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THE DEFORMATION OF LAGRANGIAN MINIMAL SURFACES IN KÄHLER-EINSTEIN SURFACES

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A Kähler manifold can be viewed both as a symplectic manifold and a Riemannian manifold. These two structures are related by the Kähler form. One can study the Lagrangian minimal submanifolds which are Lagrangian with respect to the symplectic structure and are minimal with respect to the Riemannian structure. Lagrangian minimal submanifolds have many nice properties and have been studied by several authors (see [3], [5], [13], [16], [17], [27], [30], [33], [34] etc.). There are obstructions to the existence of the Lagrangian minimal submanifolds in a general Kähler manifold [3]. These obstructions do not occur in a Kähler-Einstein manifold. But even in this case, the general existence is still unknown. Most of the discussions of the paper are on compact manifolds without boundary. We assume this from now on unless other conditions are indicated. The main result of this paper is the following:

Theorem 4. *Assume that (N, g_0) is a Kähler-Einstein surface with negative first Chern class. Let $[A]$ be a class in the second homology group $H_2(N, Z)$, which can be represented by a finite union of branched Lagrangian minimal surfaces with respect to the metric g_0. Then with respect to any other metric in the connected component of g_0 in the moduli space of Kähler-Einstein metrics, the class $[A]$ can also be represented by a finite union of branched Lagrangian minimal surfaces.*

Note that the complex structure on N is allowed to change accordingly. An immersed Lagrangian minimal submanifold in a Kähler manifold with negative Ricci curvature is strictly stable ([4], [20], [22]). Thus

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one expects to have a result as the theorem. However, there are some major difficulties due to the occurrence of branched points to realize this expectation. In this introduction we first explain how the ideas work out in the local deformation of the immersed case. Then we point out the difficulties in the branched case and how we solve the problems. When the Lagrangian minimal surface is immersed, it is strictly stable and thus the Jacobi operator is invertible. By the implicit function theory, we can find a minimal surface for any nearby metric and the tangent bundle of the surface changes smoothly. Hence the minimal surface obtained is totally real if the metric is sufficiently close to the original one. A totally real (branched) minimal surface in a Kähler-Einstein surface with negative scalar curvature is Lagrangian ([5], [33]). Therefore, we get the local deformation of an immersed Lagrangian minimal surface. If we want to continue the process, we need to take the limit of a sequence of surfaces and it is not enough to restrict to the immersed case. We need to extend each step to the branched case. It seems that there is no known result for the deformation of branched minimal surfaces except the holomorphic curves. The Jacobi operator on a branched minimal surface is degenerate and it is a delicate problem to decide the allowable variations. For our problem, it is certainly not enough to consider only the variations with support away from the branched points. Branched minimal immersions are the critical points of the energy functional. We thus study the problem in the map settings and show that the strict stability in the sense of Definition 2 works suitably for the deformation of a branched minimal immersion. In particular, we have:

Theorem 2. *Assume that $\varphi_0 : \Sigma \to (N^n, g_0)$ is a strictly stable branched minimal immersion. Then there exists a strictly stable branched minimal immersion $\varphi_t : \Sigma \to (N^n, g_t)$ for any g_t which is close to g_0. Furthermore, φ_t converges to φ_0 in C^∞ if g_t converges to g_0 in C^∞.*

Here Σ is a closed surface and N^n is a complete Riemannian n-manifold which is not necessarily compact. We show that a branched Lagrangian minimal surface in a Kähler surface with negative Ricci curvature is strictly stable. Thus we can deform the branched Lagrangian minimal surface to get a family of strictly stable branched minimal surfaces. We still need to show that these surfaces are Lagrangian. One can hardly control the behavior of the tangent bundles after perturbing the branched points. The perturbation of the holomorphic curve (z^2, z^3) reveals some of the complexity. However, there are still some control in the holomorphic case. We prove that when the branched minimal surfaces are stable, we still have the similar control. More precisely, we show:

Theorem 3. *Let $\varphi_i : \Sigma \to (N, g_i)$ be a stable branched minimal immersion from a closed surface Σ to a Riemannian 4-manifold (N, g_i). Assume that g_i converges to g_0 and φ_i converges to φ_0 in C^∞, where φ_0 is a branched minimal immersion from Σ to (N, g_0). Then*

$$a(\varphi_0(\Sigma)) = \lim_{i\to\infty} a(\varphi_i(\Sigma)).$$

The adjunction number $a(\varphi_i(\Sigma))$ in the theorem is defined to be the sum of the integral of the Gaussian curvature on the tangent bundle and the integral of the Gaussian curvature on the normal bundle. It is equal to the total number of the complex points with indices when N has an almost complex structure and the complex points on $\varphi_i(\Sigma)$ are isolated [5]. The immersed version of the theorem is proved by Chen and Tian [5] using a different approach. From this theorem we can conclude that the branched minimal surface obtained above is totally real when the metric is sufficiently close to the original one. Thus it is Lagrangian ([5], [33]). This shows the local deformation of a branched Lagrangian minimal surface. The rest of the proof for Theorem 4 follows from an area bound and standard arguments.

The organization of the paper is as follows. In section 1 we study the critical points of the energy functional and the stability. This point of view helps us to understand the branched minimal immersions and the results in this section should have their own interest. The local deformation of a strictly stable branched minimal immersion is obtained in section 2. We use the three circle theorem in section 3 to study the oscilation of the conformal harmonic maps. The adjunction number and some necessary preliminaries are introduced in section 4. In section 5 we prove the theorem about the limit of the adjunction numbers. In the last section we complete the proof of the main theorem and give one application.

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1. The energy functional

Let Σ be a closed surface of genus r and N^n be a n-manifold which is not necessarily compact. The energy functional on

$$C^\infty(\Sigma, N) \times \mathcal{M}(\Sigma) \times \mathcal{M}(N)$$

is defined to be

$$E(\varphi, h, g) = \int_\Sigma \sum h^{ij} g_{kl} \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j} \, dA,$$

where $C^\infty(\Sigma, N)$ is the set of smooth maps between Σ and N, $\mathcal{M}(\Sigma)$ and $\mathcal{M}(N)$ are the set of smooth complete metrics on Σ and N respectively, and dA is the volume form of h. The quantity

$$\sum h^{ij} g_{kl} \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j}$$

is denoted by $e(\varphi)$, which is called the energy density of φ between (Σ, h) and (N^n, g). If we fix h, g and vary the map φ only, a critical point of $E(\cdot, h, g)$ is called a harmonic map between (Σ, h) and (N^n, g). There has been a thorough study on harmonic maps. Here we only refere to [9], [10], [29] and the reference therein. If we fix φ, g and vary the metric h only, a direct computation gives the following formula:

Lemma 1. *Assume that h_t is a smooth family of metrics on Σ with $h_0 = h$. Then*

$$\frac{dE(\varphi, h_t, g)}{dt}|_{t=0} = \int \sum (\frac{1}{2} h^{ij} h^{\alpha\beta} - h^{i\alpha} h^{\beta j}) \varphi^*(g)_{ij} \dot{h}_{\alpha\beta} \, dA,$$

where

$$\varphi^*(g)_{ij} = \sum g_{kl} \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j}$$

is the pull back metric and $\dot{h} = \frac{\partial h_t}{\partial t}|_{t=0}$. In particular, a critical point of $E(\varphi, \cdot, g)$ satisfies $\varphi^*(g) = \frac{1}{2} e(\varphi) h$; that is, the map φ is weakly conformal.

Proof. Assume that x^1, x^2 are the local coordinates on Σ. Then $dA_t = \sqrt{\det h_t}\, dx^1 dx^2$ and the energy can be written as

$$E(\varphi, h_t, g) = \int \sum h_t^{ij} \varphi^*(g)_{ij} \sqrt{\det h_t}\, dx^1 dx^2.$$

Since

$$\frac{\partial (h_t^{ij} \sqrt{\det h_t})}{\partial t} = -h_t^{i\alpha} (\dot{h}_t)_{\alpha\beta} h_t^{\beta j} \sqrt{\det h_t} + h_t^{ij} \frac{1}{2} h_t^{\alpha\beta} (\dot{h}_t)_{\alpha\beta} \sqrt{\det h_t},$$

the formula follows. If $\frac{dE(\varphi,h_t,g)}{dt}|_{t=0} = 0$ for arbitrary $\dot{h}_{\alpha\beta}$, one has

$$\sum_{i,j} (\frac{1}{2} h^{ij} h^{\alpha\beta} - h^{i\alpha} h^{\beta j}) \varphi^*(g)_{ij} = 0, \quad \text{for any} \quad \alpha, \beta.$$

In the matrix expression, this becomes

$$\frac{1}{2} e(\varphi) h^{-1} - h^{-1} \varphi^*(g) h^{-1} = 0.$$

Hence $\varphi^*(g) = \frac{1}{2} e(\varphi) h$. q.e.d.

Definition 1. A map $\varphi : \Sigma \to (N^n, g)$ is called a branched immersion if φ is an immersion except isolated points at which the differential $d\varphi$ is a zero map satisfying the characterization in [11]. (See 1.2 Definition in [11].) These points are called the branched points of φ.

If φ is a branched immersion, the pull back metric $\varphi^*(g)$ can be expressed as $\lambda^2 h$, where h is a smooth metric on Σ, and λ is a smooth scalar function with isolated and finite order zeros.

Thus one has the following result of Sacks and Uhlenbeck:

Corollary 1. *[25] If (φ, h) is a critical point of $E(\cdot, \cdot, g)$, and φ is a nonconstant map, then φ is a branched minimal immersion.*

Proof. Assume that h_t is a smooth family of metrics on Σ with $h_0 = h$, $\dot{h} = \frac{\partial h_t}{\partial t}|_{t=0}$ and φ_t is a smooth family of maps with $\varphi_0 = \varphi$, $\frac{\partial \varphi_t}{\partial t}|_{t=0} = V$. By Leibniz's rule, one has

$$\frac{dE(\varphi_t, h_t, g)}{dt}|_{t=0} = \frac{dE(\varphi_t, h, g)}{dt}|_{t=0} + \frac{dE(\varphi, h_t, g)}{dt}|_{t=0}.$$

Thus (φ, h) is a critical point of $E(\cdot, \cdot, g)$ is equivalent to that the map φ is both harmonic and weakly conformal between (Σ, h) and (N^n, g). When φ is a nonconstant map, this is equivalent to that φ is a branched minimal immersion. q.e.d.

Lemma 2. *Follow the notation as in Lemma 1 and assume that h is a critical point of $E(\varphi, \cdot, g)$. Then*

$$\frac{d^2E(\varphi, h_t, g)}{dt^2}|_{t=0} = \int (\frac{1}{2}e(\varphi)(Tr\, h^{-1}\dot{h}h^{-1}\dot{h}) - \frac{1}{4}e(\varphi)(Tr\, h^{-1}\dot{h})^2)\, dA,$$

where Tr denotes the trace of a matrix.

Proof. Now we continue the computation in the proof of Lemma 1 and differentiate $\frac{dE(\varphi,h_t,g)}{dt}$. Because

$$\sum_{i,j}(\frac{1}{2}h^{ij}h^{\alpha\beta} - h^{i\alpha}h^{\beta j})\varphi^*(g)_{ij} = 0,$$

those terms which come from the differentiation on $(\dot{h}_t)_{\alpha\beta}\sqrt{\det h_t}$ have no contribuition. So we only need to differentiate

$$\sum_{i,j}(\frac{1}{2}h_t^{ij}h_t^{\alpha\beta} - h_t^{i\alpha}h_t^{\beta j})\varphi^*(g)_{ij}.$$

Now we compute the formula in terms of matrices and get

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{2}(Tr\, h_t^{-1}\varphi^*(g))h_t^{-1} - h_t^{-1}\varphi^*(g)h_t^{-1})|_{t=0} \\
&\quad = -\frac{1}{2}(Tr\, h^{-1}\dot{h}h^{-1}\varphi^*(g))h^{-1} - \frac{1}{2}e(\varphi)h^{-1}\dot{h}h^{-1} \\
&\qquad + h^{-1}\dot{h}h^{-1}\varphi^*(g)h^{-1} + h^{-1}\varphi^*(g)h^{-1}\dot{h}h^{-1} \\
&\quad = -\frac{1}{4}e(\varphi)(Tr\, h^{-1}\dot{h})h^{-1} + \frac{1}{2}e(\varphi)h^{-1}\dot{h}h^{-1}.
\end{aligned}$$

Therefore,

$$\frac{d^2E(\varphi, h_t, g)}{dt^2}|_{t=0} = \int (\frac{1}{2}e(\varphi)(Tr\, h^{-1}\dot{h}h^{-1}\dot{h}) - \frac{1}{4}e(\varphi)(Tr\, h^{-1}\dot{h})^2)\, dA.$$

q.e.d.

Lemma 3. *Assume that (φ, h) is a critical point of $E(\cdot, \cdot, g)$. Let h_t be a smooth family of metrics on Σ with $h_0 = h$, $\dot{h} = \frac{\partial h_t}{\partial t}|_{t=0}$ and let φ_t be a smooth family of maps with $\varphi_0 = \varphi$, $\frac{\partial \varphi_t}{\partial t}|_{t=0} = V$. Then we* *have*

$$
\begin{aligned}
&\frac{d^2E(\varphi_t,h_t,g)}{dt^2}|_{t=0}\\
&= 2\int (|\nabla V|^2+\sum_i < R^N(d\varphi(e_i),V)d\varphi(e_i),V >)\, dA\\
&\quad +\int (\frac{1}{2}e(\varphi)(Tr\, h^{-1}\dot{h}h^{-1}\dot{h})-\frac{1}{4}e(\varphi)(Tr\, h^{-1}\dot{h})^2)\, dA\\
&\quad +2\int\sum_{i,j}(\frac{1}{2}h^{ij}h^{\alpha\beta}-h^{i\alpha}h^{\beta j})\dot{h}_{\alpha\beta}\\
&\qquad\qquad \cdot(< \nabla_{\frac{\partial}{\partial x^i}}V, d\varphi(\frac{\partial}{\partial x^j}) > + < \nabla_{\frac{\partial}{\partial x^j}}V, d\varphi(\frac{\partial}{\partial x^i}) >)\, dA,
\end{aligned}
$$

where R^N is the curvature tensor of (N,g), and $\{e_1,\, e_2\}$ is a local frame for h.

Proof. By Leibniz's rule, one has

$$
\begin{aligned}
\frac{d^2E(\varphi_t,h_t,g)}{dt^2}|_{t=0} \quad &= \quad \frac{d^2E(\varphi_t,h,g)}{dt^2}|_{t=0}+\frac{d^2E(\varphi,h_t,g)}{dt^2}|_{t=0}\\
&\quad + 2\frac{\partial^2E(\varphi_t,h_s,g)}{\partial t\partial s}|_{t=s=0}.
\end{aligned}
$$

The formula

$$
\frac{d^2E(\varphi_t,h,g)}{dt^2}|_{t=0}=2\int(|\nabla V|^2+\sum < R^N(d\varphi(e_i),V)d\varphi(e_i),V >)\, dA
$$

is well known and can be found for instance in [9] or [29]. The formula for $\frac{d^2E(\varphi,h_t,g)}{dt^2}|_{t=0}$ is derived in Lemma 2. A direct computation shows that

$$
\frac{\partial\varphi_t^*(g)_{ij}}{\partial t} = < \nabla_{\frac{\partial}{\partial x^i}}V, d\varphi(\frac{\partial}{\partial x^j}) > + < \nabla_{\frac{\partial}{\partial x^j}}V, d\varphi(\frac{\partial}{\partial x^i}) > .
$$

This together with the computation in the proof of Lemma 1 gives the formula for $\frac{\partial^2E(\varphi_t,h_s,g)}{\partial t\partial s}|_{t=s=0}$. q.e.d.

Because the domain is two dimensional, the energy is a conformal invariant on the metric of the domain. Denote $E_g(\varphi,h)=E(\varphi,h,g)$. Then E_g can be viewed as a smooth function $E_g(\varphi,[h])$ on $C^\infty(\Sigma,N)\times\mathcal{T}_r$, where $\mathcal{T}_r$ is the Teichmüller space of Σ, and $[h]$ is the conformal class of h. Note that $[\varphi^*(g)]$ is well-defined for a branched immersion and $E_g(\varphi, [\varphi^*(g)]) = 2A(\varphi, g)$, where $A(\varphi, g)$ is the area of $\varphi(\Sigma)$ in (N, g). It is clear that the opposite direction of Corollary 1 is also true. That is, if $\varphi : \Sigma \to (N^n, g)$ is a branched minimal immersion, then $(\varphi, [\varphi^*(g)])$ is a critical point of E_g.

Remark. Assume that $(\varphi, [h])$ is a critical point of E_g and $[h_t]$ is a variation of the conformal structure. Because in our case the energy functional is a conformal invariant, and by a result of Moser [21] we can choose h_t such that they all determine the same volume form; that is, we can assume $Tr\, h^{-1}\dot{h} = 0$ in the second variational formula.

Assume that $\varphi : \Sigma \to (N^n, g)$ is a branched minimal immersion and $(\varphi, [h])$ is the corresponding critical point on E_g. Define a function f_ε on Σ:

$$f_\varepsilon(x) = \begin{cases} 0 & |x| < \varepsilon^2, \\ \dfrac{\log \frac{|x|}{\varepsilon^2}}{\log \frac{1}{\varepsilon}} & \varepsilon^2 \leq |x| \leq \varepsilon, \\ 1 & |x| > \varepsilon. \end{cases} \tag{1}$$

Then $\lim\limits_{\varepsilon \to 0} \int |\nabla f_\varepsilon|^2 \, dA = 0$. Now we choose f_ε such that it vanishes near each branched point of φ.

Lemma 4. *If we denote the second variation of E_g in the direction of V and $\dot{h}$ by $\delta^2 E_g(V, \dot{h})$, then*

$$\delta^2 E_g(V, \dot{h}) = \lim_{\varepsilon \to 0} \delta^2 E_g(f_\varepsilon V, \dot{h}).$$

Proof. A direct computation gives us

$$\nabla_{\frac{\partial}{\partial x^i}} f_\varepsilon V = f_\varepsilon \nabla_{\frac{\partial}{\partial x^i}} V + \frac{\partial f_\varepsilon}{\partial x^i} V$$

and

$$|\nabla f_\varepsilon V|^2 = f_\varepsilon^2 |\nabla V|^2 + |\nabla f_\varepsilon|^2 |V|^2 + 2 \sum < e_i(f_\varepsilon) V, f_\varepsilon \nabla_{e_i} V > .$$

Since h^{-1}, $\dot{h}$, φ, and V are smooth and fixed, the norms and the norms of their derivatives are all bounded. Therefore,

$$\begin{aligned} &|\delta^2 E_g(V, \dot{h}) - \lim_{\varepsilon \to 0} \delta^2 E_g(f_\varepsilon V, \dot{h})| \\ &\leq C_1 \lim_{\varepsilon \to 0} \int |\nabla f_\varepsilon|^2 \, dA + C_2 (\lim_{\varepsilon \to 0} \int |\nabla f_\varepsilon|^2 \, dA)^{\frac{1}{2}} \\ &= 0, \end{aligned}$$

where C_1 and C_2 are positive constants independent of ε. q.e.d.

Definition 2. A branched minimal immersion $\varphi : \Sigma \to (N^n, g)$ is called strictly stable if $\lim_{\varepsilon\to 0} \delta^2 A(f_\varepsilon V) > 0$, where $V = \frac{\partial \varphi_t}{\partial t}|_{t=0}$ for any smooth family of maps φ_t with $\varphi_0 = \varphi$. It is called stable if $\lim_{\varepsilon\to 0} \delta^2 A(f_\varepsilon V)$
≥ 0

Theorem 1. *A branched minimal immersion $\varphi : \Sigma \to (N^n, g)$ is strictly stable if and only if the correponding critical point on E_g is strictly stable.*

Proof. We first claim that for any branched immersion $\phi : \Sigma \to (N^n, g)$ and any smooth metric h on Σ, one always has

$$E_g(\phi, h) \geq 2A(\phi, g).$$

Choose x^1, x^2 to be the conformal coordinates for the pull back metric $\phi^*(g)$; that is,

$$\sum_{kl} g_{kl} \frac{\partial \phi^k}{\partial x^i} \frac{\partial \phi^l}{\partial x^j} = \mu^2 \delta_{ij},$$

where μ is nonnegative. Express the inverse matrix (h^{ij}) in this coordinates as $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$, where a and b are positive. Then we have

$$\begin{aligned} E_g(\phi, h) &= \int \sum h^{ij} g_{kl} \frac{\partial \phi^k}{\partial x^i} \frac{\partial \phi^l}{\partial x^j}\, dA \\ &= \int (a\mu + b\mu) \frac{1}{\sqrt{ab - c^2}} dx^1 dx^2 \\ &\geq 2 \int \mu \frac{\sqrt{ab}}{\sqrt{ab - c^2}} dx^1 dx^2 \\ &\geq 2A(\phi, g). \end{aligned}$$

The equalities hold if and only if $a = b$ and $c = 0$, i.e., when ϕ is a weakly conformal map.

Assume that $(\varphi, [h])$ is the corresponding critical point on E_g of the branched minimal immersion, where h is a smooth metric on Σ. Let h_t be a smooth family of metrics on Σ with $h_0 = h$, $\dot{h} = \frac{\partial h_t}{\partial t}|_{t=0}$ and φ_t be a smooth family of maps with $\varphi_0 = \varphi$, $\frac{\partial \varphi_t}{\partial t}|_{t=0} = V$. Define $\varphi_t^\varepsilon(x) = \exp_{\varphi(x)} t f_\varepsilon V(x)$, where f_ε is chosen as in (1). Then φ_t^ε is a smooth family of branched immersions with $\varphi_0^\varepsilon = \varphi$ and $\frac{\partial \varphi_t^\varepsilon}{\partial t}|_{t=0} = f_\varepsilon V$. By the claim proved above, one has

$$E_g(\varphi_t^\varepsilon, h_t) \geq 2A(\varphi_t^\varepsilon, g).$$

Define the C^2 nonnegative function F by

$$F(t) = E_g(\varphi_t^\varepsilon, h_t) - 2A(\varphi_t^\varepsilon, g).$$

Because $F(0) = \dot{F}(0) = 0$, it follows that $\ddot{F}(0) \geq 0$. Hence

$$\delta^2 E_g(f_\varepsilon V, \dot{h}) \geq 2\delta^2 A(f_\varepsilon V)$$

and thus

$$\delta^2 E_g(V, \dot{h}) \geq \lim_{\varepsilon \to 0} 2\delta^2 A(f_\varepsilon V) > 0.$$

One also has $\delta^2 E_g(0, \dot{h}) > 0$ by Lemma 3 for $\dot{h}$ which is not identically zero. This shows that $(\varphi, [h])$ is a strictly stable critical point on E_g.

Assume that $(\varphi, [h])$ is a strictly stable critical point on E_g. Then φ is a branched minimal immersion and one has

$$\delta^2 A(f_\varepsilon V) = \frac{1}{2}\delta^2 E_g(f_\varepsilon V, 0).$$

Thus

$$\lim_{\varepsilon \to 0} \delta^2 A(f_\varepsilon V) = \frac{1}{2}\delta^2 E_g(V, 0) > 0.$$

Hence φ is a strictly stable branched minimal immersion q.e.d.

2. The deformation of branched minimal surfaces

Let Σ be a closed surface and (N^n, g_0) be a complete Riemannian n-manifold which is not necessarily compact. The strict stability in the sense of Definition 2 works suitably for the deformation of a branched minimal immersion. In particular, we have:

Theorem 2. *Assume that $\varphi_0 : \Sigma \to (N^n, g_0)$ is a strictly stable branched minimal immersion. Then there exists a strictly stable branched minimal immersion $\varphi_t : \Sigma \to (N^n, g_t)$ for any g_t which is close enough to g_0. Furthermore, φ_t converges to φ_0 in C^∞ if g_t converges to g_0 in C^∞.*

Proof. Let $(\varphi_0, [h_0])$ be the corresponding critical point on E_{g_0}. By Theorem 1, one knows that $(\varphi_0, [h_0])$ is strictly stable. Particularlly, φ_0 is a strictly stable harmonic map from (Σ, h_0) to (N, g_0). It is a theorem of Eells and Lemaire [8] that there exist a neighborhood $\mathcal{V}$ of h_0 and g_0 in $\mathcal{M}(\Sigma) \times \mathcal{M}(N)$ and a unique smooth map S on $\mathcal{V}$ such that $S(h_0, g_0) = \varphi_0$ and $S(h, g)$ is a smooth harmonic map between (Σ, h) and (N, g). Let $\varphi_{t,h} = S(h, g_t)$ and $\mathcal{U}$ be the corresponding neighborhood of $[h_0]$ in the Teichmüller space $\mathcal{T}_r$. Since the energy is a conformal invariant on the domain, $\varphi_{t,h}$ is also harmonic with respect to any other representative of $[h]$. Thus $\varphi_{t,h}$ is determined by $[h]$ in $\mathcal{U}$. Define

$$G : \mathcal{U} \times (-\varepsilon, \varepsilon) \to dE_{g_t}|_{(\varphi_{t,h}, [h])},$$

where dE_{g_t} is the differential of E_{g_t}. Note that $G([h_0], 0) = 0$ and $dG|_{([h_0],0)}$ is of full rank because the $(\varphi_0, [h_0])$ is a strictly stable critical point of E_{g_0}. By applying the implicit function theory to G, there exists a smooth path $[h_t]$ in $\mathcal{T}_r$ such that $G([h_t], t) = 0$. Denote φ_{t,h_t} by φ_t. It follows that $(\varphi_t, [h_t])$ is a critical point of E_{g_t} and φ_t is a branched minimal immersion. Because the energy E_g depends smoothly on g, we can conclude that $(\varphi_t, [h_t])$ is a strictly stable critical point for t small enough. Thus φ_t is a strictly stable branched minimal immersion. By the construction of φ_t and the theorem of Eells and Lemaire in [8], one also has that φ_t converges to φ_0 in C^∞. q.e.d.

Proposition 1. *Every branched Lagrangian minimal surface in a Kähler surface N with negative Ricci curvature is strictly stable.*

Proof. The surface can be realized as the image of a branched minimal immersion φ_0 from Σ to N. Let f_ε be defined as in (1), which has support away from the branched points of φ_0, and assume that V is a vector field along φ_0 which is defined on Σ. Define the one-form β_ε on Σ by $\beta_\varepsilon(u) =< Jf_\varepsilon V, u >$, where J is the complex structure on N and $u \in T\Sigma$. By a result in [4] and [20], we have

$$\begin{aligned} \delta^2 A(f_\varepsilon V) &= \int_\Sigma (|d\beta_\varepsilon|^2 + |\delta\beta_\varepsilon|^2 - Ric\,(f_\varepsilon V, f_\varepsilon V))\, dA \\ &\geq c \int_\Sigma |f_\varepsilon V|^2\, dA, \end{aligned}$$

where Ric is the Ricci curvature of the Kähler surface satisfying

$$Ric\,(V, V) \leq -c|V|^2$$

for some positive constant c. Thus

$$\lim_{\varepsilon\to 0}\delta^2 A(f_\varepsilon V)\geq \lim_{\varepsilon\to 0} c\int_\Sigma |f_\varepsilon V|^2\, dA>0,$$

and the surface is strictly stable in the sense of Definition 2. q.e.d.

Corollary 2. *Let L be a branched Lagrangian minimal surface in a Kähler surface with negative Ricci curvature. Then there is a strictly stable branched minimal surface near L with respect to any Riemannian metric which is close to this Kähler metric.*

3. The oscillation of the conformal harmonic maps

Let $\varphi:\Sigma\to N$ be a smooth map from a Riemannian surface Σ to a complete n-dimensional Riemannian manifold N. Let $\theta^1,\ \theta^2$ be an orthonormal coframe in a neighborhood of $p\in\Sigma$ and let $\omega^1,\cdots,\omega^n$ be an orthonormal coframe in a neighborhood of $\varphi(p)\in N$. Define $\varphi^l_\alpha,\ 1\leq\alpha\leq 2,\ 1\leq l\leq n$ by

$$\varphi^*\omega^l=\sum_{\alpha=1}^2\varphi^l_\alpha\theta^\alpha \quad\text{for}\quad 1\leq l\leq n.$$

We have the structure equations for N and Σ:

$$d\omega^l=\sum_{m=1}^n\omega^l_m\wedge\omega^m \quad\text{and}\quad \omega^l_m=-\omega^m_l \quad\text{for}\quad 1\leq l,m\leq n,$$

$$d\theta^\alpha=\sum_{\beta=1}^2\theta^\alpha_\beta\wedge\theta^\beta \quad\text{and}\quad \theta^\alpha_\beta=-\theta^\beta_\alpha \quad\text{for}\quad 1\leq \alpha,\beta\leq 2.$$

Define $\varphi^l_{\alpha\beta},\ 1\leq\alpha,\beta\leq 2,\ 1\leq l\leq n$ by

$$d\varphi^l_\alpha+\sum_{m=1}^n\varphi^m_\alpha\varphi^*\omega^l_m+\sum_{\beta=1}^2\varphi^l_\beta\theta^\beta_\alpha=\sum_{\beta=1}^2\varphi^l_{\alpha\beta}\theta^\beta.$$

Choose the local coordinates at p to be 0 and let $\rho^2(y)$ be the square of the distance between y and $\varphi(0)$ in (N,g). Then $\rho^2(\varphi(x))$ is a function on Σ and

$$\Delta\rho^2(\varphi(x))=2\sum_\alpha(\sum_l\rho_l\varphi^l_\alpha)^2+2\rho\sum_{k,l,\alpha}\rho_{kl}\varphi^k_\alpha\varphi^l_\alpha+2\rho\sum_{l,\alpha}\rho_l\varphi^l_{\alpha\alpha},$$

where $1 \leq \alpha \leq 2$ and $1 \leq k, l \leq n$. The condition of φ to be harmonic is equivalent to $\sum_\alpha \varphi^l_{\alpha\alpha} = 0$ for all l. If we choose the normal coordinates $\{y^1, \ldots, y^n\}$ at $\varphi(0)$, then

$$\rho_l(y) = \frac{y^l}{\rho} \quad \text{and} \quad \rho_{kl}(y) = \frac{\delta_{kl}}{\rho} - \frac{y^k y^l}{\rho^3} - \sum_m \Gamma^m_{kl} \frac{y^m}{\rho}.$$

When φ is harmonic, one has

$$\Delta\rho^2 = 2|\nabla\varphi|^2 - 2\sum \varphi^m \Gamma^m_{kl} \varphi^k_\alpha \varphi^l_\alpha.$$

Hence $\rho^2(\varphi(x))$ is a subharmonic function on Σ when the metric on N is flat. A general Riemannian metric satisfies $\Gamma^m_{kl}(y) = O(|y|)$. By taking $y = \varphi(x)$, it follows that $\rho^2(\varphi(x))$ is subharmonic when $|x|$ is small. Further computation shows that

$$\begin{aligned} \Delta \log \rho^2 &= \frac{\Delta\rho^2}{\rho^2} - \frac{|\nabla\rho^2|^2}{\rho^4} \\ &= \frac{2}{\rho^2}\left[|\nabla\varphi|^2 - \sum \varphi^m \Gamma^m_{kl} \varphi^k_\alpha \varphi^l_\alpha - 2\sum_\alpha \left(\sum_l \rho_l \varphi^l_\alpha \right)^2 \right]. \end{aligned}$$

Lemma 5. *Assume that $\varphi : (B_2(0), \sum_{i=1}^2 (dx^i)^2) \to (N, g)$ is a conformal harmonic map from a ball of radius 2 into a normal neighborhood of $\varphi(0)$ in N. Then we have*

$$\max_{B_{r_2}(0)} \rho^2(\varphi(x)) \leq \left(\frac{r_2}{r_1} \right)^C \max_{B_{r_1}(0)} \rho^2(\varphi(x))$$

for $0 < r_1 \leq r_2 \leq \varepsilon \leq 1$, where ε is a constant depending only on the metric g, and C is a constant independent of r_1 and r_2.

Remark. The radius 2 in the Lemma does not matter, and the main point is to have a ball of fixed radius which maps into a normal neighborhood of $\varphi(0)$. The constant ε is chosen such that it is less than the fixed radius, and $\rho^2(\varphi(x))$ is subharmonic on $B_\varepsilon(0)$.

Proof. When φ is a constant map, the lemma holds trivally. So we assume that φ is a nonconstant map. Because φ is a conformal map, we have

$$\sum_l (\varphi^l_1)^2 = \sum_l (\varphi^l_2)^2 = \mu^2 \quad \text{and} \quad \sum_l \varphi^l_1 \varphi^l_2 = 0,$$

where μ is a smooth and nonnegative scalar function with isolated and finite order zeros. Hence

$$d\varphi(\frac{\partial}{\partial x^1}) = \mu e_1 \quad \text{and} \quad d\varphi(\frac{\partial}{\partial x^2}) = \mu e_2,$$

where e_1 and e_2 are orthonormal. Therefore,

$$\begin{aligned}
\Delta \log \rho^2 &= \frac{2}{\rho^2}\left[|\nabla\varphi|^2 - \sum \varphi^m \Gamma^m_{kl}\varphi^k_\alpha \varphi^l_\alpha - 2\sum_\alpha \left(\sum_l \rho_l \varphi^l_\alpha\right)^2\right] \\
&= \frac{2}{\rho^2}\left[2\mu^2 - 2\mu^2 \sum_{\alpha=1}^{2}(\nabla\rho\cdot e_\alpha)^2 - \sum \varphi^m \Gamma^m_{kl}\varphi^k_\alpha \varphi^l_\alpha\right] \\
&\geq \frac{2}{\rho^2}\left[2\mu^2 - 2\mu^2 - \sum \varphi^m \Gamma^m_{kl}\varphi^k_\alpha \varphi^l_\alpha\right] \\
&\geq -\bar{c},
\end{aligned}$$

where the positive constant $\bar{c}$ depends only on the upper bound of $|\nabla\varphi|^2$ and the metric g. We use the fact that $|\nabla\rho| = 1$ in the first inequality. A direct computation shows that $\Delta r^2 = 2$ and $\Delta \log r = 0$, where r is the distance function on the domain. Define

$$F(x) = e^{\frac{\bar{c}}{2} r(x)^2} \rho^2(\varphi(x)).$$

Then

$$\begin{aligned}
\Delta \log F(x) &= \Delta \log \rho^2 + \Delta \frac{\bar{c}}{2} r^2 \\
&\geq -\bar{c} + \bar{c} \\
&= 0,
\end{aligned}$$

so that $\log F(x)$ is a subharmonic function. Define

$$M(r) = \max_{\partial B_r(0)} F(x) = \max_{B_r(0)} F(x).$$

Then the function

$$\log F(x) - \frac{\log r - \log r_1}{\log r_2 - \log r_1}\log M(r_2) - \frac{\log r_2 - \log r}{\log r_2 - \log r_1}\log M(r_1)$$

is a subharmonic function and has nonpositive values on the circles of radius r_1 and r_2. By applying the maximun principle to the annulus between radius r_1 and r_2, we conclude that

$$\log M(r) \leq \frac{\log r - \log r_1}{\log r_2 - \log r_1}\log M(r_2) + \frac{\log r_2 - \log r}{\log r_2 - \log r_1}\log M(r_1)$$

for $r_1 < r < r_2$. This means that $\log M(r)$ is a convex function in terms of $\log r$. Since the choice of r_1 and r_2 is arbitrary, the conclusion holds for all $0 < r < 2$. Now we want to compute the derivative of $\log M(r)$ with respect to $\log r$ at $r = 1$ and bound it by a constant C. We have

$$\frac{d \log M(r)}{d \log r} = \frac{d \log M(r)}{dr} \frac{dr}{d \log r} = \frac{M'(r)}{M(r)} r,$$

where

$$M(r) = \max_{\partial B_r(0)} F(x) = e^{\frac{\bar{c}}{2} r^2} \max_{\partial B_r(0)} \rho^2(\varphi(x))$$

and

$$M'(r) \leq \max_{\partial B_r(0)} |\nabla F(x)|.$$

A direct computation shows that

$$\begin{aligned} |\nabla F(x)| &\leq \bar{c} r e^{\frac{\bar{c}}{2} r^2} \rho^2(\varphi(x)) + 2 e^{\frac{\bar{c}}{2} r^2} \rho(\varphi(x)) |\nabla \rho(\varphi(x))| \\ &\leq \bar{c} r e^{\frac{\bar{c}}{2} r^2} \rho^2(\varphi(x)) + 2 e^{\frac{\bar{c}}{2} r^2} \rho(\varphi(x)) |\nabla \varphi(x)| \end{aligned}$$

for $x \in \partial B_r(0)$. Hence

$$\frac{M'(1)}{M(1)} \leq \bar{c} + 2 \frac{\max_{\partial B_1} \rho(\varphi(x)) |\nabla \varphi|}{\max_{\partial B_1} \rho^2(\varphi(x))} \leq \bar{c} + 2 \frac{\max_{\partial B_1} |\nabla \varphi(x)|}{\max_{\partial B_1} \rho(\varphi(x))}.$$

So we can choose

$$C = \bar{c} + 2 \frac{\max_{\partial B_1} |\nabla \varphi(x)|}{\max_{\partial B_1} \rho(\varphi(x))}.$$

Because the slope of a convex function is increasing, we have that

$$\frac{\log M(r_2) - \log M(r_1)}{\log r_2 - \log r_1} \leq C,$$

for $0 < r_1 < r_2 \leq 1$. Therefore,

$$\log \frac{M(r_2)}{M(r_1)} \leq C \log \frac{r_2}{r_1},$$

or

$$\frac{M(r_2)}{M(r_1)} \leq (\frac{r_2}{r_1})^C.$$

Thus we have

$$\frac{e^{\frac{\bar{c}}{2} r_2^2} \max_{\partial B_{r_2}} \rho^2(\varphi(x))}{e^{\frac{\bar{c}}{2} r_1^2} \max_{\partial B_{r_1}} \rho^2(\varphi(x))} \leq (\frac{r_2}{r_1})^C.$$

Choose ε such that $\rho^2(\varphi(x))$ is subharmonic when $|x| \leq \varepsilon$. Hence

$$\max_{\partial B_r} \rho^2(\varphi(x)) = \max_{B_r} \rho^2(\varphi(x))$$

for $r \leq \varepsilon$. It follows that

$$\frac{\max_{B_{r_2}} \rho^2(\varphi(x))}{\max_{B_{r_1}} \rho^2(\varphi(x))} \leq \frac{e^{\frac{\bar{c}}{2} r_2^2} \max_{\partial B_{r_2}} \rho^2(\varphi(x))}{e^{\frac{\bar{c}}{2} r_1^2} \max_{\partial B_{r_1}} \rho^2(\varphi(x))} \leq (\frac{r_2}{r_1})^C,$$

when $0 < r_1 \leq r_2 \leq \varepsilon$. q.e.d.

4. The adjunction numbers

For a real surface Σ in a Riemannian 4-manifold N which has an almost complex structure J_N, one can consider the intersection of $T_x\Sigma$ and $J_N T_x\Sigma$ for points $x \in \Sigma$. There are only two possibilities: either $T_x\Sigma \cap J_N T_x\Sigma = \{0\}$ where x is called a totally real point or $T_x\Sigma = J_N T_x\Sigma$ where x is called a complex point. When the complex points are isolated, it has a well-defined index at each complex point and there are formulas which relate the total number of the complex points with indices to the topology of Σ. (See [5], [7], [31], [32], [33].) The characterization given by Chen and Tian [5] is the following:

$$a_N(\Sigma) = \int_\Sigma (K_T + K_N)\, dA = \sum ind\ x_k,$$

where K_T and K_N are the Gaussian curvatures of the tangent bundle and normal bundle of Σ in N respectively and $ind\ x_k$ is the index at a complex point x_k. The first equality is the definition of the adjunction number $a_N(\Sigma)$ of Σ in N, and the second equality is a theorem proved in [5]. The tangent planes and normal planes on a branched minimal surface are still well defined even at branched points [11]. The above discussions also hold for branched minimal surfaces and in that case the integral is understood as an improper integral. Moreover, it is proved by Webster [31] and also by Wolfson [33] that the complex points on a branched minimal surface are isolated and all of negative index when the surface is not holomorphic or antiholomorphic.

The bundle of complex structures on R^{2l} along a minimal surface Σ was discussed in Schoen's unpublished paper [26]. For the sake of completeness and the readers' reference, we adapt the argument to our settings and include a discussion here. One can identify the 2-vectors $\wedge^2 R^4$ with the anti-symmetric 4×4 matrices by associating to a 2-vector η

$$\eta = \frac{1}{2}\sum_{k,l=1}^{4} a_{kl} e_k \wedge e_l$$

the anti-symmetric matrix $A = (a_{kl})$, where $\{e_k,\ 1 \le k \le 4\}$ is an oriented orthonormal basis of R^4. The inner product of $\wedge^2 R^4$ induced on the anti-symmetric matrices is denoted by $< . , . >$, and it is

$$< A, B >= -\frac{1}{2} Tr(AB)$$

for A, B anti-symmetric matrices. Denote the set of oriented complex structures on R^4 by C_4. That is, it is the set of positively oriented $J : R^4 \to R^4$ satisfying

$$J^t J = I, \quad J^2 = -I,$$

where J^t is the transpose of the matrix J. The image of C_4 under the above identification is the sphere of radius $\sqrt{2}$ in $\wedge^2_+ R^4$ which consists of the self-dual 2-vectors in $\wedge^2 R^4$. Let $\{f_k,\ 1 \le k \le 4\}$ be another oriented orthonormal basis of R^4, where $f_k = \sum_l m_{lk} e_l$ and denote $M = (m_{kl})$. Note that $M^t M = I$, and thus $M^t = M^{-1}$. If a 2-vector η is identified with a matrix A in the basis $\{e_k,\ 1 \le k \le 4\}$, it is identified with the matrix $M^{-1}AM$ in the basis $\{f_k,\ 1 \le k \le 4\}$. If a complex structure in the basis $\{e_k,\ 1 \le k \le 4\}$ is expressed as a matrix J, it is expressed as $M^{-1}JM$ in the basis $\{f_k,\ 1 \le k \le 4\}$. Thus we have the identification as a bundle on a Riemannian 4-manifold N. Denote the total space of the restricted bundle on Σ by E. We claim that E has an almost complex structure. The fiber S^2 has an almost complex structure or we also can define the almost complex structure directly from C_4 as follows. Let $\mathcal{A}$ be the set of anti-symmetric 4×4 matrices. For $J \in C_4$, one has

$$T_J C_4 = \{A \in \mathcal{A} : AJ + JA = 0\}.$$

We define an almost complex structure

$$\mathcal{J} : T_J C_4 \to T_J C_4$$

on C_4 by $\mathcal{J}(A) = AJ$. It is easy to check that this is a right definition and it gives an almost complex structure on the fiber. The same construction gives an almost complex structure on C_{2l} for $l>2$ as well. Using the Levi-Civita connection we have a complement to the fiber which is called a horizontal space, and it can be identified with $T\Sigma$ via the projection map. The identification induces an almost complex structure on the horizontal space. Therefore, we have an almost complex structure on the total space E and we will denote it still by $\mathcal{J}$. Assume that $u(t)$ is a section along a curve $\gamma(t)$ in Σ and $\frac{d\gamma(t)}{dt} = T$. Then $(\gamma(t), u(t))$ is a curve in E and the projection of its tangent vector into the fiber is just $\nabla_T u$.

Assume that $\{e_1, e_2, e_3, e_4\}$ is a local, oriented orthonormal basis of the tangent bundle TN over Σ such that $\{e_1, e_2\}$ is an oriented basis of $T\Sigma$. We define an almost complex structure J_Σ of TN along Σ by

$$J_\Sigma(e_1) = e_2, \quad J_\Sigma(e_2) = -e_1,$$
$$J_\Sigma(e_3) = e_4, \quad J_\Sigma(e_4) = -e_3.$$

Hence J_Σ is a section of the above bundle. Chen and Tian [5] shows that

$$K_T + K_N = \Omega_{12} + \Omega_{34} + \frac{1}{2}|H|^2 - \frac{1}{4}|\nabla J_\Sigma|^2,$$

where Ω_{kl} are some ambient curvatures, H is the mean curvature on Σ satisfying

$$|H|^2 = (h^3_{11} + h^3_{22})^2 + (h^4_{11} + h^4_{22})^2,$$

and

$$|\nabla J_\Sigma|^2 = 2(h^4_{12} - h^3_{11})^2 + 2(h^3_{12} + h^4_{11})^2 + 2(h^4_{22} - h^3_{12})^2 + 2(h^3_{22} + h^4_{21})^2.$$

Thus one has

$$a_N(\Sigma) = \int_\Sigma (\Omega_{12} + \Omega_{34} + \frac{1}{2}|H|^2 - \frac{1}{4}|\nabla J_\Sigma|^2)\, dA.$$

We will consider maps from Σ into N from now on. Hence the surface on the above discussions should be replaced by the image of a map. But we will use the same notation whenever there is no confusion.

Lemma 6. *Assume that $\varphi : \Sigma \to (N, g)$ is a branched minimal immersion. Then the map $J_\Sigma : \Sigma \to E$ is holomorphic.*

Proof. Assume that x^1, x^2 are the conformal coordinates near a point p on Σ for the pull back metric. Denote the complex structure by j which satisfies

$$j\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^2} \quad \text{and} \quad j\frac{\partial}{\partial x^2} = -\frac{\partial}{\partial x^1}.$$

Let $\{e_1, e_2, e_3, e_4\}$ be a local, oriented orthonormal basis of the tangent bundle TN as described before in a neighborhood of $\varphi(p)$ on $\varphi(\Sigma)$. Therefore

$$d\varphi(\frac{\partial}{\partial x^1}) = \mu e_1 \quad \text{and} \quad d\varphi(\frac{\partial}{\partial x^2}) = \mu e_2,$$

where $\mu = |d\varphi(\frac{\partial}{\partial x^1})| = |d\varphi(\frac{\partial}{\partial x^2})|$. Note that $J_{\varphi(\Sigma)}$, which we will denote by J_Σ instead, can be identified with

$$-(e_1 \wedge e_2 + e_3 \wedge e_4).$$

When p is an unbranched point, we have

$$\begin{aligned}
&\nabla_{e_1}(e_1 \wedge e_2 + e_3 \wedge e_4)\\
&\quad = \nabla_{e_1} e_1 \wedge e_2 + e_1 \wedge \nabla_{e_1} e_2 + \nabla_{e_1} e_3 \wedge e_4 + e_3 \wedge \nabla_{e_1} e_4\\
&\quad = (h^3_{11} e_3 + h^4_{11} e_4) \wedge e_2 + e_1 \wedge (h^3_{12} e_3 + h^4_{12} e_4)\\
&\qquad + (-h^3_{11} e_1 - h^3_{12} e_2) \wedge e_4 + e_3 \wedge (-h^4_{11} e_1 - h^4_{12} e_2)\\
&\quad = (h^3_{12} + h^4_{11}) e_1 \wedge e_3 + (h^4_{12} - h^3_{11}) e_1 \wedge e_4\\
&\qquad + (-h^3_{11} + h^4_{12}) e_2 \wedge e_3 + (-h^4_{11} - h^3_{12}) e_2 \wedge e_4\\
&\quad = (h^4_{12} - h^3_{11})(e_1 \wedge e_4 + e_2 \wedge e_3) + (h^3_{12} + h^4_{11})(e_1 \wedge e_3 - e_2 \wedge e_4).
\end{aligned}$$

A similar computation gives

$$\begin{aligned}
\nabla_{e_2}(e_1 \wedge e_2 + e_3 \wedge e_4) =& (h^4_{22} - h^3_{12})(e_1 \wedge e_4 + e_2 \wedge e_3)\\
&+ (h^3_{22} + h^4_{21})(e_1 \wedge e_3 - e_2 \wedge e_4).
\end{aligned}$$

The 2-vectors are identified with the anti-symmetric matrices, so we mix the notation sometimes. It can be checked that

$$e_1 \wedge e_4 + e_2 \wedge e_3 \in T_{J_\Sigma} C_4 \quad \text{and} \quad e_1 \wedge e_3 - e_2 \wedge e_4 \in T_{J_\Sigma} C_4.$$

Furthermore, we have

$$\mathcal{J}(e_1 \wedge e_4 + e_2 \wedge e_3) = e_1 \wedge e_3 - e_2 \wedge e_4$$

and

$$\mathcal{J}(e_1 \wedge e_3 - e_2 \wedge e_4) = -(e_1 \wedge e_4 + e_2 \wedge e_3).$$

Because one has $h^3_{11} = -h^3_{22}$ and $h^4_{11} = -h^4_{22}$ on a minimal surface, it follows that

$$\begin{aligned}
&\mathcal{J}\nabla_{e_1}(e_1 \wedge e_2 + e_3 \wedge e_4)\\
&\quad = (h^4_{12} - h^3_{11})(e_1 \wedge e_3 - e_2 \wedge e_4) + (-h^3_{12} - h^4_{11})(e_1 \wedge e_4 + e_2 \wedge e_3)\\
&\quad = (h^4_{12} + h^3_{22})(e_1 \wedge e_3 - e_2 \wedge e_4) + (h^4_{22} - h^3_{12})(e_1 \wedge e_4 + e_2 \wedge e_3)\\
&\quad = \nabla_{e_2}(e_1 \wedge e_2 + e_3 \wedge e_4).
\end{aligned}$$

That is, we have $\mathcal{J}\nabla_{e_1} J_\Sigma = \nabla_{e_2} J_\Sigma$. Since the almost complex structure on the horizontal space is given by the identification with $T\Sigma$, the map also satisfies the holomorphic condition in the horizontal space. Thus

$$dJ_\Sigma(j\frac{\partial}{\partial x^1}) = dJ_\Sigma(\frac{\partial}{\partial x^2}) = \mathcal{J} dJ_\Sigma(\frac{\partial}{\partial x^1}),$$

or J_Σ is holomorphic away from the branched points. Because J_Σ is a continuous map, it then follows that J_Σ is in fact holomorphic at all points on Σ by the standard fact in complex analysis or see discussions below. q.e.d.

We would like to write the holomorphic condition in local coordinates and show that it is equivalent to satisfying a first order elliptic system. Let $\{e_1, e_2, e_3, e_4\}$ be a local, oriented orthonormal basis of the tangent bundle TN. Then

$$\begin{aligned} E_1 &= e_1 \wedge e_2 + e_3 \wedge e_4, \\ E_2 &= e_1 \wedge e_3 - e_2 \wedge e_4, \\ E_3 &= e_1 \wedge e_4 + e_2 \wedge e_3 \end{aligned}$$

becomes a local basis for the bundle of self-dual 2-vectors. Assume that $\sum J_i E_i$ is a section of this bundle. Its covariant derivative is then defined to be

$$\nabla_{\frac{\partial}{\partial x^1}} \sum J_i E_i = \sum \frac{\partial J_i}{\partial x^1} E_i + \sum J_j < \nabla_{\frac{\partial}{\partial x^1}} E_j, E_i > E_i.$$

Note that C_4 is identified with a sphere of radius $\sqrt{2}$ in $\wedge^2_+ R^4$. Thus if $\sum J_i E_i$ is in this subbundle, one has

$$\nabla_{\frac{\partial}{\partial x^1}} \sum J_i E_i = a_1\xi_1 + b_1\xi_2 \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x^2}} \sum J_i E_i = a_2\xi_1 + b_2\xi_2,$$

where ξ_1, ξ_2 are of length $\sqrt{2}$ and orthogonal. The almost complex structure on the S^2 bundle is defined to be

$$\mathcal{J}(\xi_1) = \xi_2 \quad \text{and} \quad \mathcal{J}(\xi_2) = -\xi_1.$$

A section is holomorphic is then equivalent to $a_1 = b_2$ and $b_1 = -a_2$. Assume that J_Σ is identified with

$$-(e_1 \wedge e_2 + e_3 \wedge e_4)$$

at a point p and is written as $\sum J_i E_i$ near p, where $J_1 = -\sqrt{1 - J_2^2 - J_3^2}$. Thus

$$\begin{aligned} a_1 &= < \nabla_{\frac{\partial}{\partial x^1}} \sum J_i E_i, \xi_1 > \\ &= \sum \frac{\partial J_i}{\partial x^1} < E_i, \xi_1 > + \sum J_j \Gamma^i_{1j} < E_i, \xi_1 > \end{aligned}$$

and b_1, a_2, b_2 also have similar expressions. We can choose $\xi_1 = E_2$ and $\xi_2 = -E_3$ at p. Then at p,

$$a_1 = \frac{\partial J_2}{\partial x^1} + \sum J_j \Gamma^2_{1j}, \qquad b_1 = -\frac{\partial J_3}{\partial x^1} - \sum J_j \Gamma^3_{1j},$$

and

$$a_2 = \frac{\partial J_2}{\partial x^2} + \sum J_j \Gamma^2_{2j}, \qquad b_2 = -\frac{\partial J_3}{\partial x^2} - \sum J_j \Gamma^3_{2j}.$$

The symbols for the equations $a_1 - b_2 = 0$ and $a_2 + b_1 = 0$ are nondegenerate at p. By continuity, they are still nondegenerate near p. Hence J_1, J_2 satisfy a first order elliptic system and the coefficients of the lower order terms are related to $\Gamma^i_{\alpha j}$ only.

5. The limit of the adjunction numbers

Theorem 3. *Let $\varphi_i : \Sigma \to (N, g_i)$ be a stable branched minimal immersion from a closed surface Σ to a Riemannian 4-manifold (N, g_i). Assume that g_i converges to g_0 and φ_i converges to φ_0 in C^∞, where φ_0 is a branched minimal immersion from Σ to (N, g_0). Then*

$$a(\varphi_0(\Sigma)) = \lim_{i\to\infty} a(\varphi_i(\Sigma)).$$

Proof. Without loss of generality, we can assume that φ_0 has only one branched point at x_0. Let $B_r(x_0)$ be the ball centered at x_0 of radius r with respect to the pull back metric $\varphi_0^*(g_0)$. For i large enough all the branched points of φ_i are within $B_r(x_0)$ and $K_T^i + K_N^i$ converges to $K_T^0 + K_N^0$ on $\Sigma \setminus B_r(x_0)$ uniformly. Thus

$$\begin{aligned} a(\varphi_0(\Sigma)) &= \lim_{r\to 0} \int_{\Sigma\setminus B_r(x_0)} (K_T^0 + K_N^0)\, dA_0 \\ &= \lim_{r\to 0} \lim_{i\to\infty} \int_{\Sigma\setminus B_r(x_0)} (K_T^i + K_N^i)\, dA_i \\ &= \lim_{i\to\infty} a(\varphi_i(\Sigma)) - \lim_{r\to 0} \lim_{i\to\infty} \int_{B_r(x_0)} (K_T^i + K_N^i)\, dA_i, \end{aligned}$$

where dA_i is the volume form for the pull back metric $\varphi_i^*(g_i)$. Because

$$K_T^i + K_N^i = \Omega_{12}^i + \Omega_{34}^i + \frac{1}{2}|H_i|^2 - \frac{1}{4}|\nabla J_i|^2$$

and $H_i = 0$ at unbranched points, we have

$$\begin{aligned}
&\lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} (K_T^i + K_N^i)\, dA_i \\
&\quad = \lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} (\Omega_{12}^i + \Omega_{34}^i - \frac{1}{4}|\nabla J_i|^2)\, dA_i \\
&\quad = \lim_{r\to 0}\int_{B_r(x_0)} (\Omega_{12}^0 + \Omega_{34}^0)\, dA_0 \\
&\qquad - \frac{1}{4}\lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} |\nabla J_i|^2\, dA_i \\
&\quad = -\frac{1}{4}\lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} |\nabla J_i|^2\, dA_i.
\end{aligned}$$

If we can show that $\lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} |\nabla J_i|^2\, dA_i = 0$, then the theorem will be proved. Express the pull back metric as $h_i = \varphi_i^*(g_i) = \lambda_i^2 \bar{h}_i$, where $\bar{h}_i$ is a smooth metric with the volume form $d\bar{A}_i$, and λ_i is a smooth scalar function with isolated and finite order zeros. We can choose λ_i suitably such that λ_i and $\bar{h}_i$ converge to λ_0 and $\bar{h}_0$ in C^∞ respectively. Choose r small enough such that $B_r(x_0)$ is a conformal neighborhood for all $\bar{h}_i$. Compose φ_i with a conformal transformation on $B_r(x_0)$ if necessary, we can assume that x^1, x^2 are the conformal coordinates for all $\bar{h}_i$. Because the image is minimal, by the discussions in last section, it follows that $|\nabla_{\frac{\partial}{\partial x^k}} J_i|$ is bounded for any fixed i. That is, the energy density of J_i with respect to the metric $\bar{h}_i$ is bounded for any fixed i. We change the metric on the domain to $\bar{h}_i$, but still use the same notation $|\nabla J_i|$. If $|\nabla J_i|$ is bounded in $B_r(x_0)$ by a constant c which is independent of i, then

$$\begin{aligned}
\lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} |\nabla J_i|^2 d\bar{A}_i &\le c\lim_{r\to 0}\lim_{i\to\infty}\int_{B_r(x_0)} d\bar{A}_i \\
&= c\lim_{r\to 0}\int_{B_r(x_0)} d\bar{A}_0 \\
&= 0.
\end{aligned}$$

When the domain is two-dimensional, the energy is a conformal invariant on the metric of the domain. Hence the left-hand side is exactly the quantity which we want to control. So that in this case the theorem follows.

Now assume that

$$\max_{x \in B_r(x_0)} |\nabla J_i(x)| = b_i \qquad \text{for } i > 0,$$

where b_i tends to ∞, and assume that the maximun value b_i is obtained at x_i. Because $K_T^i + K_N^i$ converges to $K_T^0 + K_N^0$ uniformly on $\Sigma \setminus B_r(x_0)$ for any r, the sequence x_i must converge to x_0. We define a new metric $h_i' = b_i^2 \bar{h}_i$ on the domain, and choose a ball of radius $\frac{b_i r}{2}$ around x_i with respect to h_i'. Now we can consider φ_i as a map from $B_{\frac{b_i r}{2}}(0)$ in R^2 to (N, g_i). If we denote the energy density with respect to h_i' still by $|\nabla J_i|^2$, then we have $|\nabla J_i(0)| = 1$ and $|\nabla J_i(x)| \leq 1$ for $x \in B_{\frac{b_i r}{2}}(0)$. Because J_i satisfies a first order elliptic system in local coordinates, by an interior Schauder estimate [1], one has

$$|J_i|_{1,\alpha;B_1} \leq C(|J_i|_{0;B_2} + |f_i|_{0,\alpha;B_2}),$$

where C is a constant, and f_i is related to the Christoffel symbol of h_i' only. (See the discussions at the end of last section.) Since the metric h_i' converges to the flat metric on B_1, it follows that $|f_i|_{0,\alpha;B_2}$ converges to 0. Thus $|J_i|_{1,\alpha;B_1}$ is uniformly bounded. By the Ascoli-Arzela convergent Theorem, we have J_i converges to a section $\bar{J}$ uniformly in C^1 and

$$|\nabla \bar{J}(0)| = \lim_{i \to \infty} |\nabla J_i(0)| = 1. \tag{2}$$

Note that $B_r(x_0)$ is a conformal neighborhood for $\bar{h}_i$ with conformal coordinates x^1, x^2. With the coordinates, we denote the ball of radius $\frac{r}{2}$ at x_i in the Euclidean metric by $D_{\frac{r}{2}}(0)$. The map φ_i is a conformal harmonic map from $(D_{\frac{r}{2}}(0), \sum_{\alpha=1}^2 (dx^\alpha)^2)$ to (N, g_i). Define $\tilde{\varphi}_i(x) = \varphi_i(\frac{x}{b_i})$. Then $\tilde{\varphi}_i(x)$ is a conformal harmonic map from $(D_{\frac{b_i r}{2}}(0), \sum_{\alpha=1}^2 (dx^\alpha)^2)$ to (N, g_i). Let $\rho^2(y, g_i)$ be the square of the distance between y and $\varphi_i(0)$ in (N, g_i). Assume that

$$\max_{D_{\frac{1}{b_i}}} \rho^2(\varphi_i(x), g_i) = \max_{D_1} \rho^2(\tilde{\varphi}_i(x), g_i) = c_i^2.$$

Because b_i tends to ∞ , it follows that c_i tends to 0 and $\rho^2(\tilde{\varphi}_i(x), g_i)$ is a subharmonic function on $D_1(0)$ for i large enough. Thus the maximun value c_i^2 for $\rho^2(\tilde{\varphi}_i(x), g_i)$ can be attained at $\bar{x}_i \in \partial D_1(0)$. By choosing a new parametrization we can assume that $\bar{x}_i$ is fixed, say at the point $q = (1,0)$. The image $\tilde{\varphi}_i(D_1)$ is a branched minimal surface in (N, g_i). Since g_i converges to g_0, the monotonicity constant for branched minimal surfaces and the radius where the bound holds can be chosen uniformly. Therefore, [24]

$$\text{area}\,(\tilde{\varphi}_i(D_1), g_i) = \text{area}\,(\varphi_i(D_{\frac{1}{b_i}}), g_i) < cc_i^2.$$

Define a new metric $g_i' = c_i^{-2} g_i$ on N. Let $||\nabla\tilde{\varphi}_i||^2$ be the energy density of $\tilde{\varphi}_i$ with respect to the metric g_i'. Then

$$\int_{D_1} ||\nabla\tilde{\varphi}_i||^2 \, dA = 2 \,\text{area}\,(\tilde{\varphi}_i(D_1), g_i') < c.$$

Because there is no harmonic map from S^2 into $(R^4, \sum_{k=1}^4 (dy^k)^2)$, there cannot be any energy concentration. Moreover, $\varphi_i(0) = x_i$ converges to x_0. Thus there exist a subsequence, which is still denoted by $\tilde{\varphi}_i$, converges to a smooth harmonic map φ from $(D_1(0), \sum_{\alpha=1}^2 (dx^\alpha)^2)$ to $(R^4, \sum_{k=1}^4 (dy^k)^2)$ in C^∞ by a result of Sacks and Uhlenbeck [25]. Moreover,

$$\rho^2(\tilde{\varphi}(q), \sum_{k=1}^{4} (dy^k)^2)) = \lim_{i\to\infty} \rho^2(\tilde{\varphi}_i(q), g_i') = 1.$$

Hence $\tilde{\varphi}$ is a nonconstant map.

For any $L > 1$ we claim that the energy $E(\tilde{\varphi}_i(D_L), g_i')$ is also unformly bounded. This follows from a modification of the proof of Lemma 5.

A modification of Lemma 5. Because g_i converges to g_0 and φ_i converges to φ_0 in C^∞, there exists a uniform ε such that $\varphi_i(D_\varepsilon)$ lies in a normal neighborhood of $\varphi_i(0)$ in (N, g_i), and $\rho^2(\varphi_i(x), g_i)$ is subharmonic on D_ε. Moreover, we also have that

$$|\nabla\varphi_i(x)| \quad \text{and} \quad \frac{|\Gamma_{kl}^m(\varphi_i(x))|}{\rho(\varphi_i(x), g_i)}$$

are uniformly bounded on $D_\varepsilon(0)$. Thus the constant $\bar{c}_i$ in Lemma 5 can be chosen uniformly. Since $\rho(\varphi_i(x), g_i)$ converges to $\rho(\varphi_0(x), g_0)$ and $\max_{\partial D_\varepsilon} \rho(\varphi_0(x), g_0)$ is positive, it follows that $\max_{\partial D_\varepsilon} \rho(\varphi_i(x), g_i)$ has a uniform positive lower bound. The constant C_i in Lemma 5 can then be chosen uniformly. In conclusion, we show that there exist positive constants ε and C such that $\varphi_i : (D_{\frac{r}{2}}(0), \sum_{\alpha=1}^{2}(dx^\alpha)^2) \to (N, g_i)$ satisfies

$$\max_{D_{r_2}} \rho^2(\varphi_i(x), g_i) \leq (\frac{r_2}{r_1})^C \max_{D_{r_1}} \rho^2(\varphi_i(x), g_i)$$

for any $0 < r_1 \leq r_2 \leq \varepsilon$. Note that a constant conformal factor on the metric of the target will not affect the conclusion. Because $\frac{L}{b_i} \leq \varepsilon$ for i sufficiently large, Lemma 5 can be applied to $\tilde{\varphi}_i$ on D_L. Thus we show that

$$\max_{D_L} \rho^2(\tilde{\varphi}_i(x), g_i') \leq L^C \max_{D_1} \rho^2(\tilde{\varphi}_i(x), g_i') \leq L^C.$$

Since $\tilde{\varphi}_i(D_L) = \varphi_i(D_{\frac{L}{b_i}})$, for i sufficiently large the image lies in a ball in (N, g_i) where the monotonicity formula holds. The same argument as above shows that area $(\tilde{\varphi}_i(D_L), g_i') < c\, L^C$ [24]. That is, the energy $E(\tilde{\varphi}_i(D_L), g_i') < C_L$, where C_L is a constant depending on L only. Therefore, there exists a subsequence of $\tilde{\varphi}_i$, which is still denoted by $\tilde{\varphi}_i$, such that $\tilde{\varphi}_i$ converges to a smooth harmonic map φ from $D_L(0)$ to $(R^4, \sum_{k=1}^{4}(dy^k)^2)$ [25]. Choose a sequence L_k which tends to ∞ and use the diagonal process to choose a subsequence which converges to a smooth harmonic map φ in any compact set of R^2 [25]. Here φ is a harmonic map from $(R^2, \sum_{\alpha=1}^{2}(dx^\alpha)^2)$ to $(R^4, \sum_{k=1}^{4}(dy^k)^2)$. Consider the variations of φ which have compact supports and vanish near the branched points. Because there are no branched points of φ_i in the support of the variation for i large enough, the stability of φ_i implies that of φ (for such variations). By the same reason as above, we can show that the area of $\tilde{\varphi}_i(D_L)$ is of quadratic growth by the monotonicity formula [24]. Thus the area of $\varphi(R^2)$ is also of quadratic growth. It is a theorem of Micallef that every complete stable branched minimal surface in R^4 which is holomorphic with respect to some complex structure on R^4 ([18], [19]). (The stability is for variations which have compact supports and vanish near the branched points.) In particular, it implies that $\nabla J = 0$, where J is the section associated with $\varphi(R^2)$. The section J_i converges to J in C^∞ on any compact set away from the branched points. On the other hand, we know that J_i converges to $\bar{J}$ in $C^{1,\alpha}$ on B_1 by (2). Thus $J = \bar{J}$ on B_1 and

$$|\nabla J(0)| = \lim_{i \to \infty} |\nabla J_i(0)| = 1.$$

It is a contradiction. Hence $|\nabla J_i(x)|$ is uniformly bounded with respect to $\bar{h}_i$ and g_i. The theorem is then proved. q.e.d.

6. The main theorem

Theorem 4. *Assume that (N, g_0) is a Kähler-Einstein surface with first negative Chern class. Let $[A]$ be a class in the second homology group $H_2(N, Z)$, which can be represented by a finite union of branched Lagrangian minimal surfaces with respect to the metric g_0. Then with respect to any other metric in the connected component of g_0 in the moduli space of Kähler-Einstein metrics, the class $[A]$ can also be represented by a finite union of branched Lagrangian minimal surfaces.*

Proof. Let g be any metric in the connected component of g_0 in the moduli space of Kähler-Einstein metrics. There exists a smooth family of Kähler-Einstein metrics g_t, $0 \leq t \leq 1$, satisfying $g_1 = g$. A metric is said to have the property P if the class $[A]$ can be represented by a finite union of branched Lagrangian minimal surfaces with respect to this metric. Let

$$T = \{\ t \mid t \in [0,1] \text{ and } g_t \text{ has the property P}\}.$$

From the assumption of the theorem, we know that T contains 0. Now assume that t_0 belongs to T. That is, the class can be written as $[A] = \cup_1^n [\varphi_i(\Sigma_i)]$, where $\varphi_i : \Sigma_i \to (N, g_{t_0})$ is a branched minimal immersion and the image is Lagrangian. We will deform each φ_i separately. So now we only work on a single map $\varphi_{t_0} : \Sigma \to (N, g_{t_0})$. It is strictly stable by Proposition 1. Thus by Theorem 2 there exists a strictly stable branched minimal immersion φ_t from Σ to (N, g_t) for $|t - t_0| < \varepsilon$ and φ_t converges to φ_{t_0} in C^∞. The Lagrangian surface $\varphi_{t_0}(\Sigma)$ satisfies $a(\varphi_{t_0}(\Sigma)) = 0$. Because the adjunction number is an integer and

$$\lim_{t \to t_0} a(\varphi_t(\Sigma)) = a(\varphi_{t_0}(\Sigma))$$

by Theorem 3, $a(\varphi_t(\Sigma)) = 0$. Since the complex points on a branched minimal surface which is not holomorphic or antiholomorphic are isolated and of negative index (see [31], [33]), it follows that $\varphi_t(\Sigma)$ is totally real. A totally real, branched minimal surface in a Kähler-Einstein surface with $C_1 < 0$ is Lagrangian ([5], [33]). Thus $\varphi_t(\Sigma)$ is a branched Lagrangian minimal surface. Because there are only finite maps, we can choose ε such that each φ_i has a deformation in $|t - t_0| < \varepsilon$. Hence the class $[A]$ can be represented by a finite union of branched Lagrangian minimal surfaces with respect to the metric g_t for $|t - t_0| < \varepsilon$. That is, the set T is open.

Next we want to show the closedness of T. We need to get an area bound and we first show this for a smooth family of branched minimal immersions $\varphi_t : \Sigma \to (N, g_t)$, $t_0 \leq t < b$, which can be thought as the maps obtained above. Denote the area of $\varphi_t(\Sigma)$ in (N, g_t) by $A(\varphi_t, g_t)$ and $h(t,x) = \varphi_t^*(g_t)(x)$ with volume form dA_t. The pull back metric $h(t,x) = \lambda(t,x)^2 \bar{h}(t,x)$ for some smooth metric $\bar{h}(t,x)$ with the volume form $d\bar{A}_t$. Then

$$\begin{aligned}\frac{dA(\varphi_t, g_t)}{dt} &= \int_\Sigma \sum_{i,j=1}^{2} h^{ij}(t,x)\dot{h}_{ij}(t,x)dA_t \\ &= \int_\Sigma \sum_{i,j=1}^{2} \bar{h}^{ij}(t,x)\dot{h}_{ij}(t,x)d\bar{A}_t.\end{aligned}$$

Note that

$$h_{ij}(t,x) = \sum_{k,l=1}^{4} g_{kl}(t, \varphi_t(x)) \frac{\partial \varphi_t(x)^k}{\partial x^i} \frac{\partial \varphi_t(x)^l}{\partial x^j}.$$

Hence

$$\begin{aligned}\frac{\partial h_{ij}(s,x)}{\partial s}\Big|_{s=t} = &\sum_{k,l=1}^{4} \frac{\partial g_{kl}(s, \varphi_t(x))}{\partial s}\Big|_{s=t} \frac{\partial \varphi_t(x)^k}{\partial x^i} \frac{\partial \varphi_t(x)^l}{\partial x^j} \\ &+ \sum_{k,l=1}^{4} \frac{\partial g_{kl}(t, \varphi_t(x))}{\partial y^m} \frac{\partial \varphi_s(x)^m}{\partial s}\Big|_{s=t} \frac{\partial \varphi_t(x)^k}{\partial x^i} \frac{\partial \varphi_t(x)^l}{\partial x^j} \\ &+ \sum_{k,l=1}^{4} g_{kl}(t, \varphi_t(x)) \frac{\partial^2 \varphi_s(x)^k}{\partial s \partial x^i}\Big|_{s=t} \frac{\partial \varphi_t(x)^l}{\partial x^j} \\ &+ \sum_{k,l=1}^{4} g_{kl}(t, \varphi_t(x)) \frac{\partial \varphi_t(x)^k}{\partial x^i} \frac{\partial^2 \varphi_s(x)^l}{\partial s \partial x^j}\Big|_{s=t}.\end{aligned}$$

Because φ_t is a branched minimal immersion, the contribution of the terms which are obtained from fixing g_t and varying φ_t is zero. Thus we only need to consider the situation where φ_t is fixed and only g_t is varied. In this case

$$\dot{h}_{ij}(t,x) = \sum_{k,l=1}^{4} \frac{\partial g_{kl}(s, \varphi_t(x))}{\partial s}\Big|_{s=t} \frac{\partial \varphi_t(x)^k}{\partial x^i} \frac{\partial \varphi_t(x)^l}{\partial x^j}.$$

For fixed t and x, we choose the conformal coordinates for $\bar{h}_t$ at x such that $h_{ij}(t,x) = \delta_{ij}\lambda^2$ or $\bar{h}_{ij}(t,x) = \delta_{ij}$. We also choose the normal coordinates for g_t at $\varphi_t(x)$ such that $g_{kl}(t,\varphi_t(x)) = \delta_{kl}$. Then at (t,x)

$$h_{ij}(t,x) = \sum_{k=1}^{4} \frac{\partial\varphi_t(x)^k}{\partial x^i}\frac{\partial\varphi_t(x)^k}{\partial x^j} = \delta_{ij}\lambda^2.$$

Because g_t is a fixed smooth family, the quantity $|\frac{\partial g_{kl}(s,\varphi_t(x))}{\partial s}|$ has a uniform bound c. Hence

$$\begin{aligned}
\frac{dA(\varphi_t,g_t)}{dt} &= \int_\Sigma \sum \bar{h}^{ij}(t,x)\frac{\partial g_{kl}(s,\varphi_t(x))}{\partial s}|_{s=t}\frac{\partial\varphi_t(x)^k}{\partial x^i}\frac{\partial\varphi_t(x)^l}{\partial x^j}d\bar{A}_t \\
&\le c\int_\Sigma \sum_{k,l,i} |\frac{\partial\varphi_t(x)^k}{\partial x^i}||\frac{\partial\varphi_t(x)^l}{\partial x^i}|d\bar{A}_t \\
&\le \frac{c}{2}\int_\Sigma \sum_{k,l,i}(|\frac{\partial\varphi_t(x)^k}{\partial x^i}|^2 + |\frac{\partial\varphi_t(x)^l}{\partial x^i}|^2)d\bar{A}_t \\
&= c\int_\Sigma \lambda^2 d\bar{A}_t \\
&\le c\,A(\varphi_t,g_t).
\end{aligned}$$

Moreover, $A(\varphi_t,g_t) \le e^{c(t-t_0)}A(\varphi_{t_0},g_{t_0})$. The Gauss equation for minimal surfaces is

$$\overline{K}_N(t,x) = K_\Sigma(t,x) + |I\!I|_t^2(x)$$

at unbranched points, where $\overline{K}_N(t,x)$ is the sectional curvature of (N,g_t) on the tangent plane of $\varphi_t(\Sigma)$ at $\varphi_t(x)$, $K_\Sigma(t,x)$ is the Gaussion curvature for the pull back metric $h(t,x)$, and $|I\!I_t|^2(x)$ is the norm of the second fundamental form of $\varphi_t(\Sigma)$ in (N,g_t) at $\varphi_t(x)$. Integrating on both sides of the equation, we get

$$\begin{aligned}
\int_\Sigma \overline{K}_N(t,x)\,dA_t &= \int_\Sigma K_\Sigma(t,x)\,dA_t + \int_\Sigma |I\!I_t|^2(x)\,dA_t \\
&= 2\pi\chi(\Sigma) + 2\pi B(t) + \int_\Sigma |I\!I_t|^2(x)\,dA_t,
\end{aligned}$$

where $B(t)$ is the total branched order of the map φ_t. Note that the integrals in the formula are all understood as improper integrals. Because we have the area bound for $\varphi_t(\Sigma)$ and $\overline{K}_N(t,x)$ is bounded for a fixed family g_t, $0 \le t \le 1$, it follows that $\int_\Sigma |I\!I_t|^2(x)\,dA_t + 2\pi B(t)$ is uniformly bounded.

Similar to the lemma of Choi and Schoen in [6], one can show the boundness of the sup norm of $|II_t|^2(x)$ in a ball away from the branched points, if the L^2 norm of $|II_t|^2(x)$ is sufficiently small in a larger ball (compare to [2]). Since $\int_\Sigma |II_t|^2(x)\ dA_t$ is uniformly bounded, by applying Sacks and Uhlenbeck's covering argument [25], one can pick up a subsequence which converges to a branched minimal surface. Because the surfaces are Lagrangian and the areas are bounded, the limit surface is Lagrangian (see [27] and compare with [15],) and the area of the limit surface is also bounded by the same constant. The area of a closed minimal surface has a lower bound which depends only on the injective radius of the ambinent manifold. Because g_t is a fixed smooth family of metrics for $0 \leq t \leq 1$, this lower bound can be chosen uniformly. Thus once we have the area bound for the union of closed minimal surfaces, the total number of closed minimal surfaces in that union will be bounded. This shows that $b \in T$ and we can use the openness to continue the deformation. Although we do not have a lower bound for the length of the interval where the deformation can go further on each step. We do have a global area bound for the family of minimal surfaces during the process. Moreover, the topology is bounded and the union is finite. The same argument as above shows the closedness of T. The nonempty set T is both open and closed. So it must be the whole set $[0, 1]$. That is, the class $[A]$ can be represented by a finite union of branched Lagrangian minimal surfaces with repect to the metric $g = g_1$. This completes the proof. q.e.d.

Now we give a simple application of the theorem. Let

$$N = (M, g) \times (M, g),$$

where (M, g) is a closed Riemannian surface with a hyperbolic metric. Assume that f is a map from a closed surface Σ to M whose induced map on the first foundamental group π_1 is injective. Then the induced map of (f, f) on π_1 is also injective and there exists a branched minimal surface in the homotopy class of (f, f) by a result of Schoen and Yau [28]. Because the metrics on the two components of N are the same, the branched minimal immersions in the homotopy class of (f, f) must be of the form $(\bar{f}, \bar{f})$ by the uniqueness of harmonic maps into a hyperbolic space [12]. Thus the branched minimal immersions are Lagrangian if we reverse the orientation on the second component. By the same argument as in [16], it follows that the branched minimal surface in the homotopy class is unique since every branched minimal surface in the class is Lagrangian.

Now we change the metric on the second component in its moduli space of hyperbolic metrics. By Theorem 4 we still have the existence of the branched Lagrangian minimal surfaces in the homotopy class with respect to the new metric. The Lagrangian minimal surfaces obtained here can be of different topology with the one we have in [16]. One can also try to combine our existence result in [16] with Theorem 4 to get the existence of the branched Lagrangian minimal surfaces in other classes.

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