k y)\right)$ of affine hyperplanes, where $\hat{E}_k$ and ν^{L_k} are as defined in Section 3. The fact that $\frac{\mathcal{H}^n\left(\widetilde{M}_k \cap B_2^{n+1}(0)\right)}{\omega_n 2^n} \leq 3 - \delta$ for sufficiently large k is easily checked using the approximate graphical decomposition (as given by the method of [**SS81**] and explained in the discussion of item (2) at the beginning of the present section) of $M_k \cap (B_{2-\epsilon}(0) \times \mathbf{R})$ for a suitably small fixed positive ϵ independent of k.

Finally, if $v \in \mathcal{F}_\delta$ and $z \in B_{3/2}(0)$ is a Lebesgue point of both v^+ and v^- with $v^+(z) = v^-(z) = y$, we let, for $\rho \in (0, 3/2 - |z|)$,

$$\mathcal{E}_{z,\rho} = \mathcal{E}_{z,\rho,y} \quad \text{and} \quad \widetilde{v}^\pm_{z,\rho} = \widetilde{v}^\pm_{z,\rho,y}. \tag{3.34}$$

Proposition 3.10.

(a) *If $v \in \mathcal{F}_\delta$, then v is (a.e. equal to) a continuous function on $B_{3/2}(0)$.*

(b) *For each $\sigma, \sigma' \in (0, 3/2)$ with $\sigma' < \sigma$, there exists a finite number $C = C(n, \sigma, \sigma')$ such that if $v \in \mathcal{F}_\delta$ and $v^+(z) = v^-(z)$ for some point $z \in B_{\sigma'}(0)$, then*

$$|v(x) - v(z)| \leq C|x - z| \left(\int_{B_\sigma(0)} |v|^2\right)^{1/2}$$

for all $x \in B_{\sigma'}(0)$.

Proof. Let $v \in \mathcal{F}_\delta$. Denote by Γ the set of points $z \in B_{3/2}(0)$ with the property that there exists $y = y_z \in \mathbf{R}$ satisfying

$$\int_{B_{\rho/2}(z)} R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^+ - y}{R}\right)\right)^2 + R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^- - y}{R}\right)\right)^2 \leq C\rho^{-n-2}\int_{B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2 \tag{3.35}$$

for all $\rho \in (0, 3/2-|z|)$, where the constant $C = C(n)$ is as in Lemma 3.8. Note that if such y exists for a given point $z \in B_{3/2}(0)$, then it is unique because if the condition (3.35) holds with y_1, y_2 in place of y, then $\int_{B_{\rho/2}(z)} R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{y_1-y_2}{R}\right)\right)^2 < \infty$, implying $y_1 = y_2$. We claim that any $z \in \Gamma$ must be a Lebesgue point of both v^+ and v^- with $v^+(z) = v^-(z) = y$ and that for $z \in \Gamma$, a local cone condition

$$|v(x) - v(z)|^2 \leq \widetilde{C}|x-z|^2\left(\rho_z^{-n-2}\int_{B_{\rho_z}(z)} |v|^2\right) \tag{3.36}$$

must hold for some $\rho_z \in (0, 3/2 - |z|)$ and a.e. $x \in B_{\rho_z/2}(z)$, where $\widetilde{C} = \widetilde{C}(n)$. In order to prove these claims, fix $z \in \Gamma$ and first note that we may suppose that at least one of v^+ or v^- is non-constant in every ball $B_\rho(z)$, $0 < \rho < 3/2 - |z|$, for if both v^+ and v^- were constant in some ball $B_{\rho'}(z)$, $\rho' \in (0, 3/2 - |z|)$, then by (3.35) the value of the constant must be y, and hence we have the claims trivially with (3.36) holding for $\rho_z = \rho'$ and $\widetilde{C} = 1$. Then we must have that

$$\int_{\partial B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2 > 0 \quad \text{for } \rho \in (0, 3/2 - |z|) \tag{3.37}$$

because otherwise, since $(v^+ - y)^2 + (v^- - y)^2$ is subharmonic in $B_{3/2}(0)$ (by Proposition 3.3, part 2), we would have by the maximum principle a $\rho > 0$ such that $(v^+(x) - y)^2 + (v^-(x) - y)^2 = 0$ for a.e. $x \in B_\rho(z)$, contrary to the preceding assumption. Hence (3.37) must hold, so that the frequency function $N_{v,z,y}(\rho)$ is defined for $\rho \in (0, 3/2 - |z|)$ and is monotonically non-decreasing. We claim that

$$\mathcal{N}_{v,y}(z) \geq 1. \tag{3.38}$$

To see this, note that by (3.35) for each $\rho \in (0, 3/2 - |z|)$,

$$\int_{B_{1/2}(0)} R^{2-n}\left(\frac{\partial\left(\widetilde{v}^+_{z,\rho,y}/R\right)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial\left(\widetilde{v}^-_{z,\rho,y}/R\right)}{\partial R}\right)^2 \leq C\int_{B_1(0)} \left(\widetilde{v}^+_{z,\rho,y}\right)^2 + \left(\widetilde{v}^-_{z,\rho,y}\right)^2 \tag{3.39}$$

where the notation is as in (3.33). Since $\widetilde{v}_{z,\rho,y} \in \mathcal{F}_\delta$, we have by Lemma 3.1 that for an arbitrary sequence $\rho_j \downarrow 0^+$, after passing to a subsequence which we continue to denote $\{j\}$, that $\widetilde{v}_{z,\rho_j,y} \to \widetilde{v} \in \mathcal{F}_\delta$, where the convergence is in $W^{1,2}(B_\sigma(0))$ for every $\sigma \in (0, 3/2)$. By (3.39),

$$\int_{B_{1/2}(0)} R^{2-n}\left(\frac{\partial\,(\widetilde{v}^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial\,(\widetilde{v}^-/R)}{\partial R}\right)^2 \le C\int_{B_1(0)} (\widetilde{v}^+)^2 + (\widetilde{v}^-)^2 < \infty \tag{3.40}$$

and we also have by Lemma 3.5 that for each $\rho \in (0,1]$, $\int_{B_\rho(0)} |\widetilde{v}_{z,\rho_j,y}|^2 \ge \left(\frac{2\rho}{3}\right)^{2(N_{v,z,y}(\frac{3}{2}-|z|)-1)}$, and hence that $\widetilde{v} \not\equiv 0$ in any ball $B_\rho(0)$, $\rho \in (0,1]$. Consequently, since $\widetilde{v}^2$ is subharmonic (by Proposition 3.3, part (2)), we have that $\int_{\partial B_\rho(0)} \widetilde{v}^2 > 0$ for $\rho \in (0,1]$, and therefore the frequency function $N_{\widetilde{v},0,0}(\rho)$ is defined for $\rho \in (0,1]$. But then

$$N_{\widetilde{v},0,0}(\rho) = \frac{\rho\int_{B_\rho(0)} |D\widetilde{v}|^2}{\int_{\partial B_\rho(0)} |\widetilde{v}|^2} = \lim_{j\to\infty} \frac{\frac{2}{3}\rho\rho_j \int_{B_{\frac{2}{3}\rho\rho_j}(z)} |Dv|^2}{\int_{\partial B_{\frac{2}{3}\rho\rho_j}(z)} (v^+ - y)^2 + (v^- - y)^2} = \mathcal{N}_{v,y}(z) \tag{3.41}$$

for $\rho \in (0,1]$, and hence by Lemma 3.6, $\widetilde{v}$ is homogeneous of degree $\mathcal{N}_{v,y}(z)$ from the origin. It then follows directly from the finiteness condition (3.40) that $\mathcal{N}_{v,y}(z) \ge 1$.

With $z \in \Gamma$ and $y = y_z$, we next claim that $\rho^{-n-2}\int_{B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2$ is monotonically non-decreasing for $\rho \in (0, 3/2 - |z|)$. To see this, we use the abbreviation $d_{v,z}(x) = \sqrt{(v^+(x) - y)^2 + (v^-(x) - y)^2}$, and compute as follows:

$$\begin{aligned}
&\frac{d}{d\rho}\,\rho^{-n-2}\int_{B_\rho(z)} d_{v,z}^2 \\
&= \frac{d}{d\rho}\int_{B_1(0)} \frac{d_{v,z}^2\,(z+\rho x)}{\rho^2}\,dx \\
&= \int_{B_1(0)} \frac{2d_{v,z}\,(z+\rho x) D\,d_{v,z}\,(z+\rho x)\cdot x}{\rho^2} - \frac{2d_{v,z}^2\,(z+\rho x)}{\rho^3} \\
&= \frac{2}{\rho^3}\int_{B_1(0)} d_{v,z}\,(z+\rho x)\,(D\,d_{v,z}\,(z+\rho x)\cdot \rho x - d_{v,z}\,(z+\rho x)) \\
&= 2\rho^{-n-3}\int_{B_\rho(z)} d_{v,z}(\widetilde{x})\,(D\,d_{v,z}(\widetilde{x})\cdot(\widetilde{x}-z) - d_{v,z}(\widetilde{x}))\,d\widetilde{x}
\end{aligned} \tag{3.42}$$

$$= 2\rho^{-n-3}\int_0^\rho \int_{\partial B_\tau(z)} \left(d_{v,z}(\widetilde{x}) D\, d_{v,z}(\widetilde{x})\cdot(\widetilde{x}-z) - d_{v,z}^2(\widetilde{x})\right)\, d\widetilde{x}\, d\tau$$

$$= 2\rho^{-n-3}\int_0^\rho \left(\frac{1}{2}\tau\int_{\partial\, B_\tau(z)} \frac{\partial}{\partial\, R} d_{v,z}^2 - \int_{\partial\, B_\tau(z)} d_{v,z}^2\right)\, d\tau$$

$$\geq 0.$$

The last inequality holds since $1 \leq \mathcal{N}_{v,y}(z) \leq N_{z,v,y}(\tau) = \frac{\tau\int_{\partial\, B_\tau(z)} \frac{\partial}{\partial R} d_{v,z}^2}{2\int_{\partial\, B_\tau(z)} d_{v,z}^2}$, by (3.38) and (3.21). Thus, in particular, $\rho^{-n-2}\int_{B_\rho(z)} d_{v,\,z}^2$ remains bounded from above as $\rho \to 0$, and consequently z must be a Lebesgue point of both v^+ and v^- with $v^+(z) = v^-(z) = y$.

Now, $|v|^2$ is subharmonic in $B_{3/2}(0)$ by Proposition 3.3, part (2), and hence by the mean value property

$$|v(z)|^2 \leq \omega_n^{-1}\rho_z^{-n}\int_{B_{\rho_z}(z)} |v|^2 \tag{3.43}$$

where $\rho_z = \frac{1}{2}(\frac{3}{2} - |z|)$. Also, since $d_{v,z}^2$ is subharmonic, again by the mean value property we have that for a.e. $x \in B_{\rho_z/2}(z)$,

$$\begin{aligned} d_{v,z}^2(x) &\leq \omega_n^{-1}(|x-z|)^{-n}\int_{B_{|x-z|}(x)} d_{v,z}^2 \\ &\leq \omega_n^{-1}(|x-z|)^{-n}\int_{B_{2|x-z|}(z)} d_{v,z}^2 \\ &= \omega_n^{-1}\, 2^{n+2}|x-z|^2(2|x-z|)^{-n-2}\int_{B_{2|x-z|}(z)} d_{v,z}^2 \\ &\leq \omega_n^{-1}\, 2^{n+2}|x-z|^2\rho_z^{-n-2}\int_{B_{\rho_z}(z)} d_{v,z}^2 \\ &\leq C|x-z|^2\rho_z^{-n-2}\int_{B_{\rho_z}(z)} |v|^2 \end{aligned} \tag{3.44}$$

where $C = C(n)$. Here we have used the monotonicity of $\rho^{-n-2}\int_{B_\rho(z)} d_{v,z}^2$ and the estimate (3.43). This is the required estimate (3.36).

We have thus shown that every $z \in \Gamma$ is a Lebesgue point of both v^+ and v^- with $v^+(z) = v^-(z) = y_z$, and that the local cone condition (3.36) holds at such z.

Now consider a point $z \in B_{3/2} \setminus \Gamma$. We claim that there exists $\sigma_z \in (0, 3/2 - |z|)$ such that $v^+|_{B_{\sigma_z}(z)}$ and $v^-|_{B_{\sigma_z}\,(z)}$ are respectively a. e. equal to harmonic functions $v^{z\,+}$ and $v^{z\,-}$ on $B_{\sigma_z}(z)$. To see this, consider a sequence of hypersurfaces $\{M_k\} \subset \mathcal{I}_b$ whose blow-up is v. There must exist $\sigma_z \in (0, 3/2 - |z|)$ such that for all sufficiently large k, $M_k \cap (B_{\sigma_z}(z) \times \mathbf{R})$ are embedded. For if not, there exists a subsequence $\{k_j\}$, $j = 1, 2, \ldots$ of $\{k\}$ and points $Z_{k_j} = (Z'_{k_j}, Z^{n+1}_{k_j}) \in$

$M_{k_j} \cap (B_{1/j}(z) \times \mathbf{R})$ with $\Theta_{M_{k_j}}(Z_{k_j}) \geq 2$ and by exactly the argument of Lemma 3.8, this implies that (3.35) holds for some $y \in \mathbf{R}$ and all $\rho \in (0, 3/2 - |z|)$, contradicting the fact that $z \in B_{3/2}(0) \setminus \Gamma$. The claim now follows from Lemma 3.2. Now define $\overline{v}^{\pm} : B_{3/2}(0) \to \mathbf{R}$ by setting $\overline{v}^{+}(z) = v^{z\,+}(z)$, $\overline{v}^{-}(z) = v^{z\,-}(z)$ if $z \in B_{3/2}(0) \setminus \Gamma$ and $\overline{v}^{+}(z) = \overline{v}^{-}(z) = y_z$ if $z \in \Gamma$. Since Γ is relatively closed in $B_{3/2}(0)$ (which follows directly from the definition of Γ), it follows by unique continuation for harmonic functions and the continuity estimate (3.36) for points $z \in \Gamma$ that $\overline{v}^{\pm}$ are well defined and are continuous in $B_{3/2}(0)$. Furthermore, $v^{\pm}$ are a. e. equal to $\overline{v}^{\pm}$. This concludes the proof of part (a) of the lemma.

To prove part (b), let $v \in \mathcal{F}_\delta$ (v now assumed to be continuous), $z \in B_{3/2}(0)$ and suppose that $v^{+}(z) = v^{-}(z) = y$. Note first that we must have that either $v^{+} \equiv v^{-} \equiv y$ in $B_{3/2}(0)$ or that $\int_{\partial B_\sigma(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$ for all $\sigma \in (0, 3/2 - |z|)$. To see this, first note that if $\int_{\partial B_{\sigma_0}(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$ for some $\sigma_0 \in (0, 3/2 - |z|)$, then by continuity, there exists $\sigma_1 \in (0, \sigma_0)$ such that $\int_{\partial B_\sigma(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$ for all $\sigma \in (\sigma_1, \sigma_0]$. Hence the frequency function $N_{v,z}(\sigma)$ is defined for $\sigma \in (\sigma_1, \sigma_0]$ and by exactly the argument leading to (3), we have the estimate

$$\frac{\int_{\partial B_{\sigma_0}(z)} (v^{+} - y)^2 + (v^{-} - y)^2}{\sigma_0^{2N+n-1}} \leq \frac{\int_{\partial B_{\sigma}(z)} (v^{+} - y)^2 + (v^{-} - y)^2}{\sigma^{2N+n-1}} \tag{3.45}$$

for each $\sigma \in (\sigma_1, \sigma_0]$, where $N = N_{v,z}(\sigma_0)$. Letting $\sigma \to \sigma_1$ in this, we see that $\int_{\partial B_{\sigma_1}(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$. This argument shows that if $\int_{\partial B_{\sigma_0}(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$ for some $\sigma_0 \in (0, 3/2 - |z|)$ then $\int_{\partial B_\sigma(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$ for all $\sigma \in (0, \sigma_0]$. On the other hand, since $(v^{+} - y)^2 + (v^{-} - y)^2$ is subharmonic, if $\int_{\partial B_\sigma(z)} (v^{+} - y)^2 + (v^{-} - y)^2 = 0$ for some $\sigma \in (0, 3/2 - |z|)$, then by the maximum principle we must have that $v^{+}(x) = v^{-}(x) = y$ for all $x \in \overline{B}_\sigma(z)$. Hence, either $\int_{\partial B_\sigma(z)} (v^{+} - y)^2 + (v^{-} - y)^2 > 0$ for all $\sigma \in (0, 3/2 - |z|)$ or $v^{+}(x) = v^{-}(x) = y$ for all $x \in B_{3/2 - |z|}(z)$. If the latter were the case, it is easy to see using the estimate (3.45) repeatedly with suitably chosen center points in place of z that we must have $v^{+}(x) = v^{-}(x) = y$ for all $x \in B_{3/2}(0)$.

If $v^{+} \equiv v^{-} \equiv y$ in $B_{3/2}(0)$, the estimate in part (b) holds trivially. Otherwise, we have by the above argument that the frequency function $N_{v,z}(\sigma)$ is well defined for $\sigma \in (0, 3/2 - |z|)$ and we claim that $\mathcal{N}_v(z) \geq 1$. This is easy to see if $v^{+}|_{B_\sigma(z)} \equiv v^{-}|_{B_\sigma(z)}$ for some $\sigma \in (0, 3/2 - |z|)$, because then $v^{+} = v^{-} = h$ in $B_\sigma(z)$ (where $h = \frac{1}{2}(v^{+} + v^{-})$) and hence, since h is harmonic (everywhere in $B_{3/2}(0)$), it follows in this case that $\mathcal{N}_v(z) = \mathcal{N}_{h-h(z)}(z) \geq 1$. Otherwise, by Lemma 3.8, we have the estimate (3.35) for each $\rho \in (0, 3/2 - |z|)$, and we may then argue exactly as in the proof of (3.38) above to conclude that $\mathcal{N}_v(z) \geq 1$. Consequently, we also have the monotonicity estimate (3.42), by the same computation, for each $\rho \in (0, 3/2 - |z|)$.

To complete the proof of part (b), let $\sigma, \sigma' \in (0, 3/2)$ with $\sigma' < \sigma$, and suppose that $z \in B_{\sigma'}(0)$ and that $v^+(z) = v^-(z)$. Since $|v|^2$ is subharmonic, we have by the mean value property that

$$\sup_{B_{\sigma'}(0)} |v|^2 \leq C \int_{B_\sigma(0)} |v|^2 \tag{3.46}$$

where $C = C(n, \sigma, \sigma')$. Also, since $d^2_{v,z} \equiv (v^+ - y)^2 + (v^- - y)^2$ is subharmonic, again by the mean value property we have that for every $x \in B_{\sigma'}(0)$ with $|x - z| \leq (\sigma - \sigma')/2$,

$$\begin{aligned} d^2_{v,z}(x) &\leq \omega_n^{-1}(|x-z|)^{-n} \int_{B_{|x-z|}(x)} d^2_{v,z} \\ &\leq \omega_n^{-1}(|x-z|)^{-n} \int_{B_{2|x-z|}(z)} d^2_{v,z} \\ &= C|x-z|^2(2|x-z|)^{-n-2} \int_{B_{2|x-z|}(z)} d^2_{v,z} \\ &\leq C|x-z|^2 \int_{B_{\sigma-\sigma'}(z)} (v^+ - y)^2 + (v^- - y)^2 \\ &\leq C|x-z|^2 \int_{B_\sigma(0)} |v|^2 \end{aligned} \tag{3.47}$$

where $C = C(n, \sigma, \sigma')$. Here we have used the monotonicity of $\rho^{-n-2}\int_{B_\rho(z)} d^2_{v,z}$ and the estimate (3.46). If, on the other hand, $x, z \in B_{\sigma'}(0)$ satisfy $|x - z| > (\sigma - \sigma')/2$, then $d_{v,z}(x)^2 \leq 2|v(x)|^2 + 2|v(z)|^2 \leq C|x-z|^2 \int_{B_\sigma(0)} |v|^2$ by (3.46). This completes the proof of part (b) and the lemma. q.e.d.

We next establish several important properties of w:

Proposition 3.11. *Suppose $v = (v^+, v^-) \in \mathcal{F}_\delta$ and recall the notation $w = \frac{1}{2}(v^+ - v^-)$. We have the following:*

(1) *$w \geq 0$.*
(2) *$\int |Dw|^2\zeta = -\int wDw \cdot D\zeta$ for every $\zeta \in C^1_c(B_{3/2}(0))$.*
(3) *$\int_{B_\sigma(z)} |Dw|^2 = \int_{\partial B_\sigma(z)} w\frac{\partial w}{\partial R}$ for each ball $B_\sigma(z)$ with $\overline{B}_\sigma(z) \subset B_{3/2}(0)$.*
(4) *$\sum_{i,j=1}^n \int_{B_\sigma(z)} \left(|Dw|^2\delta_{ij} - 2D_iwD_jw\right) D_i\zeta^j = 0$ for every ball $B_\sigma(z)$ with $\overline{B}_\sigma(z) \subset B_{3/2}(0)$ and every $\zeta^j \in C^1_c(B_\sigma(z))$, $j = 1, 2, 3, \ldots, n$.*
(5) *$\Delta w = 0$ in $B_{3/2}(0) \setminus Z_w$ where Z_w is the zero set of w.*

(6) *Either* $Z_w = \emptyset$ *or* $\mathcal{H}^{n-2}(Z_w) = \infty$.

(7) *If* $\int_{\partial B_{\rho_1}(z_1)} w^2 > 0$ *for some* $z_1 \in B_{3/2}(0)$ *and* $\rho_1 \in (0, 3/2 - |z_1|)$, *then* $\int_{\partial B_{\rho}(z_1)} w^2 > 0$ *for all* $\rho \in (0, \rho_1]$.

(8) *Either* $w \equiv 0$ *in* $B_{3/2}(0)$ *or* $\int_{\partial B_\rho(z)} w^2 > 0$ *for each* $z \in B_{3/2}(0)$ *and each* $\rho \in (0, 3/2 - |z|)$.

(9) *Either* $w \equiv 0$ *in* $B_{3/2}(0)$ *or the frequency function* $N_{w,z}(\rho) \equiv \frac{\rho \int_{B_\rho(z)} |Dw|^2}{\int_{\partial B_\rho(z)} w^2}$ *is defined for each* $z \in B_{3/2}(0)$ *and each* $\rho \in (0, 3/2 - |z|)$ *and is monotonically non-decreasing as a function of* ρ. *Hence,* $\mathcal{N}_w(z) \equiv \lim_{\rho \downarrow 0} N_{w,z}(\rho)$ *exists for each* $z \in B_{3/2}(0)$ *unless* $w \equiv 0$.

Proof. Part (1) follows from the definition of w. Part (2) follows directly by substituting $v^+ = h+w$, $v^- = h-w$ in the identity of part (2) of Proposition 3.3, and observing that h, being harmonic, satisfies the identity $\int |Dh|^2\zeta = -\int hDh\cdot D\zeta$. Similarly, part (4) follows by substituting $v^+ = h + w$, $v^- = h - w$ in the identity of part (4) of Proposition 3.3 and observing that $\sum_{i,j=1}^n \int_{B_\sigma(z)} \left(|Dh|^2\delta_{ij} - 2D_ihD_jh\right) D_i\zeta^j = 0$. Part (3) follows from part (2) by taking a smooth approximation to the characteristic function of the ball $B_\sigma(z)$. Part (5) follows from Lemma 3.9.

To see part (6), note first that it suffices to show that for each given $\sigma \in (0, 3/2)$, either $Z_w \cap B_\sigma(0) = \emptyset$ or $\mathcal{H}^{n-2}(Z_w \cap B_\sigma(0)) = \infty$. So fix $\sigma \in (0, 3/2)$ and suppose that $\mathcal{H}^{n-2}(Z_w \cap B_\sigma(0)) < \infty$. By continuity of w (Lemma 3.10), Z_w is closed, so that by exactly the same construction as in (3.8), we have for each $\tau \in (0, 3/2 - \sigma)$ a sequence of Lipschitz functions $\beta_\ell : B_{3/2}(0) \to \mathbf{R}$, $\ell = 1, 2, 3, \ldots$, with $\beta_\ell(x) \equiv 1$ for each ℓ and each x with $\operatorname{dist}(x, Z_w \cap B_\sigma(0)) > \tau$, $\beta_\ell \equiv 0$ in some neighborhood of $Z_w \cap B_\sigma(0)$, $0 \leq \beta_\ell \leq 1$ everywhere and $\int_{B_{3/2}(0)} |D\beta_\ell|^2 \to 0$ as $\ell \to \infty$. Now, given $\varphi \in C_c^\infty(B_\sigma(0))$, we have that $\beta_\ell\varphi$ is Lipschitz with compact support in $B_\sigma(0) \setminus Z_w$, and hence, since w is harmonic in $B_{3/2}(0) \setminus Z_w$,

$$\int_{B_\sigma(0)} Dw \cdot D(\beta_\ell\varphi) = 0$$

which implies that

$$\int_{B_\sigma(0)\setminus(Z_w)_\tau} Dw \cdot D\varphi = -\int_{B_\sigma(0)\cap(Z_w)_\tau} \beta_\ell Dw \cdot D\varphi + \int_{B_\sigma(0)} \varphi Dw \cdot D\beta_\ell$$

where $(Z_w)_\tau$ denotes the τ neighborhood of Z_w. Hence,

$$\begin{aligned} &\left|\int_{B_\sigma(0)\setminus(Z_w)_\tau} Dw \cdot D\varphi\right| \\ &\leq \sup|D\varphi| \left(\int_{B_\sigma(0)} |Dw|^2\right)^{1/2} \left(\mathcal{H}^n(B_\sigma(0) \cap (Z_w)_\tau)\right)^{1/2} \end{aligned} \tag{3.48}$$

$$+ \sup|\varphi| \left(\int_{B_\sigma(0)} |Dw|^2\right)^{1/2} \left(\int_{B_\sigma(0)} |D\beta_\ell|^2\right)^{1/2}.$$

Letting first $\ell \to \infty$ and then $\tau \to 0$ in this, we conclude that w is harmonic in $B_\sigma(0)$. Since $w \geq 0$ and $\mathcal{H}^{n-2}(Z_w \cap B_\sigma(0)) < \infty$, it follows from the maximum principle that $Z_w \cap B_\sigma(0) = \emptyset$. This proves the assertion in part (6).

To see part (7), first note that it follows from the identity of part (4) that

$$\frac{d}{d\rho}\left(\rho^{2-n}\int_{B_\rho(z)} |Dw|^2\right) = 2\rho^{2-n}\int_{\partial B_\rho(z)} \left|\frac{\partial\, w}{\partial\, R}\right|^2. \tag{3.49}$$

(See [**Sim96**], p. 24 for the details of this claim.) Also, the identity of part (3) and the definition of $N_{w,z}(\rho)$ directly imply that

$$N_{w,z}(\rho) = \frac{\rho\frac{d}{d\rho}\left(\rho^{1-n}\int_{\partial B_\rho(z)} w^2\right)}{2\rho^{1-n}\int_{\partial B_\rho(z)} w^2} \tag{3.50}$$

whenever $N_{w,z}(\rho)$ is defined. To prove (7), suppose $\int_{\partial B_{\rho_1}(z_1)} w^2 > 0$ for some $z_1 \in B_{3/2}(0)$ and $\rho_1 > 0$. Then by continuity, there exist ρ_0 with $0 < \rho_0 < \rho_1$ such that $\int_{\partial B_\rho(z_1)} w^2 > 0$ for all $\rho \in (\rho_0, \rho_1]$, and hence $N_{w,z}(\rho)$ is defined for all $\rho \in (\rho_0, \rho_1]$. A computation similar to that of (3.20) using the identity (3.49), the identity of part (3) of the present lemma and the Cauchy-Schwarz inequality then implies that

$$\frac{d}{d\rho}\, N_{w,z}(\rho) \geq 0 \tag{3.51}$$

for $\rho \in (\rho_0, \rho_1]$. Thus in particular, $N_{w,z}(\rho) \leq N_2 \equiv N_{w,z}(\rho_1)$ for $\rho \in (\rho_0, \rho_1]$. Using the expression (3.50) in this last inequality and integrating the resulting differential inequality then gives

$$\frac{\sigma^{1-n}\int_{\partial B_\sigma(z)} w^2}{\sigma^{2N_2}} \geq \frac{\tau^{1-n}\int_{\partial B_\tau(z)} w^2}{\tau^{2N_2}} \tag{3.52}$$

for all σ, τ with $\rho_0 < \sigma \leq \tau \leq \rho_1$. Using this with $\tau = \rho_1$ and $\sigma = \sigma_j$ where $\sigma_j \downarrow \rho_0$, we conclude that $\int_{\partial B_{\rho_0}(z)} w^2 > 0$. It follows that $\int_{\partial B_\rho(z)} w^2 > 0$ for all $\rho \in (0, \rho_1]$ as required.

To see parts (8) and (9), let $\mathcal{O} = \{z \in B_{3/2}(0) \,:\, \int_{\partial B_\rho(z)} w^2 > 0$ for each $\rho \in (0, 3/2 - |z|)\}$. Since w^2 is subharmonic (by (2)), it follows from the maximum principle that if $w(z) \neq 0$ for some $z \in B_{3/2}(0)$, then $z \in \mathcal{O}$. Thus if $w \not\equiv 0$, then $\mathcal{O} \neq \emptyset$. We argue that $\mathcal{O}$ is open as follows. Suppose $z \in \mathcal{O}$ and consider $z' \in B_{3/2}(0)$ with $|z' - z| < \frac{1}{4}(\frac{3}{2} - |z|)$. By the maximum principle and the fact that $z \in \mathcal{O}$, it follows that $\int_{\partial B_\rho(z')} w^2 > 0$ for each ρ with $|z' - z| < \rho < 3/2 - |z|$. On the other hand, it follows from part (7) that $\int_{\partial B_\rho(z')} w^2 > 0$ for each $\rho \in (0, |z'-z|]$, giving that $z' \in \mathcal{O}$. Thus $\mathcal{O}$ is open. It is easy to see by the maximum principle again that $\mathcal{O}$ is relatively closed in $B_{3/2}(0)$. Thus, we conclude that either $w \equiv 0$ in $B_{3/2}(0)$ or that $N_{w,z}(\rho)$ is defined for all $z \in B_{3/2}(0)$ and all $\rho \in (0, 3/2 - |z|)$ with (3.51) satisfied. q.e.d.

Remark. Although we shall not need it anywhere in the present paper, we point out here that w is weakly subharmonic in $B_{3/2}(0)$. To see this, choose a small positive constant ϵ, and let $\gamma_\epsilon : \mathbf{R} \to \mathbf{R}$ be a smooth cut-off function with $\gamma_\epsilon(t) = 0$ if $t \leq \epsilon$, $\gamma_\epsilon(t) = 1$ if $t > 2\epsilon$, $\gamma_\epsilon(t) \geq 0$, and $0 \leq \gamma_\epsilon'(t) \leq 2/\epsilon$ for all t. Then, since w is harmonic in $B_{3/2}(0) \setminus Z_w$, we have that for any smooth, non-negative function φ with compact support in $B_{3/2}(0)$,

$$\int_{B_{3/2}(0)} \varphi \gamma_\epsilon(w) \Delta w = 0. \tag{3.53}$$

Integrating by parts in this we get

$$\int_{B_{3/2}(0)} \gamma_\epsilon(w) D\varphi \cdot Dw = -\int_{B_{3/2}(0)} \varphi \gamma_\epsilon'(w) |Dw|^2. \tag{3.54}$$

Since the right hand side of the above is non-positive, we have that $\int_{B_{3/2}(0)} \gamma_\epsilon(w) D\varphi \cdot Dw \leq 0$. The assertion follows by letting $\epsilon \to 0$ in this.

Lemma 3.12. *Let $v = (v^+, v^-) \in \mathcal{F}_\delta$ with $v^+(0) = v^-(0) = 0$. If v is homogeneous of degree 1 from the origin, then $\operatorname{graph} v^+ \cup \operatorname{graph} v^- = P_1 \cup P_2$, where P_1, P_2 are hyperplanes of $\mathbf{R}^{n+1}$, possibly with $P_1 \equiv P_2$.*

Proof. Since $h = \frac{1}{2}(v^+ + v^-)$ is harmonic, and homogeneous of degree 1 by hypothesis, h must be a linear function. Hence, if $v^+ \equiv v^-$, the lemma holds with $P_1 \equiv P_2$. So suppose $v^+ \not\equiv v^-$. By rotating coordinates, we may and we shall assume that $h \equiv 0$. Let $w = \frac{1}{2}(v^+ - v^-)$. By Proposition 3.11, part (6) $\mathcal{H}^{n-2}(Z_w \cap B_1(0)) = \infty$. Choose an arbitrary point $z \in (Z_w \setminus \{0\}) \cap B_1(0)$ and blow up (v^+, v^-) at z. This gives

$$\widetilde{v} \equiv (\widetilde{v}^+, \widetilde{v}^-) = \lim_{j \to \infty} \widetilde{v}_{z,\sigma_j} \tag{3.55}$$

for some sequence of numbers $\sigma_j \searrow 0$, where $\widetilde{v}_{z,\sigma_j}$ is as in (3.34) with $y = v^+(z) = v^-(z) = 0$. Note that since $\widetilde{v}_{z,\sigma_j} \in \mathcal{F}_\delta$, the convergence in (3.55) is, by Lemma 3.1, in $W^{1,2}(B_\sigma(0))$ for each $\sigma \in (0, 3/2)$. Setting $\rho = \sigma_j$ and $\theta = 2\rho/3$ in Lemma 3.5 and letting $j \to \infty$, it follows that $\int_{B_\rho(0)} |\widetilde{v}|^2 \geq \left(\frac{2\rho}{3}\right)^{2(N_{v,z}(1)-1)}$ for each $\rho \in (0,1]$ so that $\widetilde{v}$ is not identically zero in any ball $B_\rho(0)$. Hence we have the assertions (3.40) and (3.41), by exactly the same reasoning. Thus, $\widetilde{v}$ is homogeneous of degree $\mathcal{N}_v(z)$ from the origin, and consequently by the finiteness of the left hand side of (3.40), we immediately have that $\mathcal{N}_v(z) \geq 1$. On the other hand, by homogeneity of v it follows that $\mathcal{N}_v(z) \leq \mathcal{N}_v(0) = 1$, and hence we conclude that

$$\mathcal{N}_v(z) = \mathcal{N}_v(0) \tag{3.56}$$

for any $z \in Z_w$. Therefore, v is invariant under translations in the direction of any element of Z_w. (See [**Wic04a**], Lemma 5.17.) Since $w \not\equiv 0$ by assumption and $\mathcal{H}^{n-2}(Z_w \cap B_1(0)) = \infty$, this means that v is invariant under translations precisely by the elements of an $(n-1)$-dimensional linear subspace, and hence each of v^+ and v^- must be a function of a single variable. Since by Proposition 3.11, part (5) $v^\pm$ are harmonic in $B_{3/2} \setminus Z_w$, it follows that the union of the graphs of v^+ and v^- must be equal to the union of four distinct closed, n-dimensional half spaces of $\mathbf{R}^{n+1}$ meeting along a common $(n-1)$-dimensional subspace.

To complete the proof, note that since $v^+ + v^- \equiv 0$, it suffices to show that the two half spaces that make up graph w make equal angles with $\mathbf{R}^n \times \{0\}$. This follows from the identity of Proposition 3.11, part (4). Specifically, suppose without loss of generality that $Z_w = \mathbf{R}^{n-1} \times \{(0,0)\}$ and $w(x) = \overline{w}(x^1)$. Setting $\zeta^2 = \zeta^3 = \ldots \zeta^n \equiv 0$ in the identity of conclusion (4) of Proposition 3.11, we get the statement that $\int \left(\frac{d\overline{w}}{dx^1}\right)^2 \frac{\partial \zeta^1}{\partial x^1} = 0$ for every $\zeta^1 \in C^1_c(B_1(0))$. If α^+ and α^- are the angles that graph $\overline{w}$ makes with the positive and negative x^1-axes respectively, this identity says that $\tan^2 \alpha^- \int_{B_1(0) \cap \{x^1 < 0\}} \frac{\partial \zeta^1}{\partial x^1} + \tan^2 \alpha^+ \int_{B_1(0) \cap \{x^1 > 0\}} \frac{\partial \zeta^1}{\partial x^1} = 0$ for every $\zeta^1 \in C^1_c(B_1(0))$. Taking a standard cut-off function for ζ^1 in this yields $\alpha^- = \alpha^+$. The lemma is thus proved. q.e.d.

The argument of the preceding lemma shows the following:

Lemma 3.13. *Suppose $v = (v^+, v^-) \in \mathcal{F}_\delta$, $v^+(z) = v^-(z)$ and that v is non constant in $B_{3/2}(0)$. Then $\mathcal{N}_v(z) \geq 1$.*

We conclude this section by stating the following upper semi-continuity result, which follows directly from the monotonicity of $N_{v,z}(\cdot)$.

Lemma 3.14. *Suppose $v_k \in \mathcal{F}_\delta$ for $k = 1, 2, 3, \ldots$, $z \in B_{3/2}(0)$, $v_k \to v$ in $W^{1,2}_{\mathrm{loc}}(B_{3/2}(0))$ and that v_k, v are not identically equal to 0 in $B_{3/2}(0)$ for all $k = 1, 2, 3, \ldots$. Then $\mathcal{N}_v(z) \geq \limsup_{k \to \infty} \mathcal{N}_{v_k}(z)$.*

4. A transverse picture

In this section, we analyze the situation where a hypersurface $M \in \mathcal{I}_b$ is weakly close to a multiplicity 2 hyperplane, but when it is scaled "vertically" (i.e., blown up) by its height excess relative to this hyperplane, it becomes close to a transversely intersecting pair of hyperplanes. The geometric meaning of this is of course that M is in fact significantly closer, in a weak sense, to a transverse pair of hyperplanes (with a small angle) than it is to the multiplicity 2 hyperplane; i.e., the "fine excess" of M measured relative to a suitably chosen transverse pair of hyperplanes is significantly smaller than the "coarse excess" of M relative to the multiplicity 2 hyperplane. We obtain in this case (in Lemma 4.1 and its variant Lemma 4.2 below) improvement of the fine excess at a fixed smaller scale. The arguments used to prove excess improvement here are in part variants of those developed by L. Simon in [**Sim93**], and are in fact carried out in detail in [**Wic04a**], although the results are not presented there in the form below. Here we state the lemmas in the form needed for the purposes of the present paper, and outline their proof, referring the reader to [**Wic04a**] and [**Sim93**] for details.

The lemmas have two applications; we shall need Lemma 4.1 to handle one case of the main excess decay lemma (Lemma 6.3) of the paper, and we shall apply Lemma 4.2 in Sections 5 to prove regularity of functions in $\mathcal{F}_\delta$ whenever their graphs are close to transversely intersecting pairs of hyperplanes. (See Lemmas 5.4 and 5.6.)

Lemma 4.1. *Let $\theta \in (0, 1/8)$, $\delta \in (0,1)$ and $\tau \in (0,1)$. There exists a number $\epsilon_0 = \epsilon_0(n, \theta, \delta, \tau) > 0$ such that the following holds. Suppose $M \in \mathcal{I}_b$ and*

(1) $\frac{\mathcal{H}^n(M \cap B_2^{n+1}(0))}{\omega_n 2^n} \leq 3 - \delta$,
(2) $\hat{E}_M^2(3/2, L) \equiv \left(\frac{3}{2}\right)^{-n-2} \int_{M \cap (B_{3/2}(0) \times \mathbf{R})} \operatorname{dist}^2(x, L) \leq \epsilon_0$ *for some affine hyperplane L with $d_{\mathcal{H}}(L \cap (B_1(0) \times \mathbf{R}), B_1(0)) \leq \epsilon_0$, and*
(3) $\int_{M \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \leq \epsilon_0 \hat{E}_M^2(3/2, L)$ *for some pair of affine hyperplanes $P = P^+ \cup P^-$ with $P^+ \cap P^- \cap (B_{\theta/4}(0) \times \mathbf{R}) \neq \emptyset$.*

Then, either

(a) *there exists an affine hyperplane $\widetilde{L}$ with $d_{\mathcal{H}}(\widetilde{L} \cap (B_1(0) \times \mathbf{R}), L \cap (B_1(0) \times \mathbf{R})) \leq C\hat{E}_M(3/2, L)$, $C = C(n)$ such that*

$$\left(\frac{1}{2}\right)^{-n-2} \int_{M \cap (B_{1/2}(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{L}) \leq \tau \hat{E}_M^2(3/2, L) \quad \text{or}$$

(b) *there exists a pair of affine hyperplanes $\widetilde{P} = \widetilde{P}^+ \cup \widetilde{P}^-$ with $\widetilde{P}^+ \not\equiv \widetilde{P}^-$ and $\widetilde{P}^+ \cap \widetilde{P}^- \cap (B_\theta(0) \times \mathbf{R}) \neq \emptyset$ such that*
(i)

$$\theta^{-2} d_{\mathcal{H}}^2(\widetilde{P} \cap (B_\theta(0) \times \mathbf{R}), P \cap (B_\theta(0) \times \mathbf{R})) \leq C \int_{M \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P),$$

(ii)

$$\theta^{-n-2} \int_{M \cap (B_\theta(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}) \leq C\theta^2 \int_{M \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \quad \text{and}$$

(iii) $M \cap ((B_\theta(0) \setminus S_{\widetilde{P}}(\theta^2/16)) \times \mathbf{R}) = \operatorname{graph} u^+ \cup \operatorname{graph} u^-$ *where, for* $\sigma \in (0,1)$,

$S_{\widetilde{P}}(\sigma) = \{x \in \mathbf{R}^n \times \{0\} \,:\, \operatorname{dist}(x, \pi(\widetilde{P}^+ \cap \widetilde{P}^-)) \leq \sigma\}$,

$u^\pm \in C^2(B_\theta(0) \setminus S_{\widetilde{P}}(\theta^2/16))$ *with* $u^+ > u^-$ *and, for* $x \in B_\theta(0) \setminus S_{\widetilde{P}}(\theta^2/16)$, $\operatorname{dist}((x, u^+(x)), \widetilde{P}) = \operatorname{dist}((x, u^+(x)), \widetilde{P}^+)$ *and* $\operatorname{dist}((x, u^-(x)), \widetilde{P}) = \operatorname{dist}((x, u^-(x)), \widetilde{P}^-)$.

Here $C = C(n) > 0$ *and* $\pi \,:\, \mathbf{R}^{n+1} \to \mathbf{R}^n \times \{0\}$ *is the orthogonal projection.*

Proof. We argue by contradiction, so consider a sequence $\{M_k\} \subset \mathcal{I}_b$ satisfying

(1) $\frac{\mathcal{H}^n(M_k \cap B_2(0))}{\omega_n 2^n} \leq 3 - \delta$

(2) $\hat{E}_k^2 \equiv \hat{E}^2_{M_k}(3/2, L_k) = \left(\frac{3}{2}\right)^{-n-2} \int_{M_k \cap (B_{3/2}(0) \times \mathbf{R})} \operatorname{dist}^2(x, L_k) \leq \frac{1}{k}$ for some affine hyperplane L_k with $d_{\mathcal{H}}(L_k \cap (B_1(0) \times \mathbf{R}), B_1(0)) \leq \frac{1}{k}$ and

(3) $\int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k) \leq \frac{1}{k} \hat{E}_k^2$ for some pair of affine hyperplanes $P_k = P_k^+ \cup P_k^-$ with $P_k^+ \cap P_k^- \cap (B_{\theta/4}(0) \times \mathbf{R}) \neq \emptyset$.

Write $P_k = P_k^{(1)} \cup P_k^{(2)}$ where $P_k^{(1)}$, $P_k^{(2)}$ are affine hyperplanes. It follows from (2) and (3) above that

$$\text{(4.1)} \quad \text{either} \quad \operatorname{dist}_{\mathcal{H}}(L_k \cap (B_1(0) \times \mathbf{R}), P_k^{(1)} \cap (B_1(0) \times \mathbf{R})) \leq C\hat{E}_k$$
$$\text{or} \quad \operatorname{dist}_{\mathcal{H}}(L_k \cap (B_1(0) \times \mathbf{R}), P_k^{(2)} \cap (B_1(0) \times \mathbf{R})) \leq C\hat{E}_k$$

where $C = C(n)$. For $i = 1, 2$, define $p_k^{(i)} \,:\, L_k \to L_k^\perp$ by $P_k^{(i)} = \operatorname{graph} p_k^{(i)} \equiv \{x + p_k^{(i)}(x) \,:\, x \in L_k\}$ (if $P_k^{(i)}$ is perpendicular to L_k, tilt $P_k^{(i)}$ slightly) and set

$$\text{(4.2)} \qquad p^{(i)} = \lim_{k \to \infty} (\hat{E}_k)^{-1} p_k^{(i)} \circ \varphi_k$$

and $P^{(i)} = \operatorname{graph} p^{(i)}$, where $\varphi_k \,:\, \mathbf{R}^n \times \{0\} \to \mathbf{R}$ is such that $\operatorname{graph} \varphi_k = L_k$. The limit exists, possibly after passing to a subsequence. Let $P = P^{(1)} \cup P^{(2)}$. Note that by (4.1), at most one of $P^{(1)}$ and $P^{(2)}$ can be perpendicular to $\mathbf{R}^n \times \{0\}$.

Now blow up the M_k's by $\hat{E}_k$, to produce $v^+, v^- \,:\, B_{3/2}(0) \to \mathbf{R}$ as described in Section 3. Condition (3) says that graph $v^+|_{B_1(0)} \cup$ graph $v^-|_{B_1(0)} \subseteq P$.

Suppose $v^+|_{B_1(0)} \equiv v^-|_{B_1(0)}$. Then $w = \frac{1}{2}(v^+ - v^-) \equiv 0$ on $\overline{B}_1(0)$, and hence by part (8) of Lemma 3.11, $w \equiv 0$ on $B_{3/2}(0)$. It follows from this and the fact that $\frac{1}{2}(v^+ + v^-)$ is harmonic everywhere that graph $v^+|_{B_{3/2}(0)} =$ graph $v^-|_{B_{3/2}(0)} = L \cap (B_{3/2}(0) \times \mathbf{R})$ for some affine hyperplane L (in fact $L = P^{(1)}$ or $L = P^{(2)}$), so that in this case, for sufficiently large k, option (a) of the conclusion of the lemma holds with M_k in place of M and $\widetilde{L}_k = \mathrm{graph}\,(\varphi_k + \hat{E}_k\varphi)$ in place of $\widetilde{L}$ where $\varphi : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ is such that $L = \mathrm{graph}\,\varphi$.

If, on the other hand, $v^+|_{B_1(0)} \not\equiv v^-|_{B_1(0)}$, then P must be the union of distinct affine hyperplanes and graph $v^+|_{B_1(0)} \cup$ graph $v^-|_{B_1(0)} = P \cap (B_1(0) \times \mathbf{R})$. Note that by Lemma 3.3, part (2), $P \cap (B_1(0) \times \mathbf{R}) \subset \{(x', x^{n+1}) \in \mathbf{R}^{n+1} : |x^{n+1}| \leq C\}$ where $C = C(n)$. If $\sup_{B_1(0)} |v^+ - v^-| < \tau/2$, we again have option (a) of the conclusion of the lemma with M_k in place of M and $\widetilde{L}_k = \mathrm{graph}\,(\varphi_k + \hat{E}_k\varphi)$ in place of $\widetilde{L}$, where $\varphi : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ is the affine function such that $\varphi|_{B_{3/2}(0)} = \frac{1}{2}(v^+ + v^-)$. So suppose

$$\sup_{B_1(0)} |v^+ - v^-| \geq \tau/2. \tag{4.3}$$

Denote by Γ the axis of P (i.e., $\Gamma = P^+ \cap P^-$) and for $\sigma \in (0,1)$, let $N(\sigma)$ be the tubular neighborhood of radius σ around Γ (i.e., $N(\sigma) = \{X \in \mathbf{R}^{n+1} : \mathrm{dist}\,(X, \Gamma) \leq \sigma\}$). We claim that for each given $\sigma \in (0, 1/2)$, $M_k \cap (B_1(0) \times \mathbf{R})$ must be embedded outside $N(\sigma)$ for all sufficiently large k. For if not, we would have a number $\sigma \in (0, 1/2)$ and a subsequence of $\{M_k\}$ which we continue to denote $\{M_k\}$ such that $(M_k \setminus N(\sigma)) \cap (B_1(0) \times \mathbf{R})$ contains a point $Z_k = (Z_k', Z_k^{n+1})$ with $\Theta_{M_k}(Z_k) \geq 2$. The argument of the proof of Lemma 3.8 (with $\rho = 1/2$) then gives that

$$\int_{B_{1/4}(z)} R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^+(x) - y}{R}\right)\right)^2 + R^{2-n}\left(\frac{\partial}{\partial R}\left(\frac{v^-(x) - y}{R}\right)\right)^2 dx < \infty$$

where $z = \lim_{k\to\infty} Z_k'$, $y = \lim_{k\to\infty} \frac{Z_k^{n+1}}{\hat{E}_k}$ (both limits exist after possibly passing to a subsequence, the latter by Lemma 3.7), $R = |x - z|$ and $\frac{\partial}{\partial R}$ denotes radial differentiation. This implies that $v^+(z) = v^-(z)$ $(= y)$, which is impossible since $z \in \overline{B_1(0)} \setminus \pi\,(N(\sigma))$ while any point $\widetilde{z}$ with $v^+(\widetilde{z}) = v^-(\widetilde{z})$ must be contained in $\pi\,(\Gamma) \cap B_1(0)$. Thus, if $\{\sigma_k\}$ is any sequence of numbers with $\tau_k \searrow 0$, we can find a subsequence of $\{M_k\}$ (which we again denote $\{M_k\}$) such that $M_k \cap (B_1(0) \times \mathbf{R})$ is embedded outside $N(\sigma_k)$.

Now blow up $M_k \cap (B_1(0) \times \mathbf{R})$ by the fine excess

$$E_k = \sqrt{\int_{M_k \cap (B_1(0) \times \mathbf{R})} \mathrm{dist}^2\,(x, P_k)}$$

exactly as described in Section 6 of [**Wic04a**], and outlined in the paragraph below. Note that although in [**Wic04a**] M_k are assumed to be free of singularities, this assumption is not necessary for the blow up argument of Section 6 of [**Wic04a**].

Thus, let $\widetilde{q}_k$ be a rigid motion of $\mathbf{R}^{n+1}$ such that $\widetilde{q}_k\,(\text{axis of } P_k) = \mathbf{R}^{n-1} \times \{0\} \times \{0\}$, $\widetilde{q}_k(a_k) = 0$, where a_k is the nearest point of the axis of P_k to the origin of $\mathbf{R}^{n+1}$, and $\widetilde{q}_k\,\widetilde{L}_k = \mathbf{R}^n \times \{0\}$, where $\widetilde{L}_k = \text{graph}\,\frac{1}{2}(p_k^+ + p_k^-)$. Following the notation of [**Wic04a**], Section 6, let $\mathbf{H}_k^{(1)} = \widetilde{q}_k\,P_k^+ \cap \{x^1 > 0\}$, $\mathbf{H}_k^{(2)} = \widetilde{q}_k\,P_k^+ \cap \{x^1 < 0\}$, $\mathbf{H}_k^{(3)} = \widetilde{q}_k\,P_k^- \cap \{x^1 < 0\}$ and $\mathbf{H}_k^{(4)} = \widetilde{q}_k\,P_k^- \cap \{x^1 > 0\}$. (Note that strictly speaking, in Section 6 of [**Wic04a**], the definitions of $\mathbf{H}_k^{(i)}$ are in terms of the blow-up $(v^+, v^-) \equiv (p^+, p^-)$, and the fine excess E_k (which is denoted β_k in [**Wic04a**]) is defined relative to the pair of affine hyperplanes $P_k^{(0)} \equiv \text{graph}\,\hat{E}_k p^+ \cup \text{graph}\,\hat{E}_k p^-$. Since here we need to prove improvement of the excess E_k defined relative to P_k—and not the improvement of excess relative to $P_k^{(0)}$—the above are the correct definitions of the half-spaces $\mathbf{H}_k^{(i)}$ to adopt.) Now, exactly as in [**Wic04a**], Section 6, we may express, by Allard's regularity theorem, $\widetilde{q}_k\,M_k \cap (B_1^{n+1}(0) \setminus T_k) = \cup_{i=1}^4 \text{graph}\,g_k^{(i)}$, where $g_k^{(i)} \in C^2(U_k^{(i)}, \mathbf{H}_k^{(i)\,\perp})$, $i = 1, \ldots, 4$ satisfy the estimates as in [**Wic04a**], Section 6, and T_k, $U_k^{(i)}$ are as defined there. Defining $\widetilde{g}_k^{(i)}$, $i = 1, \ldots, 4$ as in [**Wic04a**], Section 6, we obtain, as in [**Wic04a**], Section 6, functions (the blow-up) $w^{(1)}, w^{(4)} \in C^2(\mathbf{R}^{n\,+} \cap B_1(0))$, and $w^{(2)}, w^{(3)} \in C^2(\mathbf{R}^{n\,-} \cap B_1(0))$, where $\mathbf{R}^{n\,+} \equiv \{x \in \mathbf{R}^n \times \{0\} \,:\, x^1 > 0\}$ and $\mathbf{R}^{n\,-} \equiv \{x \in \mathbf{R}^n \times \{0\} \,:\, x^1 < 0\}$, such that $E_k^{-1}\widetilde{g}_k^{(i)} \to w^{(i)}$ for $i = 1, \ldots, 4$, where for each i, the convergence is in the C^2-norm on each compact subset of the domain of $w^{(i)}$ and also in the L^2-norm on the domain of $w^{(i)}$. By Lemma 6.23 of [**Wic04a**], the blow-up $\{w^{(i)}\}_{i=1}^4$ (restricted to a suitably smaller ball, say $B_{1/2}(0)$) consists of two harmonic functions $w^{(13)}$ and $w^{(24)}$ in the sense that the union of the closures of the graphs of $w^{(1)}$, $w^{(3)}$ in $B_{1/2}(0) \times \mathbf{R}$ is the graph of a harmonic function $w^{(13)}$ over $B_{1/2}(0)$ and similarly the union of the closures of the graphs of $w^{(2)}$, $w^{(4)}$ in $B_{1/2}(0) \times \mathbf{R}$ is the graph of a harmonic function $w^{(24)}$ over $B_{1/2}(0)$. For $x \in \mathbf{R}^n \times \{0\}$, let $l^{(13)}(x) = w^{(13)}(0) + Dw^{(13)}(0) \cdot x$, $l^{(24)}(x) = w^{(24)}(0) + Dw^{(24)} \cdot x$ and let the affine functions $h_k^{(13)}, h_k^{(24)} : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ be defined by $\text{closure}\,\mathbf{H}_k^{(1)} \cup \text{closure}\,\mathbf{H}_k^{(3)} = \text{graph}\,h_k^{(13)}$ and $\text{closure}\,\mathbf{H}_k^{(2)} \cup \text{closure}\,\mathbf{H}_k^{(2)} = \text{graph}\,h_k^{(24)}$. Set $\widetilde{P}_k = \widetilde{q}_k^{-1}\,(\text{graph}\,(h_k^{(13)} + E_k l^{(13)}) \cup \text{graph}\,(h_k^{(24)} + E_k l^{(24)}))$. Then, using standard estimates for harmonic functions, and the "non-concentration of excess" estimate of part (ii) of Lemma 6.22, [**Wic04a**], we conclude that

$$\theta^{-n-2} \int_{M_k \cap (B_\theta(0) \times \mathbf{R})} \text{dist}^2\,(X, \widetilde{P}_k) \leq C\theta^2 E_k^2 \tag{4.4}$$

for sufficiently large k, where $C = C(n)$. If we write, using our usual notation, $\widetilde{P}_k = \widetilde{P}_k^+ \cup \widetilde{P}_k^-$, then, since $\sup_{B_1(0)} |h_k^{(13)} - h_k^{(24)}| \geq \frac{1}{4}\tau \hat{E}_k$ (by (4.2) and (4.3)) and $E_k/\hat{E}_k \to 0$, we must have that $\widetilde{P}_k^+ \cap \widetilde{P}_k^- \cap (B_\theta(0) \times \mathbf{R}) \neq \emptyset$ for all sufficiently large k.

Finally, note that conclusion (b)(i) of the lemma with M_k, $\widetilde{P}_k$, P_k in place of M, $\widetilde{P}$, P follows directly from the definition of $\widetilde{P}_k$, and conclusion (b)(iii) with M_k, $\widetilde{P}_k$ in place of M, $\widetilde{P}$ and appropriate functions $u_k^{\pm} \in C^2(B_\theta(0) \setminus S_{\widetilde{P}_k}(\theta^2/16))$ in place of $u^{\pm}$ follows from Allard's regularity theorem and the fact that $E_k/\hat{E}_k \to 0$. q.e.d.

In addition to the hypotheses of Lemma 4.1, if we also assume that $0 \in \overline{M}$, $\Theta_M(0) \geq 2$ and that $P = P^+ \cup P^-$ is a pair of hyperplanes (so that $0 \in P^+ \cap P^-$), then the conclusions of the lemma hold with $\widetilde{P} = \widetilde{P}^+ \cup \widetilde{P}^-$ equal to a pair of hyperplanes (so that $0 \in \widetilde{P}^+ \cap \widetilde{P}^-$). This follows from the fact that under these additional hypotheses, we have for the fine blow-up the estimate

$$(4.5)\quad \int_{B_{1/2}(0)} R^{2-n}\left(\frac{\partial(w^{(13)}/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(w^{(24)}/R)}{\partial R}\right)^2 < C < \infty$$

where $C = C(n)$, $R = |x|$ and $\frac{\partial}{\partial R}$ denotes the radial differentiation, and $w^{(13)}$, $w^{(24)}$ are as in the proof of Lemma 4.1 above. This estimate says in particular that $w^{(13)}(0) = w^{(24)}(0) = 0$. Since we have, by hypothesis, that $0 \in P_k^+ \cap P_k^-$ for each k, we immediately conclude that $0 \in \widetilde{P}_k^+ \cap \widetilde{P}_k^-$. (Notation is as in the proof of Lemma 4.1 above.) The estimate (4.5) was first proved in [**Sim93**] (see [**Sim93**], Lemma 3.4 and [**Sim93**], Section 5.1, inequality (12)) and in view of Lemmas 6.21 and 6.22 of [**Wic04a**], the same proof as in [**Sim93**] yields it here as well.

Thus we have the following variant of Lemma 4.1.

Lemma 4.2. *Let $\theta \in (0, 1/8)$, $\delta \in (0,1)$ and $\tau \in (0,1)$. There exists $\epsilon_0 = \epsilon_0(n, \theta, \delta, \tau) > 0$ such that the following holds. Suppose $M \in \mathcal{I}_b$, $0 \in \overline{M}$ and*

(1) $\Theta_M(0) \geq 2$,
(2) $\frac{\mathcal{H}^n(M \cap B_2^{n+1}(0))}{\omega_n 2^n} \leq 3 - \delta$,
(3) $\hat{E}_M^2(3/2, L) \equiv \left(\frac{3}{2}\right)^{-n-2} \int_{M \cap (B_{3/2}(0) \times \mathbf{R})} \operatorname{dist}^2(x, L) \leq \epsilon_0$ *for some affine hyperplane L with $d_{\mathcal{H}}(L \cap (B_1(0) \times \mathbf{R}), B_1(0)) \leq \epsilon_0$, and*
(4) $\int_{M \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \leq \epsilon_0 \hat{E}_M^2(3/2, L)$ *for some pair of hyperplanes $P = P^+ \cup P^-$.*

Then, either

(a) *there exists an affine hyperplane* $\widetilde{L}$ *with* $d_{\mathcal{H}}\,(\widetilde{L}\cap(B_1(0)\times\mathbf{R}), L\cap(B_1(0)\times\mathbf{R}))\leq C\hat{E}_M(3/2,L)$, $C=C(n)$, *such that*

$$\left(\frac{1}{2}\right)^{-n-2}\int_{M\cap(B_{1/2}(0)\times\mathbf{R})}\operatorname{dist}^2(x,\widetilde{L})\leq\tau\hat{E}_M^2(3/2,L)\qquad\text{or}$$

(b) *there exists a pair of hyperplanes* $\widetilde{P}=\widetilde{P}^+\cup\widetilde{P}^-$ *with* $\widetilde{P}^+\not\equiv\widetilde{P}^-$ *such that*

(i)

$$d_{\mathcal{H}}^2(\widetilde{P}\cap(B_1(0)\times\mathbf{R}), P\cap(B_1(0)\times\mathbf{R}))\leq C\int_{M\cap B_1(0)\times\mathbf{R}}\operatorname{dist}^2(x,P),$$

(ii)

$$\theta^{-n-2}\int_{M\cap(B_\theta(0)\times\mathbf{R})}\operatorname{dist}^2(x,\widetilde{P})\leq C\theta^2\int_{M\cap(B_1(0)\times\mathbf{R})}\operatorname{dist}^2(x,P)\qquad\text{and}$$

(iii) $M\cap((B_\theta(0)\setminus S_{\widetilde{P}}(\theta^2/16))\times\mathbf{R})=\operatorname{graph}u^+\cup\operatorname{graph}u^-$ *where, for* $\sigma\in(0,1)$,

$S_{\widetilde{P}}(\sigma)=\{x\in\mathbf{R}^n\times\{0\}\,:\,\operatorname{dist}(x,\pi\,(\widetilde{P}^+\cap\widetilde{P}^-))\leq\sigma\}$,

$u^\pm\in C^2(B_\theta(0)\setminus S_{\widetilde{P}}(\theta^2/16))$ *with* $u^+>u^-$ *and, for* $x\in B_\theta(0)\setminus S_{\widetilde{P}}(\theta^2/16)$, $\operatorname{dist}((x,u^+(x)),\widetilde{P})=\operatorname{dist}((x,u^+(x)),\widetilde{P}^+)$ *and* $\operatorname{dist}((x,u^-(x)),\widetilde{P})=\operatorname{dist}((x,u^-(x)),\widetilde{P}^-)$.

Here $C=C(n)>0$ *and* $\pi\,:\,\mathbf{R}^{n+1}\to\mathbf{R}^n\times\{0\}$ *is the orthogonal projection.*

5. Regularity of blow-ups off affine hyperplanes

In order to handle one case of the proof of the main excess decay lemma (Lemma 6.3)—namely, the case in which the “fine excess” of a hypersurface $M\in\mathcal{I}_b$ (i.e., the height excess of M measured relative to a pair of affine hyperplanes) is of the same order as the “coarse excess” of M (i.e., the excess of M relative to a single affine hyperplane)—it is necessary to understand, in sufficient detail, the asymptotic behavior of the 2-valued functions belonging to the class $\mathcal{F}_\delta$. Our goal in this section is to do that. At the end of this section, we prove the following regularity theorem for any $v\in\mathcal{F}_\delta$:

Theorem 5.1. *Let* $v=(v^+,v^-)\in\mathcal{F}_\delta$. *There exists a relatively closed (possibly empty) subset* S_v *of* $B_{3/2}(0)$ (*the branch set of* v) *such that*

(a) *if* $\Omega\subset B_{3/2}(0)\setminus S_v$ *is open and simply connected, then there exist two harmonic functions* $v^1,v^2\,:\,\Omega\to\mathbf{R}$ *such that*

$$(\operatorname{graph}v^+\cup\operatorname{graph}v^-)\cap(\Omega\times\mathbf{R})=\operatorname{graph}v^1\cup\operatorname{graph}v^2$$

and

(b) *for each $z \in S_v \cap B_1(0)$, there exists an affine function $l_z : \mathbf{R}^n \to \mathbf{R}$ such that*

$$\rho^{-n-2} \int_{B_\rho(z)} (v^+ - l_z)^2 + (v^- - l_z)^2 \le C\rho^\lambda \int_{B_{5/4}(0)} (v^+)^2 + (v^-)^2$$

for all $\rho \in (0, 1/64)$, where C, λ are positive constants depending only on n and δ. In fact, $l_z(x) = h(z) + Dh(z) \cdot (x - z)$ where $h = \frac{1}{2}(v^+ + v^-)$. (Recall that h is harmonic in $B_{3/2}(0)$.)

We begin with a series of lemmas.

Lemma 5.2. *If $(v^+, v^-) \in \mathcal{F}_\delta$ and $v^+(z) = v^-(z) = y$, then $\mathcal{E}_{z,\rho} \equiv \rho^{-n-2} \int_{B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2$ is monotonically increasing as a function of ρ. Therefore, $\rho^{-n-2} \int_{B_\rho(z)} (v^+ - y)^2 + (v^- - y)^2 \le C \int_{B_1(0)} (v^+)^2 + (v^-)^2 \le C$ for all $z \in B_{1/4}(0) \cap Z_w$ and all $\rho \in (0, 1/4)$ where $C = C(n)$.*

Proof. The first assertion follows directly from Lemma 3.13 and the estimate (3.42). The second assertion follows from the first and the estimate $|y|^2 \le C \int_{B_1(0)} h^2$ which holds since $y = h(z)$ and $h = \frac{1}{2}(v^+ + v^-)$ is harmonic in $B_{3/2}(0)$. q.e.d.

Lemma 5.3. *Let $\alpha_0 \in (0, \pi/2)$, $\delta_0 \in (0, 1)$. There exists $\epsilon_1 = \epsilon_1(n, \alpha_0, \delta_0) \in (0, 1)$ such that if $P_0 = P_0^+ \cup P_0^-$ is a pair of hyperplanes with $\alpha_0 \le \angle P_0 < \pi$, $(v^+, v^-) \in \mathcal{F}_\delta$, $v^+(0) = v^-(0)$ and*

$$\int_{B_1(0)} (v^+ - p_0^+)^2 + (v^- - p_0^-)^2 \le \epsilon_1$$

then $1 \le N_{v,0}(1) \le 1 + \delta_0$.

Proof. Since $v^+(0) = v^-(0)$, the lower bound $N_{v,0}(1) \ge 1$ follows from the monotonicity of $N_{v,0}$ and Lemma 3.13. If the upper bound fails to hold for some $\delta_0 \in (0, 1)$, there exists a sequence $v_k = (v_k^+, v_k^-) \in \mathcal{F}_\delta$, $k = 1, 2, \ldots$, with $v_k^+(0) = v_k^-(0)$, and a sequence of pairs of hyperplanes $P_k = P_k^+ \cup P_k^-$ with $\alpha_0 \le \angle P_k < \pi$ satisfying

$$\int_{B_1(0)} (v_k^+ - p_k^+)^2 + (v_k^- - p_k^-)^2 \le \frac{1}{k}, \tag{5.1}$$

and yet $N_{v_k,0}(1) > 1 + \delta_0$ for all k. In view of Proposition 3.3, part (2), the inequality (5.1) implies that $\int_{B_1(0)} (p_k^+)^2 + (p_k^-)^2 \le C$ for each k where $C = C(n)$. Passing to a subsequence, $P_k \to P$ for some pair of hyperplanes $P = P^+ \cup P^-$ with $\alpha_0 \le \angle P < \pi$. By Lemma 3.1, after passing to a further subsequence $v_k \to v$ for some $v \in \mathcal{F}_\delta$, where the convergence is in $W^{1,2}(B_\sigma(0))$ for every $\sigma \in (0, 3/2)$. Thus $N_{v_k,0}(1) \to N_{v,0}(1)$. By (5.1), $v \equiv p = (p^+, p^-)$ on $B_1(0)$, so that $N_{v,0}(\sigma) = N_{p,0}(\sigma) = 1$ for each $\sigma \in (0, 1)$, and hence $N_{v,0}(1) = 1$. This proves the lemma. q.e.d.

Lemma 5.4. *Let $\overline{\theta} \in (0, 1/8)$, $\delta \in (0,1)$ and $\overline{\alpha} \in (0,\pi)$. There exists a number $\overline{\epsilon} = \overline{\epsilon}(n, \overline{\theta}, \delta, \overline{\alpha}) \in (0,1)$ such that the following holds. If $\overline{P} = \overline{P}^+ \cup \overline{P}^-$ is a pair of hyperplanes of $\mathbf{R}^{n+1}$ with $\pi > \angle \overline{P} \geq \overline{\alpha}$, $v = (v^+, v^-) \in \mathcal{F}_\delta$, $v^+(0) = v^-(0) = 0$,*

$$\int_{B_1(0)} \mathrm{dist}^2\left((x, v^+(x)), \overline{P}\right) + \mathrm{dist}^2\left((x, v^-(x)), \overline{P}\right) \leq \overline{\epsilon} \quad \textit{and}$$
$$\int_{B_1(0)\setminus S_{\overline{P}}(\overline{\theta}/16)} (v^+ - \overline{p}^+)^2 + (v^- - \overline{p}^-)^2 \leq \overline{\epsilon},$$

where $S_{\overline{P}}(\sigma) = \left\{x \in \mathbf{R}^n \times \{0\} \,:\, \mathrm{dist}\left(x, \pi\left(\overline{P}^+ \cap \overline{P}^-\right)\right) \leq \sigma\right\}$, then there exists a pair of hyperplanes $\widetilde{P} = \widetilde{P}^+ \cup \widetilde{P}^-$ of $\mathbf{R}^{n+1}$ such that

$$(\overline{\theta})^{-n-2} \int_{B_{\overline{\theta}}(0)} \mathrm{dist}^2\left((x, v^+(x)), \widetilde{P}\right) + \mathrm{dist}^2\left((x, v^-(x)), \widetilde{P}\right)$$
$$\leq \overline{C}\overline{\theta}^2 \int_{B_1(0)} \mathrm{dist}^2\left((x, v^+(x)), \overline{P}\right) + \mathrm{dist}^2\left((x, v^-(x)), \overline{P}\right)$$

and

$$(\overline{\theta})^{-n-2} \int_{B_{\overline{\theta}}(0)\setminus S_{\widetilde{P}}(\overline{\theta}^2/16)} (v^+ - \widetilde{p}^+)^2 + (v^- - \widetilde{p}^-)^2$$
$$\leq \overline{C}\overline{\theta}^2 \int_{B_1(0)} \mathrm{dist}^2\left((x, v^+(x)), \overline{P}\right) + \mathrm{dist}^2\left((x, v^-(x)), \overline{P}\right).$$

Here $\overline{C} = \overline{C}(n) \in (0, \infty)$.

Proof. By the definition of $\mathcal{F}_\delta$, there exists a sequence M_k of hypersurfaces in $\mathcal{I}_b$ with $\frac{\mathcal{H}^n\left(M_k \cap B_2^{n+1}(0)\right)}{\omega_n 2^n} \leq 3 - \delta$ and a sequence L_k of affine hyperplanes converging to $\mathbf{R}^n \times \{0\}$ such that $\hat{E}_{M_k}(3/2, L_k) \to 0$ and the blow-up of $\{M_k\}$ off $\{L_k\}$ (as described in Section 3) is (v^+, v^-). Since $\int_{B_1(0)\setminus S_{\overline{P}}(\overline{\theta}/16)} (v^+ - \overline{p}^+)^2 + (v^- - \overline{p}^-)^2 \leq \overline{\epsilon}$, if $\overline{\epsilon} = \overline{\epsilon}(n, \overline{\alpha})$ is sufficiently small, it follows from Lemma 3.11 part (8) that $v^+ \not\equiv v^-$ in any ball $B_\sigma(0)$, $0 < \sigma \leq 1$. Thus, since $v^+(0) = v^-(0)$, we have by the remark following Lemma 3.8 that possibly after taking a subsequence of $\{k\}$ which we continue to denote $\{k\}$, there exists $Z_k = (Z_k', Z_k^{n+1}) \in M_k \cap (B_1(0) \times \mathbf{R})$ such that $\Theta_{M_k}(Z_k) \geq 2$ and $\left(Z_k', \frac{Z_k^{n+1}}{\hat{E}_k}\right) \to (0,0)$. Let $\widetilde{M}_k \equiv \eta_{Z_k, 1-|Z_k|}\, M_k$. By the monotonicity of mass ratio, for sufficiently large k, $\frac{\mathcal{H}^n\left(\widetilde{M}_k \cap B_2^{n+1}(0)\right)}{\omega_n 2^n} \leq 3 - \delta/2$ and the blow-up, as in Section 3, of the sequence of hypersurfaces $\widetilde{M}_k$ off the sequence $(1 - |Z_k|)^{-1}(L_k - Z_k)$ of affine hyperlanes is also (v^+, v^-). Thus, by replacing the original sequence M_k with $\widetilde{M}_k$, we may assume that $0 \in M_k$ and $\Theta_{M_k}(0) \geq 2$ for all k so that the hypotheses (1) and (2) of Lemma 4.2 are satisfied with M_k in place of M and $\delta/2$ in place of δ.

By hypothesis we have

$$\int_{B_1(0)} \operatorname{dist}^2((x, v^+(x)), \overline{P}) + \operatorname{dist}^2((x, v^-(x)), \overline{P}) \leq \overline{\epsilon},$$

which together with the squared triangle inequality $\operatorname{dist}^2(X, \overline{P}) \leq 2\operatorname{dist}^2(Y, \overline{P}) + 2|X-Y|^2$ implies, for sufficiently large k, that

(5.2)
$$\int_{B_1(0)} \operatorname{dist}^2\left(\left(x, \frac{\overline{\psi}_k(x)u_k^+(x)}{\hat{E}_k}\right), \overline{P}\right) + \operatorname{dist}^2\left(\left(x, \frac{\overline{\psi}_k(x)u_k^-(x)}{\hat{E}_k}\right), \overline{P}\right) \leq 4\overline{\epsilon}$$

where the notation is as in (3.5) and (3.14). Let $\overline{P}_k = \operatorname{graph} \hat{E}_k\overline{p}^+ \cup \operatorname{graph} \hat{E}_k\overline{p}^-$. Then

$$\begin{aligned}
(5.3)\qquad & \int_{M_k \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(X, \overline{P}_k) \\
&= \int_{G_k^+ \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(X, \overline{P}_k) \\
&\quad + \int_{G_k^- \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(X, \overline{P}_k) \\
&\quad + \int_{(M_k \setminus G_k) \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(X, \overline{P}_k) \\
&\leq C \int_{B_1(0)} \operatorname{dist}^2((x, \overline{\psi}_k(x)u_k^+(x)), \overline{P}_k) \\
&\quad + \operatorname{dist}^2((x, \overline{\psi}_k(x)u_k^-(x)), \overline{P}_k) + C\hat{E}_k^{2+\mu}
\end{aligned}$$

where $C = C(n)$. The inequality in the above follows from the estimates (3.11) and (3.12). In view of the general fact that if $L = \operatorname{graph} \ell$ is a hyperplane of $\mathbf{R}^{n+1}$, where $\ell : \mathbf{R}^n \to \mathbf{R}$ is given by $\ell(x') = a \cdot x'$ for some $a \in \mathbf{R}^n$, then for any point $(x', x^{n+1}) \in \mathbf{R}^{n+1}$ and any number $\lambda > 0$,

$$(5.4)\qquad \operatorname{dist}^2((x', \lambda x^{n+1}), L^\lambda) = \frac{\lambda^2(1+|a|^2)}{1+\lambda^2|a|^2} \operatorname{dist}^2((x', x^{n+1}), L),$$

where $L^\lambda = \operatorname{graph} \lambda\ell$, we have by the inequalities (5.2) and (5.3) that for all sufficiently large k,

$$(5.5)\qquad \int_{M_k \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(X, \overline{P}_k) \leq C\overline{\epsilon}\hat{E}_k^2.$$

Now note that there exists a constant $C_1 \in (0,1)$ depending only on $\overline{\alpha}$ such that

$$(5.6)\qquad d_{\mathcal{H}}(\overline{P} \cap (B_{1/2}(0)\times \mathbf{R}), L \cap (B_{1/2}(0)\times \mathbf{R})) \geq C_1$$

for any affine hyperplane L. In view of (5.5), given any $\tau \in (0,1)$, if $\overline{\epsilon} = \overline{\epsilon}(n, \overline{\theta}, \delta, \tau)$ is sufficiently small, we may apply Lemma 4.2 with $\theta = \overline{\theta}$, $\overline{P}$ in place of P^0, $\delta/2$ in place of δ and M_k in place of M. Lemma 4.2 then gives for each k either a pair of hyperplanes $\widetilde{P}_k = \widetilde{P}_k^+ \cup \widetilde{P}_k^-$ with

(5.7)
$$d_{\mathcal{H}}^2\,(\widetilde{P}_k \cap (B_1(0) \times \mathbf{R}), \overline{P}_k \cap (B_1(0) \times \mathbf{R}) \leq C \int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, \overline{P}_k)$$

such that

(5.8)
$$\overline{\theta}^{-n-2} \int_{M_k \cap (B_{\overline{\theta}}(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}_k) \leq C\overline{\theta}^2 \int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, \overline{P}_k)$$

where $C = C(n)$, or an affine hyperplane $\widetilde{L}_k$ with $d_{\mathcal{H}}\,(\widetilde{L}_k \cap (B_1(0) \times \mathbf{R}), L_k \cap (B_1(0) \times \mathbf{R})) \leq C\hat{E}_k$, $C = C(n)$, satisfying

$$\int_{M_k \cap (B_{1/2}(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{L}_k) \leq \tau \hat{E}_k^2. \tag{5.9}$$

However, if (5.9) holds for infinitely many k, we see by dividing (5.9) by $\hat{E}_k^2$ and passing to the limit as $k \to \infty$ that $\int_{B_1(0)} (v^+ - \ell)^2 + (v^- - \ell)^2 \leq \tau$ for some affine function ℓ, which, in view of (5.6), contradicts the hypothesis $\int_{B_1(0) \setminus S_{\overline{P}}(\overline{\theta}/16)} (v^+ - \overline{p}^+)^2 + (v^- - \overline{p}^-)^2 \leq \overline{\epsilon}$ provided $\tau = \tau(n, C_1) \in (0,1)$ (hence $\tau = \tau(n, \overline{\alpha})$) is chosen sufficiently small. (Here C_1 is as in (5.6).) Thus if $\overline{\epsilon} = \overline{\epsilon}(n, \overline{\theta}, \delta, \overline{\alpha})$ is chosen sufficiently small, option (5.9) cannot occur for infinitely many k, and hence we must have (5.8) for all sufficiently large k. It follows, upon dividing the inequality (5.8) by $\hat{E}_k^2$ and letting $k \to \infty$ after possibly passing to a subsequence, (and using the estimates (5.3), (5.7) and $\mathcal{H}^n((M_k \setminus G_k) \cap (B_1(0) \times \mathbf{R})) \leq C\,\hat{E}_k^{2+\mu}$) that for some pair of hyperplanes $\widetilde{P}$,

$$\begin{aligned} \overline{\theta}^{-n-2} \int_{B_{\overline{\theta}}(0)} & \operatorname{dist}^2((x, v^+(x)), \widetilde{P}) + \operatorname{dist}^2((x, v^-(x)), \widetilde{P}) \\ & \leq C\overline{\theta}^2 \textstyle\int_{B_1(0)} \operatorname{dist}^2((x, v^+(x)), \overline{P}) + \operatorname{dist}^2((x, v^-(x)), \overline{P}) \end{aligned} \tag{5.10}$$

where $C = C(n)$. The remaining claim follows directly from conclusion (b)(iii) of Lemma 4.2. The lemma is thus proved. q.e.d.

The next lemma says that if the graph of $w = \frac{1}{2}(v^+ - v^-)$ stays close, in $B_1(0) \times \mathbf{R}$, to a pair of n-dimensional half-spaces of $\mathbf{R}^{n+1}$ meeting at an angle $< \pi$ along an $(n-1)$-dimensional axis, and if Z_w is the zero set of w, then $Z_w \cap B_{1/2}(0)$ cannot have too large a gap.

Lemma 5.5. *Let $(v^+, v^-) \in \mathcal{F}_\delta$, $w = \frac{1}{2}(v^+ - v^-)$ and $\gamma \in (0, 1/2)$. Suppose that $\int_{B_1(0)} (w - L)^2 \leq \gamma$ where $L : \mathbf{R}^n \to \mathbf{R}^+ \cup \{0\}$ is such that* graph L *is equal to the union of two n-dimensional half-spaces of* $\mathbf{R}^{n+1}$

meeting along $\mathbf{R}^{n-1} \times \{(0,0)\}$*, each making the same angle* $\beta \in (0, \pi/2)$ *with* $\mathbf{R}^n \times \{0\}$*. If* $B_r(q) \cap Z_w = \emptyset$ *for some* $q \in (\mathbf{R}^{n-1} \times \{(0,0)\}) \cap B^{n+1}_{1/2}(0)$ *and* $r \in (0,1)$*, then* $r \leq C\gamma^{1/2n}$ *where* C *depends only on* n *and* β*.*

Proof. Let $Q = \{x \in B_1(0) : |w(x) - L(x)| \geq \gamma^{1/4}\}$. Since $\int_{B_1(0)} (w - L)^2 \leq \gamma$, it follows that

$$\mathcal{L}^n(Q) \leq \gamma^{1/2}. \tag{5.11}$$

Suppose $B_r(q) \cap Z_w = \emptyset$ for some $q \in (\mathbf{R}^{n-1} \times \{(0,0)\}) \cap B_{1/2}(0)$ and $r \in (0,1)$. Then by Proposition 3.11, part (6), w is harmonic (and positive) in $B_r(q)$, so that by the Harnack inequality we have that

$$\sup\nolimits_{B_{r/2}(q)} w \leq c_0 \inf\nolimits_{B_{r/2}(q)} w \tag{5.12}$$

where $c_0 = c_0(n)$. With $\mu = \mu(n) \in (0, 1/2)$ to be chosen, let $\Lambda = \mathcal{L}^n(B_1(0) \cap (\mathbf{R}^{n-1} \times [-\mu, \mu]))$. If r is such that $\Lambda \left(\frac{r}{2}\right)^n > \gamma^{1/2}$, then in view of (5.11), there must exist a point $x_0 \in B_{r/2}(q) \cap (\mathbf{R}^{n-1} \times [-\mu r/2, \mu r/2])$ with $|w(x_0) - L(x_0)| < \gamma^{1/4}$. Then, $w(x_0) \leq \gamma^{1/4} + C_1 \mu r$ where $C_1 = C_1(\beta)$, so that

$$\inf\nolimits_{B_{r/2}(q)} w \leq \gamma^{1/4} + C_1 \mu r. \tag{5.13}$$

On the other hand, choosing $\mu' = \mu'(n) \in (0, 1/2)$ such that $\Lambda' \equiv \mathcal{L}^n(B_1(0) \cap (\mathbf{R}^{n-1} \times [-\mu', \mu'])) < \frac{\omega_n}{4}$, if r also satisfies $(\omega_n - \Lambda') \left(\frac{r}{2}\right)^n > \gamma^{1/2}$, then, again in view of (5.11), there must exist a point $x_1 \in B_{r/2}(q) \setminus (\mathbf{R}^{n-1} \times [-\mu' r/2, \mu' r/2])$ such that $|w(x_1) - L(x_1)| < \gamma^{1/4}$. Then $w(x_1) > L(x_1) - \gamma^{1/4} \geq C_1 \mu' r - \gamma^{1/4}$ and hence

$$\sup\nolimits_{B_{r/2}(q)} w \geq C_1 \mu' r - \gamma^{1/4}. \tag{5.14}$$

Taking $\mu = \frac{\mu'}{2c_0}$ and combining the inequalities (5.12), (5.13) and (5.14), we then have that $r \leq C\gamma^{1/4}$ where $C = C(\beta, n)$. Thus in all cases, $r \leq C\gamma^{1/2n}$. q.e.d.

Lemma 5.6. *Let* $\alpha \in (0, \pi)$ *and* $\delta \in (0,1)$*. There exist numbers* $\epsilon = \epsilon(n, \delta, \alpha) \in (0,1)$ *and* $\kappa = \kappa(n, \alpha) \in (0,1)$ *such that the following is true. If* $\widetilde{P}_0 = \widetilde{P}_0^+ \cup \widetilde{P}_0^-$ *is a pair of hyperplanes with* $\alpha \leq \angle \widetilde{P}_0 < \pi$*,* $\widetilde{p}_0^+ + \widetilde{p}_0^- \equiv 0$*, and if* $(v^+, v^-) \in \mathcal{F}_\delta$ *satisfies* $v^+(0) = v^-(0) = 0$*, and* $\int_{B_1(0)} (v^+ - \widetilde{p}_0^+)^2 + (v^- - \widetilde{p}_0^-)^2 \leq \epsilon$*, then there exist two harmonic functions* $v_1, v_2 : B_\kappa(0) \to \mathbf{R}$ *such that* $v^+|_{B_\kappa(0)} = max\{v_1, v_2\}$ *and* $v^-|_{B_\kappa(0)} = min\{v_1, v_2\}$*. Furthermore, the vanishing order of* $v_1 - v_2$ *at any point* $z \in B_\kappa(0)$ *where* $v_1(z) = v_2(z)$ *is equal to* 1*.*

Proof. The hypotheses

$$\int_{B_1(0)} (v^+ - \widetilde{p}_0^+)^2 + (v^- - \widetilde{p}_0^-)^2 \leq \epsilon \tag{5.15}$$

and $\alpha \leq \angle \widetilde{P}_0$ together with the fact that $\mathcal{E}_1^2 = \int_{B_1(0)} (v^+)^2 + (v^-)^2 \leq \left(\frac{3}{2}\right)^{n+2}$ (Proposition 3.3, part (2)) imply that

$$\Lambda \leq \int_{B_1(0)} (\widetilde{p}_0^+)^2 + (\widetilde{p}_0^-)^2 \leq 2\left(\frac{3}{2}\right)^{n+2} + 2\epsilon \tag{5.16}$$

for some $\Lambda = \Lambda(n, \alpha) > 0$, and consequently that

$$\mathcal{E}_1^2 = \int_{B_1(0)} (v^+)^2 + (v^-)^2 \geq \frac{\Lambda}{2} - \epsilon \geq \frac{\Lambda}{4}, \tag{5.17}$$

provided $\epsilon = \epsilon(n, \alpha) < \Lambda/4$.

Set $\overline{P}_0 = \text{graph}\, \frac{1}{\mathcal{E}_1}\widetilde{p}_0^+ \cup \text{graph}\, \frac{1}{\mathcal{E}_1}\widetilde{p}_0^-$ and $S^{(0)} = \{x \in \mathbf{R}^n \times \{0\} : \text{dist}\,(x, \pi\,(\overline{P}_0^+ \cap \overline{P}_0^-)) \leq \theta/16\}$. Note that inequality (5.15) implies that

$$\begin{aligned}
&\int_{B_1(0)} \text{dist}^2\,((x, \widetilde{v}_1^+(x)), \overline{P}_0) + \text{dist}^2\,((x, \widetilde{v}_1^-(x)), \overline{P}_0) \\
&\leq \int_{B_1(0)} (\widetilde{v}_1^+ - \overline{p}_0^+)^2 + (\widetilde{v}_1^- - \overline{p}_0^-)^2 \\
&\leq \left(\frac{2}{3}\right)^{-n-2} \frac{4\epsilon}{\Lambda}
\end{aligned} \tag{5.18}$$

(notation as in (3.32)), which of course in particular says that

$$\int_{B_1(0)\setminus S^{(0)}} (\widetilde{v}_1^+ - \overline{p}_0^+)^2 + (\widetilde{v}_1^- - \overline{p}_0^-)^2 \leq \left(\frac{2}{3}\right)^{-n-2} \frac{4\epsilon}{\Lambda}. \tag{5.19}$$

Since $\angle \widetilde{P}_0 \in [\alpha, \pi)$ and $\mathcal{E}_1^2 \leq \left(\frac{3}{2}\right)^{n+2}$, we have that

$$\overline{\alpha}_0 \leq \angle \overline{P}_0 < \pi \tag{5.20}$$

for some $\overline{\alpha}_0 = \overline{\alpha}_0(n, \alpha) > 0$. Now choose $\theta = \theta(n) \in (0, 1)$ such that

$$\overline{C}\theta < \frac{1}{4}$$

where $\overline{C} = \overline{C}(n)$ is as in Lemma 5.4. If we then choose $\epsilon = \epsilon(n, \delta, \alpha)$ so that

$$\left(\frac{2}{3}\right)^{-n-2} \frac{4\epsilon}{\Lambda} < \overline{\epsilon}(n, \theta, \delta, \overline{\alpha}_0) \tag{5.21}$$

where $\overline{\epsilon}$ is as in Lemma 5.4, we may apply Lemma 5.4 with $\overline{P}_0$ in place of $\overline{P}$, $\overline{\alpha}_0$ in place of $\overline{\alpha}$, θ in place of $\overline{\theta}$ and $\widetilde{v}_1$ in place of v to conclude that there exists a pair of hyperplanes $\widetilde{P}_1$ such that

$$\begin{aligned}
(5.22)\quad &\theta^{-n-2} \int_{B_\theta(0)} \text{dist}^2((x, \widetilde{v}_1^+(x)), \widetilde{P}_1) + \text{dist}^2((x, \widetilde{v}_1^-(x)), \widetilde{P}_1) \\
&\leq \overline{C}\theta^2 \int_{B_1(0)} \text{dist}^2((x, \widetilde{v}_1^+(x)), \overline{P}_0) + \text{dist}^2((x, \widetilde{v}_1^-(x)), \overline{P}_0) \quad \text{and}
\end{aligned}$$

$$\theta^{-n-2}\int_{B_\theta(0)\setminus S_{\widetilde{P}_1}(\theta^2/16)} (\widetilde{v}_1^+ - \widetilde{p}_1^+)^2 + (\widetilde{v}_1^- - \widetilde{p}_1^-)^2 \tag{5.23}$$
$$\leq \overline{C}\theta^2 \int_{B_1(0)} \mathrm{dist}^2((x,\widetilde{v}_1^+(x)),\overline{P}_0) + \mathrm{dist}^2((x,\widetilde{v}_1^-(x)),\overline{P}_0)$$

where $\overline{C} = \overline{C}(n)$ is as in Lemma 5.4 and $S_{\widetilde{P}_1}(\sigma) = \{x \in \mathbf{R}^n \times \{0\} : \mathrm{dist}\,(x, \pi\,(\widetilde{P}_1^+ \cap \widetilde{P}_1^-)) \leq \sigma\}$.

Now, if $\epsilon \leq \epsilon_1(n,\alpha,1/2)$ where ϵ_1 is as in Lemma 5.3, we have by Lemmas 5.2, 3.5 and 5.3 that

$$1 \geq \frac{\mathcal{E}_\theta^2}{\mathcal{E}_1^2} \geq \theta^{2(N_v(1)-1)} \geq \theta. \tag{5.24}$$

Setting

$$\overline{P}_1 = \mathrm{graph}\,\frac{\mathcal{E}_1}{\mathcal{E}_\theta}\widetilde{p}_1^+ \cup \mathrm{graph}\,\frac{\mathcal{E}_1}{\mathcal{E}_\theta}\widetilde{p}_1^-,$$

we conclude from (5.22), (5.23), (5.24) and (5.4) (with $\lambda = \frac{\mathcal{E}_1}{\mathcal{E}_\theta} \in [1,\theta^{-1/2}]$, so that $\mathrm{dist}^2\,((x',\lambda x^{n+1}),L^\lambda) \leq \theta^{-1}\mathrm{dist}^2\,((x',x^{n+1}),L)$, where L, L^λ are as in (5.4)) that

$$\int_{B_1(0)} \mathrm{dist}^2((x,\widetilde{v}_\theta^+(x)),\overline{P}_1) + \mathrm{dist}^2\,((x,\widetilde{v}_\theta^-(x)),\overline{P}_1) \tag{5.25}$$
$$\leq \overline{C}\theta \int_{B_1(0)} \mathrm{dist}^2\,((x,\widetilde{v}_1^+(x)),\overline{P}_0) + \mathrm{dist}^2\,((x,\widetilde{v}_1^-(x)),\overline{P}_0)$$
$$\leq 4^{-1}\epsilon_2 \quad \text{and}$$

$$\int_{B_1(0)\setminus S^{(1)}} (\widetilde{v}_\theta^+ - \overline{p}_1^+)^2 + (\widetilde{v}_\theta^- - \overline{p}_1^-)^2 \tag{5.26}$$
$$\leq \overline{C}\theta \int_{B_1(0)} \mathrm{dist}^2\,((x,\widetilde{v}_1^+(x)),\overline{P}_0) + \mathrm{dist}^2\,((x,\widetilde{v}_1^-(x)),\overline{P}_0)$$
$$\leq 4^{-1}\epsilon_2$$

where $S^{(1)} = \{x \in \mathbf{R}^n \times \{0\} \;:\; \mathrm{dist}\,(x,\pi\,(\overline{P}_1^+ \cap \overline{P}_1^-)) \leq \theta/16\}$ and $\epsilon_2 = \left(\frac{2}{3}\right)^{-2n-2}\frac{4\epsilon}{\Lambda}$.

We claim that for each $j = 1, 2, \ldots$, we can find a pair of hyperplanes $\overline{P}_j$ such that

(5.27)
$$\int_{B_1(0)} \mathrm{dist}^2((x,\widetilde{v}_{\theta^j}^+(x)),\overline{P}_j) + \mathrm{dist}^2((x,\widetilde{v}_{\theta^j}^-(x)),\overline{P}_j)$$
$$\leq \overline{C}\theta \int_{B_1(0)} \mathrm{dist}^2((x,\widetilde{v}_{\theta^{j-1}}^+(x)),\overline{P}_{j-1}) + \mathrm{dist}^2((x,\widetilde{v}_{\theta^{j-1}}^-(x)),\overline{P}_{j-1}) \quad \text{and}$$

(5.28)
$$\begin{aligned}&\int_{B_1(0)\setminus S^{(j)}} (\widetilde{v}^+_{\theta^j} - \overline{p}^+_j)^2 + (\widetilde{v}^-_{\theta^j} - \overline{p}^-_j)^2 \\ &\leq \overline{C}\theta \int_{B_1(0)} \mathrm{dist}^2((x, \widetilde{v}^+_{\theta^{j-1}}(x)), \overline{P}_{j-1}) + \mathrm{dist}^2((x, \widetilde{v}^-_{\theta^{j-1}}(x)), \overline{P}_{j-1})\end{aligned}$$

where $S^{(j)} = \{x \in \mathbf{R}^n \times \{0\} \, : \, \mathrm{dist}\,(x, \pi\,(\overline{P}^+_j \cap \overline{P}^-_j)) \leq \theta/16\}$. We prove this by induction. Note that by (5.25) and (5.26), the assertion is true for $j = 1$. Suppose that it holds for all $j = 1, 2, \ldots, i$ for some i. Thus

(5.29)
$$\begin{aligned}&\int_{B_1(0)} \mathrm{dist}^2((x, \widetilde{v}^+_{\theta^i}(x)), \overline{P}_i) + \mathrm{dist}^2((x, \widetilde{v}^-_{\theta^i}(x)), \overline{P}_i) \\ &\leq \overline{C}\theta \int_{B_1(0)} \mathrm{dist}^2\,((x, \widetilde{v}^+_{\theta^{i-1}}(x)), \overline{P}_{i-1}) + \mathrm{dist}^2((x, \widetilde{v}^-_{\theta^{i-1}}(x)), \overline{P}_{i-1}) \\ &\leq \left(\overline{C}\theta\right)^i \int_{B_1(0)} \mathrm{dist}^2((x, \widetilde{v}^+_1(x)), \overline{P}_0) + \mathrm{dist}^2((x, \widetilde{v}^-_1(x)), \overline{P}_0) \\ &\leq 4^{-i}\epsilon_2,\end{aligned}$$

(5.30)
$$\begin{aligned}&\int_{B_1(0)\setminus S^{(j)}} (\widetilde{v}^+_{\theta^j} - \overline{p}^+_j)^2 + (\widetilde{v}^-_{\theta^j} - \overline{p}^-_j)^2 \\ &\leq \overline{C}\theta \int_{B_1(0)} \mathrm{dist}^2\,((x, \widetilde{v}^+_{\theta^{j-1}}(x)), \overline{P}_{j-1}) + \mathrm{dist}^2((x, \widetilde{v}^-_{\theta^{j-1}}(x)), \overline{P}_{j-1}) \\ &\leq 4^{-j}\epsilon_2 \quad \text{ and}\end{aligned}$$

$$\int_{B_1(0)\setminus S^{(j-1)}} (\widetilde{v}^+_{\theta^{j-1}} - \overline{p}^+_{j-1})^2 + (\widetilde{v}^-_{\theta^{j-1}} - \overline{p}^-_{j-1})^2 \leq 4^{-(j-1)}\epsilon_2 \tag{5.31}$$

for $j = 1, 2, \ldots, i$. Writing $\widetilde{P}_j = \mathrm{graph}\, \frac{\mathcal{E}_{\theta^j}}{\mathcal{E}_{\theta^{j-1}}} \overline{p}^+_j \cup \mathrm{graph}\, \frac{\mathcal{E}_{\theta^j}}{\mathcal{E}_{\theta^{j-1}}} \overline{p}^-_j$ and using the fact that $\mathcal{E}_{\theta^j} \leq \mathcal{E}_{\theta^{j-1}}$ (by Lemma 5.2), we see from the inequality (5.30) that

$$\theta^{-n-2} \int_{B_\theta(0)\setminus S_{\widetilde{P}_j}(\theta^2/16)} (\widetilde{v}^+_{\theta^{j-1}} - \widetilde{p}^+_j)^2 + (\widetilde{v}^-_{\theta^{j-1}} - \widetilde{p}^-_j)^2 \leq 4^{-j}\epsilon_2 \tag{5.32}$$

for $j = 1, 2, \ldots, i$, which together with the inequality (5.31) implies, by the triangle inequality and homogeneity of $\widetilde{P}_j$, $\overline{P}_{j-1}$, that

$$\int_{B_1(0)} (\widetilde{p}^+_j - \overline{p}^+_{j-1})^2 + (\widetilde{p}^-_j - \overline{p}^-_{j-1})^2 \leq \widetilde{C}_1 4^{-(j-1)}\epsilon_2 \tag{5.33}$$

for $j = 1, 2, \ldots, i$, where $\widetilde{C}_1 = \widetilde{C}_1(n, \alpha)$. Therefore,

$$\|\widetilde{p}^+_j - \widetilde{p}^-_j\|_{L^2(B_1(0))} \geq \|\overline{p}^+_{j-1} - \overline{p}^-_{j-1}\|_{L^2(B_1(0))} - \sqrt{2\widetilde{C}_1\epsilon_2}\, 2^{-(j-1)} \tag{5.34}$$

and hence, by the definition of $\widetilde{p}_j^{\pm}$ and the fact that $\mathcal{E}_{\theta^j} \leq \mathcal{E}_{\theta^{j-1}}$,

$$\|\overline{p}_j^+ - \overline{p}_j^-\|_{L^2(B_1(0))} \geq \|\overline{p}_{j-1}^+ - \overline{p}_{j-1}^-\|_{L^2(B_1(0))} - \sqrt{2\widetilde{C}_1\epsilon_2}\, 2^{-(j-1)}. \tag{5.35}$$

Summing over j, we conclude from this that

$$\|\overline{p}_i^+ - \overline{p}_i^-\|_{L^2(B_1(0))} \geq \|\overline{p}_0^+ - \overline{p}_0^-\|_{L^2(B_1(0))} - 2\sqrt{\widetilde{C}_1\epsilon_2}. \tag{5.36}$$

By inequality (5.30), Proposition 3.3, part (2) and homogeneity of $\overline{p}_j^{\pm}$, it follows that $\int_{B_1(0)}(\overline{p}_j^+ + \overline{p}_j^-)^2 \leq C$ for some fixed constant $C = C(n) \in (0,\infty)$, and hence, provided $\epsilon = \epsilon(n,\alpha)$ is sufficiently small, we have from the estimate (5.36) that

$$\pi > \angle \overline{P}_i \geq \beta$$

where $\beta = \beta(n,\alpha) \in (0,\pi/2)$ is a fixed angle. Thus, since $(\widetilde{v}_{\theta^j}^+, \widetilde{v}_{\theta^j}^-) \in \mathcal{F}_\delta$, we may apply Lemma 5.4 with θ in place of $\overline{\theta}$, β in place of $\overline{\alpha}$, $(\widetilde{v}_{\theta^i}^+, \widetilde{v}_{\theta^i}^-)$ in place of (v^+, v^-) and $\overline{P}_i$ in place of $\overline{P}$ to conclude that there exists a pair of hyperplanes $\widetilde{P}_{i+1}$ such that

$$\begin{aligned} \theta^{-n-2}\int_{B_\theta(0)} &\mathrm{dist}^2((x,\widetilde{v}_{\theta^i}^+(x)),\widetilde{P}_{i+1}) + \mathrm{dist}^2((x,\widetilde{v}_{\theta^i}^-(x)),\widetilde{P}_{i+1}) \\ &\leq \overline{C}\theta^2\int_{B_1(0)} \mathrm{dist}^2((x,\widetilde{v}_{\theta^i}^+(x)),\overline{P}_i) + \mathrm{dist}^2((x,\widetilde{v}_{\theta^i}^-(x)),\overline{P}_i) \quad \text{and} \end{aligned} \tag{5.37}$$

$$\begin{aligned} \theta^{-n-2}\int_{B_\theta(0)\setminus S_{\widetilde{P}_{i+1}}(\theta^2/16)} &(\widetilde{v}_{\theta^i}^+ - \widetilde{p}_{i+1}^+)^2 + (\widetilde{v}_{\theta^i}^- - \widetilde{p}_{i+1}^-)^2 \\ &\leq \overline{C}\theta^2\int_{B_1(0)} \mathrm{dist}^2((x,\widetilde{v}_{\theta^i}^+(x)),\overline{P}_i) + \mathrm{dist}^2((x,\widetilde{v}_{\theta^i}^-(x)),\overline{P}_i). \end{aligned} \tag{5.38}$$

It follows from the triangle inequality, the inequalities (5.30), (5.38) and homogeneity of $\widetilde{P}_{i+1}$, $\overline{P}_i$ that

$$\int_{B_1(0)} (\widetilde{p}_{i+1}^+ - \overline{p}_i^+)^2 + (\widetilde{p}_{i+1}^- - \overline{p}_i^-)^2 \leq \widetilde{C}_1 4^{-i}\epsilon_2 \tag{5.39}$$

where $\widetilde{C}_1 = \widetilde{C}_1(n,\alpha)$ is as in (5.33).

Note again that by Lemmas 5.2, 3.5, the monotonicity of the frequency function $N_v(\cdot)$ and Lemma 5.3, we have

$$1 \geq \frac{\mathcal{E}_{\theta^{i+1}}^2}{\mathcal{E}_{\theta^i}^2} \geq \theta^{2(N_v(1)-1)} \geq \theta \tag{5.40}$$

so setting $\overline{P}_{i+1} = \text{graph}\,\frac{\mathcal{E}_{\theta^i}}{\mathcal{E}_{\theta^{i+1}}}\widetilde{p}^{+}_{i+1} \cup \text{graph}\,\frac{\mathcal{E}_{\theta^i}}{\mathcal{E}_{\theta^{i+1}}}\widetilde{p}^{-}_{i+1}$ and using the bound (5.40), we obtain from (5.37), (5.38) and (5.4) that

$$\begin{aligned}(5.41)\quad &\int_{B_1(0)} \text{dist}^2((x,\widetilde{v}^{+}_{\theta^{i+1}}(x)),\overline{P}_{i+1}) + \text{dist}^2((x,\widetilde{v}^{-}_{\theta^{i+1}}(x)),\overline{P}_{i+1})\\ &\leq \overline{C}\theta \int_{B_1(0)} \text{dist}^2((x,\widetilde{v}^{+}_{\theta^{j}}(x)),\overline{P}_{j}) + \text{dist}^2((x,\widetilde{v}^{-}_{\theta^{j}}(x)),\overline{P}_{j}) \quad \text{and}\end{aligned}$$

$$\begin{aligned}(5.42)\quad &\int_{B_1(0)\setminus S^{(i+1)}} (\widetilde{v}^{+}_{\theta^{i+1}} - \overline{p}^{+}_{i+1})^2 + (\widetilde{v}^{-}_{\theta^{i+1}} - \overline{p}^{-}_{i+1})^2\\ &\leq \overline{C}\theta \int_{B_1(0)} \text{dist}^2((x,\widetilde{v}^{+}_{\theta^{i}}(x)),\overline{P}_{i}) + \text{dist}^2((x,\widetilde{v}^{-}_{\theta^{i}}(x)),\overline{P}_{i})\end{aligned}$$

where $S^{(i+1)} = \{x \in \mathbf{R}^n \times \{0\} \,:\, \text{dist}\,(x, \pi\,(\overline{P}^{+}_i \cap \overline{P}^{-}_i)) \leq \theta/16\}$. This completes the induction.

We thus obtain a sequence of pairs of hyperplanes $\overline{P}_j$, $j = 1, 2, 3, \ldots$ satisfying (5.29) and (5.30). Now let $P^{\pm}_j = \text{graph}\,\mathcal{E}_{\theta^j}\overline{p}^{\pm}_j$. Then (5.29), (5.30) and (5.4) say that

(5.43)

$$\begin{aligned}&\left(\frac{2}{3}\theta^j\right)^{-n-2} \int_{B_{\frac{2}{3}\theta^j}(0)} \text{dist}^2((x,v^{+}(x)),P_j) + \text{dist}^2((x,v^{-}(x)),P_j)\\ &\leq 4^{-j}\left(\frac{3}{2}\right)^{n+2}\epsilon_2 \qquad \text{and}\end{aligned}$$

$$\begin{aligned}(5.44)\quad &\left(\frac{2}{3}\theta^j\right)^{-n-2} \int_{B_{\frac{2}{3}\theta^j}(0)\setminus S_{P_j}(\theta^{j+1}/24)} (v^{+} - p^{+}_j)^2 + (v^{-} - p^{-}_j)^2\\ &\leq 4^{-j}\left(\frac{2}{3}\right)^{n+2}\epsilon_2\end{aligned}$$

for all $j = 0, 1, 2, \ldots$, where we have used the fact that $\mathcal{E}_{\theta^j} \leq \mathcal{E}_1 \leq \left(\frac{3}{2}\right)^{n+2}$. By the triangle inequality and the homogeneity of P_j, P_{j-1}, (5.44) implies that

$$(5.45)\qquad \|(p^{+}_j, p^{-}_j) - (p^{+}_{j-1}, p^{-}_{j-1})\|_{L^2(B_1(0))} \leq C4^{-(j-1)}\epsilon_2$$

where $C = C(n, \alpha)$. i.e., that (p^{+}_j, p^{-}_j) is a Cauchy sequence. Hence there exists a pair of hyperplanes P such that $P_j \to P$. We then have by the triangle inequality and the inequalities (5.43), (5.44) and (5.45) that

(5.46)

$$\left(\frac{2}{3}\theta^j\right)^{-n-2} \int_{B_{\frac{2}{3}\theta^j}(0)} \text{dist}^2\left((x,v^{+}(x)),P\right) + \text{dist}^2\left((x,v^{-}(x)),P\right) \leq C4^{-j}\epsilon,$$

(5.47)
$$\left(\frac{2}{3}\theta^j\right)^{-n-2}\int_{B_{\frac{2}{3}\theta^j}(0)\setminus S_{P_j}(\theta^{j+1}/24)}(v^+-p^+)^2+(v^--p^-)^2\leq C4^{-j}\epsilon \quad \text{and}$$

$$\|(p_j^+,p_j^-)-(p^+,p^-)\|_{L^2(B_1(0))}\leq C4^{-j}\epsilon \tag{5.48}$$

for all $j=0,1,2,\ldots$, where $C=C(n,\alpha)$. Now, given any $\rho\in(0,1/4)$, there exists a unique non-negative integer $j^\star$ such that $\frac{2}{3}\theta^{j^\star+1}\leq\rho<\frac{2}{3}\theta^{j^\star}$. Using the estimates (5.46), (5.47) and (5.48) with $j=j^\star$, we obtain that

$$\rho^{-n-2}\int_{B_\rho(0)}\operatorname{dist}^2((x,v^+(x)),P)+\operatorname{dist}^2((x,v^-(x)),P)\leq C\rho^\mu\epsilon, \tag{5.49}$$

$$\rho^{-n-2}\int_{B_\rho(0)\setminus S_{P_{j^\star}}(\rho/16)}(v^+-p^+)^2+(v^--p^-)^2\leq C\rho^\mu\epsilon \quad \text{and} \tag{5.50}$$

$$\|(p_{j^\star}^+,p_{j^\star}^-)-(p^+,p^-)\|_{L^2(B_1(0))}\leq C\rho^\mu\epsilon \tag{5.51}$$

where $C=C(n,\alpha)>0$ and $\mu=\mu(n,\alpha)>0$. Since (5.51) implies

$$d_{\mathcal{H}}(T_{P_{j^\star}}\cap B_\rho(0),T_P\cap B_\rho(0))\leq C\rho^{1+\mu}\epsilon \tag{5.52}$$

where $C=C(n,\alpha)$ and T_P denotes the orthogonal projection of the axis $P^+\cap P^-$ of P onto $\mathbf{R}^n\times\{0\}$, we deduce from (5.50) that

$$\rho^{-n-2}\int_{B_\rho(0)\setminus S_P(\rho/8)}(v^+-p^+)^2+(v^--p^-)^2\leq C\rho^\mu\epsilon \quad \text{and} \tag{5.53}$$

provided $C\epsilon\leq 1/16$, where C is as in (5.52). Thus, we have the estimates (5.49) and (5.53) for all $\rho\in(0,1/4]$ provided $\epsilon=\epsilon(n,\alpha)\in(0,1)$ is sufficiently small. Note also that (5.48) in particular says

$$\|(p^+,p^-)-(\widetilde{p}_0^+,\widetilde{p}_0^-)\|_{L^2(B_1(0))}\leq C\epsilon \tag{5.54}$$

where $C=C(n,\alpha)$, which implies that if $\epsilon=\epsilon(n,\alpha)$ is sufficiently small, P must be a transverse pair of hyperplanes with $\alpha/2\leq\angle P<\pi$. Hence, provided $\epsilon=\epsilon(n,\alpha)\in(0,1)$ is chosen sufficiently small, the estimate of Lemma 3.10, part (b) together with the estimate (5.53) implies that

$$Z_w\cap(B_\rho(0)\setminus S_P(\rho/8))=\emptyset \quad \text{for each } \rho\in(0,1/4] \tag{5.55}$$

where $Z_w=\{z\,:\,v^+(z)=v^-(z)\}$, i.e., that $Z_w\cap B_{1/4}(0)$ is contained in a cone with vertex at the origin, axis the orthogonal projection of the axis of P onto $\mathbf{R}^n\times\{0\}$ and with a fixed cone angle depending only on n.

Next we argue that provided $\epsilon=\epsilon(n,\alpha)$ is sufficiently small, the decay estimates (5.49), (5.53) and the cone condition (5.55) hold uniformly for each "base point" $z\in Z_w$ sufficiently close to the origin, with a unique choice of a pair of affine hyperplanes P_z depending on z. So let

$z \in B_{1/4}(0)$ be such that $v^+(z) = v^-(z)$. Set $V^{(z)\,\pm}(x) = \widetilde{v}^{\pm}_{z,1/2}(x)$ for $x \in B_1(0)$ where the notation is as in (3.33). Then $(V^{(z)\,+}, V^{(z)\,-}) \in \mathcal{F}_\delta$ and $V^{(z)\,\pm}(0) = 0$. Note that by the standard estimates for harmonic functions we have that, since $y = h(z)$,

$$|y|,\ |Dh(z) - Dh(0)| \leq C|z| \left(\int_{B_1(0)} (v^+)^2 + (v^-)^2\right)^{1/2} \leq C|z| \tag{5.56}$$

for all $z \in B_{1/4}(0)$, where $C = C(n)$. Also note that it follows from the inequality (5.15) that provided $\epsilon = \epsilon(n,\alpha) \in (0,1)$ is sufficiently small,

$$\widetilde{C} \geq \mathcal{E}^2_{z,1/2} \geq C > 0 \tag{5.57}$$

where $\widetilde{C} = \widetilde{C}(n)$ and $C = C(n,\alpha)$.

Now set $\widetilde{p}^{(z)\,\pm}(x) = \frac{1}{2\mathcal{E}_{z,1/2}}\widetilde{p}^{\pm}_0(x)$. Then $\pi > \angle\widetilde{P}^{(z)} \geq \widetilde{\alpha}$, where $\widetilde{\alpha} = \widetilde{\alpha}(n,\alpha) > 0$. It is then easy to see directly from the definition of $V^{(z)\,\pm}$ and the estimates (5.56) that there exists $\gamma = \gamma(n,\alpha) > 0$ and $\kappa = \kappa(n,\alpha) > 0$ such that for all $z \in B_\kappa(0)$ with $v^+(z) = v^-(z)$,

$$\begin{aligned}
&\int_{B_1(0)} \left(V^{(z)\,+} - \widetilde{p}^{(z)\,+}\right)^2 \\
&\quad + \left(V^{(z)\,-} - \widetilde{p}^{(z)\,-}\right)^2 \\
&\leq \frac{3^{n+2}}{\mathcal{E}^2_{z,1/2}} \int_{B_{1/3}(z)} (v^+(x) - \frac{1}{3}\widetilde{p}^{+}_0(3x-z))^2 \\
&\quad + (v^-(x) - \frac{1}{3}\widetilde{p}^{-}_0(3x-z))^2\,dx \\
&\leq \frac{2\cdot 3^{n+2}}{\mathcal{E}^2_{z,1/2}} \int_{B_1(0)} (v^+(x) - \widetilde{p}^{+}_0(x))^2 + (v^-(x) - \widetilde{p}^{-}_0(x))^2 \\
&\quad + \left(\widetilde{p}^{+}_0(x) - \widetilde{p}^{+}_0\left(x - \frac{z}{3}\right)\right)^2 + \left(\widetilde{p}^{-}_0(x) - \widetilde{p}^{-}_0\left(x - \frac{z}{3}\right)\right)^2\,dx \\
&\leq \frac{2\cdot 3^{n+2}}{\mathcal{E}^2_{z,1/2}} \int_{B_1(0)} (v^+ - \widetilde{p}^{+}_0)^2 + (v^- - \widetilde{p}^{-}_0)^2 + C|z|^2 \\
&\leq \epsilon
\end{aligned} \tag{5.58}$$

where $C = C(n,\alpha)$ and $\epsilon = \epsilon(n,\widetilde{\alpha})$ is as in the argument (with $\widetilde{\alpha}$ in place of α) leading to the estimates (5.49) and (5.53), provided $\int_{B_1(0)} (v^+ - \widetilde{p}^{+}_0)^2 + (v^- - \widetilde{p}^{-}_0)^2 \leq \gamma\epsilon$.

Therefore, if the hypotheses of the lemma are satisfied with $\gamma\epsilon$ in place of ϵ, we may repeat the argument leading to the estimates (5.49), (5.53), (5.54) and the cone condition (5.55) with $V^{(z)\,\pm}$ in place of $v^\pm$ and $\widetilde{p}^{(z)\,\pm}$ in place of $\widetilde{p}^{\pm}_0$. This will yield for each $z \in B_\kappa(0)$ with $v^+(z) = v^-(z)$ a pair of transverse hyperplanes $P_z = P_z^+ \cup P_z^-$ satisfying

$$\rho^{-n-2} \int_{B_\rho(z)} \operatorname{dist}^2((x, v^+(x)), (z, y) + P_z) + \operatorname{dist}^2((x, v^-(x)), (z, y) + P_z) \leq C\rho^\mu \epsilon, \tag{5.59}$$

$$\rho^{-n-2} \int_{B_\rho(z) \setminus S_{(z,y)+P_z}(\rho/8)} (v^+(x) - (y + p_z^+(x-z)))^2 + (v^-(x) - (y + p_z^-(x-z)))^2 \leq C\rho^\mu \epsilon, \tag{5.60}$$

$$\|(p_z^+, p_z^-) - (\widetilde{p}_0^+, \widetilde{p}_0^-)\|_{L^2(B_1(0))} \leq C\epsilon \quad \text{and} \tag{5.61}$$

$$Z_w \cap (B_\rho(z) \setminus S_{(z,y)+P_z}(\rho/8)) = \emptyset \tag{5.62}$$

for all $\rho \in (0, 1/12)$. Here $y = v^+(z) = v^-(z)$ and $C = C(n, \alpha)$.

Note that by the estimates (5.60), (5.61), Lemma 5.2 and the triangle inequality, it follows that for each $z \in B_\kappa(0) \cap Z_w$ and $\rho \in (0, 1/12)$,

$$C \leq \mathcal{E}_{z,\rho} \leq \widetilde{C} \tag{5.63}$$

for fixed $C = C(n, \alpha) > 0$ and $\widetilde{C} = \widetilde{C}(n) < \infty$.

Next we assert that $Z_w \cap \{x \in \mathbf{R}^n : |\pi_{\widetilde{P}_0^+ \cap \widetilde{P}_0^-}(x)| < \kappa/2\}$ projects fully onto the axis $\widetilde{P}_0^+ \cap \widetilde{P}_0^- \cap B_{\kappa/2}(0)$. To see this, first note that since $\widetilde{p}_0^+ + \widetilde{p}_0^- \equiv 0$ by hypothesis, we have that $\widetilde{P}_0^+ \cap \widetilde{P}_0^- \subset \mathbf{R}^n \times \{0\}$. For notational convenience (and without loss of generality, by making an orthogonal rotation of $\mathbf{R}^n \times \{0\}$), let us assume that $\widetilde{P}_0^+ \cap \widetilde{P}_0^- = \mathbf{R}^{n-1} \times \{(0,0)\}$. If there is a point $(\xi, 0, 0) \in (\mathbf{R}^{n-1} \times \{(0,0)\}) \cap B_{\kappa/2}(0)$ with $\mathbf{p}^{-1}(\xi, 0, 0) \cap Z_w = \emptyset$, where $\mathbf{p} : \mathbf{R}^n \times \{0\} \to \mathbf{R}^{n-1} \times \{(0,0)\}$ is the orthogonal projection, then, since Z_w is a closed set, there must exist $r > 0$ such that

$$\begin{aligned} &(B_r^{n-1}(\xi, 0, 0) \times \mathbf{R} \times \{0\}) \cap Z_w = \emptyset \quad \text{and} \\ &(\overline{B}_r^{n-1}(\xi, 0, 0) \times \mathbf{R} \times \{0\}) \cap Z_w \neq \emptyset. \end{aligned} \tag{5.64}$$

Choose $z \in (\overline{B}_r^{n-1}(\xi, 0, 0) \times \mathbf{R} \times \{0\}) \cap Z_w$.

Note next the following fact: Let $\alpha_1 \in (0, \pi)$. Then for any given η, there exists $\zeta = \zeta(\alpha_1, \eta)$ with $\zeta \downarrow 0$ as $\eta \downarrow 0$ such that if $v = (v^+, v^-) \in \mathcal{F}_\delta$ satisfies $\int_{B_1(0)} \operatorname{dist}^2((x, v^+(x)), P_1) + \operatorname{dist}^2((x, v^-(x)), P_1) \leq \zeta$ and $\int_{B_1(0) \setminus S_{P_1}(1/8)} (v^+ - p_1^+)^2 + (v^- - p_1^-)^2 \leq \zeta$ for some pair of hyperplanes P_1 with $\alpha_1 \leq \angle P_1 < \pi$, then $\int_{B_1(0)} (w - L_1)^2 \leq \eta$ where $w = \frac{1}{2}(v^+ - v^-)$ and $L_1 = \frac{1}{2}(p_1^+ - p_1^-)$. (This can easily be seen by arguing by contradiction.) Since the estimates (5.59) and (5.60) say that for each $\rho \in (0, 1/8)$, $\int_{B_1(0)} \operatorname{dist}^2((x, \widetilde{v}_{z,\rho}^+(x)), P_z^{(\rho)}) + \operatorname{dist}^2((x, \widetilde{v}_{z,\rho}^-(x)), P_z^{(\rho)}) \leq$

$C\rho^{\mu}\epsilon$ and $\int_{B_1(0)\setminus S_{P_z^{(\rho)}}(1/8)} (\widetilde{v}_{z,\rho}^{+} - p_z^{(\rho)\,+})^2 + (\widetilde{v}_{z,\rho}^{-} - p_z^{(\rho)\,-})^2 \le C\rho^{\mu}\epsilon$ where $p_z^{(\rho)\,\pm} = \frac{1}{\mathcal{E}_{z,\rho}} p_z^{\pm}$ and the estimates (5.63) say that $P_z^{(\rho)}$ satisfies $\alpha_1 \le \angle\, P_z^{(\rho)} < \pi$ for some $\alpha_1 = \alpha_1(n,\alpha) > 0$, it follows that for any given $\eta \in (0,1/2)$, there exists $\rho = \rho(n,\alpha,\eta) \in (0,1/2)$ such that $\int_{B_1(0)} (\widetilde{w}_{z,\rho} - L_z^{(\rho)})^2 \le \eta$ where $\widetilde{w}_{z,\rho} = \frac{1}{2}(\widetilde{v}_{z,\rho}^{+} - \widetilde{v}_{z,\rho}^{-})$ and $L_z^{(\rho)} = \frac{1}{2}(P_z^{(\rho)\,+} - P_z^{(\rho)\,-})$. Thus, we may apply Lemma 5.5 with $\widetilde{v}_{z,\rho}^{\pm}$ in place of $v^{\pm}$ for a suitable choice of sufficiently small $\rho \in (0, r/4)$ to arrive at a contradiction of (5.64). (Note that here we have also used the fact that $\pi\,(P_z^{(\rho)\,+} \cap P_z^{(\rho)\,-})$ remains close to $\widetilde{P}_0^{+} \cap \widetilde{P}_0^{-}$ as $\rho \downarrow 0$, which follows from the estimate (5.61).) Hence $Z_w \cap \{x \in \mathbf{R}^n : |\pi_{\widetilde{P}_0^{+} \cap \widetilde{P}_0^{-}}(x)| < \kappa/2\}$ must have full projection onto $\widetilde{P}_0^{+} \cap \widetilde{P}_0^{-} \cap B_{\kappa/2}(0)$.

It then follows first from the estimates (5.59), (5.61), and (5.62) that $Z_w \cap \{x \in \mathbf{R}^n : |\pi_{\widetilde{P}_0^{+} \cap \widetilde{P}_0^{-}}(x)| < \kappa/2\}$ is equal to a Lipschitz graph (over $\widetilde{P}_0^{+} \cap \widetilde{P}_0^{-} \cap B_{\kappa/2}(0)$) and then by the estimate (5.60) that this graph is $C^{1,\mu}$. This implies directly that the union of the graphs of v^{+}, v^{-} over $B_{\kappa/2}(0)$ is equal to the union of the graphs of two harmonic functions $v^1, v^2 : B_{\kappa/2}(0) \to \mathbf{R}$. Specifically, if we let $\Omega^{\pm}$ denote the two components of $B_{\kappa/2}(0) \setminus Z_w$ and define a function v^1 on $B_{\kappa/2}(0)$ by setting $v^1(x) = v^{+}(x)$ if $x \in \overline{\Omega}^{+}$, and $v^1(x) = v^{-}(x)$ if $x \in \Omega^{-}$, we see first by (5.59) that $v^1 \in C^1(B_{\kappa/2}(0))$ and then by integration by parts that $\int_{B_{\kappa/2}(0)} Dv^1 \cdot D\zeta = \int_{B_{\kappa/2}(0)\cap\Omega^{+}} Dv^1 \cdot D\zeta + \int_{B_{\kappa/2}(0)\cap\Omega^{-}} Dv^1 \cdot D\zeta = 0$ for every $\zeta \in C_c^1(B_{\kappa/2}(0))$. Thus v^1 is harmonic. Similarly, we may define $v^2 : B_{\kappa/2}(0) \to \mathbf{R}$ by setting $v^2(x) = v^{-}(x)$ if $x \in \overline{\Omega}^{+}$, and $v^2(x) = v^{+}(x)$ if $x \in \Omega^{-}$, and check that v^2 is also harmonic.

Finally, since by (5.59) and (5.60) the tangent planes to the graphs of v^1 and v^2 at any point (z,y) where $v^1(z) = v^2(z) = y$ are transversely intersecting, it follows that the vanishing order of $v^1 - v^2$ at such a point must be equal to 1. Thus the lemma holds with $\kappa/2$ in place of κ and $\gamma\epsilon$ in place of ϵ. q.e.d.

Definition. Given $v = (v^{+}, v^{-}) \in \mathcal{F}_{\delta}$, we shall call a point $z \in B_{3/2}(0)$ a *branch point* of v if there exists no $\sigma > 0$ such that $(\mathrm{graph}\, v^{+} \cup \mathrm{graph}\, v^{-}) \cap (B_{\sigma}(z) \times \mathbf{R})$ is equal to the union of the graphs of two harmonic functions over $B_{\sigma}(z)$.

Remark. It follows directly from Proposition 3.3(2) and Proposition 3.11 (5) that if z is a branch point of $v = (v^{+}, v^{-})$, then $z \in Z_w$, i.e., that $v^{+}(z) = v^{-}(z)$. Furthermore, if $v^{+} \equiv v^{-}$, then $v^{\pm}$ are each harmonic, so no point $z \in B_{3/2}(0)$ is a branch point in this case.

Using Lemma 5.6 and adapting techniques due to L. Simon [**Sim93**], we establish in the next two lemmas crucial uniform asymptotic decay estimates for any function $v \in \mathcal{F}_\delta$ at a branch point.

Lemma 5.7. *Let $\delta \in (0,1)$. There exists a constant $c = c(n,\delta) > 0$ such that the following is true. If $v = (v^+, v^-) \in \mathcal{F}_\delta$, $v^+(0) = v^-(0) = 0$, $Dh(0) = 0$, where $h = \frac{1}{2}(v^+ + v^-)$, and if either*

(a) *the origin is a branch point of v or*
(b) *$w \not\equiv 0$ and $\mathcal{N}_w(0) > 1$, where $w = \frac{1}{2}(v^+ - v^-)$,*

then

$$\int_{B_1(0)\setminus B_{1/2}(0)} \left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + \left(\frac{\partial(v^-/R)}{\partial R}\right)^2 \geq c\int_{B_1(0)} (v^+)^2 + (v^-)^2.$$

Proof. If the lemma is not true, there exists a sequence of functions $v_k = (v_k^+, v_k^-) \in \mathcal{F}_\delta$ satisfying $v_k^+(0) = v_k^-(0) = 0$ and $Dh_k(0) = 0$ where $h_k = \frac{1}{2}(v_k^+ + v_k^-)$, such that for each k, either the origin is a branch point of v_k (in which case $w_k \not\equiv 0$, where $w_k = \frac{1}{2}(v_k^+ - v_k^-)$) or $w_k \not\equiv 0$ and $\mathcal{N}_{w_k}(0) > 1$, and

(5.65)
$$\int_{B_1(0)\setminus B_{1/2}(0)} \left(\frac{\partial(v_k^+/R)}{\partial R}\right)^2 + \left(\frac{\partial(v_k^-/R)}{\partial R}\right)^2 \leq \frac{1}{k}\int_{B_1(0)} (v_k^+)^2 + (v_k^-)^2.$$

Let $\widetilde{v}_k^{\pm}(x) = \frac{3}{2}\frac{v_k^{\pm}(2x/3)}{\left(\int_{B_1(0)}(v_k^+)^2+(v_k^-)^2\right)^{1/2}}$. Then $\widetilde{v}_k \equiv (\widetilde{v}_k^+, \widetilde{v}_k^-) \in \mathcal{F}_\delta$, and by Lemma 3.1, after passing to a subsequence which we continue to denote $\{k\}$, $(\widetilde{v}_k^+, \widetilde{v}_k^-) \to v = (v^+, v^-) \in \mathcal{F}_\delta$ where the convergence is in $W^{1,2}(B_\sigma(0))$ for every $\sigma \in (0, 3/2)$. By (5.65),

$$\int_{B_{3/2}(0)\setminus B_{3/4}(0)} \left(\frac{\partial(\widetilde{v}_k^+/R)}{\partial R}\right)^2 + \left(\frac{\partial(\widetilde{v}_k^-/R)}{\partial R}\right)^2 \leq \frac{1}{k}\left(\frac{3}{2}\right)^{n-2}. \tag{5.66}$$

We claim that v cannot be identically equal to zero on $B_1(0)$. To see this, first note that for any $r, s \in (3/4, 3/2)$ and $\omega \in \mathbf{S}^{n-1}$, we have that

$$\begin{aligned}\left|\frac{\widetilde{v}_k(r\,\omega)}{r} - \frac{\widetilde{v}_k(s\,\omega)}{s}\right| &= \left|\int_s^r \frac{\partial\left(\widetilde{v}_k(R\,\omega)/R\right)}{\partial\, R}dR\right| \\ &\leq \int_{3/4}^{3/2}\left|\frac{\partial\left(\widetilde{v}_k(R\,\omega)/R\right)}{\partial\, R}\right| dR\end{aligned}$$

which implies, by the triangle inequality, Cauchy-Schwarz inequality and the fact that $r, s \in (3/4, 3/2)$ that

$$|\widetilde{v}_k(r\,\omega)|^2 \leq c\left(|\widetilde{v}_k(s\,\omega)|^2 + \int_{3/4}^{3/2} R^{n-1}\left|\frac{\partial\left(\widetilde{v}_k(R\,\omega)/R\right)}{\partial\, R}\right|^2 dR\right)$$

where $c = c(n) \in [1, \infty)$. Integrating this with respect to ω yields

$$\begin{aligned}&\int_{\mathbf{S}^{n-1}} |\widetilde{v}_k(r\,\omega)|^2 d\omega \\ &\leq c\left(\int_{\mathbf{S}^{n-1}} |\widetilde{v}_k(s\,\omega)|^2 d\omega + \int_{B_{3/2}(0)\setminus B_{3/4}(0)} \left|\frac{\partial\,(\widetilde{v}_k/R)}{\partial\,R}\right|^2\right)\end{aligned}$$

where $c = c(n) \in [1, \infty)$. First multiplying both sides of the above by r^{n-1} and integrating with respect to r over the interval $(3/4, 3/2)$, and then multiplying both sides of the resulting inequality by s^{n-1} and integrating it with respect to s over the interval $(3/4, 1)$ gives

$$\begin{aligned}&\int_{B_{3/2}(0)\setminus B_{3/4}(0)} |\widetilde{v}_k|^2 \\ &\leq c\left(\int_{B_1(0)\setminus B_{3/4}(0)} |\widetilde{v}_k|^2 + \int_{B_{3/2}(0)\setminus B_{3/4}(0)} \left|\frac{\partial\,(\widetilde{v}_k/R)}{\partial\,R}\right|^2\right)\end{aligned}$$

where $c = c(n) \in [1, \infty)$. Since $\|\widetilde{v}_k\|^2_{L^2(B_{3/2}(0))} = \left(\frac{3}{2}\right)^{n+2}$, this implies that

$$\left(\frac{3}{2}\right)^{n+2} \leq c\left(\int_{B_1(0)} |\widetilde{v}_k|^2 + \int_{B_{3/2}(0)\setminus B_{3/4}(0)} \left|\frac{\partial\,(\widetilde{v}_k/R)}{\partial\,R}\right|^2\right),$$

which in view of (5.66) immediately implies that $v \not\equiv 0$ in $B_1(0)$. Hence, by Lemma 3.11, part (8), $\int_{\partial\,B_\rho(0)} |v|^2 > 0$ for all $\rho \in (0, 3/2)$. By (5.66) again, v is homogeneous of degree one in the region $B_1 \setminus B_{3/4}$ which implies that $N_{v,0}(\rho) = 1$ for $3/4 \leq \rho \leq 1$. (This can be seen easily by the expression $N_{v,0}(\rho) = \frac{\rho\frac{d}{d\rho}\int_{\mathbf{S}^{n-1}}|\hat{v}_\rho|^2}{2\int_{\mathbf{S}^{n-1}}|\hat{v}_\rho|^2}$ where $\hat{v}_\rho(x) = v(\rho x)$). By Lemma 3.13, $\mathcal{N}_{\widetilde{v}_k}(0) \geq 1$, and hence by Lemma 3.14, we have that $\mathcal{N}_v(0) \geq 1$. Hence by monotonicity of $N_{v,0}(\cdot)$, it follows that $N_{v,0}(\rho) = 1$ for every $\rho \in (0,1)$. By Lemma 3.6, this means that v is homogeneous of degree 1 from the origin, and hence by Lemma 3.12, $\operatorname{graph} v^+ \cup \operatorname{graph} v^- = P_1 \cup P_2$ for hyperplanes P_1, P_2. Thus, if v^+ is not identically equal to v^-, by Lemma 5.6, for sufficiently large k, $v_k^+ = \max\{v_k^1, v_k^2\}$ and $v_k^- = \min\{v_k^1, v_k^2\}$ in $B_\kappa(0)$ for some $\kappa > 0$, where v_k^1, v_k^2 are harmonic functions in $B_\kappa(0)$, each equal to zero at the origin, and with the difference $v_k^1 - v_k^2$ having vanishing order at the origin equal to 1. But this contradicts either of the hypotheses that v_k has a branch point at 0 or that $\mathcal{N}_{w_k}(0) > 1$. Thus we must have that $v^+ \equiv v^- \equiv L$ for some linear function L. But then since $D\widetilde{h}_k(0) = 0$ for every k, where $\widetilde{h}_k = \frac{1}{2}(\widetilde{v}_k^+ + \widetilde{v}_k^-)$, and $\widetilde{h}_k \to \frac{1}{2}(v^+ + v^-)$ smoothly in $B_1(0)$ (since $\widetilde{h}_k$ are harmonic with uniformly bounded $L^2(B_{3/2}(0))$ norm), L would have to be identically zero, which is impossible. The lemma is thus proved.
q.e.d.

Lemma 5.8. *Let $v = (v^+, v^-) \in \mathcal{F}_\delta$, $v^+(0) = v^-(0) = 0$, and suppose either that the origin is a branch point of v or that $\mathcal{N}_w(0) > 1$. Then*

$$\rho^{-n-2} \int_{B_\rho(0)} (v^+ - l)^2 + (v^- - l)^2 \le C\rho^\nu \int_{B_1(0)} (v^+)^2 + (v^-)^2$$

for some linear function $l : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ and all $\rho \in (0, 1/16)$. In fact, $l(x) = Dh(0) \cdot x$ where $h = \frac{1}{2}(v^+ + v^-)$. Here $C = C(n, \delta) \in (0, \infty)$ and $\nu = \nu(n, \delta) \in (0, 1)$.

Proof. Let $l(x) = Dh(0) \cdot x$. Note that there exists $C = C(n)$ such that

$$|Dh(0)| \le C \left(\int_{B_1(0)} (v^+)^2 + (v^-)^2 \right)^{1/2} \le C. \tag{5.67}$$

By the definition of $\mathcal{F}_\delta$, there exists a sequence of hypersurfaces $M_k \in \mathcal{I}_b$ and a sequence of affine hyperplanes $L_k \to \mathbf{R}^n \times \{0\}$ such that the blow-up of $\{M_k\}$ by the height excesses $\hat{E}_k$ of M_k relative to L_k (as in Section 3) is (v^+, v^-). For each k, let $l_k : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ be the affine function such that $L_k = \operatorname{graph} l_k$. Let $\left(v^+_{(l)}, v^-_{(l)}\right)$ be the blow-up produced by blowing up the M_k's by their height excesses $\hat{E}_k^{(l)}$ relative to the affine hyperplanes given by graph $(l_k + \hat{E}_k l)$. Since by (5.67), $\frac{\hat{E}_k^{(l)}}{\hat{E}_k} \le C$, where $C = C(n) < \infty$, we have that

$$C_l \left(v^+_{(l)}, v^-_{(l)}\right) = (v^+ - l, v^- - l)$$

where $0 < C_l \le C = C(n) < \infty$. (Note that here we are assuming that not both v^+, v^- are identical to l; if this were the case, the lemma is trivially true.) It then follows that since $\left(v^+_{(l)}, v^-_{(l)}\right) \in \mathcal{F}_\delta$ (by the definition of $\mathcal{F}_\delta$), all the properties and estimates we have established for (v^+, v^-) will hold with $v^\pm - l$ in place of $v^\pm$. In particular, Lemma 3.8 (with $z = 0$, $y = 0$) holds with $v^+ - l$, $v^- - l$ in place of v^+, v^-. Thus

$$\begin{aligned} \int_{B_{\rho/2}(0)} R^{2-n} \left(\frac{\partial\,(v^+/R)}{\partial R}\right)^2 + R^{2-n} \left(\frac{\partial\,(v^-/R)}{\partial R}\right)^2 \\ \le C\rho^{-n-2} \int_{B_\rho(0)} (v^+ - l)^2 + (v^- - l)^2 \end{aligned} \tag{5.68}$$

for all $\rho \in (0, 1/8)$, where $C = C(n)$. On the other hand, applying Lemma 5.7 with $(\widetilde{V}^+, \widetilde{V}^-) \equiv \frac{(v^+_\rho - l,\, v^-_\rho - l)}{\left(\rho^{-n-2} \int_{B_\rho(0)} (v^+ - l)^2 + (v^- - l)^2\right)^{1/2}} \in \mathcal{F}_\delta$ in place of (v^+, v^-), where $v^\pm_\rho(x) = \frac{1}{(2\rho/3)} v^\pm(\frac{2}{3}\rho x)$ (noting that, by definition of l, $D\widetilde{H}(0) = 0$ where $\widetilde{H} = \frac{1}{2}(\widetilde{V}^+ + \widetilde{V}^-)$), we have that

$$\int_{B_{2\rho/3}(0)\setminus B_{\rho/3}(0)} \left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + \left(\frac{\partial(v^-/R)}{\partial R}\right)^2 \geq c\rho^{-4}\int_{B_{2\rho/3}(0)} (v^+ - l)^2 + (v^- - l)^2$$

where $c = c(n, \delta) > 0$. This gives, since $R = |X| \leq 2\rho/3$ for $X \in B_{2\rho/3}(0)$, that

$$\int_{B_{2\rho/3}(0)\setminus B_{\rho/3}(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2 \geq c\rho^{-n-2}\int_{B_{2\rho/3}(0)} (v^+ - l)^2 + (v^- - l)^2. \tag{5.69}$$

Replacing ρ with $3\rho/2$ in the inequalities (5.68) and (5.69), and combining them gives

$$\int_{B_\rho(0)\setminus B_{\rho/2}(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2 \geq \frac{c}{C}\int_{B_{\rho/2}(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2$$

for $\rho \in (0, 1/12)$. This implies that

$$\int_{B_{\rho/2}(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2 \leq \kappa\int_{B_\rho(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2$$

where $\kappa = \kappa(n, \delta) = \frac{1}{1+\frac{c}{C}} \in (0, 1)$ (here C and c are as in inequalities (5.68) and (5.69)). By iterating this starting with $\rho = 1/16$, we obtain that

$$\int_{B_{\frac{2^{-j}}{16}}(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2 \leq \kappa^j\int_{B_{1/16}(0)} R^{2-n}\left(\frac{\partial(v^+/R)}{\partial R}\right)^2 + R^{2-n}\left(\frac{\partial(v^-/R)}{\partial R}\right)^2$$

for every $j = 1, 2, \ldots$. Combining this with inequalities (5.68) and (5.69), we have

(5.70)
$$\left(\frac{2^{-j}}{16}\right)^{-n-2} \int_{B_{\frac{2^{-j}}{16}}(0)} (v^+ - l)^2 + (v^- - l)^2 \leq C\kappa^j \int_{B_{1/8}(0)} (v^+ - l)^2 + (v^- - l)^2$$

for all j. Now given any $\rho \in (0, 1/16)$, there exists a unique non-negative integer j such that $\frac{2^{-j-1}}{16} \leq \rho < \frac{2^{-j}}{16}$, and using (5.70) with this j gives

(5.71)
$$\begin{aligned}\rho^{-n-2} \int_{B_\rho(0)} (v^+ - l)^2 + (v^- - l)^2 &\leq C\rho^\nu \int_{B_{1/8}(0)} (v^+ - l)^2 + (v^- - l)^2 \\ &\leq C\rho^\nu \int_{B_1(0)} (v^+)^2 + (v^-)^2\end{aligned}$$

for all $\rho \in (0, 1/16)$, where $C = C(n, \delta) \in (0, \infty)$ and $\nu = \nu(n, \delta) \in (0, 1)$. (The last inequality in (5.71) follows from (5.67).) This is the desired estimate. q.e.d.

The preceding two lemmas imply the existence of a fixed positive "frequency gap" for the functions in $\mathcal{F}_\delta$. Specifically, we have the following:

Lemma 5.9. *Let $\delta \in (0, 1)$. There exists a fixed constant $\nu_0 > 0$ depending only on n and δ such that if $v = (v^+, v^-) \in \mathcal{F}_\delta$, $w = \frac{1}{2}(v^+ - v^-)$, $z \in Z_w \cap B_{1/2}(0)$ and either $\mathcal{N}_w(z) > 1$ or z is a branch point of v, then $\mathcal{N}_w(z) \geq 1 + \nu_0$.*

Proof. Recall that $N_{w,z}(\rho) = \frac{\rho \frac{d}{d\rho} \int_{\mathbf{S}^{n-1}} w_{z,\rho}^2}{2 \int_{\mathbf{S}^{n-1}} w_{z,\rho}^2}$. Fix any $\rho \in (0, 1/2)$. Then we have by the monotonicity of $N_{w,z}(\cdot)$ that $\frac{\sigma \frac{d}{d\sigma} \int_{\mathbf{S}^{n-1}} w_{z,\sigma}^2}{2 \int_{\mathbf{S}^{n-1}} w_{z,\sigma}^2} \leq N_{w,z}(\rho)$ for all $\sigma \in (0, \rho]$. Integrating this differential inequality (cf. Lemma 3.5) gives

$$\sigma^{-n} \int_{B_\sigma(z)} w^2 \geq \left(\rho^{1-n-2N_{w,z}(\rho)} \int_{\partial B_\rho(z)} w^2\right) \sigma^{2N_{w,z}(\rho)} \tag{5.72}$$

for all $\sigma \in (0, \rho]$. On the other hand, Lemma 5.8, applied with $v^\pm(z + (\cdot))$ in place of $v^\pm(\cdot)$, implies that

$$\sigma^{-n-2} \int_{B_\sigma(z)} w^2 \leq C\sigma^\nu \tag{5.73}$$

for all $\sigma \in (0, 1/8)$. The estimates (5.72) and (5.73) readily imply that

$$N_{w,z}(\rho) \geq 1 + \frac{\nu}{2}$$

for all $\rho \in (0, 1/2)$. This gives $\mathcal{N}_w(z) \geq 1 + \frac{\nu}{2}$. q.e.d.

Lemma 5.10. *Let* $(v^+, v^-) \in \mathcal{F}_\delta$ *and* $w = \frac{1}{2}(v^+ - v^-)$. *Suppose* $z \in B_1(0)$ *and* $v^+(z) = v^-(z)$. *If* $\mathcal{N}_w(z) = 1$, *then there exists* $\sigma = \sigma(z) > 0$ *and two harmonic functions* v^1, $v^2 : B_\sigma(z) \to \mathbf{R}$ *such that* $v^+|_{B_\sigma(z)} = \max\{v^1, v^2\}$ *and* $v^-|_{B_\sigma(z)} = \min\{v^1, v^2\}$.

Proof. This follows immediately from Lemma 5.9 and the definition of branch point. q.e.d.

Proof of Theorem 5.1. Let $v = (v^+, v^-) \in \mathcal{F}_\delta$ and $w = \frac{1}{2}(v^+ - v^-)$. Let $S_v = \{z \in B_1(0) : z \text{ is a branch point of } v\}$. Then S_v is a relatively closed subset of $B_1(0)$ by definition. Also by the definition of S_v, if $z \in B_1(0) \setminus S_v$, then the graphs of $v^\pm$ decompose, locally near z, as the union of the graphs of two harmonic functions, and hence, the same is true over any open, simply connected subset $\Omega \subseteq B_1(0) \setminus S_v$. This proves part (a) of the lemma.

Part (b) follows by applying Lemma 5.8 to the function $\widetilde{v}_{z, \frac{3}{8}}$ (notation as in 3.33) and changing variables. Note that $\widetilde{v}_{z, \frac{3}{8}} \in \mathcal{F}_\delta$. q.e.d.

6. Improvement of excess relative to pairs of hyperplanes

In this section, we prove the main excess decay lemma (Lemma 6.3 below) needed for the proof of Theorem 1.1. Roughly speaking, this lemma says that whenever a hypersurface $M \in \mathcal{I}_b$ satisfying $0 \in \overline{M}$ and $\frac{\mathcal{H}^n(M \cap (B_1(0) \times \mathbf{R}))}{\omega_n} \leq 3 - \delta$ for some fixed $\delta \in (0,1)$ is sufficiently L^2-close, in the cylinder $B_1(0) \times \mathbf{R}$, to a pair of affine hyperlanes of $\mathbf{R}^{n+1}$—i.e., has small height excess relative to a pair of affine hyperplanes—then, at one of three possible smaller scales, it is closer by a fixed factor to a new pair of affine hyperplanes; i.e., the height excess improves. By iterating this result, we shall prove in the next section our main regularity theorem, Theorem 1.1. The principal quantity we are interested in keeping track of that measures the height excess of M at scale $\rho \in (0,1)$ and that is improving is $E_M(\rho, P) \equiv \sqrt{\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P)}$, where P denotes a pair of affine hyperplanes. However, in the proof of Lemma 6.3 (see case (a) of the proof), we need to make sure that the "sheets" of M separate whenever this excess is significantly smaller than a certain "coarse excess," which measures the L^2 deviation of M from a *single* affine hyperplane. In order to achieve this, it is necessary to modify the definition of the improving quantity and consider the sum of $E_M^2(\rho, P)$ and a quantity that measures the squared L^2-distance *of* P *from* M (see the statement of Lemma 6.3 for the precise definition of this quantity). The main point that necessitates this is simply that smallness of $E_M(\rho, P)$ alone need not imply separate closeness of the "individual sheets" of M to each of the two affine hyperplanes that make up P; M may consist of two sheets both of which are close to the same single affine hyperplane of P.

In the proof of Lemma 6.3, we shall need the elementary facts asserted in Lemmas 6.1 and 6.2 below. But first we need to recall/introduce some notation we shall use in this section and the next. The purpose of items (1) through (6) below is to fix notation that will enable us to define in a convenient way the "second term" of the improving quantity of Lemma 6.3 referred to in the preceding paragraph, and facilitate statement and proof of Lemma 6.3.

Fix $\delta \in (0,1)$. Let $\rho \in (0,1]$, $M \in \mathcal{I}_b$ and suppose that $0 \in \overline{M}$ and $\frac{\mathcal{H}^n\,(M \cap (B_\rho(0)\times \mathbf{R}))}{\omega_n \rho^n} \leq 3-\delta$.

(1) $\mathcal{A}(M,\rho)$ denotes the set of affine hyperplanes L of $\mathbf{R}^{n+1}$ satisfying $L \cap (B_1(0)\times \mathbf{R}) \subset \{(x', x^{n+1}) \in \mathbf{R}^{n+1} \,:\, |x^{n+1}| \leq 1/8\}$ and

$$\begin{aligned}\hat{E}^2_M(\rho, L) &\equiv \rho^{-n-2}\int_{M\cap(B_\rho(0)\times\mathbf{R})} \mathrm{dist}^2(x, L)\\ &\leq \frac{3}{2}\inf_{L'} \rho^{-n-2}\int_{M\cap(B_\rho(0)\times\mathbf{R})} \mathrm{dist}^2(x, L'),\end{aligned}$$

where the inf is taken over all affine hyperplanes L' of $\mathbf{R}^{n+1}$ satisfying $L' \cap (B_1(0)\times\mathbf{R}) \subset \{(x', x^{n+1}) \in \mathbf{R}^{n+1} \,:\, |x^{n+1}| \leq 1/8\}$.

(2) Given an affine hyperplane L of $\mathbf{R}^{n+1}$ with $L \cap (B_1(0)\times\mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$, let $\mathcal{R}\,(M, L, \rho)$ denote the set of regular values $t \in (1/4, 1/2)$ of the function $g(X) \equiv 1 - (\nu(X)\cdot\nu^L)^2$ on M, satisfying

$$\mathcal{H}^{n-1}\left(M \cap (B_{3\rho/4}(0)\times\mathbf{R}) \cap \{X \,:\, g(X) = t\}\right) \leq C\hat{E}^2_M\,(\rho, L)$$

where ν, ν^L are the unit normals to M, L respectively, and $C = C(n)$ is the constant as in inequality (3.7). Note that $\mathcal{R}(M, L, \rho)$ contains infinitely many numbers (see the argument of [**SS81**], p. 753.)

(3) Given affine hyperplane L of $\mathbf{R}^{n+1}$ with $L\cap(B_1(0)\times\mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$ and $t \in \mathcal{R}(M, L, \rho)$, and assuming $\hat{E}_M\,(\rho, L) \leq \overline{\epsilon}$ where $\overline{\epsilon} = \epsilon(n) \in (0,1)$ is a sufficiently small fixed constant depending only on n, let $G_M^{(L,\,t)}(\rho)$ denote the graphical part, relative to L, of $M \cap q_L^{-1}\,(B_{3\rho/4}(0)\times\mathbf{R})$ chosen in the sense of [**SS81**]. (See item (2) of the discussion at the beginning of Section 3.) Here q_L denotes a rigid motion of $\mathbf{R}^{n+1}$ with $q_L(a_L) = 0$ and $q_L(L) = \mathbf{R}^n \times \{0\}$ where a_L is the nearest point of L to $0 \in \mathbf{R}^{n+1}$. Thus, for any given radius $\rho \in (0,1]$ and choices of an affine hyperplne L with $L \cap (B_1(0)\times\mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$ and $t \in \mathcal{R}(M, L, \rho)$, provided $\hat{E}_M\,(\rho, L) \leq \overline{\epsilon}$, $G_M^{(L,\,t)}(\rho)$ is uniquely determined, and is the union of two Lipschitz graphs over a domain $\subset L$ with

Lipschitz constants ≤ 1; moreover,

$$\mathcal{H}^n\left((M \setminus G_M^{(L,\,t)}(\rho)) \cap q_L^{-1}\left(B_{3\rho/4}(0) \times \mathbf{R}\right)\right) \leq C(\hat{E}_M(\rho, L))^{2+\mu}, \tag{6.1}$$

where $C = C(n)$ and $\mu = \mu(n)$ are fixed positive constants depending only on n. (However, we remark here that in the proof of Lemma 6.3, we do not need such precise control of the size of the complement of $G_M^{(L,\,t)}(\rho)$ as is given by the estimate (6.1); all we need is that $G_M^{(L,\,t)}(\rho)$ has n-dimensional measure larger than a fixed fraction of the measure of $B_{\rho/2}(0)$. See Lemma 6.1 below.)

(4) Given affine hyperplanes L, U of $\mathbf{R}^{n+1}$ with $L \cap (B_1(0) \times \mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$, $U \cap (B_1(0) \times \mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$ such that $\hat{E}_M(\rho, L) \leq \overline{\epsilon}$ ($\overline{\epsilon}$ as in (3) above), and $t \in \mathcal{R}(M, L, \rho)$, let

$$U^\star(M, L, t, \rho) = U \cap \pi^{-1}(\pi\, G_M^{(L,t)}(\rho)).$$

Given L as above and a pair of affine hyperplanes $P = P_1 \cup P_2$ of $\mathbf{R}^{n+1}$ (with P_1, P_2 affine hyperplanes of $\mathbf{R}^{n+1}$) such that $P \cap (B_1(0) \times \mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$, define

$$P^\star(M, L, t, \rho) = P_1^\star(M, L, t, \rho) \cup P_2^\star(M, L, t, \rho).$$

(5) If $P = P_1 \cup P_2$ is a pair of affine hyperplanes of $\mathbf{R}^{n+1}$ (with P_1, P_2 affine hyperplanes) such that $P \cap (B_1(0) \times \mathbf{R}) \subset \{|x^{n+1}| < 1/8\}$, we set, for $\tau \in (0, 1/2)$, $S_P(\tau) = \{x \in \mathbf{R}^n \times \{0\} : \operatorname{dist}(x, \pi(P_1 \cap P_2)) \leq \tau\}$ if P_1 and P_2 are distinct with $\pi(P_1 \cap P_2) \cap B_{1/8}(0) \neq \emptyset$, and $S_P(\tau) = \emptyset$ otherwise.

(6) If U is an affine hyperplane of $\mathbf{R}^{n+1}$, we shall denote by U^T the hyperplane obtained by translating U parallel to itself. If $P = P_1 \cup P_2$ is a pair of affine hyperplanes, with P_1, P_2 affine hyperplanes, then we shall let $P^T = P_1^T \cup P_2^T$.

Lemma 6.1. *Let $\delta \in (0, 1)$. There exist constants $c_1 = c_1(n, \delta) \in (0, \infty)$, $c_2 = c_2(n, \delta) \in (0, \infty)$ and $\zeta_0 = \zeta_0(n, \delta) \in (0, 1)$ such that the following is true. If $M \in \mathcal{I}_b$, $\frac{\mathcal{H}^n\,(M \cap (B_1(0) \times \mathbf{R}))}{\omega_n} \leq 3 - \delta$, $L \in \mathcal{A}(M, 1)$, $t \in \mathcal{R}(M, L, 1)$, $\hat{E}_M(1, L) \leq 1$, $P = P^+ \cup P^-$ is a pair of affine hyperplanes with* $\operatorname{dist}_{\mathcal{H}}(P \cap (B_1(0) \times \mathbf{R}), B_1(0)) \leq \zeta_0$ *and*

$$\int_{M \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) + \int_{P^\star \cap ((B_{1/2}(0) \setminus S_P(1/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(L,\,t)}(1)) \leq \zeta_0 \hat{E}_M^2(1, L)$$

where $P^\star \equiv P^\star(M, L, t, 1)$, then

$$c_1 \hat{E}_M(1, L) \leq \sup\nolimits_{B_1(0)} |p^+ - p^-| \leq c_2\, \hat{E}_M(1, L).$$

Proof. Note first that it follows from the conditions

$$\operatorname{dist}_{\mathcal{H}}(P \cap (B_1(0) \times \mathbf{R}), B_1(0)) \leq \zeta_0,$$

$$\int_{M \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \leq \zeta_0,$$

and the triangle inequality that $\int_{M\cap(B_1(0)\times\mathbf{R})} |x^{n+1}|^2 \leq C\zeta_0$, where $C = C(n)$, so that by the definition of $\mathcal{A}(M,1)$, we have that

$$\hat{E}_M^2(1, L) \leq \frac{3}{2} C\zeta_0. \tag{6.2}$$

Thus, if $\zeta_0 = \zeta_0(n)$ is sufficiently small, $G_M^{(L,t)}(1) \neq \emptyset$, and in fact by the estimate (6.1),

$$\mathcal{H}^n\,(G_M^{(L,t)}(1) \cap (B_{1/2}(0) \times \mathbf{R})) \geq \frac{1}{2}\omega_n \left(\frac{1}{2}\right)^n. \tag{6.3}$$

To see the lower bound of the asserted inequalities in the conclusion of the lemma, let $U = \operatorname{graph} \frac{1}{2}(p^+ + p^-)$. Then, by the definition of $\hat{E}_M(1, L)$ and the triangle inequality, we have that

$$\begin{aligned} \frac{2}{3}\hat{E}_M^2(1,L) &\leq \int_{M\cap(B_1(0)\times\mathbf{R})} \operatorname{dist}^2(x, U) \\ &\leq 2\int_{M\cap(B_1(0)\times\mathbf{R})} \operatorname{dist}^2(x,P) + c\sup|p^+ - p^-|^2 \end{aligned} \tag{6.4}$$

where $c = c(n)$. Provided we take $\zeta_0 < 1/4$, the lower bound follows directly from this since $\int_{M\cap(B_1(0)\times\mathbf{R})} \operatorname{dist}^2(x,P) \leq \zeta_0 \hat{E}_M^2(1,L)$ by hypothesis.

To see the upper bound, we argue by contradiction. If the assertion is not true, then there exist a sequence of hypersurface $M_k \in \mathcal{I}_b$, $k = 1, 2, 3, \ldots$, with $\frac{\mathcal{H}^n\,(M_k\cap(B_1(0)\times\mathbf{R}))}{\omega_n} \leq 3 - \delta$, a sequence of affine hyperplanes L_k with $L_k \cap (B_1(0) \times \mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$ and

$$\hat{E}_k^2 \equiv \int_{M_k\cap(B_1(0)\times\mathbf{R})} \operatorname{dist}^2(x, L_k) \leq \frac{3}{2}\inf_{L'} \int_{M_k\cap(B_1(0)\times\mathbf{R})} \operatorname{dist}(x, L'), \tag{6.5}$$

where for each k, the inf is taken over all affine hyperplanes L' satisfying $L' \cap (B_1(0) \times \mathbf{R}) \subset \{|x^{n+1}| \leq 1/8\}$, a sequence of numbers $t_k \in \mathcal{R}(M_k, L_k, 1)$, a sequence $P_k = P_k^+ \cup P_k^-$ of pairs of affine hyperplanes with

$$\operatorname{dist}_{\mathcal{H}}(P_k \cap (B_1(0) \times \mathbf{R}), B_1(0)) \to 0 \quad \text{as } k \to \infty \text{ and} \tag{6.6}$$

$$(6.7) \quad \int_{M_k\cap(B_1(0)\times\mathbf{R})} \operatorname{dist}^2(x,P_k) + \int_{P_k^\star\cap((B_{1/2}(0)\setminus S_{P_k}(1/16))\times\mathbf{R})} \operatorname{dist}^2(x,G_k) \leq \frac{1}{k}\hat{E}_k^2\,;$$

and yet, for each k,

$$(6.8) \quad \sup_{B_1(0)} |p_k^+ - p_k^-| \geq k\hat{E}_k.$$

Here we are using the abbreviations $G_k = G_{M_k}^{(L_k,\,t_k)}(1)$ and $P_k^\star = P_k^\star(M_k, L_k, t_k, 1)$. Note then by (6.2), $\hat{E}_k \to 0$, and by (6.6) and (6.7), $M_k\cap(B_{1/2}(0)\times\mathbf{R}) \to B_{1/2}(0)\times\{0\}$ in Hausdorff distance. Consequently, $L_k \to \mathbf{R}^n \times \{0\}$. Note also that by (6.3),

$$(6.9) \quad \mathcal{H}^n\left(G_k \cap (B_{1/2}(0)\times\mathbf{R})\right) \geq \frac{1}{2}\omega_n\left(\frac{1}{2}\right)^n$$

for all sufficiently large k. Let $v \in L^2(B_1(0);\mathbf{R}^2)\cap W^{1,2}_{\mathrm{loc}}(B_1(0);\mathbf{R}^2)$ be the blow-up, in the sense of Section 3, of M_k by $\hat{E}_k$. In view of Proposition 3.3, part (2), it follows from (the bound on the first term on the left hand side of) (6.7) and (6.8) that $v^+ \equiv v^- \equiv l$ for some affine function l. Indeed, if we write $P_k = P_k^{(1)} \cup P_k^{(2)}$ where $P_k^{(1)}, P_k^{(2)}$ are affine hyperplanes, and define functions $p_k^{(1)}, p_k^{(2)} : \mathbf{R}^n\times\{0\} \to \mathbf{R}$ by $P_k^{(i)} = \operatorname{graph} p_k^{(i)}$, $i = 1, 2$, then, after possibly passing to a subsequence, $l = \lim_{k\to\infty}(p_k^{(1)} - \phi_k)/\hat{E}_k$ or $l = \lim_{k\to\infty}(p_k^{(2)} - \phi_k)/\hat{E}_k$ where $\phi_k : \mathbf{R}^n\times\{0\} \to \mathbf{R}$ is such that $L_k = \operatorname{graph}\phi_k$. (The existence of one of these two limits is guaranteed by Lemma 3.3, part (2) and the bound on the first term on the left hand side of (6.7).) By relabeling if necessary, we assume that $l = \lim_{k\to\infty}(p_k^{(1)} - \phi_k)/\hat{E}_k$. Note then that by (6.8),

$$(6.10) \quad \lim_{k\to\infty}\sup_{B_1(0)} \frac{|p_k^{(2)} - \phi_k|}{\hat{E}_k} = \infty$$

and that (6.7) in particular says that

$$(6.11) \quad \int_{P_k^{(2)\,\star}\cap((B_{1/2}(0)\setminus S_{P_k}(1/16))\times\mathbf{R})} \operatorname{dist}^2(x, G_k) \leq \frac{1}{k}\hat{E}_k^2.$$

If we let $\widetilde{L}_k = \operatorname{graph}(\phi_k + \hat{E}_k l)$, we have

$$(6.12) \quad \int_{M_k\cap(B_{1/2}(0)\times\mathbf{R})} \operatorname{dist}^2(x, \widetilde{L}_k) \leq \frac{1}{16}\hat{E}_k^2$$

for infinitely many k, which implies by the triangle inequality that

$$(6.13) \quad \int_{G_k\cap(B_{1/2}(0)\times\mathbf{R})} \operatorname{dist}^2(x, P_k^{(1)}) \leq \frac{1}{6}\hat{E}_k^2$$

for infinitely many k. Now let $\widetilde{G}_k = \{x \in G_k \cap (B_{1/2}(0) \times \mathbf{R}) : \operatorname{dist}(x, P_k^{(1)}) \leq \sqrt{\frac{2^{n-1}}{\omega_n} \hat{E}_k}\}$. Then by (6.13) and (6.9),

$$\mathcal{H}^n(\widetilde{G}_k) \geq \frac{1}{6}\omega_n \left(\frac{1}{2}\right)^n. \tag{6.14}$$

Since G_k is the union of two Lipschitz graphs with Lipschitz constants ≤ 1, for any $x = (x', x^{n+1}) \in \pi\, G_k \times \mathbf{R}$, $\operatorname{dist}(x, G_k)$ is bounded below by a fixed positive constant times the "vertical distance" $\min\{|x^{n+1} - y_1^{n+1}|, |x^{n+1} - y_2^{n+1}| : (x', y_1^{n+1}), (x', y_2^{n+1}) \in G_k\}$. Moreover, by (6.14), $\mathcal{H}^n(P_k^{(2)\star} \cap ((B_{1/2}(0) \setminus S_{P_k}(1/16)) \times \mathbf{R})) \geq C = C(n) > 0$ which, in view of (6.8), contradicts (6.11). This completes the proof of the lemma. q.e.d.

Lemma 6.2. *Let $\delta \in (0,1)$, $\eta \in (0,1)$ and $c_1 \in (0, \infty)$ be given. There exists a number $\zeta = \zeta(n, \delta, \eta, c_1) \in (0,1)$ such that the following holds. If $P = P^+ \cup P^-$ is a pair of affine hyperplanes of $\mathbf{R}^{n+1}$ with $\sup_{B_1(0)} |p^+ - p^-| \geq c_1$, and $v = (v^+, v^-) \in \mathcal{F}_\delta$ satisfies*

$$\int_{B_1(0)} \operatorname{dist}^2((x, v^+(x)), P) + \operatorname{dist}^2((x, v^-(x)), P)$$
$$+ \int_{B_{1/2}(0) \setminus S_P(1/8)} \operatorname{dist}^2((x, p^+(x)), V) + \operatorname{dist}^2((x, p^-(x)), V) \leq \zeta$$

where $V = \operatorname{graph} v^+ \cup \operatorname{graph} v^-$, then

$$\int_{B_1(0)} (v^+ - p^+)^2 + (v^- - p^-)^2 \leq \eta.$$

Proof. If the assertion is false, then there exist numbers $\delta \in (0,1)$, $\eta \in (0,1)$, $c_1 \in (0, \infty)$, a sequence of functions $v_k = (v_k^+, v_k^-) \in \mathcal{F}_\delta$ and a sequence of affine hyperplanes $P_k = (P_k^+, P_k^-)$ of $\mathbf{R}^{n+1}$ such that

$$\sup_{B_1(0)} |p_k^+ - p_k^-| \geq c_1 \qquad \text{and} \tag{6.15}$$

(6.16)
$$\int_{B_1(0)} \operatorname{dist}^2((x, v_k^+(x)), P_k) + \operatorname{dist}^2((x, v_k^-(x)), P_k)$$
$$+ \int_{B_{1/2}(0) \setminus S_{P_k}(1/8)} \operatorname{dist}^2((x, p_k^+(x)), V_k) + \operatorname{dist}^2((x, p_k^-(x)), V_k) \leq \frac{1}{k}$$

where $V_k = \operatorname{graph} v_k^+ \cup \operatorname{graph} v_k^-$, and yet

$$\int_{B_1(0)} (v_k^+ - p_k^+)^2 + (v_k^- - p_k^-)^2 \geq \eta \tag{6.17}$$

for all $k = 1, 2, 3, \ldots$. After passing to a subsequence, we have by Lemma 3.1 that $v_k \to v$ for some $v \in \mathcal{F}_\delta$, where the convergence is in $W^{1,2}(B_1(0);\mathbf{R}^2)$, and that $P_k \to P$ for some affine pair of hyperplanes of $\mathbf{R}^{n+1}$ satisfying $\sup_{B_1(0)} |p^+ - p^-| \geq c_1$. Note that since $v^\pm$ are bounded in $B_1(0)$ (by Proposition 3.3; part (3) says $|v|^2$ is subharmonic in $B_{3/2}(0)$, and the mean value property and part (2) say $|v|^2$ is bounded in $B_1(0)$) and continuous (by Proposition 3.10), (6.16) says that $v^+ \equiv p^+$ and $v^- \equiv p^-$ on $B_1(0)$. This immediately contradicts (6.17) for sufficiently large k. q.e.d.

Lemma 6.3. *Let $\theta \in (0, 1/16)$, $\beta \in (0, \theta/16)$ and $\gamma \in (0, \beta/16)$. Let $\delta \in (0,1)$. There exist numbers $\epsilon_0 = \epsilon_0(n, \delta, \theta, \beta, \gamma) \in (0, 1/2)$ and $\lambda = \lambda(n, \delta) \in (0,1)$ such that the following is true. Suppose $M \in \mathcal{I}_b$, $0 \in \overline{M}$, $\rho \in (0, 1]$,*

$$\frac{\mathcal{H}^n(M \cap (B_\rho(0) \times \mathbf{R}))}{\omega_n \rho^n} \leq 3 - \delta \qquad \textit{and}$$

$$\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) + \rho^{-n-2} \int_{P^\star \cap ((B_{\rho/2}(0) \setminus S_P(\rho/16)) \times \mathbf{R})} \operatorname{dist}^2\left(x, G_M^{(L,t)}(\rho)\right) \leq \epsilon_0$$

for some affine hyperplane $L \in \mathcal{A}(M, \rho)$, number $t \in \mathcal{R}(M, L, \rho)$ and some pair of affine hyperplanes P of $\mathbf{R}^{n+1}$ satisfying $\operatorname{dist}_{\mathcal{H}}(P \cap (B_1(0) \times \mathbf{R}), B_1(0)) \leq \epsilon_0$. Here we have used the notation $P^\star \equiv P^\star(M, L, t, \rho)$. Then there exists a pair of affine hyperplanes $\widetilde{P}$, an affine hyperplane $\widetilde{L}$ and a number $\widetilde{t} \in (1/4, 1/2)$ such that

(1)
$$\rho^{-2} d_{\mathcal{H}}^2(\widetilde{P} \cap (B_\rho(0) \times \mathbf{R}), P \cap (B_\rho(0) \times \mathbf{R})) \leq C\bigg(\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) + \rho^{-n-2} \int_{P^\star \cap ((B_{\rho/2}(0) \setminus S_P(\rho/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(L,t)}(\rho))\bigg),$$

(2)
$$d_{\mathcal{H}}^2(\widetilde{P}^T \cap (B_1(0) \times \mathbf{R}), P^T \cap (B_1(0) \times \mathbf{R})) \leq C\bigg(\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) + \rho^{-n-2} \int_{P^\star \cap ((B_{\rho/2}(0) \setminus S_P(\rho/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(L,t)}(\rho))\bigg)$$

and

(3) *one of the following options (A), (B) or (C) holds:*

(A) $\widetilde{L} \in \mathcal{A}(M, \theta\rho)$, $\widetilde{t} \in \mathcal{R}(M, \widetilde{L}, \theta\rho)$,

$$(\theta\rho)^{-n-2} \int_{M \cap (B_{\theta\rho}(0) \times \mathbf{R})} \operatorname{dist}^2 (x, \widetilde{P}) + (\theta\rho)^{-n-2} \int_{\widetilde{P}^\star \cap ((B_{\theta\rho/2}(0) \setminus S_{\widetilde{P}}(\theta\rho/16)) \times \mathbf{R})} \operatorname{dist}^2 (x, G_M^{(\widetilde{L}, \widetilde{t})}(\theta\rho))$$
$$\leq C_1 \theta^\lambda \Big(\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2 (x, P) + \rho^{-n-2} \int_{P^\star \cap ((B_{\rho/2}(0) \setminus S_P(\rho/16)) \times \mathbf{R})} \operatorname{dist}^2 (x, G_M^{(L,\tau)}(\rho)) \Big),$$

where $\widetilde{P}^\star \equiv \widetilde{P}^\star(M, \widetilde{L}, \widetilde{t}, \theta\rho)$, *and*

$$\frac{\mathcal{H}^n(M \cap (B_{\theta\rho}(0) \times \mathbf{R}))}{\omega_n (\theta\rho)^n} \leq 3 - \delta.$$

(B) $\widetilde{L} \in \mathcal{A}(M, \beta\rho)$, $\widetilde{t} \in \mathcal{R}(M, \widetilde{L}, \beta\rho)$,

$$(\beta\rho)^{-n-2} \int_{M \cap (B_{\beta\rho}(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}) + (\beta\rho)^{-n-2} \int_{\widetilde{P}^\star \cap ((B_{\beta\rho/2}(0) \setminus S_{\widetilde{P}}(\beta\rho/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(\widetilde{L}, \widetilde{t})}(\beta\rho))$$
$$\leq C_2 \beta^\lambda \Big(\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) + \rho^{-n-2} \int_{P^\star \cap ((B_{\rho/2}(0) \setminus S_P(\rho/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(L,t)}(\rho)) \Big),$$

where $\widetilde{P}^\star \equiv \widetilde{P}^\star(M, \widetilde{L}, \widetilde{t}, \beta\rho)$, *and*

$$\frac{\mathcal{H}^n(M \cap (B_{\beta\rho}(0) \times \mathbf{R}))}{\omega_n (\beta\rho)^n} \leq 3 - \delta.$$

(C) $\widetilde{L} \in \mathcal{A}(M, \gamma\rho)$, $\widetilde{t} \in \mathcal{R}(M, \widetilde{L}, \gamma\rho)$,

$$(\gamma\rho)^{-n-2} \int_{M \cap (B_{\gamma\rho}(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}) + (\gamma\rho)^{-n-2} \int_{\widetilde{P}^\star \cap ((B_{\gamma\rho/2}(0) \setminus S_{\widetilde{P}}(\gamma\rho/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(\widetilde{L}, \widetilde{t})}(\gamma\rho))$$
$$\leq C_3 \gamma^\lambda \Big(\rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) + \rho^{-n-2} \int_{P^\star \cap ((B_{\rho/2}(0) \setminus S_P(\rho/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_M^{(L,t)}(\rho)) \Big),$$

where $\widetilde{P}^\star \equiv \widetilde{P}^\star(M, \widetilde{L}, \widetilde{t}, \gamma\rho)$, *and*

$$\frac{\mathcal{H}^n(M \cap (B_{\gamma\rho}(0) \times \mathbf{R}))}{\omega_n (\gamma\rho)^n} \leq 3 - \delta.$$

Here the dependence of the constants C, C_i, $i = 1, 2, 3$ *on the parameters is as follows:* $C = C(n, \delta, \theta, \beta, \gamma)$, $C_1 = C_1(n, \delta)$, $C_2 = C_2(n, \delta, \theta)$ *and* $C_3 = C_3(n, \delta, \theta, \beta)$.

Proof. Note first that conclusion (2) follows from conclusion (1). Since the hypotheses and the conclusions of the lemma are scale invariant, it suffices to prove the lemma assuming $\rho = 1$, and we shall make this assumption in what follows. Let $\{M_k\} \subset \mathcal{I}_b$ be an arbitrary sequence of hypersurfaces with $0 \in \overline{M}_k$,

$$\frac{\mathcal{H}^n(M_k \cap (B_1(0) \times \mathbf{R}))}{\omega_n} \leq 3 - \delta, \tag{6.18}$$

(6.19)
$$\int_{M_k \cap (B_1(0) \times \mathbf{R})} \mathrm{dist}^2\,(x, P_k) + \int_{P_k^\star \cap ((B_{1/2}(0) \setminus S_{P_k}(1/16)) \times \mathbf{R})} \mathrm{dist}^2\,(x, G_k) \searrow 0$$

for a sequence of affine hyperplanes $L_k \in \mathcal{A}(M_k, 1)$, a sequence of numbers $t_k \in \mathcal{R}(M_k, L_k, 1)$ and a sequence of pairs of affine hyperplanes $P_k = P_k^1 \cup P_k^2$ (where P_k^1, P_k^2 are affine hyperplanes, possibly with $P_k^1 \equiv P_k^2$), satisfying

$$d_{\mathcal{H}}\,(P_k \cap (B_1(0) \times \mathbf{R}), B_1(0)) \searrow 0. \tag{6.20}$$

Here we use the notation $G_k \equiv G_{M_k}^{(L_k, t_k)}(1)$ and $P_k^\star \equiv P_k^\star(M_k, L_k, t_k, 1)$. Note that (6.19) and (6.20) imply that $M_k \cap (B_{1/2}(0) \times \mathbf{R}) \to B_{1/2}(0) \times \{0\}$ in Hausdorff distance and that $\int_{M_k \cap (B_1(0) \times \mathbf{R})} |x^{n+1}|^2 \to 0$. By the definition of $\mathcal{A}(M_k, 1)$, it then follows that

$$\hat{E}_k \to 0, \tag{6.21}$$

where we use the notation $\hat{E}_k \equiv \hat{E}_{M_k}\,(1, L_k)$. This in turn says that $\mathrm{dist}_{\mathcal{H}}\,(M_k \cap (B_{1/2}(0) \times \mathbf{R}), L_k \cap (B_{1/2}(0) \times \mathbf{R})) \to 0$, so that $L_k \to \mathbf{R}^n \times \{0\}$. Note also that (6.21) in particular implies that for all sufficiently large k,

$$\mathcal{H}^n\,(G_k) \geq \frac{1}{2}\omega_n \left(\frac{1}{2}\right)^n \tag{6.22}$$

and hence that

$$\mathcal{H}^n\,(P_k^\star \cap (B_{1/2}(0) \setminus S_{P_k}(1/16)) \times \mathbf{R})) \geq \frac{1}{4}\omega_n \left(\frac{1}{2}\right)^n. \tag{6.23}$$

We show that for infinitely many k, we can find pairs of affine hyperplanes $\widetilde{P}_k$, affine hyperplanes $\widetilde{L}_k$ and numbers $\widetilde{t}_k \in (1/4, 1/2)$ such that the conclusions of the lemma hold with $M_k, P_k, L_k, t_k, \widetilde{P}_k, \widetilde{L}_k, \widetilde{t}_k$ in place of M, P, L, t, $\widetilde{P}$, $\widetilde{L}$ and $\widetilde{t}$ respectively, and with the constants C, C_i, $i = 1, 2, 3$ and λ fixed depending only on the specified parameters as in the statement of the lemma. In view of the arbitrariness of $\{M_k\}$, this will prove the lemma.

First notice that for any given $\tau \in (0, 1/2)$ we have, since $\hat{E}_k \to 0$, that for all sufficiently large k depending on τ, $\mathcal{H}^n\,(G_k \cap (B_\tau(0) \times \mathbf{R})) \to 2\omega_n \tau^n$ and $\mathcal{H}^n\,((M_k \setminus G_k) \cap (B_\tau(0) \times \mathbf{R})) \to 0$, so that the last of the conclusions in each of the options $(3)(A)$, $(3)(B)$ and $(3)(C)$ hold with M_k in place of M for all sufficiently large k. It only remains to show that the other conclusions hold with M_k, P_k, L_k, t_k in place of M, P, L, t respectively and with suitable choices of $\widetilde{P}_k$, $\widetilde{L}_k$ and $\widetilde{t}_k$, in place of $\widetilde{P}$, $\widetilde{L}$ and $\widetilde{t}$ respectively.

Let $\zeta = \zeta(n, \theta, \delta) \in (0, 1/8)$ be a small number to be determined depending only on n, θ and δ. We divide the rest of the proof of the lemma into two cases according to the following two possibilities, one of which must hold for infinitely many k:

(a) $\int_{M_k \cap (B_1(0) \times \mathbf{R})} \mathrm{dist}^2\,(x, P_k) + \int_{P_k^\star \cap ((B_{1/2}(0) \setminus S_{P_k}(1/16)) \times \mathbf{R})} \mathrm{dist}^2\,(x, G_k) < \zeta\,\hat{E}_k^2$.

(b) $\int_{M_k \cap (B_1(0) \times \mathbf{R})} \mathrm{dist}^2\,(x, P_k) + \int_{P_k^\star \cap ((B_{1/2}(0) \setminus S_{P_k}(1/16)) \times \mathbf{R})} \mathrm{dist}^2\,(x, G_k) \geq \zeta\,\hat{E}_k^2$.

Suppose first that possibility (a) occurs. By Lemma 6.1, provided we choose $\zeta \leq \zeta_0$, where $\zeta_0 = \zeta_0(n, \delta)$ is as in Lemma 6.1, we have in this case that

$$P_k = \mathrm{graph}\,\hat{E}_k p_k^{0\,+} \cup \mathrm{graph}\,\hat{E}_k p_k^{0\,-} \tag{6.24}$$

for infinitely many k, with $P_k^0 = P_k^{0\,+} \cup P_k^{0\,-}$ ($P_k^{0\,\pm} = \mathrm{graph}\,p_k^{0\,\pm}$) equal to a pair of affine hyperplanes satisfying

$$c_1 \leq \sup\nolimits_{B_1(0)} |p_k^{0\,+} - p_k^{0\,-}| \leq c_2, \tag{6.25}$$

where $c_1 = c_1(n, \delta)$, $c_2 = c_2(n, \delta)$ are the positive constants given by Lemma 6.1. Note that (6.24) and (6.25) say that the blow-up by $\hat{E}_k$ of a subsequence of the sequence $\{P_k\}$ is a transverse pair of planes. So let $P^0 = P^{0\,+} \cup P^{0\,-}$ be a subsequential limit of $\{P_k^0\}$ and consider the blow-up $v = (v^+, v^-)$ of M_k by $\hat{E}_k$. We have directly from the defining condition of case (a) and the identity (5.4) that

(6.26)
$$\int_{B_{2/3}(0)} \mathrm{dist}^2((x, v^+(x)), P^0) + \mathrm{dist}^2((x, v^-(x)), P^0) + \int_{B_{1/2}(0) \setminus S_{P^0}(1/8)} \mathrm{dist}^2((x, p^{0\,+}(x)), V) + \mathrm{dist}^2((x, p^{0\,-}(x)), V) \leq C\zeta$$

where $V = \operatorname{graph} v^+ \cup \operatorname{graph} v^-$ and $C = C(n, \delta)$. In view of the lower bound of (6.25), we then have by Lemma 6.2 that for any given $\eta \in (0,1)$,

$$\int_{B_{2/3}(0)} (v^+ - p^{0\,+})^2 + (v^- - p^{0\,-})^2 \leq \eta \tag{6.27}$$

provided $\zeta = \zeta(n, \delta, \eta) \in (0,1)$ is sufficiently small.

We now separate the analysis of case (a) into two further possibilities depending on the nature of P^0. Precisely one of the following must hold:

(a)(i) $P^{0\,+} \cap P^{0\,-} \cap (B_\theta(0) \times \mathbf{R}) = \emptyset$ or

(a)(ii) $P^{0\,+} \cap P^{0\,-} \cap (B_\theta(0) \times \mathbf{R}) \neq \emptyset$.

Suppose first that (a)(i) holds. Taking $\eta = \eta(n, \delta, \theta) > 0$ in (6.27) sufficiently small, we see by the estimate of Proposition 3.10, part (b) and the fact that $P^{0\,+} \cap P^{0\,-} \cap (B_\theta(0) \times \mathbf{R}) = \emptyset$ that (6.27) implies, provided only that $\zeta = \zeta(n, \delta, \theta)$ is chosen sufficiently small, that we have $Z_w \cap B_{3\theta/4}(0) = \emptyset$, where $w = \frac{1}{2}(v^+ - v^-)$ and Z_w is the zero set of w. By the remark following Lemma 3.9, this means that $M_k \cap (B_{\theta/2}(0) \times \mathbf{R})$ are embedded for all sufficiently large k, and hence by Schoen-Simon regularity theorem ([**SS81**], Theorem 1), $M_k \cap (B_{\theta/4}(0) \times \mathbf{R})$ decomposes as the disjoint union of minimal graphs $\mathcal{U}_k^{(1)}, \mathcal{U}_k^{(2)}$ (over the affine hyperplanes P_k^1 and P_k^2). By standard elliptic estimates, we then have that

$$\begin{aligned}(\sigma\theta)^{-n-2} &\int_{M_k \cap (B_{\sigma\theta}(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}_k) \\ &\leq C\sigma^2\theta^{-n-2} \int_{M_k \cap (B_\theta(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k)\end{aligned} \tag{6.28}$$

for all $\sigma \in (0, 1/4)$, where $C = C(n)$ and $\widetilde{P}_k$ is the union of the tangent planes $\widetilde{P}_k^1, \widetilde{P}_k^2$ to $\mathcal{U}_k^{(1)}, \mathcal{U}_k^{(2)}$ respectively at points $Z_k^{(1)} \in \mathcal{U}_k^{(1)}$, $Z_k^{(2)} \in \mathcal{U}_k^{(2)}$ with $\pi(Z_k^{(i)}) = 0$ for $i = 1, 2$. Taking $\sigma = \beta/\theta$ in this, we conclude that

$$\beta^{-n-2} \int_{M_k \cap (B_\beta(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}_k) \leq C_2\beta^2 \int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k) \tag{6.29}$$

where $C_2 = C_2(n, \theta)$. Note that by the definition of $\widetilde{P}_k^i$ and elliptic estimates again, it follows that $(\theta/8)^{-2}\operatorname{dist}^2_{\mathcal{H}}(\widetilde{P}_k \cap (B_{\theta/8}(0) \times \mathbf{R}), P_k \cap (B_{\theta/8}(0) \times \mathbf{R})) \leq C\theta^{-n-2} \int_{M_k \cap (B_{\theta/4}(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k)$ where $C = C(n)$, which implies that $\operatorname{dist}^2_{\mathcal{H}}(\widetilde{P}_k \cap (B_1(0) \times \mathbf{R}),\ P_k \cap (B_1(0) \times \mathbf{R})) \leq C \int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k)$ where $C = C(n, \theta)$.

Now for each k, take any $\widetilde{L}_k \in \mathcal{A}(M_k, \beta)$ and any $\widetilde{t}_k \in \mathcal{R}(M_k, \widetilde{L}_k, \beta)$. Since $\beta^{-n-2} \int_{M_k \cap (B_\beta(0) \times \mathbf{R})} |x^{n+1}|^2 \to 0$ as $k \to \infty$ (by Hausdorff convergence), it follows from the definition of $\mathcal{A}(M_k, \beta)$ that $\hat{E}_{M_k}(\beta, \widetilde{L}_k) \to 0$

as $k \to \infty$, which in turn implies that $\operatorname{dist}_{\mathcal{H}} (\widetilde{L}_k \cap (B_1(0) \times \mathbf{R}), B_1(0) \times \{0\}) \to 0$ as $k \to \infty$. Thus, we have in the present case (i.e., case (a)(i)) that $G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\beta) \cap (B_\beta(0) \times \mathbf{R}) = (\mathcal{U}_k^{(1)} \cup \mathcal{U}_k^{(2)}) \cap (B_\beta(0) \times \mathbf{R})$. If we write $\mathcal{U}_k^{(i)} \cap (B_\beta(0) \times \mathbf{R}) = \operatorname{graph} \widetilde{u}_k^i$, where $\widetilde{u}_k^i : B_\beta(0) \to \mathbf{R}$, we have by (6.27) that provided $\zeta = \zeta(n, \delta, \theta)$ is sufficiently small, for each $x = (x', x^{n+1}) \in (\widetilde{P}_k^i)^\star$ $(= \widetilde{P}_k^i \cap (B_\beta(0) \times \mathbf{R}))$,

$$\begin{aligned} \operatorname{dist}(x, G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\beta)) \leq |x^{n+1} - \widetilde{u}_k^i(x')| &\leq 2 \operatorname{dist}((x', u_k^i(x')), \widetilde{P}_k^i) \\ &= 2\operatorname{dist}((x', u_k^i(x')), \widetilde{P}_k) \end{aligned} \tag{6.30}$$

for $i = 1, 2$. This implies that

(6.31)
$$\int_{\widetilde{P}_k^\star \cap (B_\beta(0) \times \mathbf{R})} \operatorname{dist}^2(x, G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\beta)) \leq 4 \int_{M_k \cap (B_\beta(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}_k).$$

Thus, we conclude in case (a)(i) that for infinitely many k, the conclusions of the lemma hold with option (3)(B), with M_k, P_k, L_k, t_k, $\widetilde{P}_k$, $\widetilde{L}_k$ and $\widetilde{t}_k$ in place of M, P, L, t, $\widetilde{P}$, $\widetilde{L}$ and $\widetilde{t}$ respectively, and with $\lambda = 2$.

If (a)(ii) holds for infinitely many k, then we have

$$\int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k) < \zeta\, \hat{E}_k^2$$

for infinitely many k, where P_k is as in (6.24) with $P_k^0 = \operatorname{graph} p_k^{0\,+} \cup \operatorname{graph} p_k^{0\,-}$ equal to a transverse pair of affine hyperplanes satisfying (6.25) and $P_k^{0\,+} \cap P_k^{0\,-} \cap (B_{3\theta/2}(0) \times \mathbf{R}) \neq \emptyset$. Thus $\pi - \overline{\alpha} > \angle P_k^0 > \overline{\alpha}$ for some fixed angle $\overline{\alpha} = \overline{\alpha}(n, \delta) \in (0, \pi)$. Let $\tau \in (0, 1)$ be arbitrary for the moment. Choosing the constant $\zeta = \zeta(n, \theta, \delta, \tau) > 0$ so that, in addition to the restrictions already imposed upon ζ, we also have

$$\zeta \leq \left(\frac{2}{3}\right)^{n+2} \epsilon_0 \tag{6.32}$$

where $\epsilon_0 = \epsilon_0(n, \delta, \overline{\alpha}, 6\theta, \tau)$ is as in Lemma 4.1, we have by Lemma 4.1 (with $\overline{\alpha}$ in place of α_0, 6θ in place of θ and $\eta_{0,\,2/3}\, M_k$ in place of M) that for infinitely many k, either there exists a pair of hyperplanes $\widetilde{P}_k$ with

$$d_{\mathcal{H}}(\widetilde{P}_k \cap (B_1(0) \times \mathbf{R}), P_k \cap (B_1(0) \times \mathbf{R}) \leq C \int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k)$$

satisfying

(6.33)
$$\theta^{-n-2} \int_{M_k \cap (B_\theta(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}_k) \leq C\theta^2 \int_{M_k \cap (B_1(0) \times \mathbf{R})} \operatorname{dist}^2(x, P_k),$$

where $C = C(n)$, or that

$$\int_{M_k \cap (B_{1/2}(0) \times \mathbf{R})} \operatorname{dist}^2(x, L_k') \leq \tau \hat{E}_k^2 \tag{6.34}$$

for some affine hyperplane L_k' with $d_{\mathcal{H}}(L_k' \cap (B_1(0) \times \mathbf{R}), L_k \cap (B_1(0) \times \mathbf{R})) \leq C\hat{E}_k$, $C = Cn)$. However, if (6.34) holds for infinitely many k, we must have that

$$\int_{B_{1/2}(0)} (v^+ - \ell')^2 + (v^- - \ell')^2 \leq \tau \tag{6.35}$$

for some affine function $\ell' : \mathbf{R}^n \times \{0\} \to \mathbf{R}$, which contradicts (6.27) provided we choose $\eta = \eta(c_1) \in (0,1)$ and $\tau = \tau(c_1) \in (0,1)$ sufficiently small depending only on c_1 (hence only on n and δ), where c_1 is as in (6.25). Thus, provided $\zeta = \zeta(n, \theta, \delta) \in (0,1)$ is sufficiently small depending only on n, θ and δ, we must have the option (6.33) for infinitely many k.

Next in this case, we check that

$$\begin{aligned} &\int_{\widetilde{P}_k^\star \cap ((B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16)) \times \mathbf{R})} \operatorname{dist}^2(x, G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\theta)) \\ &\quad \leq 4 \int_{M_k \cap (B_\theta(0) \times \mathbf{R})} \operatorname{dist}^2(x, \widetilde{P}_k) \end{aligned} \tag{6.36}$$

for arbitrary choices of $\widetilde{L}_k \in \mathcal{A}(M_k, \theta)$ and $\widetilde{t}_k \in \mathcal{R}(M_k, \widetilde{L}_k, \theta)$. Reasoning as in case (a)(i) (see paragraph preceding inequalities (6.30)), we see that $\hat{E}_{M_k}(\theta, \widetilde{L}_k) \to 0$ as $k \to \infty$, and by Lemma 4.1, part (b)(iii), that

$$G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\theta) \cap ((B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16)) \times \mathbf{R}) = \operatorname{graph} u_k^+ \cup \operatorname{graph} u_k^-$$

where $u_k^\pm \in C^2(B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16))$ (in fact $u_k^\pm$ solve the minimal surface equation), $u_k^+ > u_k^-$, and $\operatorname{dist}((x', u_k^\pm(x')), \widetilde{P}_k) = \operatorname{dist}((x', u_k^\pm(x')), \widetilde{P}_k^\pm) \geq \frac{1}{2}|u_k^\pm(x') - \widetilde{p}_k^\pm(x')|$ for every $x' \in B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16)$, where $\widetilde{P}_k^\pm = \operatorname{graph} \widetilde{p}_k^\pm$. Hence we have in this case for any $x = (x', x^{n+1}) \in \widetilde{P}_k^\star \cap ((B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16)) \times \mathbf{R})(= \widetilde{P}_k \cap ((B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16) \times \mathbf{R})$, provided $\zeta = \zeta(n, \theta, \delta) \in (0,1)$ is chosen sufficiently small (so as to ensure that $\operatorname{dist}(x, \operatorname{graph} u_k^\pm) \leq \operatorname{dist}(x, \operatorname{graph} u_k^\mp)$ whenever $x \in \widetilde{P}_k^{\star\,\pm} \cap ((B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16)) \times \mathbf{R})$), that

$$\operatorname{dist}(x, G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\theta)) \leq 2\operatorname{dist}((x', u_k^\pm(x')), \widetilde{P}_k) \tag{6.37}$$

for $x \in \widetilde{P}_k^\star \cap ((B_{\theta/2}(0) \setminus S_{\widetilde{P}_k}(\theta/16)) \times \mathbf{R})$, where the sign $\pm$ is chosen according to whether $x \in \widetilde{P}_k^\pm$. This of course implies (6.36). We thus have in case (a)(ii), for infinitely many k, the conclusions of the lemma with option (3)(A), with M_k, P_k, L_k, t_k, $\widetilde{P}_k$, $\widetilde{L}_k$, $\widetilde{t}_k$ in place of M, P, L, t, $\widetilde{P}$, $\widetilde{L}$, $\widetilde{t}$ respectively and with $\lambda = 2$.

It now remains to analyze possibility (b). We shall take $\zeta = \zeta(n, \theta, \delta) \in (0, 1)$ to be fixed (chosen as specified above) for the remainder of the proof. If possibility (b) holds for infinitely many k, consider the blow-up $v = (v^+, v^-)$ of $\{M_k\}$ by the excess $\hat{E}_k$ off L_k, as described in Section 3. (To be precise, since the excess $\hat{E}_k$ is at scale 1 here, we are in fact applying the analysis of Section 3 with $\eta_{0,\,2/3}\, M_k$ in place of M_k.) Thus $v^+, v^- \in L^2(B_1(0)) \cap W^{1,\,2}_{\text{loc}}\,(B_1(0))$ satisfy the asymptotic decay properties as given by Theorem 5.1. Let $w = \frac{1}{2}(v^+ - v^-)$, and Z_w be the zero set of w. One of the following 2 possibilities must occur:

(b)(i) v has no branch point in $B_{2\beta}(0)$.
(b)(ii) v has a branch point $z \in B_{2\beta}(0)$.

If (b)(i) occurs, then the union of the graphs of v^+, v^- over $B_{2\beta}(0)$ is, locally near every point of $B_{2\beta}(0)$, the union of the graphs of two harmonic functions. Hence, since $B_{2\beta}(0)$ is simply connected, the union of the graphs of v^+, v^- over $B_{2\beta}(0)$ is globally the union of the graphs of two harmonic functions v^1, v^2 : $B_{2\beta}(0) \to \mathbf{R}$. Let l^i, $i = 1, 2$ be the affine part of the Taylor series of v^i around 0 (i.e., $l^i(x) = v^i(0) + x \cdot Dv^i(0)$ for $x \in B_{2\beta}(0)$), let $P_k^{(i)} = \text{graph}\,(\varphi_k + \hat{E}_k l^i)$ where $L_k = \text{graph}\,\varphi_k$ and set $\widetilde{P}_k = P_k^{(1)} \cup P_k^{(2)}$. Then

$$
\begin{aligned}
(6.38)\qquad & \gamma^{-n-2} \int_{M_k \cap (B_\gamma(0) \times \mathbf{R})} \text{dist}^2\,(X, \widetilde{P}_k) \\
&= \gamma^{-n-2} \int_{G_k^+ \cap (B_\gamma(0) \times \mathbf{R})} \text{dist}^2\,(X, \widetilde{P}_k) \\
&\quad + \gamma^{-n-2} \int_{G_k^- \cap (B_\gamma(0) \times \mathbf{R})} \text{dist}^2\,(X, \widetilde{P}_k) \\
&\quad + \gamma^{-n-2} \int_{(\eta_{0,\,2/3}\, M_k \setminus (G_k^+ \cup G_k^-)) \cap (B_\gamma(0) \times \mathbf{R})} \text{dist}^2\,(X, \widetilde{P}_k) \\
&\leq c\, \gamma^{-n-2} \int_{B_\gamma(0)} \text{dist}^2\,((x, \varphi_k(x) + \overline{\psi}_k u_k^+(x)), \widetilde{P}_k) \\
&\quad + c\gamma^{-n-2} \int_{B_\gamma(0)} \text{dist}^2\,((x, \varphi_k(x) + \overline{\psi}_k u_k^-(x)), \widetilde{P}_k) \\
&\quad + c\, \gamma^{-n-2} \hat{E}_k^{2+\mu} \\
&\leq c\, \gamma^{-n-2} \int_{B_\gamma(0)} (\overline{\psi}_k u_k^+ - \hat{E}_k v^+)^2 \\
&\quad + c\, \gamma^{-n-2} \int_{B_\gamma(0)} (\overline{\psi}_k u_k^- - \hat{E}_k v^-)^2
\end{aligned}
$$

$$
\begin{aligned}
&+ c\,\gamma^{-n-2}\int_{B_\gamma(0)} \operatorname{dist}^2\left((x,\varphi_k(x)+\hat{E}_k v^+(x)),\widetilde{P}_k\right)\\
&+ c\,\gamma^{-n-2}\int_{B_\gamma(0)} \operatorname{dist}^2\left((x,\varphi_k(x)+\hat{E}_k v^-(x)),\widetilde{P}_k\right)\\
&+ c\,\gamma^{-n-2}\hat{E}_k^{2+\mu}\\
&= c\,\gamma^{-n-2}\int_{B_\gamma(0)} (\overline{\psi}_k u_k^+ - \hat{E}_k v^+)^2\\
&+ c\,\gamma^{-n-2}\int_{B_\gamma(0)} (\overline{\psi}_k u_k^- - \hat{E}_k v^-)^2\\
&+ c\,\gamma^{-n-2}\int_{B_\gamma(0)} \operatorname{dist}^2\left((x,\varphi_k(x)+\hat{E}_k v^1(x)),\widetilde{P}_k\right)\\
&+ c\,\gamma^{-n-2}\int_{B_\gamma(0)} \operatorname{dist}^2\left((x,\varphi_k(x)+\hat{E}_k v^2(x)),\widetilde{P}_k\right)\\
&+ c\,\gamma^{-n-2}\hat{E}_k^{2+\mu}\\
&\leq c\,\gamma^{-n-2}q(\hat{E}_k)\hat{E}_k^2 + c\,\gamma^{-n-2}\hat{E}_k^2\int_{B_\gamma(0)}(v^1-l^1)^2\\
&+ c\,\gamma^{-n-2}\hat{E}_k^2\int_{B_\gamma(0)}(v^2-l^2)^2 + c\,\gamma^{-n-2}\hat{E}_k^{2+\mu}\\
&\leq c\,\gamma^{-n-2}q(\hat{E}_k)\hat{E}_k^2 + c\gamma^2\beta^{-n-4}\hat{E}_k^2\left(\int_{B_{1/2}(0)}(v^+)^2+(v^-)^2\right)\\
&+ c\,\gamma^{-n-2}\hat{E}_k^{2+\mu}\\
&\leq c\,\hat{E}_k^2\left(\gamma^{-n-2}q(\hat{E}_k) + \gamma^2\beta^{-n-4} + \gamma^{-n-2}\hat{E}_k^{\mu}\right)
\end{aligned}
$$

where $q(t) \to 0$ as $t \to 0$ and c depends only on n and δ. It follows from this that for all sufficiently large k,

$$
\begin{aligned}
\gamma^{-n-2}\int_{M_k\cap(B_\gamma(0)\times\mathbf{R})} \operatorname{dist}^2(X,\widetilde{P}_k) &\leq C\gamma^2\beta^{-n-4}\hat{E}_k^2\\
&\leq C_3\gamma^2\bigg(\int_{M_k\cap(B_1(0)\times\mathbf{R})}\operatorname{dist}^2(x,P_k)\\
&\qquad+\int_{P_k^\star\cap((B_{1/2}(0)\setminus S_{P_k}(1/16))\times\mathbf{R})}\operatorname{dist}^2(x,G_k)\bigg)
\end{aligned}
\tag{6.39}
$$

where $C = C(n) > 0$ and we have set $C_3 = \frac{C\beta^{-n-4}}{\zeta}$, with $\zeta = \zeta(n,\theta,\delta)$ as in the definitions of cases (a) and (b), so that $C_3 = C_3(n,\delta,\theta,\beta)$.

Notice next that by the definition of $\widetilde{P}_k$, we see that

$$
d_{\mathcal{H}}^2\left(\widetilde{P}_k\cap(B_1(0)\times\mathbf{R}), L_k\cap(B_1(0)\times\mathbf{R})\right) \leq C\hat{E}_k^2
$$

where $C = C(n)$. On the other hand, it follows from the inequality

$$\zeta \hat{E}_k^2 \leq \int_{M_k \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(x, P_k) + \int_{P_k^\star \cap ((B_{1/2}(0)\setminus S_{P_k}(1/16))\times \mathbf{R})} \operatorname{dist}^2(x, G_k)$$

and the triangle inequality that

$$\begin{aligned} &d_{\mathcal{H}}^2\left(L_k \cap (B_1(0)\times \mathbf{R}), P_k \cap (B_1(0)\times \mathbf{R})\right) \\ &\leq C\bigg(\int_{M_k \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(x, P_k) \\ &\qquad + \int_{P_k^\star \cap ((B_{1/2}(0)\setminus S_{P_k}(1/16))\times \mathbf{R})} \operatorname{dist}^2(x, G_k)\bigg) \end{aligned}$$

with $C = C(n, \theta, \delta)$, and therefore, by the triangle inequality again, we have that

$$\begin{aligned} &d_{\mathcal{H}}^2\left(\widetilde{P}_k \cap (B_1(0)\times \mathbf{R}), P_k \cap (B_1(0)\times \mathbf{R})\right) \\ &\leq C\bigg(\int_{M_k \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(x, P_k) \\ &\qquad + \int_{P_k^\star \cap ((B_{1/2}(0)\setminus S_{P_k}(1/16))\times \mathbf{R})} \operatorname{dist}^2(x, G_k)\bigg) \end{aligned} \tag{6.40}$$

where $C = C(n, \theta, \delta)$.

We next show in case (b)(i) that for any choice of $\widetilde{L}_k \in \mathcal{A}(M_k, \gamma)$ and $\widetilde{t}_k \in \mathcal{R}(M_k, \widetilde{L}_k, \gamma)$,

$$\begin{aligned} &\gamma^{-n-2} \int_{\widetilde{P}_k^\star \cap (B_{\gamma/2}(0)\setminus S_{\widetilde{P}_k}(\gamma/16))\times \mathbf{R})} \operatorname{dist}^2\left(x, G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\gamma)\right) \\ &\leq C_3\gamma^2\bigg(\int_{M_k \cap (B_1(0)\times \mathbf{R})} \operatorname{dist}^2(x, P_k) \\ &\qquad + \int_{P_k^\star \cap ((B_{1/2}(0)\setminus S_{P_k}(1/16))\times \mathbf{R})} \operatorname{dist}^2(x, G_k)\bigg) \end{aligned} \tag{6.41}$$

where $C_3 = C_3(n, \delta, \theta, \beta)$. For this, recall first that since v^i are harmonic in $B_{2\beta}(0)$, we have the estimates

$$\sup_{B_{4\gamma}(0)} |v^i - l^i|^2 \leq C\gamma^4\beta^{-n-4} \int_{B_{2\beta}(0)} |v^i|^2,$$

$C = C(n)$, so that by Proposition 3.3, part (2), we have that

$$\sup_{B_{4\gamma}(0)} |v^i - l^i| \leq \Gamma\gamma^2\beta^{\frac{-n-4}{2}} \tag{6.42}$$

for $i = 1, 2$, where $\Gamma = \Gamma(n)$. Consider first the case when

$$\sup_{B_{4\gamma}(0)} |l^1 - l^2| \geq \alpha\Gamma\gamma^2\beta^{\frac{-n-4}{2}}, \tag{6.43}$$

where $\alpha > 1$ is to be chosen depending only on n. In this case, if $\alpha > 68$, the estimates (6.42) say that for each k, there is no point $x \in B_{4\gamma}(0) \setminus S_{\widetilde{P}_k}(\gamma/32)$ such that $v^1(x) = v^2(x)$, and hence by the argument of Lemma 3.9, it follows that for infinitely many k, $M_k \cap ((B_{3\gamma}(0) \setminus S_{\widetilde{P}_k}(\gamma/28)) \times \mathbf{R})$ must be embedded. But then by Schoen-Simon regularity theorem ([**SS81**], Theorem 1), $M_k \cap ((B_{2\gamma}(0) \setminus S_{\widetilde{P}_k}(\gamma/24)) \times \mathbf{R}) = \operatorname{graph} \widetilde{u}_k^+ \cup \operatorname{graph} \widetilde{u}_k^-$ where $\widetilde{u}_k^\pm : B_{2\gamma}(0) \setminus S_{\widetilde{P}_k}(\gamma/24) \to \mathbf{R}$ are smooth solutions of the minimal surface equation in their domain, with $\widetilde{u}_k^+ > \widetilde{u}_k^-$. Hence we have by elliptic theory the pointwise estimates

(6.44)
$$\sup_{B_\gamma(0)\setminus S_{\widetilde{P}_k}(\gamma/16)} |\widetilde{u}_k^\pm - \widetilde{p}_k^\pm|^2 \leq C\gamma^{-n} \int_{B_{3\gamma/2}(0)\setminus S_{\widetilde{P}_k}(\gamma/20)} |\widetilde{u}_k^+ - \widetilde{p}_k^+|^2 + |\widetilde{u}_k^- - \widetilde{p}_k^-|^2$$

where $C = C(n)$. Recall our notation that $\widetilde{p}_k^\pm : B_1(0) \to \mathbf{R}$ are such that $\operatorname{graph} \widetilde{p}_k^\pm = \widetilde{P}_k^\pm$. Note also that by elliptic estimates again, $\sup_{B_{7\gamma/4}(0)\setminus S_{\widetilde{P}_k}(\gamma/22)} |D\widetilde{u}_k^\pm| \to 0$ as $k \to \infty$ (since $M_k \cap (B_1(0) \times \mathbf{R}) \to B_1(0) \times \{0\}$ in Hausdorff distance), and hence

$$M_k \cap ((B_{3\gamma/2}(0) \setminus S_{\widetilde{P}_k}(\gamma/20)) \times \mathbf{R}) \subset G_k$$

for infinitely many k. (This follows from the way G_k is defined.) Hence, $\hat{E}_k^{-1}(\widetilde{u}_k^\pm - \varphi_k) \to v^\pm$ in $L^2(B_{3\gamma/2}(0) \setminus S^v(\gamma/18))$, where $\varphi_k : \mathbf{R}^n \times \{0\} \to \mathbf{R}$, $\operatorname{graph} \varphi_k = L_k$ and $S^v(\gamma/18)$ denotes the set $\{x \in B_1(0) : \operatorname{dist}(x, A) \leq \gamma/18\}$ with $A = \{l^1(x) = l^2(x)\}$. Hence, by the estimates (6.42) and (6.44), we have that

$$\sup_{B_\gamma(0)\setminus S_{\widetilde{P}_k}(\gamma/16)} |\widetilde{u}_k^\pm - \widetilde{p}_k^\pm|^2 \leq 2C\omega_n \Gamma^2 \gamma^4 \beta^{-n-4} \hat{E}_k^2 \tag{6.45}$$

where $C = C(n)$ is as in (6.44). Thus, if $\alpha = \alpha(n)$ in (6.43) is chosen sufficiently large, the estimates (6.45) imply, by exactly the same reasoning used to justify inequality (6.37), that for each $x = (x', x^{n+1}) \in \widetilde{P}_k^\star \cap (B_{\gamma/2}(0) \setminus S_{\widetilde{P}_k}(\gamma/16)) \times \mathbf{R})$,

$$\operatorname{dist}(x, G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\gamma)) \leq 2\operatorname{dist}((x', \widetilde{u}_k^\pm(x')), \widetilde{P}_k) \tag{6.46}$$

where the sign $\pm$ is chosen according to whether $x \in \widetilde{P}_k^\pm$. In view of the estimate (6.39), this gives (6.41).

Suppose the condition (6.43) fails to hold. Note that we have

$$G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\gamma) = \operatorname{graph} \widetilde{u}_k^+ \cup \operatorname{graph} \widetilde{u}_k^+ \tag{6.47}$$

where $\widetilde{u}_k^\pm : \pi(G_{M_k}^{(\widetilde{L}_k, \widetilde{t}_k)}(\gamma)) \to \mathbf{R}$ are Lipschitz with Lipschitz constant $\leq 3/2$ and $\widetilde{u}_k^+ \geq \widetilde{u}_k^-$. From this we see that for $x = (x', x^{n+1}) \in \widetilde{P}_k^\star \cap$

$((B_{\gamma/2}(0) \setminus S_{\widetilde{P}_k}(\gamma/16)) \times \mathbf{R})$,

$$\begin{aligned} \operatorname{dist}(x, G_{M_k}^{(\widetilde{L}_k,\widetilde{t}_k)}(\gamma)) &\leq \min\{|x^{n+1} - \widetilde{u}_k^+(x')|, |x^{n+1} - \widetilde{u}_k^-(x')|\} \\ &\leq 2\operatorname{dist}((x', \widetilde{u}_k^{\pm}(x')), \widetilde{P}_k) + 2\hat{E}_k|l^1(x') - l^2(x')| \\ &\leq 2\operatorname{dist}((x', \widetilde{u}_k^{\pm}(x')), \widetilde{P}_k) + 2\alpha\Gamma\gamma^2\beta^{\frac{-n-4}{2}}\hat{E}_k, \end{aligned} \tag{6.48}$$

where in the second of the inequalities here we have used the fact that $\widetilde{u}_k^{\pm}$ are Lipschitz functions with Lipschitz constants $\leq 3/2$, and the sign $\pm$ there is chosen according to whether $x \in \widetilde{P}_k^{\pm}$. By the estimate (6.39) and the defining property of case (b)(i), we again have from this the required estimate (6.41). We have thus shown that in case (b)(i), for infinitely many k, the conclusions of the lemma with option (3)(C) hold, with M_k, P_k, L_k, t_k, $\widetilde{P}_k$, $\widetilde{L}_k$ and $\widetilde{t}_k$ in place of M, P, L, t, $\widetilde{P}$, $\widetilde{L}$ and $\widetilde{t}$ respectively and with $\lambda = 2$.

Finally, suppose (b)(ii) occurs. Then $v^+(z) = v^-(z)$ and by Lemma 5.8, we have that

$$\rho^{-n-2}\int_{B_\rho(z)} (v^+ - l_z)^2 + (v^- - l_z)^2 \leq C\rho^\nu \int_{B_1(0)} (v^+)^2 + (v^-)^2 \tag{6.49}$$

for some affine function l_z and all $\rho \in (0, 1/64)$. Here $C = C(n,\delta) > 0$ and $\nu = \nu(n,\delta) > 0$. Now fix this z. We obtain from (6.49) that

$$\begin{aligned} &\rho^{-n-2}\int_{B_\rho(0)} (v^+ - l_z)^2 + (v^- - l_z)^2 \\ &\leq \left(1 + \frac{|z|}{\rho}\right)^{n+2} (\rho + |z|)^{-n-2}\int_{B_{\rho+|z|}(z)} (v^+ - l_z)^2 + (v^- - l_z)^2 \\ &\leq C\left(1 + \frac{|z|}{\rho}\right)^{n+2}\left(1 + \frac{|z|}{\rho}\right)^{\nu}\rho^\nu\int_{B_1(0)} (v^+)^2 + (v^-)^2, \end{aligned} \tag{6.50}$$

provided $\rho + |z| \leq 1/64$. In particular, taking $\rho = \beta$ in this and using the fact that $z \in B_{2\beta}(0)$ (so that $1 + \frac{|z|}{\beta} \leq 3$), and since $3\beta < 1/64$, we have that

$$\beta^{-n-2}\int_{B_\beta(0)} (v^+ - l_z)^2 + (v^- - l_z)^2 \leq C\beta^\nu \int_{B_1(0)} (v^+)^2 + (v^-)^2 \tag{6.51}$$

where $C = C(n, \delta)$. With this, we can estimate as in (6.38) to conclude that if possibility (b)(ii) occurs, then we must have that

$$(6.52)\qquad \begin{aligned}&\beta^{-n-2}\int_{M_k\cap(B_\beta(0)\times\mathbf{R})} \operatorname{dist}^2(X, \widetilde{P}_k) \leq C\beta^\nu \hat{E}_k^2\\ &\leq C_2\beta^\nu\Big(\int_{M_k\cap(B_1(0)\times\mathbf{R})}\operatorname{dist}^2(x, P_k)\\ &\qquad + \int_{P_k^\star\cap((B_{1/2}(0)\setminus S_{P_k}(1/16))\times\mathbf{R})}\operatorname{dist}^2(x, G_k)\Big)\end{aligned}$$

for all sufficiently large k, where $\widetilde{P}_k = \operatorname{graph}(\varphi_k + \hat{E}_k l_z)$. Here $C = C(n, \delta)$ is as in the estimate (6.51) and we have set $C_2 = \frac{C}{\zeta}$ where $\zeta = \zeta(n, \delta, \theta)$ is as in the definition of cases (a) and (b), so that $C_2 = C_2(n, \delta, \theta)$.

Arguing exactly as in the proof of the estimate (6.40), we also have in this case that

$$(6.53)\qquad \begin{aligned}&d_{\mathcal{H}}^2(\widetilde{P}_k\cap(B_1(0)\times\mathbf{R}), P_k\cap(B_1(0)\times\mathbf{R}))\\ &\leq C\Big(\int_{M_k\cap(B_1(0)\times\mathbf{R})}\operatorname{dist}^2(x, P_k)\\ &\qquad + \int_{P_k^\star\cap((B_{1/2}(0)\setminus S_{P_k}(1/16))\times\mathbf{R})}\operatorname{dist}^2(x, G_k)\Big)\end{aligned}$$

where $C = C(n, \theta, \delta)$.

To complete the proof of the lemma, we now check in case (b)(ii) that for any choice of $\widetilde{L}_k \in \mathcal{A}(M_k, \beta)$ and $\widetilde{t}_k \in \mathcal{R}(M_k, \widetilde{L}_k, \beta)$,

$$(6.54)\qquad \begin{aligned}&\beta^{-n-2}\int_{\widetilde{P}_k^\star\cap(B_{\beta/2}(0)\setminus S_{\widetilde{P}_k}(\beta/16))\times\mathbf{R})}\operatorname{dist}^2(x, G_{M_k}^{(\widetilde{L}_k,\widetilde{t}_k)}(\beta))\\ &\leq C_2\beta^\nu\Big(\int_{M_k\cap(B_1(0)\times\mathbf{R})}\operatorname{dist}^2(x, P_k)\\ &\qquad + \int_{P_k^\star\cap((B_{1/2}(0)\setminus S_{P_k}(1/16))\times\mathbf{R})}\operatorname{dist}^2(x, G_k)\Big)\end{aligned}$$

where $C_2 = C_2(n, \delta, \theta)$ is as in the estimate (6.52). But this follows directly from the pointwise estimate that for each $x = (x', x^{n+1}) \in \widetilde{P}_k^\star \cap (B_{\beta/2}(0) \setminus S_{\widetilde{P}_k}(\beta/16)) \times \mathbf{R})$,

$$(6.55)\qquad \begin{aligned}&\operatorname{dist}(x, G_{M_k}^{(\widetilde{L}_k,\widetilde{t}_k)}(\beta))\\ &\leq \min\{|x^{n+1} - \widetilde{u}_k^+(x')|, |x^{n+1} - \widetilde{u}_k^-(x')|\}\\ &\leq 2\min\{\operatorname{dist}((x', \widetilde{u}_k^+(x')), \widetilde{P}_k), \operatorname{dist}((x', \widetilde{u}_k^-(x')), \widetilde{P}_k)\}\end{aligned}$$

where $\widetilde{u}_k^\pm$ are defined exactly as in (6.47) with β in place of γ. In the second of the inequalities above, we have used the fact that $\widetilde{u}_k^\pm$ are

Lipschitz functions with Lipschitz constants $\leq 3/2$, and that $\widetilde{P}_k$ is a single affine hyperplane. The required estimate (6.54) follows from this and the estimate (6.52). We have thus shown that in case (b)(ii), for infinitely many k, the conclusions of the lemma with option (3)(B) hold, with M_k, P_k, L_k, t_k, $\widetilde{P}_k$, $\widetilde{L}_k$ and $\widetilde{t}_k$ in place of M, P, L, t, $\widetilde{P}$, $\widetilde{L}$ and $\widetilde{t}$ respectively and with $\lambda = \nu$. This completes the proof of the lemma.

q.e.d.

7. Main regularity theorems

We are now ready to prove Theorems 1.1, 1.2 and 1.4.

Proof of Theorem 1.1. First choose $\theta = \theta(n, \delta) \in (0, 1/16)$ such that $C_1\theta^{\lambda} < 1/4$, then choose $\beta = \beta(n, \delta) \in (0, \theta/16)$ such that $C_2\beta^{\lambda} < 1/4$, and finally choose $\gamma = \gamma(n, \delta) \in (0, \beta/16)$ such that $C_3\gamma^{\lambda} < 1/4$, where C_1, C_2, C_3 and λ are as in Lemma 6.3.

Suppose M satisfies the hypotheses of Theorem 1.1. Note first that since L^2 closeness of M to a hyperplane implies closeness in Hausdorff distance, the hypothesis $\int_{M\cap(B_1(0)\times\mathbf{R})} |x^{n+1}|^2 \leq \epsilon$ implies that $d_{\mathcal{H}}\left(L_0 \cap (B_1(0) \times \mathbf{R}), B_1(0)\right) \leq \tau(\epsilon)$ and $\int_{B_{1/2}(0)} \mathrm{dist}^2\left(x, G_M^{(L_0, t_0)}(1)\right) \leq \tau(\epsilon)$ for any $L_0 \in \mathcal{A}\left(M, 1\right)$ and any $t_0 \in \mathcal{R}(M, L_0, 1)$, where $\tau(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. Fix such L_0 and t_0.

In what follows, let us use the notation

$$
\begin{aligned}
Q(\rho, P, L, t) = \rho^{-n-2} &\int_{M\cap(B_\rho(0)\times\mathbf{R})} \mathrm{dist}^2(x, P) + \rho^{-n-2} \\
&\cdot \int_{P^\star\cap((B_{\rho/2}(0)\setminus S_P(\rho/16))\times\mathbf{R})} \mathrm{dist}^2\left(x, G_M^{(L,t)}(\rho)\right).
\end{aligned}
$$

If $\epsilon = \epsilon(n, \delta) \in (0, 1)$ is sufficiently small, by iterating Lemma 6.3 starting with $P = P_0 \equiv \mathbf{R}^n \times \{0\}$, $L = L_0$ and $t = t_0$, we get a sequence of pairs of affine hyperplanes P_j, a sequence of affine hyperplanes $L_j \in \mathcal{A}\left(M, \theta^{k_j}\beta^{l_j}\gamma^{m_j}\right)$ and a sequence of numbers $t_j \in \mathcal{R}\left(M, L_j, \theta^{k_j}\beta^{l_j}\gamma^{m_j}\right)$ satisfying at the jth iteration either

$$
\begin{aligned}
Q(\theta^{k_j}\beta^{l_j}\gamma^{m_j}, P_j, L_j, t_j) &\leq 4^{-1}Q(\theta^{k_j-1}\beta^{l_j}\gamma^{m_j}, P_{j-1}, L_{j-1}, t_{j-1}) \\
&\leq 4^{-j}Q_1 \qquad (7.1)
\end{aligned}
$$

or

$$
\begin{aligned}
Q(\theta^{k_j}\beta^{l_j}\gamma^{m_j}, P_j, L_j, t_j) &\leq 4^{-1}Q(\theta^{k_j}\beta^{l_j-1}\gamma^{m_j}, P_{j-1}, L_{j-1}, t_{j-1}) \\
&\leq 4^{-j}Q_1 \qquad (7.2)
\end{aligned}
$$

or

$$
\begin{aligned}
Q(\theta^{k_j}\beta^{l_j}\gamma^{m_j}, P_j, L_j, t_j) &\leq 4^{-1}Q(\theta^{k_j}\beta^{l_j}\gamma^{m_j-1}, P_{j-1}, L_{j-1}, t_{j-1}) \\
&\leq 4^{-j}Q_1, \qquad (7.3)
\end{aligned}
$$

where k_j, l_j, m_j are non-negative integers with $k_j + l_j + m_j = j$ and

$$Q_1 = \int_{M\cap(B_1(0)\times\mathbf{R})} |x^{n+1}|^2 + \int_{B_1(0)} \mathrm{dist}^2\,(x, G_M^{(L_0,t_0)}(1)).$$

Let us denote the sequence of scales so generated $\{s_j\}$. Thus, for each $j = 0, 1, 2, \ldots$, $s_j = \theta^{k_j}\beta^{l_j}\gamma^{m_j}$ for some non-negative integers k_j, l_j, m_j with $k_j + l_j + m_j = j$, and, $s_{j+1} = \theta s_j$ or βs_j or γs_j. Then (7.1)–(7.4) may be rewritten as

$$\begin{aligned} Q(s_j, P_j, L_j, t_j) &\leq 4^{-1}\, Q(s_{j-1}, P_{j-1}, L_{j-1}, t_{j-1}) \\ &\leq 4^{-j} Q_1. \end{aligned} \tag{7.4}$$

The lemma also gives us that

$$\begin{aligned} &\mathrm{dist}^2_{\mathcal{H}}\,(P_j \cap (B_1(0)\times\mathbf{R}), P_{j-1}\cap(B_1(0)\times\mathbf{R})) \\ &\quad\leq C\, Q(s_{j-1}, P_{j-1}, L_{j-1}, t_{j-1}) \\ &\quad\leq C4^{-j}Q_1 \end{aligned} \tag{7.5}$$

and that

$$\begin{aligned} &\mathrm{dist}^2_{\mathcal{H}}\,(P_j^T \cap (B_1(0)\times\mathbf{R}), P_{j-1}^T\cap(B_1(0)\times\mathbf{R})) \\ &\quad\leq C\, Q(s_{j-1}, P_{j-1}, L_{j-1}, t_{j-1}) \\ &\quad\leq C4^{-j}Q_1 \end{aligned} \tag{7.6}$$

where C depends only on n and δ. Thus, $\{P_j\}$ is a Cauchy sequence of pairs of affine hyperplanes, and hence there exists a pair of affine hyperplanes P such that $P_j \to P$. By (7.5), (7.6) and (7.1) respectively, we have that

$$\mathrm{dist}^2_{\mathcal{H}}\,(P \cap (B_1(0)\times\mathbf{R}), P_{j-1}\cap(B_1(0)\times\mathbf{R})) \leq C4^{-j}Q_1, \tag{7.7}$$

$$\mathrm{dist}^2_{\mathcal{H}}\,(P^T \cap (B_1(0)\times\mathbf{R}), P_{j-1}^T\cap(B_1(0)\times\mathbf{R})) \leq C4^{-j}Q_1, \quad \text{and} \tag{7.8}$$

$$s_j^{-n-2}\int_{M\cap(B_{s_j}(0)\times\mathbf{R})} \mathrm{dist}^2\,(x, P) \leq C4^{-j}Q_1 \tag{7.9}$$

where C depends only on n and δ. Note that (7.7) and (7.8) in particular say that

$$\mathrm{dist}^2_{\mathcal{H}}\,(P \cap (B_1(0)\times\mathbf{R}), B_1(0)) \leq CQ_1 \quad \text{and} \tag{7.10}$$

$$\mathrm{dist}^2_{\mathcal{H}}\,(P^T \cap (B_1(0)\times\mathbf{R}), B_1(0)) \leq CQ_1. \tag{7.11}$$

Now, given any $\rho \in (0, 1/8)$, there exists a unique j with $s_{j+1} \leq \rho < s_j$. Since $\gamma < \beta < \theta$, this implies that $\gamma^{j+1} \leq \rho < \theta^j$, or, equivalently, that $\frac{\log \rho}{\log \theta} > j \geq \frac{\log \rho}{\log \gamma} - 1$. Hence, by (7.9), we conclude that

$$\begin{aligned}(7.12) \qquad & \rho^{-n-2} \int_{M \cap (B_\rho(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \\ & \leq s_{j+1}^{-n-2} \int_{M \cap (B_{s_j}(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \\ & = \left(\frac{s_j}{s_{j+1}}\right)^{n+2} s_j^{-n-2} \int_{M \cap (B_{s_j}(0) \times \mathbf{R})} \operatorname{dist}^2(x, P) \\ & \leq C \rho^\kappa Q_1\end{aligned}$$

for all $\rho \in (0, 1/8)$, where $\kappa = -\log 4 / \log \gamma$ and C depends only on n.

Next observe that we can move the base point and repeat the entire argument leading to the estimates (7.10), (7.11) and (7.12). Specifically, for any given $X \in M \cap B_{3/4}^{n+1}(0)$, we have

$$\begin{aligned}(7.13) \qquad & \frac{\mathcal{H}^n\left(M \cap B_{7/8}^{n+1}(X)\right)}{\omega_n (7/8)^n} \\ & = \frac{\mathcal{H}^n\left(G \cap B_{7/8}^{n+1}(X)\right) + \mathcal{H}^n((M \setminus G) \cap B_{7/8}^{n+1}(X))}{\omega_n (7/8)^n} \\ & \leq \frac{1}{\omega_n (7/8)^n} \int_\Omega \sqrt{1 + |Du^+|^2} + \sqrt{1 + |Du^-|^2} + C \hat{E}^{2+\mu} \\ & \leq 2\sqrt{2} + C\hat{E}^{2+\mu} \\ & \leq 3 - \delta/16\end{aligned}$$

provided $\epsilon = \epsilon(n, \delta) \in (0, 1)$ is sufficiently small. Here G denotes the graphical part of $M \cap (B_{7/8}(0) \times \mathbf{R})$ as described in Section 3, $\Omega \subset B_{7/8}(0)$, $u^\pm : \Omega \to \mathbf{R}$ are such that $G = \operatorname{graph} u^+ \cup \operatorname{graph} u^-$ and $\hat{E}^2 = \int_{M \cap (B_1(0) \times \mathbf{R})} |x^{n+1}|^2$, $C = C(n, \delta)$ and $\mu = \mu(n)$.

Thus, provided ϵ is sufficiently small, we can repeat the argument leading to the estimates (7.10), (7.11) and (7.12), iteratively applying Lemma 6.3 with $\eta_{X, 7/16}\, M$ in place of M and starting with $P = \eta_{X, 7/16}\, (\mathbf{R}^n \times \{0\})$ and arbitrary $L \in \mathcal{A}\,(\eta_{X, 7/16}\, M, 1)$ and $t \in \mathcal{R}\,(\eta_{X, 7/16}\, M, L, 1)$ to conclude that for every $X \in \overline{M} \cap B_{3/4}^{n+1}(0)$, there exists a pair of affine hyperplanes P_X such that

$$(7.14) \qquad \mathrm{d}_{\mathcal{H}}^2\left(P_X - X \cap (B_1(0) \times \mathbf{R}), B_1(0)\right) \leq C Q_1,$$

$$(7.15) \qquad \mathrm{d}_{\mathcal{H}}^2\left(P_X^T \cap (B_1(0) \times \mathbf{R}), B_1(0)\right) \leq C Q_1 \quad \text{and}$$

$$(7.16) \qquad \rho^{-n-2} \int_{M \cap (B_\rho(X') \times \mathbf{R})} \operatorname{dist}^2(x, P_X) \leq C \rho^\kappa Q_1$$

for all $\rho \in (0, 1/8)$, where $X' = \pi(X)$. It follows from this that provided $\epsilon = \epsilon(n, \delta)$ is sufficiently small, $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R})$ is the graph of a 2-valued $C^{1,\kappa}$ function. The proof of this claim is as follows:

First note that by choosing $\epsilon = \epsilon(n, \delta)$ sufficiently small, we may assume that $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R}) \subseteq \overline{M} \cap B^{n+1}_{3/4}(0)$. For $X \in M \cap B^{n+1}_{3/4}(0)$, let P_X be as in (7.14)—(7.16). Note that by (7.14), $P_X \cap \pi^{-1}(X')$ consists precisely of two (possibly coinciding) points X and $\widetilde{X}$. Multiplying the inequality (7.16) by ρ^2 and letting $\rho \to 0$, we see that for each $X \in M \cap (B_{1/2}(0) \times \mathbf{R})$, $M \cap \pi^{-1}(X') = P_X \cap \pi^{-1}(X')$ so that $M \cap \pi^{-1}(X')$ consists of (possibly coinciding) two points. Furthermore, (7.16) says that the two tangent planes to M at X and $\widetilde{X}$ are the two hyperplanes whose union is P_X. Thus, in view of (7.15), we have that for each $X \in M \cap (B_{1/2}(0) \times \mathbf{R})$, $|\nu_1(X) - e_{n+1}|$, $|\nu_2(\widetilde{X}) - e_{n+1}| \leq CQ_1$, where ν_1, ν_2 denote the (locally defined) upward pointing unit normals to M. (Thus, in case $X = \widetilde{X}$, $\nu_1(X), \nu_2(X)$ are the two upward pointing unit normals at X to the respective smooth sheets whose union is $M \cap B^{n+1}_\sigma(X)$ for some $\sigma > 0$.) This means that

$$M \cap (B_{1/2}(0) \times \mathbf{R}) = \operatorname{graph} u^+ \cup \operatorname{graph} u^- \tag{7.17}$$

where $u^\pm : B_{1/2} \setminus \pi(\operatorname{sing} M) \to \mathbf{R}$ are Lipschitz functions with $u^+ \geq u^-$ and Lipschitz constants $\leq CQ_1$. The functions u^+, u^- then extend uniquely as Lipschitz functions $\overline{u}^+, \overline{u}^- : B_{1/2}(0) \to \mathbf{R}$ respectively, with the same Lipschitz constants, and we have that

$$\overline{M} \cap (B_{1/2}(0) \times \mathbf{R}) = \operatorname{graph} \overline{u}^+ \cup \operatorname{graph} \overline{u}^-. \tag{7.18}$$

Now note that since $M \cap (B_{1/2}(0) \times \mathbf{R})$ is a Lipschitz graph with Lipschitz constant ≤ 1, it follows that $Q_1 \leq C\hat{E}^2$ for some fixed constant $C = C(n)$, where $\hat{E} = \int_{M \cap (B_1(0) \times \mathbf{R})} |x^{n+1}|^2$, and hence we may replace Q_1 with $\hat{E}^2$ in all of the above estimates. Note also that since $0 \in \overline{M}$, the estimate for the Lipschitz constant implies the height bound

$$|u^+(x)|, |u^-(x)| \leq C\hat{E}, \quad x \in B_{1/2}(0). \tag{7.19}$$

It now remains to show that the union of the two Lipschitz graphs in (7.18) is the graph of a single 2-valued $C^{1,\kappa}$ function, with its $C^{1,\kappa}$ norm bounded by a constant times $\hat{E}$. We proceed as follows:

Take any two points $X_1, X_2 \in M \cap (B_{1/2}(0) \times \mathbf{R})$ with $X'_1 \neq X'_2$ and let $r = |X'_1 - X'_2|$. By (7.16), we have that

$$(2r)^{-n-2} \int_{M \cap (B_{2r}(X'_2) \times \mathbf{R})} \operatorname{dist}^2(x, P_{X_2}) \leq Cr^\kappa \hat{E}^2 \tag{7.20}$$

and hence, since $B_r(X'_1) \subset B_{2r}(X'_2)$ it follows that

$$r^{-n-2} \int_{M \cap (B_r(X'_1) \times \mathbf{R})} \operatorname{dist}^2(x, P_{X_2}) \leq Cr^\kappa \hat{E}^2. \tag{7.21}$$

Also by (7.14) and (7.15) we have

$$d^2_{\mathcal{H}}\,(P_{X_2} \cap (B_1(X_1') \times \mathbf{R}), B_1(X_1')) \leq C\hat{E}^2. \tag{7.22}$$

This means that provided $\epsilon = \epsilon(n, \delta)$ is sufficiently small, we may use Lemma 6.3 exactly as it was used in the argument leading to (7.10), (7.11) and (7.12), with $\eta_{X_1,\,r}\, M$ in place of M, $\eta_{X_1,\,r}\, P_{X_2}$ in place of P of the lemma (which was taken to be the multiplicity 2 hyperplane corresponding to $\mathbf{R}^n \times \{0\}$ in the argument leading to (7.10), (7.11) and (7.12) above) to conclude that there exists a pair of affine hyperplanes P'_{X_1} such that

$$\begin{aligned}(\rho r)^{-n-2} &\int_{M \cap B_{\rho r}(X_1') \times \mathbf{R})} \operatorname{dist}^2(x, P'_{X_1}) \\ &\leq C\rho^\kappa r^{-n-2} \int_{M \cap (B_r(X_1') \times \mathbf{R})} \operatorname{dist}^2(x, P_{X_2})\end{aligned} \tag{7.23}$$

for all $\rho \in (0, 1/8)$ and

$$\begin{aligned}d^2_{\mathcal{H}}\,&(P'^{T}_{X_1} \cap (B_1(0) \times \mathbf{R}), P^T_{X_2} \cap (B_1(0) \times \mathbf{R})) \\ &\leq Cr^{-n-2} \int_{M \cap (B_r(X_1') \times \mathbf{R})} \operatorname{dist}^2(x, P_{X_2}).\end{aligned} \tag{7.24}$$

In view of (7.16) (with $X = X_1$), (7.23) implies that $P'_{X_1} \equiv P_{X_1}$, and hence, (7.24) combined with (7.21) gives that

$$\begin{aligned}d^2_{\mathcal{H}}\,&(P^T_{X_1} \cap (B_1(0) \times \mathbf{R}), P^T_{X_2} \cap (B_1(0) \times \mathbf{R})) \\ &\leq C\hat{E}^2 |X_1' - X_2'|^\kappa\end{aligned} \tag{7.25}$$

for all $X_1,\, X_2 \in \overline{M} \cap (B_{\sigma/4}(0) \times \mathbf{R})$. This says that, in the notation introduced in Section 2, the 2-valued function $u\,:\, B_{1/2}(0) \to \mathbf{Q}_2(\mathbf{R})$ defined by $u(x) = \{\overline{u}^+(x), \overline{u}^-(x)\}$ satisfies

$$\mathcal{G}\,(Du(x_1), Du(x_2)) \leq C\hat{E}|x_1 - x_2|^{\kappa/2} \tag{7.26}$$

for all $x_1, x_2 \in B_{1/2}(0)$. i.e., that u is a $C^{1,\kappa/2}(B_{1/2}(0))$ function with $[u]_{1,\,\kappa/2;\,B_{1/2}(0)} \leq C\hat{E}$. This together with the estimates (7.11) and (7.19) imply the $\|u\|_{C^{1,\kappa/2}(B_{1/2}(0))} \leq C\hat{E}$. The theorem is thus proved. q.e.d.

Proof of Theorem 1.2. By Theorem 1.1, $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R})$ is either the graph of a single $C^{1,\alpha}$ function u^0 or the graph of a 2-valued $C^{1,\alpha}$ function u, with the appropriate estimate for the $C^{1,\alpha}$ norm in either case. In case $\overline{M} \cap (B_{1/2}(0) \times \mathbf{R})$ is the graph of a 2-valued function u, we have that locally in a neighborhood Ω_x of any point x of the open set $B_{1/2}(0) \setminus \pi\,(\operatorname{sing} M)$, u is given by two functions, each satisfying the minimal surface equation in Ω_x. Since $\mathcal{H}^{n-2}\,(\operatorname{sing} M) = 0$ by assumption, $B_{1/2}(0) \setminus \pi\,(\operatorname{sing} M)$ is simply connected, and hence

$M \cap ((B_{1/2}(0) \setminus \pi(\operatorname{sing} M)) \times \mathbf{R})$ is equal to the union of the graphs of two functions $\widetilde{u}_1, \widetilde{u}_2 : B_{1/2}(0) \setminus \pi(\operatorname{sing} M) \to \mathbf{R}$, each satisfying the minimal surface equation. But then by the removable singularity theorem of L. Simon [**Sim77**], $\widetilde{u}_1, \widetilde{u}_2$ extend as functions $u_1, u_2 : B_{1/2}(0) \to \mathbf{R}$, satisfying the minimal surface equation. q.e.d.

Proof of Theorem 1.4. Were the assertion false, there would exist a sequence of hypersurfaces $M_k \in \mathcal{I}_b$, $k = 1, 2, 3, \ldots$, with $0 \in \overline{M}_k$, $\Theta_{M_k}(0) \geq 2$ and

$$\frac{\mathcal{H}^n(M_k)}{\omega_n 2^n} \leq 2 + 1/k \tag{7.27}$$

such that for each k, the conclusion of the theorem fails with M_k in place of M and with any choice of hyperplanes P, $P^{(1)}$, $P^{(2)}$ of $\mathbf{R}^{n+1}$. By Allard's integer varifold compactness theorem, we obtain, possibly after passing to a subsequence of $\{k\}$ which we continue to denote $\{k\}$, that $M_k \to V$ as varifolds for some integer multiplicity stationary varifold of $B_2^{n+1}(0)$. By (7.27), upper semicontinuity of density, and the continuity of mass under varifold convergence, it follows that $2 \leq \Theta(\|V\|, (0)) \leq \frac{\mathcal{H}^n(\operatorname{spt}\|V\| \cap B_2^{n+1}(0))}{\omega_n 2^n} \leq 2$, so that by the monotonicity formula, V must be a cone with $\Theta(\|V\|, (0)) = 2$. By Lemma 8.1, part (b) below, V must either be a pair of transverse multiplicity 1 hyperplanes or a hyperplane with multiplicity 2. Thus for infinitely many k of the original sequence, the conclusions of the theorem hold, by Theorem 1.1 or Theorem 1 of [**Wic**], with M_k in place of M. This proves the theorem. q.e.d.

8. Compactness and decomposition theorems

In this section we prove Theorems 1.3 and 1.5. First we need the following lemma, in which we shall use the following notation. Given a p dimensional rectifiable varifold $V = (\Sigma, \theta)$ in $\mathbf{R}^{p+1}$, where θ is the multiplicity of V (see [**Sim83**], Chapter 4 for an exposition of the the theory of rectifiable varifolds), we let $V \times \mathbf{R}^{n-p}$ denote the rectifiable varifold $(\Sigma \times \mathbf{R}^{n-p}, \theta_1)$ of $\mathbf{R}^{n+1}$ where $\theta_1(x, y) = \theta(x)$ for $(x, y) \in \Sigma \times \mathbf{R}^{n-p}$.

Lemma 8.1.

(a) *Suppose* $\mathbf{C}$ *is a cone with* $\Theta(\|\mathbf{C}\|, 0) \leq 3$ *belonging to the varifold closure of immersed, stable minimal hypersurfaces* M *of* $B_2^{n+1}(0)$ *with* $\mathcal{H}^{n-2}(\operatorname{sing} M) = 0$. *If* $\mathbf{C}$ *has the form* $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}^{n-p}$ *for some* $p \leq 6$ (*which holds automatically if* $n \leq 6$), *then* $\mathbf{C}$ *must be the sum of at most* 3 *multiplicity* 1 *hyperplanes of* $\mathbf{R}^{n+1}$.

(b) *There exists a fixed number* $\epsilon \in (0, 1)$ *such that if* $\mathbf{C}$ *is a cone with* $\Theta(\|\mathbf{C}\|, 0) \leq 2+\epsilon$ *belonging to the varifold closure of immersed stable minimal hypersurfaces* M *of* $B_2^{n+1}(0)$ *satisfying* $\mathcal{H}^{n-2}(\operatorname{sing} M)$

$< \infty$, *and if* $\mathbf{C}$ *has the form* $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}^{n-p}$ *for some* $p \leq 6$ (*which holds automatically if* $n \leq 6$), *then* $\mathbf{C}$ *is equal to the sum of at most* 2 *multiplicity* 1 *hyperplanes of* $\mathbf{R}^{n+1}$.

Proof. First recall the following standard facts about any stationary cone W; namely, that $\Theta(\|W\|, X) \leq \Theta(\|W\|, 0)$ for any $X \in \text{spt}\,\|W\|$ and, that if $\Theta(\|W\|, X) = \Theta(\|W\|, 0)$ for some $X \in \text{spt}\,\|W\| \setminus \{0\}$, then $\text{spt}\,\|W\|$ is invariant under translations by elements of the line $\{tX : t \in \mathbf{R}\}$. Furthermore, the set $\mathcal{S}(W) = \{X \in \text{spt}\,\|W\| : \Theta(\|W\|, X) = \Theta(\|W\|, 0)\}$ is a linear subspace of $\mathbf{R}^{n+1}$ and $\text{spt}\,\|W\|$ is invariant under translations by the elements of $\mathcal{S}(W)$. In view of these facts, by rotating we may assume (in both parts (a) and (b)) that $\mathbf{C} = \mathbf{C}' \times \mathbf{R}^{n-d}$ for some $d \in \{1, 2, \ldots, 6\}$, where $\mathbf{C}'$ is a d-dimensional stationary cone in $\mathbf{R}^{d+1}$ with $\Theta(\|\mathbf{C}'\|, X) < \Theta(\|\mathbf{C}'\|, 0) \leq 3$ for every $X \in \text{spt}\,\|\mathbf{C}'\| \setminus \{0\}$.

Suppose $\mathbf{C}$ satisfies the hypotheses of part (a). We first consider the case $d = 1$. In this case, $\text{spt}\,\|\mathbf{C}'\|$ must be a union of at most 6 rays. Since $\mathbf{C}$ is the limit of a sequence of minimal hypersurfaces M_k having no current boundary in $B_2^{n+1}(0)$, it follows that $\Theta(\|\mathbf{C}\|, 0) \in \{1, 2, 3\}$, and since M_k are stable, by the argument of Lemma 6.19 of [**Wic04a**], it follows that $\mathbf{C}$ is the sum of hyperplanes.

Suppose now that $d \geq 2$. If $\text{spt}\,\|\mathbf{C}'\| \setminus \{0\}$ is a regular (i.e., properly immersed) submanifold of $\mathbf{R}^{d+1}$, then, since $\mathbf{C}$ is the limit of stable minimal hypersurfaces, $\text{spt}\,\|\mathbf{C}'\| \setminus \{0\}$ is a stable minimal hypersurface of $\mathbf{R}^{d+1}$ (in the sense that the stability inequality (2.2) holds with $\text{spt}\,\|\mathbf{C}'\| \setminus \{0\}$ in place of M) so J. Simons' theorem ([**SJ68**], see also [**Sim83**], appendix B) concerning the non-existence of non-trivial stable minimal hypercones of dimension ≤ 6 implies that $\mathbf{C}$ must be a union of at most 3 hyperplanes.

We now rule out the possibility that $\text{spt}\,\|\mathbf{C}'\| \setminus \{0\}$ is not immersed at some point $X_1 \in \text{spt}\,\|\mathbf{C}'\| \setminus \{0\}$. If this were the case, we may take a tangent cone $\widetilde{\mathbf{C}}$ to $\mathbf{C}$ at $(X_1, 0)$ and $\widetilde{\mathbf{C}}$ after rotating must then have the form $\widetilde{\mathbf{C}} = \widetilde{\mathbf{C}}' \times \mathbf{R}^{n-d+1}$ where $\widetilde{\mathbf{C}}'$ is a $(d-1)$-dimensional stationary cone in $\mathbf{R}^d$. Applying the previous reasoning to the cone $\widetilde{\mathbf{C}}$, and keeping in mind that $\Theta(\|\widetilde{\mathbf{C}}\|, 0) = \Theta(\|\mathbf{C}\|, (X_1, 0)) < 3$, we conclude that either $\text{spt}\,\|\widetilde{\mathbf{C}}\|$ is the union of at most 2 (not necessarily distinct) hyperplanes, or $\text{spt}\,\|\widetilde{\mathbf{C}}'\| \setminus \{0\}$ is not immersed at some point $\widetilde{X}_1 \in \text{spt}\,\|\widetilde{\mathbf{C}}'\| \setminus \{0\}$. The former is not possible, since if it were the case, by Allard's regularity theorem (applied to suitably translated and rescaled $\mathbf{C}$) or Theorem 1 of [**Wic**] or Theorem 1.2 of the present paper (applied, in either case, to suitably translated and rescaled members of the sequence of stable minimal hypersurfaces approximating $\mathbf{C}$), depending respectively on whether $\widetilde{\mathbf{C}}$ is the multiplicity 1 varifold associated with a hyperplane or the multiplicity 1 varifold associated with the union of 2 distinct hyperplanes or the multiplicity 2 varifold associated with a hyperplane, we see that $\mathbf{C}$ would have to be immersed at $(X_1, 0)$ contrary to our assumption concerning X_1. Thus, there must be a point $\widetilde{X}_1 \in \operatorname{spt} \|\widetilde{\mathbf{C}}'\| \setminus \{0\}$ at which $\operatorname{spt} \|\widetilde{\mathbf{C}}'\| \setminus \{0\}$ is not immersed, so we may take a tangent cone to $\widetilde{\mathbf{C}}$ at $(\widetilde{X}_1, 0)$ and repeat the argument. After doing this a finite number of times, we produce a cylindrical stationary cone not equal to the union of hyperplanes but having cross sectional dimension 1 and equal to the limit of a sequence stable minimal hypersurfaces with singular sets of vanishing $(n-2)$-dimensional Hausdorff measure. But we have established above that such a cone does not exist. This concludes the proof of part (a).

To see the assertion in part (b), first consider the case when $\Theta\left(\|\mathbf{C}\|, 0\right) \leq 2$. In this case, $\Theta\left(\|\mathbf{C}'\|, 0\right) \leq 2$ and $\Theta\left(\|\mathbf{C}'\|, X\right) < 2$ for every $X \in \operatorname{spt} \|\mathbf{C}'\| \setminus \{0\}$, and hence if $\operatorname{spt} \|\mathbf{C}'\| \setminus \{0\}$ is regular (in which case it would be embedded), J. Simons' theorem says that $\mathbf{C}$ must be a multiplicity 1 hyperplane unless $d = 1$, in which case, since $\Theta\left(\|\mathbf{C}\|, 0\right) = 2$, the desired conclusion follows from the same argument as for the case $d = 1$ in part (a) above. We can rule out the possibility that $d \geq 2$ and there is a point $X_1 \in \operatorname{spt} \|\mathbf{C}'\|$ where $\operatorname{spt} \|\mathbf{C}'\|$ is not immersed, by arguing exactly as before.

To show the existence of an ϵ as asserted in the lemma, we argue by contradiction. If there were no such ϵ, then there would exist a sequence of cones $\mathbf{C}_k$, $k = 1, 2, \ldots$ in $\mathbf{R}^{n+1}$, each arising as the varifold limit of a sequence of immersed stable minimal hypersurfaces M_k^j, $j = 1, 2, \ldots$ of $B_2^{n+1}(0)$ with $\mathcal{H}^{n-2}\left(\operatorname{sing} M_k^j\right) < \infty$ for each k and j, such that $\Theta\left(\|\mathbf{C}_k\|, 0\right) \leq 2 + k^{-1}$ and each $\mathbf{C}_k$ has the form $\mathbf{C}_k = \mathbf{C}_0^{(k)} \times \mathbf{R}^{n-p_k}$ for some $p_k \leq 6$; yet $\mathbf{C}_k$ is not a union of hyperplanes for any k. In view of the uniform mass bound (implied by the density hypothesis), we may extract a subsequence, which we will continue to denote $\mathbf{C}_k$, such that $\mathbf{C}_k \to \mathbf{C}$ for some cone $\mathbf{C}$ where the convergence is as varifolds. By continuity of mass under varifold convergence, we have that $\Theta\left(\|\mathbf{C}\|, 0\right) \leq 2$. Furthermore, $\mathbf{C}$ has the form $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}^{n-p}$ for some $p \leq 6$, so by the discussion of the previous paragraph we see that $\mathbf{C}$ is either a multiplicity 1 hyperplane or the sum of two multiplicity 1 hyperplanes. Hence by Allard's regularity theorem, Theorem 1.1 of [**Wic**] or Theorem 1.1 of the present paper, we must have that for each sufficiently large k and each sufficiently large j (depending on k) $M_k^j \cap B_1^{n+1}(0)$ must either be a (single valued) $C^{1,\alpha}$ graph over a hyperplane, or the union of two (single valued) $C^{1,\alpha}$ graphs over a pair of hyperplanes, or a 2 valued $C^{1,\alpha}$ graph over a hyperplane, with an interior estimate, in each case, for the $C^{1,\alpha}$ norm of the function(s) defining the graph(s) over a ball in terms of the L^2 norm of the function(s) over a larger ball. But this means that for all sufficiently large k, $\operatorname{spt} \|\mathbf{C}_k\| \cap B_1^{n+1}(0)$ must either be immersed or equal to a 2 valued $C^{1,\alpha}$ graph. In all cases, by taking the tangent cone at the origin (which on the one hand must be equal to the tangent plane(s) to the graph at the origin and on the other hand coincide with $\mathbf{C}_k$ since $\mathbf{C}_k$ is already a cone), we see that $\mathrm{spt}\,\|\mathbf{C}_k\|$ must be the union of at most 2 hyperplanes, contrary to the assumption. The lemma is thus proved. q.e.d.

Proof of Theorem 1.3. First note that by Allard's varifold compactness theorem ([**All72**], [**Sim83**]), we obtain a stationary integral varifold V of $B_2^{n+1}(0)$ such that for some subsequence of $\{M_k\}$ which we continue to denote $\{M_k\}$, we have $M_k \to V$ as varifolds. Next we claim that there exists $\sigma = \sigma(n, \delta) \in (0, 1/2)$ such that

$$\frac{\mathcal{H}^n(M_k \cap B_1^{n+1}(X))}{\omega_n} \leq 3 - \delta/2 \tag{8.1}$$

for all k and all $X \in M_k \cap B_\sigma^{n+1}(0)$. To see this, fix any k and suppose that $X \in M_k \cap B_{1/2}^{n+1}(0)$. Then by the monotonicity of mass ratio, we have that

$$\begin{aligned} \frac{\mathcal{H}^n\left(M_k \cap B_1^{n+1}(X)\right)}{\omega_n} &\leq \frac{\mathcal{H}^n\left(M_k \cap B_{1+|X|}^{n+1}(0)\right)}{\omega_n} \\ &= (1+|X|)^n \frac{\mathcal{H}^n\left(M_k \cap B_{1+|X|}^{n+1}(0)\right)}{\omega_n(1+|X|)^n} \\ &\leq (1+|X|)^n \frac{\mathcal{H}^n\left(M_k \cap B_2^{n+1}(0)\right)}{\omega_n 2^n} \\ &\leq (1+|X|)^n(3-\delta), \end{aligned} \tag{8.2}$$

which readily implies (8.1) provided $X \in M_k \cap B_\sigma^{n+1}(0)$ for a suitable choice of $\sigma = \sigma(n, \delta) \in (0, 1/2)$. It then follows that $\frac{\mathcal{H}^n(\mathrm{spt}\,\|V\| \cap B_1^{n+1}(X))}{\omega_n} \leq 3 - \delta/2$ for all $X \in \mathrm{spt}\,\|V\| \cap B_\sigma^{n+1}(0)$, so that $\Theta\left(\|V\|, X\right) \leq 3 - \delta/2$ for all $X \in \mathrm{spt}\,\|V\| \cap B_\sigma^{n+1}(0)$. Hence, if $X \in \mathrm{spt}\,\|V\| \cap B_\sigma^{n+1}(0)$ is a singular point of $\mathrm{spt}\,\|V\| \cap B_\sigma^{n+1}(0)$ and $\mathbf{C}$ is any tangent cone to V at X having, after a rotation, the form $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}^{n-p}$ for some $p \in \{1, 2, \ldots, n\}$, then by Lemma 8.1, we must have, in case $n \geq 7$, that $p \geq 7$; otherwise, Lemma 8.1 says that $\mathbf{C}$ must be a union of hyperplanes, and since $\Theta\left(\|\mathbf{C}\|, 0\right) = \Theta\left(\|V\|, X\right) < 3$, it must either be a multiplicity 1 hyperplane, a multiplicity 2 hyperplane or a transverse pair of hyperplanes, in all of which cases, by Allard's regularity theorem, Theorem 1.2 of the present paper or Theorem 1 of [**Wic**], $\mathrm{spt}\,\|V\|$ would be a regular immersed submanifold near X, contrary to the hypothesis that X is a singular point. Hence, in case $n \geq 7$, Federer's dimension reducing principle implies that $\dim \mathrm{sing}\,\mathrm{spt}\,\|V\| \cap B_\sigma^{n+1}(0) \leq n - 7$. In case $2 \leq n \leq 6$, Lemma 8.1 says that any tangent cone at any point $X \in \mathrm{spt}\,\|V\| \cap B_\sigma^{n+1}(0)$ is either a multiplicity 1 hyperplane, a multiplicity 2 hyperplane or a transverse pair of hyperlanes, so that X must be a regular point of spt $\|V\|$.

It remains to show that when $n = 7$, $\operatorname{sing}\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(0)$ is discrete. This follows from the standard argument. Were it not true, there exist singular points X and X_j, $j = 1, 2, \ldots$, of spt $\|V\| \cap B^{n+1}_\sigma(0)$ such that $X_j \neq X$ for all j and $X_j \to X$. Let $\rho_j = |X - X_j|$. Then after passing to a subsequence, $\eta_{X,\rho_j\,\#}\, V \to \mathbf{C}$, where $\mathbf{C}$ is a cone with singularities at the origin and at a point $Y = \lim_{j\to\infty} \frac{X - X_j}{\rho_j} \in \operatorname{spt}\|\mathbf{C}\| \cap \mathbf{S}^{n-1}$. (This last claim that Y is a singular point of $\mathbf{C}$ follows from the appropriate regularity theorem—i.e., Allard's theorem, Theorem 1.2 of the present paper or Theorem 1 of [**Wic**].) But then since $\mathbf{C}$ is a cone, this means that the entire ray defined by Y consists of singularities of $\mathbf{C}$, which is impossible since in dimension $n = 7$, we have just shown that the singular set is 0-dimensional. This concludes the proof of the theorem. q.e.d.

Proof of Theorem 1.5. Let $\epsilon = \epsilon(n) \in (0, 1)$ be as in Lemma 8.1, part (b), and choose $\sigma = \sigma(n) \in (0, 1/2)$ as in the proof of Theorem 1.3 (i.e., via the estimate (8.2)), so that $\Theta(\|V\|, X) \leq 2 + \epsilon/2$ for all $X \in \operatorname{spt}\|V\| \cap B^{n+1}_\sigma(0)$. Let B be the set of branch points of $\operatorname{spt}|V\| \cap B^{n+1}_\sigma(0)$. Thus

$$B = \{Z \in \operatorname{sing} V \cap B^{n+1}_\sigma(0) \;:\; V \text{ has a (unique) multiplicity 2 tangent plane at } Z\}.$$

Set $S = \operatorname{sing} V \cap B^{n+1}_\sigma(0) \setminus B$. Then $\operatorname{sing} V \cap B^{n+1}_\sigma(0) = B \cup S$, $B \cap S = \emptyset$ by definition, and by Theorem 1.1, S is relatively closed in spt $\|V\| \cap B^{n+1}_\sigma(0)$. By Theorem 1.2, it follows readily that if $\mathcal{H}^{n-2}(B) = 0$, then $B = \emptyset$. To estimate the Hausdorff dimension of S, we proceed as follows. Consider an arbitrary point $Z \in S$. Let $\mathbf{C}$ be any tangent cone to V at Z. Then by the definition of S and Theorem 1 of [**Wic**], $\mathbf{C}$ cannot be equal to a pair of hyperplanes. Hence, if $2 \leq n \leq 6$, it follows from Lemma 8.1, part (b) that $S = \emptyset$. If $n \geq 7$, Lemma 8.1, part (b) says that, after a rotation, $\mathbf{C} = \mathbf{C}_0 \times \mathbf{R}^{n-p}$ for some $p \geq 7$. It then follows by the dimension reducing principle of Federer that

$$\mathcal{H}^{n-7+\gamma}(S) = 0 \tag{8.3}$$

for every $\gamma > 0$.

It only remains to show that S is finite when $n = 7$. To see this, suppose S is an infinite set. Then there exists a point $Z \in \operatorname{spt}\|V\| \cap \overline{B}^{n+1}_\sigma(0)$ and a sequence of points $Z_j \in S$ with $Z_j \neq Z$ for each j, such that $Z_j \to Z$ as $j \to \infty$. Let $r_j = |Z_j - Z|$ and $V_j = \eta_{Z,r_j\,\#}\, V$. Then, after passing to a subsequence, $V_j \to \mathbf{C}$ as varifolds, where $\mathbf{C}$ is a cone. Let $\zeta_j = r_j^{-1}(Z_j - Z)$. Then $\zeta_j \in \operatorname{sing} V_j \cap \mathbf{S}^n$, and hence, after passing to a further subsequence, $\zeta_j \to \zeta \in \operatorname{sing} \mathbf{C} \cap \mathbf{S}^n$. Now write $\operatorname{sing} \mathbf{C} \cap B_\sigma^{n+1}(0) = B_{\mathbf{C}} \cup S_{\mathbf{C}}$, where $B_{\mathbf{C}}$ is the set of branch points of $\mathbf{C}$ in $B_\sigma^{n+1}(0)$ (thus each point of $B_{\mathbf{C}}$ is a singular point of $\mathbf{C}$ where $\mathbf{C}$ has a unique multiplicity 2 tangent plane) and $S_{\mathbf{C}}$ is the complement of $B_{\mathbf{C}}$ in $\operatorname{sing} \mathbf{C} \cap B_\sigma^{n+1}(0)$. Similarly, write $\operatorname{sing} V_j \cap B_\sigma^{n+1}(0) = B_{V_j} \cup S_{V_j}$ with B_{V_j}, S_{V_j} having analogous meaning. Then $\zeta_j \in S_{V_j}$ since $Z_j \in S$. By (8.3),

$$\mathcal{H}^\gamma \left(S_{\mathbf{C}}\right) = 0 \tag{8.4}$$

for each $\gamma > 0$. On the other hand, since $\mathbf{C}$ is a cone and $\zeta \in \operatorname{sing} \mathbf{C} \cap \mathbf{S}^n$, we have that $\{t\zeta \,:\, t > 0\} \subset \operatorname{sing} \mathbf{C}$. In fact, we must have that

$$\{t\zeta \,:\, t > 0\} \cap B_\sigma^{n+1}(0) \subset S_{\mathbf{C}}. \tag{8.5}$$

For if not, $\zeta \in B_{\mathbf{C}}$ in which case $\mathbf{C}$ would have a (unique) multiplicity 2 tangent plane at ζ, and since $V_j \to \mathbf{C}$, by Theorem 1.1, it follows that for all sufficiently large j, $\operatorname{spt} \|V_j\|$ is a 2-valued $C^{1,\alpha}$ graph in some neighborhood of ζ. But this contradicts the fact that $\zeta_j \in S_{V_j}$. Hence we must have (8.5), but this contradicts the dimension estimate (8.4). This concludes the proof of the lemma. q.e.d.

9. Some further corollaries

Theorem 9.1. *Let $\delta \in (0,1)$. There exist positive numbers Γ and σ depending only on δ such that if $2 \leq n \leq 6$ and M is a an immersed, stable minimal hypersurface of $B_2^{n+1}(0)$ satisfying $\mathcal{H}^{n-2}(\operatorname{sing} M) = 0$ and $\frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 3 - \delta$, then $\operatorname{sing} M \cap B_\sigma^{n+1}(0) = \emptyset$ and*

$$\sup\nolimits_{M \cap B_\sigma^{n+1}(0)} |A| \leq \Gamma,$$

where A denotes the second fundamental form of M and $|A|$ the length of A.

Remark. If M is assumed to be embedded, this result holds with mass bound arbitrary, and is due to R. Schoen and L. Simon [**SS81**]. In dimensions $2 \leq n \leq 5$, provided we *assume* $\operatorname{sing} M = \emptyset$, the result (for M immersed) holds with mass bound arbitrary, and is due to R. Schoen, L. Simon and S.-T. Yau [**SSY75**].

Proof. Set $\sigma_1 = \min\{\sigma(1,\delta), \ldots, \sigma(6,\delta)\}$ and $\sigma = \sigma_1/4$, where $\sigma(n,\delta)$ is as in Theorem 1.3. Then it follows directly by taking $M_k = M$ in Theorem 1.3 that $\operatorname{sing} M \cap B_{\sigma_1}^{n+1}(0) = \emptyset$, so we only need to prove the curvature estimate.

If there is no such Γ, then for some n with $2 \leq n \leq 6$ and some $\delta \in (0,1)$, there exists a sequence $\{M_k\}$ of stable minimal hypersurfaces immersed in $B_2^{n+1}(0)$ with $0 \in M_k$, satisfying $\mathcal{H}^{n-2}(\operatorname{sing} M_k) = 0$ (or we may assume $\operatorname{sing} M_k \cap B_1^{n+1}(0) = \emptyset$ if we wish, in view of the preceding paragraph) and $\frac{\mathcal{H}^n(M_k)}{\omega_n 2^n} \leq 3 - \delta$ for each k; yet there exists a point $Z_k \in M_k \cap B^{n+1}_\sigma(0)$ for each k with

$$|A_k|(Z_k) \to \infty, \tag{9.1}$$

where A_k denotes the second fundamental form of M_k and $|A_k|$ its length. By Theorem 1.3, there exists a stationary varifold V of $B^{n+1}_2(0)$ such that after passing to a subsequence, which we continue to denote $\{M_k\}$, we have that $M_k \to V$ as varifolds, and that $\operatorname{spt}\|V\| \cap B^{n+1}_{\sigma_1}(0) = M$ where M is a smooth (i.e., having $\operatorname{sing} M = \emptyset$) stable minimal hypersurface of $B^{n+1}_{\sigma_1}(0)$; since varifold convergence implies convergence (of the supports of the weight measures) in Hausdorff distance, we also have that $Z_k \to Z$ for some $Z \in M \cap \overline{B}^{n+1}_\sigma(0)$. But since M is a regular immersed hypersurface, and the density of M at X is $\leq 3 - \delta$ for every $X \in M \cap B^{n+1}_{\sigma_1/2}(0)$, the tangent cone to M at Z is either a multiplicity 1 plane, or a multiplicity 2 plane, or a transversely intersecting pair of hyperplanes. Applying respectively Allard's regularity theorem, Theorem 1.2 or Theorem 1 of [**Wic**], we conclude that there exists a fixed radius $\rho > 0$ independent of k such that in each of these cases, for all sufficiently large k, we have that

$$\sup{}_{M_k \cap B^{n+1}_\rho(Z)} |A_k| \leq \frac{C}{\rho}$$

for some fixed constant $C = C(n)$ independent of k. But this contradicts (9.1). The theorem is thus proved. q.e.d.

Theorem 9.2 (A Bernstein type theorem)**.** *Let $\delta \in (0,1)$. Suppose $2 \leq n \leq 6$, M is a complete, non-compact stable minimal hypersurface of $\mathbf{R}^{n+1}$ satisfying $\frac{\mathcal{H}^n(M \cap B^{n+1}_R(0))}{\omega_n R^n} \leq 3 - \delta$ for all $R > 0$. Then M must be a union of affine hyperplanes.*

Remark. This is a slight generalization of the Bernstein type theorem in [**Wic04c**], which asserts the existence of a number $\epsilon \in (0,1)$ such that the conclusion of the theorem is true whenever $2 \leq n \leq 6$ and $\frac{\mathcal{H}^n M \cap B^{n+1}_R(0)}{\omega_n R^n} \leq 2 + \epsilon$ for all $R > 0$.

Proof. By Theorem 9.1, $\sup_{B^{n+1}_{\sigma R}(0)} |A| \leq \frac{\Gamma}{R}$ for all $R > 0$, where $\sigma > 0$ and Γ are independent of R. Let $R \to \infty$. q.e.d.

The following result is an improvement of Lemma 1 of [**SS81**]. Note that our proof of it below uses the regularity theory; Lemma 1 of [**SS81**] on the other hand was used in *proving* the regularity theorem of [**SS81**], and it would be interesting to see if the result below has a proof independent of the regularity theory.

Theorem 9.3. *Let $p \in (0, 4 + \sqrt{8/n})$, $\Lambda > 0$ and $\theta \in (0,1)$. There exists a constant $C = C(n,p,\Lambda,\theta)$ such that if M is an embedded, stable* *minimal hypersurface of* $B_1^{n+1}(0)$ *with* $0 \in \overline{M}$, $\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty$ *and* $\mathcal{H}^n(M) \leq \Lambda$, *then*

$$\int_{M \cap B_\theta^{n+1}(0)} |A|^p \leq C \left(\int_{M \cap B_1^{n+1}(0)} 1 - (\nu \cdot \nu_0)^2 \right)^{p/2}$$

for any unit vector $\nu_0 \in \mathbf{R}^{n+1}$. *Here* A *denotes the second fundamental form of* M *and* ν *the unit normal vector to* M.

The estimate continues to hold if M *is immersed provided* $\Lambda = \omega_n(3-\delta)$ *for some* $\delta \in (0,1)$ *and* $\mathcal{H}^{n-2}(\operatorname{sing} M) = 0$.

Proof. We argue by contradiction. If the estimate were not true for some Λ, $p \in [4, 4+\sqrt{8/n})$ and $\theta \in (0,1)$, then there exists a sequence of stable minimal hypersurfaces M_k of $B_1^{n+1}(0)$ with $0 \in \overline{M}_k$, $\mathcal{H}^n(M_k) \leq \Lambda$ and

$$\int_{M_k \cap B_\theta^{n+1}(0)} |A_k|^p \geq k \left(\int_{M_k \cap B_1^{n+1}(0)} 1 - (\nu_k \cdot \nu_0^k)^2 \right)^{p/2}, \tag{9.2}$$

where ν_0^k are unit vectors in $\mathbf{R}^{n+1}$. Note that under the assumptions of the theorem, $\mathcal{H}^{n-7+\gamma}(\operatorname{sing} M_k) = 0$ for each $\gamma > 0$ if $n \geq 7$ and $\operatorname{sing} M_k = \emptyset$ if $2 \leq n \leq 6$, which follows from Theorem 3 of [**SS81**] if M_k are embedded, and from Theorem 1.3 above if M_k are immersed and $\Lambda = \omega_n(3-\delta)$. By the Schoen-Simon-Yau ([**SSY75**]) integral curvature estimate (which was originally proved for smooth, stable minimal hypersurfaces but continues to hold for stable minimal hypersurfaces M with singularities provided $\mathcal{H}^{n-p}(\operatorname{sing} M) < \infty$, as can be seen using an easy cut-off function argument), we have that

$$\int_{M_k \cap B_\theta^{n+1}(0)} |A_k|^p \leq C \tag{9.3}$$

where C is a constant that depends only on n, p, Λ and θ. From (9.2) and (9.3), it follows that

$$\int_{M_k \cap B_1^{n+1}(0)} 1 - (\nu_k \cdot \nu_0^k)^2 \to 0. \tag{9.4}$$

Since mass of M_k is uniformly bounded, Allard's compactness theorem says that after passing to a subsequence, $M_k \to V$ for some stationary varifold V of $B_1^{n+1}(0)$, and (9.4) says that V must be a hyperplane with some positive integer multiplicity. Let us assume without loss of generality that this hyperplane is $\mathbf{R}^n \times \{0\}$. Now in case M_k are embedded, by the Schoen-Simon regularity theorem, this means that for all sufficiently large k, $M_k \cap (B_{\frac{1+\theta}{2}}(0) \times \mathbf{R})$ decomposes as the (disjoint) union of graphs of m_k functions $u_i^k : B_{\frac{1+\theta}{2}}(0) \to \mathbf{R}$, $1 \leq i \leq m_k$, (with m_k bounded independently of k by a number depending on Λ), each having small gradient and solving the minimal surface equation. In the immersed case under the stronger mass bound, by Theorem 1.2, the same conclusion holds with $m_k \leq 2$.

Now let L_k be the hyperplane determined by the unit vector ν_0^k, and $l_k : \mathbf{R}^n \times \{0\} \to \mathbf{R}$ be the linear function whose graph is L_k. (Note that $\nu_0^k \cdot e^{n+1} \to 1$.) Then $u_i^k - l_k$ solves a uniformly elliptic equation over $B_{\frac{1+\theta}{2}}(0)$, and so by elliptic estimates, we have a constant $C = C(n, \theta)$ such that $\sup_{B_{\frac{1+3\theta}{4}}(0)} |D^2 u_i^k| \leq C \|Du_i^k - Dl_k\|_{L^2(B_{\frac{1+\theta}{2}}(0))}$ and $\sup_{B_{5/8}(0)} |Du_i^k - Dl_k| \leq C\|Du_i^k - Dl_k\|_{L^2(B_{3/4}(0))}$ for each i. But this means that $\sup_{M_k \cap B_\theta^{n+1}(0)} |A_k| \leq C \left(\int_{M_k \cap B_1^{n+1}(0)} 1 - (\nu_k \cdot \nu_0^k)^2\right)^{1/2}$ where $C = C(n, \Lambda, \theta)$, which contradicts (9.2). q.e.d.

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