J. DIFFERENTIAL GEOMETRY
2 (1968) 67-75

DEFORMATIONS OF CLOSED SPACE CURVES

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1. Introduction

In this note we will be interested in closed space curves, that is C^k ($k \geq 2$) immersions of S^1 into R^3. We say a space curve is *non-degenerate* if the square of the curvature is never zero. This non-degeneracy condition is the classical hypothesis used to insure the existence of the moving Frenet frame along the curve. The question we would like to ask is the following one. Given any two closed non-degenerate space curves, when are they homotopic through a homotopy composed entirely of closed non-degenerate space curves? More precisely we want to study the space N of non-degenerate closed space curves, considered as a subspace of $C^k(S^1, R^3)$, the C^k maps from S^1 into R^3, with the C^k-topology ($k \geq 2$) [4]. We ask: what are the arc components of N? A continuous path in N will be called a *non-degenerate homotopy*.

It will be convenient to make all homotopies *based*. To make this specific let us first define the Frenet frame for $\gamma(s) \in N$. This is done by fixing the sign of the curvature to be positive, and letting the *principal normal* $t_2(s)$ be defined by $dt_1(s)/ds = k(s)t_2(s)$, where s is arc-length parameter, $t_1(s) = d\gamma(s)/ds$ is the *unit tangent* vector, and $k(s)$ is the curvature of the given space curve γ. One then defines the *binormal vector* $t_3(s)$ by the formula $t_3(s) = t_1(s) \times t_2(s)$. Now let us fix a base point $\theta_0 \in S^1$, and let

$$N_0 = \{\gamma \in N \mid \gamma(\theta_0) = 0,\ t_1(\theta_0) = e_1,\ t_2(\theta_0) = e_2,\ t_3(\theta_0) = e_3\}\,,$$

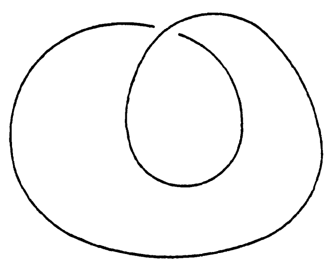
where (t_1, t_2, t_3) is the Frenet frame of γ, and the e_i are the unit coordinate vectors of R^3 (i.e., $e_1 = (1, 0, 0)$ etc.). An element of N_0 will be called a *based non-degenerate* curve, and a continuous path in N_0 a *based non-degenerate homotopy*. By using rotations and translations the arc components of N are determined by those of N_0, because the group of rigid motions is connected. Given any $\gamma \in N_0$, we define $F(\gamma): S^1 \to SO(3)$, by associating to each point of γ, its Frenet frame, where $SO(3)$ is the special orthogonal group. We see $F(\gamma)$ is of class C^{k-2}, and $F(\gamma)(\theta_0) = f_0 = (e_1, e_2, e_3) \in SO(3)$. Our main result is the following theorem.

Communicated by J. Eells, Jr., November 17, 1967. This research was partially supported by NSF grant GP-5227.

Theorem I. *Let γ and $\eta \in N_0$. Then γ and η are based non-degenerately homotopic if and only if $F(\gamma)$ is based homotopic to $F(\eta)$ (that is they determine the same element of $\pi_1(SO(3), f_0)$).*

The "if" part is obvious; hence the main task will be to show the converse. We will now apply the theorem to get the desired classification.

It is well known that $\pi_1(SO(3), f_0) \cong Z_2$ and that the generator is represented by the curve of 3-frames $f(\theta) = (f_1(\theta), f_2(\theta), e_3)$ where $f_1(\theta) = (\cos\theta, \sin\theta, 0)$, and $f_2(\theta) = (-\sin\theta, \cos\theta, 0)$ for $0 \leq \theta \leq 2\pi$ (see [6]). Let $\alpha(\theta) = (\sin\theta, 1 - \cos\theta, 0)$. Then $\alpha \in N_0$, and $F(\alpha)(\theta) = f(\theta)$. Therefore we see N_0 has two connected components, the first determined by traversing the circle α once, the second by traversing α twice (i.e., by $\alpha(2\theta), 0 \leq \theta \leq 2\pi$). Hence, if we deform the latter curve a bit so it is an embedded circle, we see that any $\gamma \in N(N_0)$ is (based) non-degenerately homotopic to either the circle or curve 2 pictured below.



Curve 2

Besides the intrinsic geometric interest of this problem, it is also the most elementary open part of the following more general problem. Let X be a Riemanian manifold. We say an immersed circle is non-degenerate if the geodesic curvature never vanishes. When are two non-degenerate immersed circles homotopic through non-degenerate immersions? Let us restrict ourselves to curves which are "based", that is, fix $\theta_0 \in S^1$, $x_0 \in X$, f_1 and f_2 mutually perpendicular unit tangent vectors at x_0, and restrict ourselves to $f: S^1 \to X$, such that $f(\theta_0) = x_0$, and the unit tangent (unit principal normal) to f at θ_0 is $f_1(f_2)$. By associating with each $\theta \in S^1$ the unit tangent and unit principal normal of f at $f(\theta)$ one obtains a curve in $V_2(X)$ the orthonormal two frame bundle of X. Hence the non-degenerate immersion f defines an element $V(f)$ of $\pi_1(V_2(X), f_0)$, where $f_0 = (x_0, f_1, f_2)$. If f and g are based non-degenerate immersions of S^1 in X, is the condition that $V(f) = V(g)$ sufficient to insure that f is based nondegenerate regularly homotopic to g? When $\dim X \geq 4$ it is indeed the case since $\pi_1(V_{2,k}) = 0$ $(k \geq 4)$ where $V_{2,k}$ is the Stiefel manifold of 2 frames in k-space. Therefore $\pi_1(V_2(X), f_0) \cong \pi_1(X, x_0)$. We showed in [1] that in this case f and g are (based) non-degenerately homotopic if and only if they were (based) homotopic. This leaves only two and there dimensional manifolds, and this note supplies an affirmative answer to the question in case $X = R^3$, since $V_{2,3} \cong SO(3)$. For $X = R^2$, the question is quite easily answered. The winding number (normal degree) is the only invariant, and every winding is realized except of course 0. The proof is much easier than that of the Whitney-Graustein theorem [8], since the positivity (negativity) of the curvature guarantees that the tangent map is a covering map of S^1 by S^1. If we take the above question as a conjecture I would assume it is true for $\dim X = 3$, and false for $\dim X = 2$. One can clearly ask the analogous questions which arise by demanding that higher and higher order curvatures never vanish.

Finally I would like to thank R. Sacksteder for making an essential simplification in the proof of Fenchel's theorem [3], and C. Weaver for pointing out a way of simplifying the original proof of this theorem.

2. The factorization and spherical curves

Let us first identify $SO(3)$, $V_{2,3}$ and $T_0(S^2)$, the unit tangent circle bundle of S^2, as follows. Note for any 3-frame $(f_1, f_2, f_3) \in SO(3)$, $f_3 = f_1 \times f_2$, and $(f_1, f_2) \in V_{2,3}$. Thus we can identify $V_{2,3}$ and $SO(3)$. We then view $(f_1, f_2) \in V_{2,3}$ as defining a point $f_1 \in S^2$, and a unit tangent vector to S^2 at f_1 by parallel translating the foot of f_2 from 0 to f_1. With the above identifications in mind we will use these three spaces interchangeably. Now let us fix $\theta_0 \in S^1$, and $f_0 = (e_1, e_2) \in V_{2,3}$, where e_i is the i-th coordinate vector. Let $V_0 = \{f \in C^{k-2}(S^1, V_{2,3}) \mid f(\theta_0) = f_0\}$. Then by associating to each $\gamma \in N_0$, the moving 2-frame defined by its unit tangent and principal normal vectors, we have defined a continuous map $F: N_0 \to V_0$, where V_0 is given the C^{k-2} topology. Let $I_0 = \{\lambda \in C^{k-1}(S^1, S^2) \mid \lambda$ is an immersion, $\lambda(\theta_0) = e_1$, and $\lambda'(\theta_0)/|\lambda'(\theta_0)| = e_2\}$, and give I_0 the C^{k-1} topology. Then we can factor F through I_0 as follows. Let $\gamma \in N_0$, and set $G(\gamma)(\theta) = \gamma'(\theta)/|\gamma'(\theta)|$. We could define $G(\gamma)$ for an arbitrary C^1 space curve γ; however $G(\gamma)$ is an immersion if and only if γ is non-degenerate. G clearly maps N_0 into I_0 continuously.

$$\begin{tikzcd} N_0 \arrow[rr, "F"] \arrow[dr, "G"'] & & V_0 \\ & I_0 \arrow[ur, "S"'] & \end{tikzcd}$$

Let $\lambda \in I_0$, and set $S(\lambda)(\theta) = (\lambda(\theta), \lambda'(\theta)/|\lambda'(\theta)|)$. Then S is continuous and $F = S \circ G$. We now quote the following theorem of Smale [5].

Theorem S. *Let λ and $\bar{\lambda} \in I_0$. Then λ and $\bar{\lambda}$ are based regularly homotopic* (i.e. lie in the same arc component of I_0), *if and only if $S(\lambda)$ and $S(\bar{\lambda})$ represent the same element of $\pi_1(V_{2,3}; f_0)$. If $S(\lambda)$ is homotopic to $S(\bar{\lambda})$, and furthermore there exists a neighborhood U of θ_0 on which λ and $\bar{\lambda}$ agree, then we can find a neighborhood $U' \subset U$ of θ_0, and a regular homotopy λ_s joining λ to $\bar{\lambda}$* (i.e. a path in I_0) *such that λ_s and λ agree on U' for $0 \leq s \leq 1$.*

Remark. This is a special case of Smale's theorem and it admits a rather easy elementary proof, by first applying the Whitney-Graustein theorem [8] to the punctured sphere, and then "cancelling" loops. By applying theorem S we see that the following Theorem I′ is equivalent to Theorem I.

Theorem I′. *Let γ and $\bar{\gamma} \in N_0$. Then γ is based non-degenerately homotopic to $\bar{\gamma}$ if and only if $G(\gamma)$ is based regularly homotopic to $G(\bar{\gamma})$.*

If we are to prove I′, we have to know which closed spherical curves are tangent indicatrices of closed space curves. This question is answered by the following proposition of Fenchel [2], [3]. In the "if" part we will merely sketch Loewner's well known remarks [2]. In the "only if" part Fenchel's argument is somewhat simplified by a remark of Sacksteder; we include this proof because the argument is central in the proof of Theorem I′.

Let us fix the following notation. If A is a subset of R^3, let $[A]$, A^c, A^i denote respectively the convex hull of A, the closure of A, and the interior of A.

Proposition 2.1. a) *Let $\gamma(\theta)$ be a closed non-plane space curve of class $C^k (k \geq 1)$, and $G(\theta) = \gamma'(\theta)/|\gamma'(\theta)|$ the tangent map. Then $0 \in [\{G(\theta) \mid \theta \in S^1\}]^i$.*

b) *Let $\lambda(\theta)$ be a closed curve of class C^{k-1} $(k \geq 1)$ on $S^2 \subseteq R^3$. If $0 \in [\{\lambda(\theta) \mid \theta \in S^1\}]^i$ then $\lambda(\theta)$ is the tangent indicatrix of a closed C^k, nonplane, space curve.*

Proof. a) Let P be a plane in R^3 which does not meet γ. Then there must be both a maximum and a minimum of the distance from P to γ. If $p_* = \gamma(\theta_*)$ is either a maximum or minimum point of this distance function, then $G(\theta_*)$ is parallel to P and therefore $G(\theta_*)$ lies on the great circle determined by the intersection of S^2 with the plane parallel to P which passes through the origin. The fact that we have both maxima and minima implies that the set $\{G(\theta) \mid \theta \in S^1\}$ meets both open hemispheres determined by the aforementioned great circle.

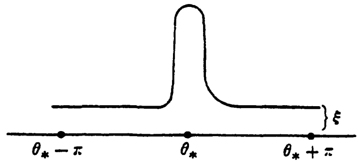
b) Let $\lambda(\theta)$ be viewed as a periodic vector valued function of period 2π, C be the vector space of real valued C^∞ functions on S^1, viewed as periodic functions with period 2π, and $P = \{\rho \in C \mid \rho(\theta) > 0,\ 0 \leq \theta \leq 2\pi\}$. We note that it suffices to find a $\rho \in P$ such that $\int_0^{2\pi} \rho(\theta)\lambda(\theta)d\theta = 0$, because we can then set $\gamma(t) = \int_0^t \rho(\theta)\lambda(\theta)d\theta$, and γ will be the desired space curve.

Let $K = \{y \in R^3 \mid$ there exists $\rho \in P$ such that $y = \int_0^{2\pi} \rho(\theta)\lambda(\theta)d\theta\}$. We see that K is a convex cone because P is such. At this point Sacksteder noticed that it suffices to show $\{\lambda(\theta) \mid 0 \leq \theta \leq 2\pi\} \subseteq K^c$, because $0 \in [\{\lambda(\theta) \mid 0 \leq \theta \leq 2\pi\}]^i$, and therefore $0 \in K$ provided the above inclusion holds. The remainder of the argument is essentially Fenchel's.

Let

$$\tilde{\varphi}(x) = \begin{cases} e(1/x^2 - 1) & -1 \leq x \leq 1\,, \\ 0 & \text{elsewhere}\,, \end{cases}$$

where $e(t) = e^t$. Let $b = \int_{-\infty}^{\infty} \tilde{\varphi}(x)dx$; therefore we set $\varphi(x) = \frac{1}{b}\tilde{\varphi}(x)$. Pick θ_* such that $0 \leq \theta_* \leq 2\pi$, and ξ such that $0 < 2\pi\,\xi < 1$. We then define a function $f_\xi(\theta, \theta_*)$ on an interval of length 2π with θ_* as midpoint as follows. $f_\xi(\theta, \theta_*) = \xi + h\varphi\left(\frac{\theta - \theta_*}{\xi}\right)$ where h is so chosen that $\int_{\theta_*-\pi}^{\theta_*+\pi} f_\xi(\theta, \theta_*)d\theta = 1$. We extend f_ξ to all of R by making it periodic of period 2π. $f_\xi(\theta, \theta_*) \in P$. Let $\lambda_\xi(\theta_*) = \int_0^{2\pi} \lambda(\theta)f_\xi(\theta, \theta_*)d\theta$. Then $\lambda_\xi(\theta_*) \in K$. Note that $\int_0^{2\pi} f_\xi = 1$ implies that



$$\left|\lambda_\xi(\theta_*) - \lambda(\theta_*)\right| = \left|\int_0^{2\pi}[\lambda(\theta) - \lambda(\theta_*)](f_\xi(\theta, \theta_*)d\theta\right|.$$

Thus we see $\lambda_\xi(\theta_*) \to \lambda(\theta_*)$ as $\xi \to 0$ by the continuity of λ and the construction of f_ξ. Hence $\lambda(\theta_*) \in K^c$.

We can prove Theorem I$'$, if we can carry out the following procedures. First, let $G_0 = G(\gamma_0)$ and $G_1 = G(\gamma_1)$ be the tangent images of γ_0 and $\gamma_1 \in N_0$. If G_0 and G_1 are regularly homotopic, does there exist a base regular homotopy G_s joining G_0 to G_1 such that $0 \in [\{G_s(\theta) \mid 0 \leq \theta \leq 2\pi\}]^i$ for each s, $0 < s < 1$? Second, if we have the desired homotopy G_s above, can we find a continuous 1-parameter family of weighting functions $\rho_s \in P$ such that $\int_0^{2\pi} \rho_s(\theta)G_s(\theta)d\theta = 0$ where $\rho_0(\theta) = |\gamma_0'(\theta)|$ and $\rho_1(\theta) = |\gamma_1'(\theta)|$?

3. First deformations

From now on we will view all maps with S^1 as source, as periodic maps of period 2π with R as source.

Proposition 3.1. *Let $\gamma \in N_0$. Then there exists a based non-degenerate homotopy between γ and an element $\bar{\gamma} \in N_0$, where $\bar{\gamma}$ has the following property: For some parametrization t of γ, we can find a number $l > 0$ depending on γ, such that $G(\bar{\gamma})(t) = (\cos t, \sin t, 0)$ for $0 \leq t \bmod 2\pi \leq l$.*

Proof. Let us note that γ is non-degenerate if and only if the vectors $\gamma'(t)$ and $\gamma''(t)$ are linearly independent for each t, $0 \leq t \leq 2\pi$. By reparametrizing γ if necessary we can assume $\gamma'(0) = e_1$, and $\gamma''(0) = |\gamma''(0)|e_2$ where e_i is the unit vector in the i-th direction. If we let $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, we see that there exists a number $l_1 > 0$ such that if $|t| \bmod 2\pi < l_1$, then the vectors $(\gamma_1'(t), \gamma_2'(t), 0)$ and $(\gamma_1''(t), \gamma_2''(t), 0)$ are linearly independent. Let $\psi(t)$ be a C^∞ periodic function of period 2π, such that $\psi(t) \equiv 1$ for $0 \leq |t| \bmod 2\pi \leq l_1/2$,

$\psi(t) \equiv 0$ for $|t| \bmod 2\pi \geq l_1$, and $0 \leq \psi \leq 1$. We define a based non-degenerate homotopy by the formula

$$\gamma_u(t) = (\gamma_1(t), \gamma_2(t), (1 - u\psi(t))\gamma_0(t))$$

such that $\gamma_0(t) = \gamma(t)$, and $\gamma_u(t) = \gamma(t)$ for $|t| \bmod 2\pi \geq l_1$. Then $\gamma_u(t) \in N_0$ by our choice of l_1. Let $\bar{\gamma}(t) = \gamma_1(t)$. For $0 \leq |t| \bmod 2\pi \leq l_1/2$, $\bar{\gamma}(t)$ lies in the (x, y) plane and has positive curvature. Hence we can reparametrize $\bar{\gamma}$ so that $G(\bar{\gamma})(\tau) = (\cos\tau, \sin\tau, 0)$ for $0 \leq |\tau| \bmod 2\pi \leq l$ for some $l > 0$, which depends upon l_1.

Proposition 3.2. *Let $\gamma(t) \in N_0$, and assume γ lies in the (x, y) plane. Then for an arbitrarily small ξ, $0 < \xi < \pi$, γ is based non-degenerately homotopic to a nonplane curve η, such that $\gamma(t) = \eta(t)$ for $0 \leq |t| \bmod 2\pi \leq \pi - \xi$.*

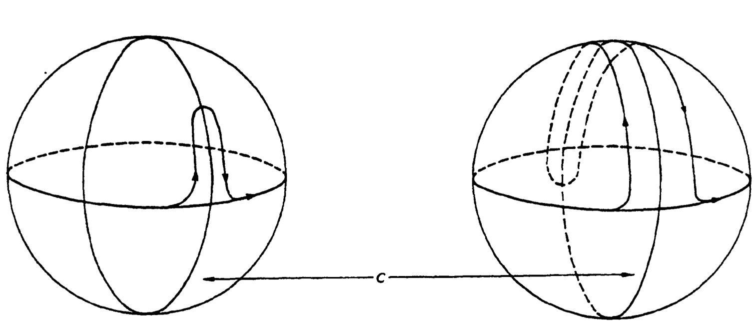
Proof. Let $\varphi_0(t) = \begin{cases} e(1/t^2 - 1), & -1 \leq t \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$ Extend $\varphi_0\left(\frac{t-\pi}{\xi}\right)\Big| [0, 2\pi]$ for $0 < \xi < \pi$ to all of R, by making it periodic of period 2π, and denote this function by φ. Let $\psi(u) : R \to R$ by a C^∞ function, such that $\psi(u) = 0$ for $u \leq 0$, $\psi(u) = 1$ for $u \geq 1$ and $0 \leq \psi(u) \leq 1$. Set $\gamma_u(t) = (\gamma_1(t), \gamma_2(t), \psi(u)\varphi(t))$. Then $\gamma_0(t) = \gamma(t)$ and define $\eta(t) = \gamma_1(t)$. We thus see that $\gamma_u(t)$ is the desired non-degenerate homotopy.

Lemma 3.3. *Let $\lambda(t)$ and $\eta(t)$ be in I_0, and assume that λ and η are based regularly homotopic and that*

$$0 \in [\{\lambda(t) \,|\, 0 \leq t \leq 2\pi\}]^i \cap [\{\eta(t) \,|\, 0 \leq t \leq 2\pi\}]^i .$$

Finally we assume that $\lambda(t) = \eta(t) = (\cos t, \sin t, 0)$ for $0 \leq t \leq l_1$, for some $l_1 > 0$, $l_1 < \pi$. Then there exists a based regular homotopy $\lambda_u(t)$ joining λ to η such that $0 \in [\{\lambda_u(t) \,|\, 0 \leq t \leq 2\pi\}]^i$ for each u, $0 \leq u \leq 1$.

Proof. First pick a regular homotopy $\nu_u(t)$ such that $\nu_0 = \lambda$, $\nu_1 = \eta$, and $\nu_u(t) = (\cos t, \sin t, 0)$ for $0 \leq t \leq l$ where $0 < l < l_1$, and is determined by Theorem S of §2. Let us look at the great circle c which passes through $\lambda\left(\frac{1}{2}l\right)$ and the north pole. Let $\Omega_u(t)$ be a regular homotopy of immersions



$\Omega_{\frac{1}{2}}(t)$ Figure 1. $\Omega_1(t)$

of the interval $[0, l]$ into S^2 which we will describe as follows. Let $\Omega_u(t) = (\cos t, \sin t, 0)$ for $0 \leq t \leq \frac{1}{4}l$ and for $\frac{3l}{4} \leq t \leq l$, and let $\Omega_0(t) = (\cos t, \sin t, 0)$ for all t. Finally as u varies from 0 to 1 we pull out a long skinny "bump" which is symmetric about the great circle c, until finally at $u = 1$ the bump has gone more than halfway around the circle c, as we see in figure 1.

We can now define the desired homotopy $\lambda_u(t)$.

$$\lambda_u(t) = \begin{cases} \begin{cases} \Omega_{3u}(t) & 0 \leq t \leq l \\ \lambda(t) & l \leq t \leq 2\pi \end{cases} & 0 \leq u \leq \frac{1}{3} \\ \begin{cases} \Omega_1(t) & 0 \leq t \leq l \\ \nu_{3u-1}(t) & l \leq t \leq 2\pi \end{cases} & \frac{1}{3} \leq u \leq \frac{2}{3} \\ \begin{cases} \Omega_{3-3u}(t) & 0 \leq t \leq l \\ \nu_1(t) & l \leq t \leq 2\pi \end{cases} & \frac{2}{3} \leq u \leq 1 \,. \end{cases}$$

This homotopy clearly has the correct properties as the long hump guarantees the convexity property as we deform through the homotopy ν_u.

4. Deforming the weighting functions

Let $P = \{\rho \in C^{k-1}(S^1, R) \mid \rho(\theta) > 0 \text{ for } 0 \leq \theta \leq 2\pi\}$. If we give $C^{k-1}(S^1, R)$ the C^{k-1} topology, then P is an open convex cone. Denote $C^{k-1}(S^1, R)$ by C.

Lemma 4.1. *Let λ_0 and λ_1 be non-plane elements of I_0, such that for some $l > 0$, $0 \leq \theta \leq l < \pi$, $\lambda_0(\theta) = \lambda_1(\theta) = (\cos\theta, \sin\theta, 0) = \alpha(\theta)$. Assume furthermore that*

$$0 \in [\{\lambda_0(\theta) \mid 0 \leq \theta \leq 2\pi\}]^i \cap [\{\lambda_1(\theta) \mid 0 \leq \theta \leq 2\pi\}]^i \,,$$

and that λ_0 is based regularly homotopic to λ_1. Pick $\rho_i(\theta) \in P$, $i = 0, 1$, such that $\int_0^{2\pi} \rho_i(\theta)\lambda_i(\theta)d\theta = 0$ for $i = 0, 1$. We can then find a regular homotopy $\lambda_w(\theta)$, $0 \leq w \leq 1$, between λ_0 and λ_1, an interval $[0, l_1]$, $0 < l_1 < l$, and a continuous path $\rho_w(\theta) \in P$ (continuous as a map from $[0, 1]$ into P) with the following properties: $\lambda_w(\theta) = \alpha(\theta)$ for $0 \leq w \leq 1$ and $\theta \in [0, l_1]$, each $\lambda_w(\theta)$ is nonplaner, $0 \in [\{\lambda_w(\theta) \mid 0 \leq \theta \leq 2\pi\}]^i$ for each w, and $\int_0^{2\pi} \lambda_w(\theta)\rho_w(\theta)d\theta = 0$ for each w.

Before proving Lemma 4.1 we note that this lemma and the results of § 3 will imply theorem I′ as follows. Let η_0 and $\eta_1 \in N_0$, and assume $G(\eta_0)$ is based regularly homotopic to $G(\eta_1)$. By applying Propositions 3.1 and 3.2 we can find curves $\gamma_i \in N_0$, $i = 0, 1$, such that γ_i is based non-degenerately homotopic to η_i, $i = 0, 1$, and such that the curves $\lambda_i = G(\gamma_i)$, $i = 0, 1$, have the properties of the λ_i of Lemma 4.1. Let $\rho_i(\theta) = |\gamma_i'(\theta)|$, and $\lambda_w(\theta)$ and $\rho_w(\theta)$ be the regular homotopy and path of weighting functions determined by Lemma 4.1. We then set $\gamma_w(t) = \int_0^t \lambda_w(\theta)\rho_w(\theta)d\theta$ which determines the desired path in N_0.

Proof of Lemma 4.1. By Lemma 3.3 we can find a based regular homotopy $\lambda_u(\theta)$ joining λ_0 to λ_1, and an interval $[0, l_1]$, $0 < l_1 < l$, such that $0 \in [\{\lambda_u(\theta) \mid 0 \leq \theta \leq 2\pi\}]^i$ for each fixed u, and such that $\lambda_u(\theta) = \alpha(\theta)$ for $\theta \in [0, l_1]$, $0 \leq u \leq 1$. Let $L = L^2(S^1, R)$ denote the space of square integrable periodic real-valued functions, and write $\lambda_u(\theta) = (\lambda_{u,1}(\theta), \lambda_{u,2}(\theta), \lambda_{u,3}(\theta))$. For each fixed j, $\lambda_{u,j} \in C \subseteq L$, and in fact $\lambda_{u,1}$, $\lambda_{u,2}$ and $\lambda_{u,3}$ are linearly independent elements of L as well as of C, for each u. We will now adopt the following notation. If $x(\theta)$ and $y(\theta) \in C$ we will set $\langle x(\theta), y(\theta)\rangle = \int_0^{2\pi} x(\theta)y(\theta)d\theta$. We will suppress the circle variable θ, and we will write $\lambda_{u,j}(\theta)$ as $\lambda_j(u)$ for $1 \leq j \leq 3$, and $\rho_i(\theta)$ as $\rho(i)$ for $i = 1, 2$. Hence we are given three continuous curves $\lambda_j(u)$ in C, which for each fixed u, determines 3 elements linearly independent in both C and L. Therefore we want to find a curve $\rho(u)$ in P joining $\rho(0)$ to $\rho(1)$ such that $\langle \rho(u), \lambda_j(u)\rangle = 0$ for $j = 1, 2$ and 3.

By the Gram-Schmidt process we can replace the curves $(\lambda_j(u))$, $1 \leq j \leq 3$, by curves $(\mu_j(u))$, $1 \leq j \leq 3$, such that $\langle \mu_i(u), \mu_j(u)\rangle = \delta_{ij}$ for $1 \leq i \leq j \leq 3$, and such that for each u, $\mu_1(u)$, $\mu_2(u)$ and $\mu_3(u)$ span the same subspace of C as $\lambda_1(u)$, $\lambda_2(u)$ and $\lambda_3(u)$. Hence $\langle \rho, \lambda_j(u)\rangle = 0$, $1 \leq j \leq 3$, if and only if $\langle \rho, \mu_j(u)\rangle = 0$, $1 \leq j \leq 3$. Therefore it suffices to find a curve $\rho(u)$ in P joining $\rho(0)$ to $\rho(1)$ such that $\langle \rho(u), \mu_j(u)\rangle = 0$, $1 \leq j \leq 3$. For each $v \in [0, 1]$ let us pick by Proposition 2.1 an element $\rho_v \in P$ such that $\langle \rho_v, \mu_j(v)\rangle = 0$, $1 \leq j \leq 3$, and such that $\rho_0 = \rho(0)$ and $\rho_1 = \rho(1)$. Let $\rho_v(u) = \rho_v - \sum_{j=1}^{3} \langle \mu_j(u), \rho_v\rangle \mu_j(u)$. We see that $\langle \rho_v(u), \mu_j(u)\rangle = 0$ for $1 \leq j \leq 3$. We also see that there exists a real number $\varepsilon_v > 0$ depending upon ρ_v, such that if $|u - v| < \varepsilon_v$ then $\rho_v(u) \in P$. Let $I_v = \{u \mid |u - v| < \varepsilon_v\}$. The I_v form an open covering of $[0, 1]$, and therefore there exists a finite subcovering $I_{v_1} \cdots I_{v_k}$. If I_0 is not in this list throw it in. Hence by relabeling these intervals if necessary, we can find a sequence of intervals $I_0, \cdots, I_k$, where $I_n = I_{v_n}$, and points $0 = u_{-1} < u_0 < \cdots < u_{k-1} < u_k = 1$ with the following properties: $I_0 = I_0$, $u_n \in I_n \cap I_{n+1}$, and $1 \in I_k$. Therefore $[0, u_0] \subseteq I_0$, $[u_0, u_1] \subseteq I_1, \cdots$, $[u_{k-2}, u_{k-1}] \subseteq I_{k-1}$ and $[u_{k-1}, 1] \subseteq I_k$. Let $\rho_n(u) = \rho_{v_n}(u)$ be the curve in P defined by ρ_{v_n} on the interval $I_n = I_{v_n}$, and reparametrize the curves $\mu_j(u)$ as follows:

$$\mu_j(w) = \begin{cases} \mu_j(u_{n-1} + 2(w - u_{n-1})) & \text{for } u_{n-1} \leq w \leq u_{n-1} + \frac{1}{2}(u_n - u_{n-1}) \\ \mu_j(u_n) \quad \text{for } u_{n-1} + \frac{1}{2}(u_n - u_{n-1}) \leq w \leq u_n, & 0 \leq n \leq k\,. \end{cases}$$

Let us formally set $\rho_{k+1}(u_k) = \rho_{k+1}(1) = \rho(1)$. We then define $\rho(w)$ by the formulas:

$$\rho_n[u_{n-1} + 2(w - u_{n-1})] \quad \text{for } u_{n-1} \leq w \leq u_{n-1} + \tfrac{1}{2}(u_n - u_{n-1}),$$

$$\rho_n(u_n)\frac{2(u_n - w)}{u_n - u_{n-1}} + \rho_{n+1}(u_n)\frac{2w - u_n - u_{n-1}}{u_n - u_{n-1}}$$
$$\text{for } u_{n-1} + \tfrac{1}{2}(u_n - u_{n-1}) \leq w \leq u_n .$$

Note that the above second formula is a convex sum, and therefore represents a line in P joining $\rho_n(u_n)$ to $\rho_{n+1}(u_n)$. Hence $\rho(w)$ is a continuous path in P. Finally we see that $<\rho(w), \mu_j(w)> = 0$ for $1 \leq j \leq 3$ because this is true for $u_{n-1} \leq w \leq u_{n-1} + \frac{1}{2}(u_n - u_{n-1})$, and because

$$<\mu_j(u_n), \rho_n(u_n)> = <\mu_j(u_n), \rho_{n+1}(u_n)> = 0 \quad \text{for } 1 \leq j \leq 3$$

implies

$$<\mu_j(u_n), t\rho_n(u_n) + (1 - t)\rho_{n+1}(u_n)> = 0 \quad \text{for } 0 \leq t \leq 1, \text{ and } 1 \leq j \leq 3 .$$

This completes our proof.

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