ON THE TOPOLOGY OF BIRATIONAL MINIMAL MODELS

CHIN-LUNG WANG

Dedicated to Hui-Wen and Chung-Yang

Introduction

In the study of higher dimensional algebraic geometry, an important reduction step is to study certain good birational models of a given algebraic manifold. This leads to the famous "minimal model program" initiated by Mori – the search for birational models with numerically effective canonical divisors and at most terminal singularities. The existence problem is still completely open in dimensions higher than three, but even worse, in contrast to the two-dimensional case, the minimal model is not unique in higher dimensions. It is then an important question to see what kind of invariants are shared by all the birationally equivalent minimal models, and more generally, are preserved under certain elementary birational transformations. In this paper, some results in this direction are given:

Theorem A. *Let* $f \colon X \dashrightarrow X'$ *be a birational map between two smooth complex projective varieties such that the canonical bundles are numerically effective along the exceptional loci. Then X and X' have the same Betti numbers. In particular, birational smooth minimal models have the same Betti numbers.*

Theorem A, in the smooth case, generalizes previous results of Kollár on the invariance of cohomologies under flops in dimension three (cf. 5.1). Another interesting corollary via the Mayer-Vietoris argument shows that the exceptional loci of the given birational map also share the same Betti numbers (Corollary 4.5).

The proof of Theorem A is based on general considerations in birational geometry given in §1 and Grothendieck-Deligne's solution to the Weil conjecture [4], [5]. The bridge to connect these two is the theory of p-adic integrals.

The idea to use the Weil conjecture via p-adic integrals to compute cohomologies can be dated back to Harder and Narasimhan in the 70's [9]. However, it was used there in a somewhat different way. In the context of Igusa-Weil local zeta functions, the p-adic integral has been studied extensively by Denef and Loeser since late 80's [12], [6]. Recently this was taken up again by Batyrev by developing Weil's idea of p-adic measure [1]. In fact, he established Theorem A in the special case of projective Calabi-Yau manifolds.

By extending this idea further, an argument based on birational correspondences is developed here in order to deal with the general case. Namely, we introduce in §1 the notion of "K-partial ordering" and relate it to interesting geometric situations. The applicability of the Weil conjecture is largely clarified in terms of this notion (cf. Proposition 2.16 and Theorem 3.1). Moreover, this approach also provides a natural setting in the singular case.

In this paper, We have tried to develop this, together with the p-adic measure, as far as possible so that it could fit the need of the minimal model theory. In fact, an easy but very interesting result observed here is that the integral points of a p-adic variety has finite p-adic measure if and only if it has at most log-terminal singularities (Proposition 2.12). This gives the basic reason why p-adic integrals fit into the framework of minimal model theory naturally. But due to technical reasons, we have restricted ourself to the smooth case when we state and prove Theorem A. (See however 5.3 for the singular case.)

Theorem A is still not all satisfactory in two aspects – the torsion elements are not considered, and no natural maps between cohomologies have been mentioned. Although there is one obvious candidate for this map – the cohomology correspondence induced from the birational correspondence, it is not clear how to show directly that it induces isomorphisms. In fact, there is no strong evidence why this should be true. The next result only deals with the simplest cases. However, it is included to emphasize this important aspect.

Theorem B. *Smooth minimal models minimize $H^2(X, \mathbf{Z})$ compatible with the Hodge structure among birational smooth projective varieties. In the singular case, the minimal models minimize the group of*

Weil divisors among birational projective varieties with at most terminal singularities.

The proof, which is elementary (does not use the Weil conjecture), is contained in §4 together with some related results. In fact, it is simply another application of the notion of K-partial ordering.

To finish the introduction, it is worth pointing out that in stating both theorems, what we have in mind is that there should be a "minimal cohomology theory" among birational varieties. Moreover, it should be realized exactly by the minimal models.

1. Birational geometry

We begin with some standard definitions. For a complete treatment of minimal model theory, the reader should consult [13].

Let X be an n-dimensional complex normal $\mathbf{Q}$-Gorenstein variety, that is, the canonical divisor K_X is $\mathbf{Q}$-Cartier. Recall that X has (at most) terminal (resp. canonical, resp. log-terminal) singularities if there is a resolution $\phi : Y \to X$ such that in the canonical bundle relation

$$(1.1) \qquad\qquad K_Y =_{\mathbf{Q}} \phi^* K_X + \sum a_i E_i,$$

we have that $a_i > 0$ (resp. $a_i \geq 0$, resp. $a_i > -1$) for all i. Here, the E_i's vary among the prime components of all the exceptional divisors. Although (1.1) holds only up to $\mathbf{Q}$-linear equivalence, the divisor $\sum a_i E_i \in Z_{n-1} \otimes \mathbf{Q}$ is uniquely determined. Moreover, the condition on a_i's is readily seen to be independent of the chosen resolution. It is also elementary to see that smooth points are all terminal.

Let Z be a proper subvariety of X. A $\mathbf{Q}$-Cartier divisor D is called numerically effective (nef) along Z if $D.C := \deg_{\tilde{C}}(f^* D) \geq 0$ for all effective curves $C \subset Z$, where $f: \tilde{C} \to C$ is the normalization of C. D is simply called nef if $Z = X$. A projective variety X is called a minimal model if X is terminal and K_X is nef.

Two normal varieties X and X' are birational if they have isomorphic function fields $K(X) \cong K(X')$ (over $\mathbf{C}$). Geometrically, this means that there is a rational map $f: X \dashrightarrow X'$ such that f^{-1} is also rational. The exceptional loci of f are defined to be the smallest subvarieties $Z \subset X$ and $Z' \subset X'$ such that f induces an isomorphism $X - Z \cong X' - Z'$.

Among the class of birational $\mathbf{Q}$-Gorenstein varieties, we have the notion of **K-partial ordering**, where the "K" is for canonical divisors:

Definition 1.2. For two $\mathbf{Q}$-Gorenstein varieties X and X', we say that $X \leq_K X'$ (resp. $X <_K X'$) if there is a birational correspondence $(\phi, \phi') : X \leftarrow Y \rightarrow X'$ with Y smooth, such that $\phi^* K_X \leq_{\mathbf{Q}} \phi'^* K_{X'}$ (resp. "$<_{\mathbf{Q}}$"). Moreover, "$X \leq_K X'$" plus "$X \geq_K X'$" implies that "$X =_K X'$", i.e., $\phi^* K_X =_{\mathbf{Q}} \phi'^* K_{X'}$. In this case, we say that X and X' are K-equivalent.

The well-definedness of this notion follows from the canonical bundle relations

$$(1.3) \qquad K_Y =_{\mathbf{Q}} \phi^* K_X + E =_{\mathbf{Q}} \phi'^* K_{X'} + E',$$

since we know that $X \leq_K X'$ if and only if $E \geq E'$. In the terminal case, this means that ϕ has more exceptional divisors than ϕ' (so heuristically, X is "smaller" than X').

Here is the typical geometric situation that we can compare their K-partial order:

Theorem 1.4. *Let $f : X \dashrightarrow X'$ be a birational map between two varieties with canonical singularities. Suppose that the exceptional locus $Z \subset X$ is proper and that K_X is nef along Z. Then $X \leq_K X'$. Moreover, if X' is terminal, then Z has codimension at least two.*

Proof. Let $\phi : Y \rightarrow X$ and $\phi' : Y \rightarrow X'$ be a good common resolution of singularities of f so that the union of the exceptional set of ϕ and ϕ' is a normal crossing divisor of Y. This can be done by considering $\bar{\Gamma}_f \subset X \times X'$, the closure of the graph of f, blowing up the exceptional set of $\bar{\Gamma}_f \rightarrow X$ and $\bar{\Gamma}_f \rightarrow X'$ and then taking Y to be a Hironaka (embedded) resolution [10].

Consider the canonical bundle relations:

$$(1.5) \qquad \begin{aligned} K_Y &=_{\mathbf{Q}} \phi^* K_X + E \equiv \phi^* K_X + F + G \\ &=_{\mathbf{Q}} \phi'^* K_{X'} + E' \equiv \phi'^* K_{X'} + F' + G'. \end{aligned}$$

Here F and F' denote the sum of divisors (with coefficients ≥ 0) which are both ϕ and ϕ' exceptional. G (resp. G') denotes the part which is ϕ exceptional but not ϕ' exceptional (resp. ϕ' but not ϕ exceptional). Notice that $\phi(G') \subset Z$.

To proceed, we write

$$(1.6) \qquad \phi'^* K_{X'} =_{\mathbf{Q}} \phi^* K_X + G + (F - F' - G').$$

It is enough to prove that $F - F' - G' \geq 0$, because this implies that $F - F' \geq 0$ and $G' = 0$, and so $E \geq E'$.

By taking a generic hyperplane section H of Y $(n-2)$-times, the problem is reduced to a problem on surfaces. Namely,

$$(1.7) \qquad H^{n-2}.\phi'^{*}K_{X'} =_{\mathbf{Q}} H^{n-2}.\phi^{*}K_{X} + \zeta + (\xi - \xi' - \zeta'),$$

where $\xi = H^{n-2}.F$ and $\zeta = H^{n-2}.G$ etc. If $\xi - \xi' - \zeta'$ is not effective, write it as $H^{n-2}.(A - B) = a - b$ with A and B effective. Then by taking the intersection of (1.5) with b, we get

$$(1.8) \qquad B.H^{n-2}.\phi'^{*}K_{X'} =_{\mathbf{Q}} B.H^{n-2}.\phi^{*}K_{X} + b.\zeta + b.a - b^{2}.$$

The left-hand side is zero since $B \subset E'$ is ϕ' exceptional. Moreover, if $B \subset F'$, then $B.H^{n-2}.\phi^{*}K_{X} = 0$. If $B \subset G'$, then the curve $\phi(B.H^{n-2}) \subset \phi(G') \subset Z$ is inside the exceptional locus. So the first three terms on the right-hand side are non-negative since K_{X} is nef along Z and a, b and ξ are different components. However, since b is a nontrivial combination of exceptional curves in H^{n-2}, from the Hodge index theorem we have that $b^{2} < 0$, a contradiction! Hence $F - F' - G' \geq 0$.

For the second statement, from the construction of Y, we know that all components of the exceptional sets, denoted by $\operatorname{Exc}\phi$ and $\operatorname{Exc}\phi'$ respectively, are divisors. If X' is assumed to be terminal, then all ϕ' exceptional divisors occur as components of E'. So $G' = 0$ implies that $\operatorname{Exc}\phi' \subset \operatorname{Exc}\phi$. With this understood, from

$$(1.9) \quad X - \phi(\operatorname{Exc}\phi) \cong Y - \operatorname{Exc}\phi \cong X' - \phi'(\operatorname{Exc}\phi) \subset X' - \phi'(\operatorname{Exc}\phi'),$$

we conclude that $Z \subset \phi(\operatorname{Exc}\phi)$ is of codimension at least two.   q.e.d.

Corollary 1.10. *Let* $f\colon X \dashrightarrow X'$ *be a birational map between two terminal varieties such that* K_{X} *(resp.* $K_{X'}$*) is nef along the exceptional locus* $Z \subset X$ *(resp.* $Z' \subset X'$*). Then* $X =_{K} X'$ *and* f *is an isomorphism in codimension one. This applies, in particular, if both* X *and* X' *are minimal models.*

Variant 1.11. Instead of assuming that the exceptional locus in X is proper, one can generalize Theorem 1.4 to the relative case, namely f is a S-birational map and that $X \to S$ and $X' \to S$ are proper S-schemes. The proof is identical to the one given above by changing notation.

Remark 1.12. This type of argument is familiar in the minimal model theory. Notably, in analyzing the log-flip diagram (eg. [13, 5-1-11]) or more specially, the flops. Theorem 1.4 implies that if X' is a flip

of X, then $X \geq_K X'$ (in fact, more is true: $X >_K X'$). Corollary 1.10 yields that flop induces K-equivalence. Since flip/flop will not be used in any essential way in this paper, we will refer the interested reader to [13] for the definitions. The proof given above is inspired by Kollár's treatment of flops in [14].

2. The Weil Conjecture and p-adic integrals

To prove Theorem A, we will show that X and X' have the same number of rational points over certain finite fields when a suitable good reduction is taken. That is, we prove that they have the same "zeta function". The theorem will then follow from the statement of the Weil conjecture.

2.1. The reduction procedure. This is standard in algebraic geometry and in number theory: as long as we perform only a finite number of "algebraic constructions" in the complex case, e.g. consider morphisms, since all the objects involved can by defined by a finite number of polynomials, we can take $S \subset \mathbf{C}$ a finitely generated subring over $\mathbf{Z}$ so that everything is defined over S. S has the property that the residue field S/m of any maximal ideal $m \subset S$ is finite.

If we start with "smooth objects", general reduction theory then implies that for an infinite number of "good primes" (in fact, Zariski dense in $\mathrm{Spec}\,(S)$), we may get good reductions so that everything is defined smoothly over the finite residue field $\mathbf{F}_q$ with $q = p^r$ for some prime number p. We may also assume that this reduction has a lifting such that everything is defined smoothly over R, the maximal compact subring of a p-adic local field $\mathbf{K}$, i.e., a finite extension field of $\mathbf{Q}_p$, with residue field $\mathbf{F}_q$.

More precisely, let F be the quotient field of S. Based on the fact (and others) that $\mathbf{Q}_p$ has infinite transcendence degree, the "embedding theorem" (see for example [3, p.82]) yields that for an infinite number of p's, there is an embedding of fields $i : F \to \mathbf{Q}_p$ such that $i(S) \subset \mathbf{Z}_p$. Moreover, i may be chosen so that a prescribed finite subset of S, say the coefficients of those defining polynomials, is mapped into the set of p-adic units. This embedding then gives the desired lifting.

Let P be the unique maximal ideal of R (so $R/P \cong \mathbf{F}_q$). We denote by $\bar{X}$, $\bar{U}$, ... those objects constructed from X, U ... via reductions mod P, that is, objects lie over the point $\mathrm{Spec}\,R/P \to \mathrm{Spec}\,R$ – they are defined over $\mathbf{F}_q$. We also denote the reduction map by $\pi : X(R) \to$

$\bar{X}(\mathbf{F}_q)$ etc.

2.2. The Weil conjecture.

Let $\bar{X}$ be a variety defined over a finite field $\mathbf{F}_q$. After fixing an algebraic closure, the Weil zeta function of $\bar{X}$ is defined by

$$(2.3) \qquad Z(\bar{X}, t) := \exp\left(\sum_{k \geq 1} |\bar{X}(\mathbf{F}_{q^k})| \frac{t^k}{k} \right).$$

In 1949, Weil conjectured several nice properties of this zeta function for smooth projective varieties and expalined how some of these would follow once a suitable cohomology theory exists [16]. This led Grothendieck to his creation of étale cohomology theory.

More precisely, Grothendieck proved a "Lefschetz fixed point formula" in a very general context (eg. constructible sheaves over seperated schems of finite type ...) [5], which in particular implies that the zeta function is a rational function:

$$(2.4) \qquad Z(\bar{X}, t) = \frac{P_1(t) \cdots P_{2n-1}(t)}{P_0(t) P_2(t) \cdots P_{2n}(t)},$$

where $P_j(t)$ is a polynomial with integer coefficients such that $P_j(0) = 1$ and $\deg P_j(t) = h^j$, the j-th Betti number of certain étale cohomologies. Moreover, when $\bar{X}$ comes from a good reduction of a smooth complex projective variety X in the sense described in (2.1), h^j coincides with the j-th Betti number of the singular cohomologies of $X(\mathbf{C})$.

Deligne [4] completed the proof of the Weil conjecture by proving the important "Riemann Hypothesis" that all roots of $P_j(t)$ have absolute value $q^{-j/2}$. In particular, the complete information about the $\mathbf{F}_{q^k}$-rational points determines the h^j's and all the roots.

2.5. Counting points via p-adic integrals.

How do we count $\bar{X}(\mathbf{F}_q)$? If $\bar{X}$ comes from the good reduction of a smooth R-scheme, we will see that such a counting can be achieved by using p-adic integrals (cf. Theorem 2.8). We will first recall some elementary aspects of the p-adic integral over $\mathbf{K}$-analytic manifolds and over R-schemes.

Consider the Haar measure on the locally compact field $\mathbf{K}$, normalized so that the compact open "disk" R has volume 1:

$$(2.6) \qquad \int_R |dz| = 1.$$

We may extend this to the multivariable case and define the p-adic integral of any regular n form $\Psi = \psi(z_1, \cdots, z_n) dz_1 \wedge \cdots \wedge dz_n$ by

$$(2.7) \qquad \int_{R^n} |\Psi| := \int_{R^n} |\psi(z)| |dz_1 \wedge \cdots \wedge dz_n|.$$

Here $|a| := q^{-\nu_p(\mathrm{N}_{\mathbf{K}/\mathbf{Q}_p}(a))}$ is the usual p-adic norm.

We may define an integral slightly more general than (2.7): suppose that Ψ is a r-pluricanonical form such that in local analytic coordinates $\Psi = \psi(z_1, \cdots, z_n)(dz_1 \wedge \cdots \wedge dz_n)^{\otimes r}$. We define the integration of a "r-th root of $|\Psi|$" by

$$(2.7') \qquad \int_{R^n} |\Psi|^{1/r} := \int_{R^n} |\psi(z)|^{1/r} |dz_1 \wedge \cdots \wedge dz_n|.$$

This is independent of the choice of coordinates, as can be checked easily by the same method as in [17, p.14]. So we can extend the definition to (not necessarily complete) $\mathbf{K}$-analytic manifolds with Ψ a (possibly meromorphic) pluricanonical form. Certainly then the integral defined may not be finite.

The key property we need is the following (slightly more general form of a) formula of Weil [17, 2.2.5]. We briefly sketch its proof.

Theorem 2.8. *Let U be a smooth R-scheme and Ω a nowhere zero r-pluricanonical form on U. Then*

$$\int_{U(R)} |\Omega|^{1/r} = \frac{|\bar{U}(\mathbf{F}_q)|}{q^n}.$$

Proof. The proof given by Weil in [17] goes through without difficulties. One first observes that the reduction map $\pi \colon U(R) \to \bar{U}(\mathbf{F}_q)$ induces an isomorphism between $\pi^{-1}(\bar{t})$ and PR^n for any $\bar{t} \in \bar{U}(\mathbf{F}_q)$ (Hensel's lemma) such that there is a function ψ with $|\psi(z)| = 1$ and

$$(2.9) \qquad \Omega = \psi(z) \cdot (dz_1 \wedge \cdots \wedge dz_n)^{\otimes r}$$

in the $\mathbf{K}$-analytic chart PR^n. This implies that $\int_{\pi^{-1}(\bar{t})} |\Omega|^{1/r} = 1/q^n$ for any $\bar{t} \in \bar{U}(\mathbf{F}_q)$. Summing over $\bar{t}$ then gives the result.   q.e.d.

The right-hand side of (2.8) shows that the integral is independent of the choice of the form Ω. One may also see this by observing that any two such forms differ by a nowhere vanishing function on U (over

R) which takes values in the units on all R-points. This allows one to define a canonical p-adic measure on the R-points of smooth R-schemes by "gluing" the local integrals. We will define it in the singular case with the hope that it may be useful for later development.

2.10. Canonical measure on Q-Gorenstein R-schemes.

We will only consider those R-schemes, eg. X, that come from complex **Q**-Gorenstein varieties as in (2.1). Let $r \in \mathbf{N}$ such that rK_X is Cartier (locally free). We may assume that we have a R-resolution of singularities $\phi\colon Y \to X$, which is a projective R-morphism, so that the reduced part of the exceptional set is a simple normal crossing R-variety. We will define a measure on $X(R)$ such that the measurable sets are exactly the compact open subsets in the **K**-analytic topology.

Let U_i's be a Zariski open cover of X such that $rK_X|_{U_i}$ is actually free. Then for a compact open subset $S \subset U_i(R) \subset X(R)$, we define its measure by

$$(2.11) \qquad m_X(S) \equiv \int_S |\Omega_i|^{1/r} := \int_{\phi^{-1}(S)} |\phi^*\Omega_i|^{1/r},$$

where Ω_i is an arbitrary generator of $rK_X|_{U_i}$. Notice that the properness of ϕ implies that $\phi^{-1}(S) \subset Y(R)$. This allows us to operate the integral entirely on R-points.

For general compact open $S \subset X(R)$, we may break S into disjoint pieces S_j so that S_j is contained in some $U_i(R)$ (in fact, S_j may be chosen to lie entirely in a fiber of the reduction map π), and then define $m_X(S) = \sum_i m_X(S_i)$. Notice that $m_X(S)$ is again independent of the choice of U_i, Ω_i and Y.

The following proposition explains the possible connection between the canonical measure and the minimal model theory:

Proposition 2.12. *For a* **Q**-*Gorenstein R-variety X, $X(R)$ has finite measure if and only if X has at most log-terminal singularities.*

Proof. Consider the canonical bundle relation for $\phi\colon Y \to X$,

$$(2.13) \qquad rK_Y = \phi^* rK_X + \sum_i e_i E_i,$$

with rK_X being Cartier and $e_i \in \mathbf{Z}$. To determine the finiteness of $m_X(X(R))$, we only need to consider those R-points on the exceptional fibers. Locally, $\operatorname{div} \phi^*\Omega = \sum_i e_i E_i$ for a generator Ω of rK_X. So the integral is a product of one-dimensional integrals of the form

$$(2.14) \qquad I_i := \int_R |z^{e_i} dz^{\otimes r}|^{1/r} = \int_R |z|^{e_i/r} |dz|.$$

If this is finite, then

$$(2.15) \qquad I_i = \int_{PR} |z|^{e_i/r} \, |dz| + (q-1)\frac{1}{q} = q^{-(e_i/r+1)} I_i + \frac{q-1}{q}.$$

Since $I_i > 0$, this makes sense only if $q^{e_i/r+1} > 1$. That is, $e_i/r > -1$, which is exactly the definition of log-terminal singularities.   q.e.d.

Since the measure is defined Zariski-locally via p-adic integrals, for smooth X, from Weil's formula (2.8) we have

Corollary 2.16. *Let X be an n-dimensional smooth R-variety with finite residue field $\mathbf{F}_q$. Then*

$$m_X(X(R)) = \frac{|X(\mathbf{F}_q)|}{q^n}.$$

Remark 2.17. If X is singular, $m_X((X(R))$ is a weighted counting of the rational points. By definition, if $\phi\colon Y \to X$ is a crepant R-morphism, i.e., $K_Y =_{\mathbf{Q}} \phi^* K_X$, then $m_X((X(R)) = m_Y((Y(R))$. In particular, $m_X((X(R))$ counts the rational points of $\bar{Y}$ if Y is smooth! This applies to many interesting "pure canonical" singularities and to terminal singularities having small resolutions. However, further investigation on the precise "geometric meaning" of this weighted counting is still needed for the general case (cf. 5.3).

3. The Proof of Theorem A

We will in fact prove a result which connects the notion of K-partial ordering and the canonical measure. This will largely clarify the role played by the Weil conjecture.

Theorem 3.1. *Let X and X' be two birational log-terminal R-varieties. Then $m_X(X(R)) \leq m_{X'}(X'(R))$ if $X \leq_K X'$.*

Proof. Consider as before, a birational correspondence $(\phi, \phi') : X \leftarrow Y \to X'$ over R with Y a smooth R-variety. Let $r \in \mathbf{N}$ be such that both rK_X and $rK_{X'}$ are Cartier. Then $X \leq_K X'$ if and only if in the canonical bundle relations $rK_Y = \phi^* rK_X + E = \phi'^* rK_{X'} + E'$, we have $E \geq E'$.

By the properness of ϕ and ϕ', $\phi^{-1}(X(R)) = Y(R) = \phi'^{-1}(X'(R))$. So from the definition of the measure (2.11), it suffices to show that for

any compact open subset $T \subset Y(R)$ with $\pi(T)$ a single point $\bar{y} \in \bar{Y}(\mathbf{F}_q)$, we have

$$(3.2) \qquad \int_T |\phi^* \Omega|^{1/r} \leq \int_T |\phi'^* \Omega'|^{1/r}.$$

Here Ω is an arbitrary local generator of rK_X on a Zariski open set U where rK_X is actually free and such that $\bar{\phi}(\bar{y}) \in \bar{U}$ (and with similar conditions for Ω').

Clearly, (3.2) can fail to be an equality only if $\bar{y} \in \bar{E} \cup \bar{E}'$. However, in this case $E \geq E'$ implies that the order of $\phi^* \Omega$ is no less than that of $\phi^* \Omega$. (3.2) then follows from the definition of the p-adic integral $(2.7')$ (see also (2.15)).   q.e.d.

If X and X' are smooth, combining this with (2.16) gives

Corollary 3.3. *Let X and X' be two birational smooth R-schemes. Then $|\bar{X}(\mathbf{F}_q)| \leq |\bar{X}'(\mathbf{F}_q)|$ if $X \leq_K X'$.*

With this been done, by working on cyclotomic extensions of $\mathbf{K}$, the same proof shows that $|\bar{X}(\mathbf{F}_{q^k})| \leq |\bar{X}'(\mathbf{F}_{q^k})|$ for all $k \in \mathbf{N}$. In particular, $Z(\bar{X}, t) \leq Z(\bar{X}', t)$ for all $t > 0$. The same is true for all the derivatives, but it is not clear how to make use of these.

Corollary 3.4. *Let X and X' be two birational complex smooth varieties. They have the same Euler number for the compactly supported cohomologies if $X =_K X'$.*

Proof. Apply the reduction procedure (2.1) to reduce this to the p-adic case. The statement then follows from Grothendieck's Lefschetz fixed point formula (2.4) and the above comparison of zeta functions.

$$\text{q.e.d.}$$

What kind of geometric situation can we have $X \leq_K X'$? Theorem 1.4 provides such a typical case inspired by the minimal model theory. Namely, let $f \colon X \dashrightarrow X'$ be a birational map between two varieties with at most canonical singularities and proper exceptional locus $Z \subset X$ such that K_X is nef along Z. Then $X \leq_K X'$.

So far we have not used Deligne's theorem on the "Riemann Hypothesis". To use it, we need to impose the projective assumption.

Theorem 3.5. *Let X and X' be two birational smooth projective R-schemes. If $X =_K X'$ then $m_X(X(R)) = m_{X'}(X'(R))$. Equivalently, $Z(\bar{X}, t) = Z(\bar{X}', t)$. In particular, they have the same étale Betti numbers by the Weil conjecture.*

Now Theorem A simply follows from the reduction procedure (2.1), Corollary 1.10 and Theorem 3.5.   q.e.d.

Remark 3.6. In the preliminary version of this paper (dated October 1997), Theorem A was stated with the assumption that the canonical bundle is semi-ample, that is, rK_X is generated by global sections for some $r \in \mathbf{N}$. The proof proceeds by cutting out the pluri-canonical divisors and applying p-adic integrals to the birational correspondence, where the notion of K-equivalence is essential for this step to work.

By using Weil's formula (2.8), the proof is then concluded by induction on dimensions. In this approach, the usage of integration of a r-th root of the absolute value of a pluricanonical form was suggested to the author by C.-L. Chai in order to deal with the case $r > 1$.

Remark 3.7. The equivalence of zeta functions is a stronger statement than the equivalence of Betti numbers. Moreover, we have in fact established the equivalence of zeta functions for a dense set of primes. From the theory of motives, this suggests that we may in fact have the equivalence of Hodge structures. Further investigation in this should be interesting and important.

Question 3.8. Is Theorem A true for Kähler manifolds?

4. Miscellaneous results and the Proof of Theorem B

Now we come back to the complex number field and begin with an elementary observation:

Lemma 4.1. *If the exceptional loci of a birational map $f \colon X \dashrightarrow X'$ between two smooth projective varieties have codimension at least two, then for $i \leq 2$ we have $\pi_i(X) \cong \pi_i(X')$ and $H^i(X, \mathbf{Z}) \cong H^i(X', \mathbf{Z})$ which is compatible with the rational Hodge structures.*

Proof. The real codimension-four statement plus the transversality argument shows that $\pi_i(X) \cong \pi_i(X')$, $H_i(X, \mathbf{Z}) \cong H_i(X', \mathbf{Z})$ and $H^i(X, \mathbf{Z}) \cong H^i(X', \mathbf{Z})$ canonically for $i \leq 2$. Moreover, by Hartog's extension we know that the Hodge groups $H^0(\Omega^i)$ are all birational invariants among smooth varieties. The orthogonality of Hodge filtrations then shows that $H^i(X, \mathbf{Q})$ and $H^i(X', \mathbf{Q})$ share the same rational Hodge structures for $i \leq 2$.   q.e.d.

A slightly deeper result is given by

Proposition 4.2. *If the exceptional loci $Z \subset X$ and $Z' \subset X'$ of a birational map f between two smooth varieties have codimension at least two, then $h^i(X) - h^i(Z) = h^i(X') - h^i(Z')$.*

Proof. Construct a birational correpondence $X \leftarrow Y \rightarrow X$ as in §1 and denote the exceptional divisor of $\phi\colon Y \rightarrow X$ (resp. $\phi'\colon Y \rightarrow X'$) by E (resp. E'). Since Hironaka's resolution process only blows up smooth centers inside the singular set of the graph of f, the isomorphism $X - Z \cong X' - Z'$ implies that $\phi(E \cup E') \subset Z$ and $\phi'(E \cup E') \subset Z'$, hence that $E_{\mathrm{red}} = E'_{\mathrm{red}}$, $Z = \phi(E)$ and $Z' = \phi'(E')$.

Consider an open cover $\{V, W\}$ of X by letting $V := X - Z$ and $W \supset Z$ be a deformation retract neighborhood. Let $\tilde{V} := \phi^{-1}(V)$ and $\tilde{W} := \phi^{-1}(W) \supset E$ be the corresponding open cover of Y. Then we have the following commutative diagram of integral cohomologies:

$$(4.3) \quad \begin{array}{ccccccc} H^{i-1}(\tilde{V} \cap \tilde{W}) & \rightarrow & H^i(Y) & \rightarrow & H^i(\tilde{V}) \oplus H^i(E) & \rightarrow & H^i(\tilde{V} \cap \tilde{W}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H^{i-1}(V \cap W) & \rightarrow & H^i(X) & \rightarrow & H^i(V) \oplus H^i(Z) & \rightarrow & H^i(V \cap W) \end{array}$$

It is a general fact that $\phi^*\colon H^i(X) \rightarrow H^i(Y)$ is injective by the projection formula which states that ϕ is proper of degree one implies that $\phi_! \circ \phi^*(a) = a$ for all $a \in H^i(X)$. Since $\tilde{V} \cong V$ and $\tilde{V} \cap \tilde{W} \cong V \cap W$, simple diagram chasing shows that $H^i(Z) \rightarrow H^i(E)$ is also injective. We may then break (4.3) into short exact sequences

$$(4.4) \qquad 0 \rightarrow \phi^* H^i(X) \rightarrow H^i(Y) \rightarrow H^i(E)/\phi^* H^i(Z) \rightarrow 0.$$

Similarly, for $\phi'\colon Y \rightarrow X'$ we have

$$(4.4') \qquad 0 \rightarrow \phi'^* H^i(X') \rightarrow H^i(Y) \rightarrow H^i(E')/\phi'^* H^i(Z') \rightarrow 0.$$

Since $E_{\mathrm{red}} = E'_{\mathrm{red}}$, the proposition follows immedeately.   q.e.d.

Combining this with Theorem A gives

Corollary 4.5. *Let $f\colon X \cdots\rightarrow X'$ be a birational map between two smooth complex projective varieties such that the canonical bundles are numerically effective along the exceptional loci. Then the exceptional loci also have the same Betti numbers. In particular, this applies to birational smooth minimal models.*

Remark 4.6. The proof of Theorem A in fact also shows that $\bar{Z}$ and $\bar{Z}'$ have the same number of $\mathbf{F}_q$-rational points. This is simply

because $|\bar{X}(\mathbf{F}_q)| = |\bar{X}'(\mathbf{F}_q)|$ and $\bar{X} - \bar{Z} \cong \bar{X}' - \bar{Z}'$. In particular, if Z and Z' are smooth, then they have the same Betti numbers. Although this argument apparently only works for smooth Z and Z', which is very restricted, it is more than just a special case of (4.5) – since it carries certain nontrivial arithmetic information.

Now we begin the proof of Theorem B. Let $f: X \dashrightarrow X'$ be a birational map between two n-dimensional smooth projective varieties where only X is assumed to be minimal. In the notation of §1, Theorem 1.4 says that $E \geq E'$. So we obtain canonical morphisms $H^i(E) \to H^i(E')$ induced from $E' \subset E$. Since $Z := \phi(E)$ and $Z' := \phi'(E')$ are of codimension at least two, $H^{2n-2}(Z) = 0 = H^{2n-2}(Z')$. By comparing (4.4) and (4.4') via the surjective map $H^{2n-2}(E) \to H^{2n-2}(E')$, we obtain a canonical embedding:

$$(4.7) \qquad \phi^* H^{2n-2}(X, \mathbf{Z}) \subset \phi'^* H^{2n-2}(X', \mathbf{Z}).$$

which respects the Hodge structures. This induces an injective map

$$(4.8) \qquad \phi'_! \circ \phi^* : H^{2n-2}(X, \mathbf{Z}) \to H^{2n-2}(X', \mathbf{Z}),$$

which by the projection formula is easily seen to be independent of the choice of Y, and hence canonical. Poincaré duality then concludes the first statement of Theorem B.

For the second statement, we may simply copy the above proof by replacing (4.4) with the similar formula for the Weil divisors.   q.e.d.

One can also interpret this result in terms of the Picard group if the terminal varieties considered are assumed to be factorial or $\mathbf{Q}$-factorial.

5. Further Remarks

We conclude this paper with two historical and two technical remarks:

5.1. Birational geometry. A version of Theorem 1.4, or rather the Corollary 1.10, was used before by Kollár in his study of three-dimensional flops. In fact, he proved that three-dimensional birational $\mathbf{Q}$-factorial minimal models all share the same singularities, singular cohomologies and intersection cohomologies with pure Hodge structures (for deep reasons). See [14] for the details.

More recently, the author used a relative version of (1.10) to study the degenerations of minimal projective threefolds [15, §4] and obtained

a negative answer to the so called "filling-in problem" in dimension three. Namely, there exist degenerating projective families of smooth threefolds which are C^∞ trivial over the punctured disk, but can not be completed into smooth projective families.

5.2. Previous results.   After Kollár's result on threefolds, the problem on the equivalence of Betti numbers seemed to be ignored for a while until recently when Batyrev treated the case of projective Calabi-Yau manifolds [1].

In the special case of projective hyperkähler manifolds, Theorem A has also been proved recently by Huybrechts [11] using quite different methods. In fact, he proves more – these manifolds are all inseparable points in the moduli space (hence are diffeomorphic and share the same Hodge structures)!

This problem on general minimal models, to the best of the author's knowledge, has not been studied before our paper. In our case, the homotopy types will generally be different. In fact, it is well known that for a single elementary transform of threefolds, although the singular cohomologies are canonically identified, the cup product must change. However, inspired by Kollár's result and Remark 3.7, we still expect that the (non-polarized) Hodge structures will turn out to be the same.

5.3. Singular case.   In order to generalize Theorem A to the singular case, our approach works equally well in the log-terminal case, with the only problem being that we need a good interpretation like Weil's formula (2.8) for the precise meaning of the weighted counting, which is the key to relate p-adic integrals to the Weil conjecture.

Since a suitable version of the Weil conjecture for singular varieties has already been proved by Deligne in [2] in terms of the intersection cohomologies introduced by Goresky and MacPherson [8], this problem is thus reduced to the calculation of local Lefschetz numbers.

More precisely, one needs to evaluate the p-adic integrals over a singular point and to reconstruct the "constructible complexes of sheaves" which it may correspond to. If luckily enough, it is the intersection cohomology complexes, then we may get our conclusion again via Deligne's theorem. A detailed discussion on this will be continued in a subsequent paper.

5.4. Minimal cohomology.   For Theorem B, it is likely that a similar argument works for proving that terminal minimal models also minimize the second intersection cohomology groups and that they all share the same pure Hodge structures. The important injectivity of

$\phi^* : IH^i(X) \to IH^i(Y)$ needed to conclude (4.4) is now a consequence of the so called "decomposition theorem" of projective morphisms. ([2] again!)

An interesting question arising: is the Picard number (or the second Betti number) of a non-minimal model always strictly bigger than the one attained by the minimal models?

Mazur raised the following question: can one extract the expected "minimal cohomology piece" directly from any smooth model without referring to the minimal models?

5.5. Recent development. We first notice that the proof of Theorem A can be formally separated into three parts:

1. Geometric situations lead to the conclusion of K-equivalence. This is done in Theorem 1.4, or Corollary 1.10. In particular, this applies to birational minimal models.

2. A reasonable integration/measure theory attached to a variety. Here we deal with p-adic integrals, or equivalently, the number of rational points in the case of smooth varieties. Theorem 3.1 shows that K-equivalence implies measure equivalence. In the notation used there, E and E' are exactly the Jacobian factor occuring in the changing of variables formula from X and X' to Y respectively.

3. Topological/geometrical interpretation of the integral. In our case, this corresponds to Grothendieck-Deligne's solution to the Weil conjecture.

We can then formulate a meta theorem via the above steps by considering more general integrals.

Recently, based on an idea of Kontsevich, Denef and Loeser [7] has constructed a motivic integration on the space of arcs of an algebraic variety, which generalizes the p-adic integral. Using this new integration theory in step 2 and Deligne's theorem on the existence of functorial mixed Hodge structures on compactly supported homologies of algebraic varieties in step 3, Theorem A can be strengthened to the statement that X and X' also have the same Hodge numbers. Moreover, the usage of motivic integration allows much better understanding of the exceptional loci. However, like the case of p-adic integrals, the topological meaning of the full measure in the singular case is still not well understood.

After the present work was completed, their preprint [7] and then the preprint version of the paper became avaliable in the network. Afterwards, the above implication was also observed and pointed out to

the author by Loeser. Since their construction of motivic integration is quite delicate, we will not try to say anything about it here. The interested reader is referred to [7] for the details of this wonderful theory.

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HARVARD UNIVERSITY