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SUTURED MANIFOLD HIERARCHIES, ESSENTIAL LAMINATIONS, AND DEHN SURGERY

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0. Introduction

A compact orientable surface F with nonnegative Euler characteristic is either a sphere, a disk, a torus, or an annulus. If a 3-manifold M contains such an essential surface, then it is said to be reducible, ∂-reducible, toroidal, or annular, respectively. Any such surface can be used to decompose the manifold further into simpler manifolds. We say that M is a *simple manifold* if it has no such surfaces. A simple manifold is expected to have a nice geometric structure. If M has nonempty boundary, then the Geometrization Theorem of Thurston for Haken manifolds says that M with boundary tori removed admits a finite volume hyperbolic structure with totally geodesic boundary. When M has no boundaries, Thurston's Geometrization Conjecture asserts that M is either hyperbolic, or is a Seifert fiber space with orbifold a sphere with at most 3 cone points.

Suppose T is a torus boundary component of M. We use $M(\gamma)$ to denote the manifold obtained by Dehn filling on T so that the slope γ on T bounds a disk in the Dehn filling solid torus. When $M = E(K)$ is the exterior of a knot K in S^3, denote $M(\gamma)$ by $K(\gamma)$, and call it the manifold obtained by γ surgery on the knot K. It is well known that if M is simple, then there are only finitely many Dehn fillings on

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T which produce nonsimple manifolds. If $M(\gamma_1)$ and $M(\gamma_2)$ are nonsimple manifolds, then the geometric intersection number between γ_1 and γ_2, denoted by $\Delta(\gamma_1, \gamma_2)$, is proved to be at most 8 by Gordon [10]. There are 10 different cases, according to the types of nonnegative Euler characteristic surfaces in $M(\gamma_i)$. In many cases, the upper bounds for $\Delta(\gamma_1, \gamma_2)$ have been established; see Table 0.1.

M(r_1) \ M(r_2)	A	T	D	S
S	2	3	0	1
D	5* (2)	2	1	
T	5* (3)	8		
A	5* (2)			

TABLE 0.1. Upper bounds of $\Delta(\gamma_1, \gamma_2)$

In the table, the left column and the top row denote the types of $M(\gamma_1)$ and $M(\gamma_2)$; D, S, A, T mean that $M(\gamma_i)$ contains an essential disk, sphere, annulus, or torus, respectively. The numbers without star are known to be the best possible, while those with star are best results so far (till June 1996). The numbers in parenthesis are the maximal values of known examples. The results can be found in the following papers: T–T, T–A, A–A and D–A are proved by Gordon [10]; S–T is proved independently by Wu [32] and Oh [24]; D–D by Wu [31]; S–D by Scharlemann [28]; S–S and D–T by Gordon and Luecke [13], [15]; S–A is to be proved in this paper:

Theorem 5.1. *Suppose M is a simple manifold with torus T as a boundary component. If $M(\gamma_1)$ is annular, and $M(\gamma_2)$ is reducible, then $\Delta(\gamma_1, \gamma_2) \leq 2$.*

The theorem is sharp. Hayashi and Motegi [19] gave an example of a hyperbolic 3-manifold M, such that $M(\gamma_1)$ is reducible and ∂-reducible, $M(\gamma_2)$ is toroidal and annular, and $\Delta(\gamma_1, \gamma_2) = 2$.

Remark. More recently, Gordon and I [16], [17] settled the remaining cases in Table 0.1. It turns out that for type D–A the bound is 2, and for types A–T and A–A the bound 5 given by Gordon in [10] is the best possible. Furthermore, we showed that if $M(r_1)$ is annular and $M(r_2)$ is toroidal, then $\Delta(r_1, r_2) \leq 3$, except for three specific manifolds M.

There are many examples showing that a generic hyperbolic manifold admits very few nonhyperbolic Dehn fillings. The following theorem shows that if the manifold is "large", then stronger results than those in Table 0.1 hold.

Theorems 4.1 and 4.6. *Let M be a simple 3-manifold with torus T as a boundary component, such that $H_2(M, \partial M - T) \neq 0$. If γ_1 and γ_2 are slopes on T such that*
(1) $M(\gamma_1)$ is annular and $M(\gamma_2)$ is reducible, or
(2) $M(\gamma_1)$ is toroidal, and $M(\gamma_2)$ is reducible, or
(3) $M(\gamma_1)$ is toroidal, and $M(\gamma_2)$ is ∂-reducible, then $\Delta(\gamma_1, \gamma_2) \leq 1$.

Note that the condition $H_2(M, \partial M - T) \neq 0$ is true unless M is either a rational homology solid torus or a rational homology cobordism between two tori. In particular, it is true if M either has a boundary component with genus ≥ 2, or if it has more than two boundary tori. Similar to Table 0.1, we have the following table of upper bounds of $\Delta(\gamma_1, \gamma_2)$ for such manifolds:

$M(r_1)$ \ $M(r_2)$	A	T	D	S
S	1	1	0	0
D	?	1	1	
T	?	?		
A	?			

TABLE 0.2. Upper bounds of $\Delta(\gamma_1, \gamma_2)$ when $H_2(M, \partial M - T) \neq 0$

The entries with question marks are unsettled. The case S–S is proved by Luecke [22], others follow from Theorems 4.1, 4.6 and Table 0.1. The results here are also sharp; see Examples 4.7 and 4.8.

The following is an application of the above theorems to Dehn surgery on knots K in S^3. It was conjectured that if K is hyperbolic and $K(\gamma)$ is nonhyperbolic then $r = p/q$ with $|q| \leq 2$. Corollary 4.5 proves this conjecture in the case that the knot complement contains an incompressible surface F cutting $E(K)$ into anannular manifolds.

Corollary 4.5. *Let K be a hyperbolic knot in S^3. Suppose there is an incompressible surface F in $E(K)$, cutting $E(K)$ into anannular manifolds X and Y. Then $K(\gamma)$ is hyperbolic for all non-integral slopes γ.*

The proofs of these results use a combination of sutured manifold theory, essential laminations, essential branched surfaces, and some combinatorial arguments. In Section 1 we defined cusped manifolds, and show that if there are some essential annuli connecting T to some components of $\partial_h M$, then $M(\gamma)$ has some nice properties whenever γ has high intersection number with the boundary slope of those annuli (Theorems 1.6, 1.8 and 1.9). Section 2 is devoted to the study of intersections between essential surfaces and essential branched surfaces. We show that they can be modified to intersect essentially on both of them.

Theorem 2.2. *Suppose B is an essential branched surface which fully carries a lamination λ, and suppose F is an essential surface in M. Then there is an essential branched surface B' which is a λ-splitting of B, and a surface F' isotopic to F, such that $F' \cap B'$ is an essential train track on F'.*

This theorem is fundamental in our proofs, and should be useful in the future. Combining with a result of Brittenham, it gives the following theorem. Recall that a closed orientable 3-manifold is *hyperbolike* if it is atoroidal, irreducible, and is not a small Seifert fiber space [11].

Theorem 2.5. *Suppose M is a hyperbolic manifold with torus boundary T. Let B be an essential branched surface in M such that $M - \mathrm{Int}N(B)$ contains an essential annulus with one boundary on $\partial_h N(B)$ and the other a curve on T of slope γ_0. Then $M(\gamma)$ is hyperbolike for all γ with $\Delta(\gamma_0, \gamma) > 2$.*

Gabai and Mosher [8] showed that any hyperbolic manifold M with torus boundary contains an essential branched surface satisfying the condition in the theorem. This implies that all nonhyperbolike surgery slopes lie on 5 lines in the Dehn surgery space.

In Section 3 we use sutured manifold hierarchy to prove Theorems 4.1 and 4.6 in the case that ∂M consists of tori. The general version of these theorems is proved in Section 4, by using a construction of

Luecke. In Section 5 we prove Theorem 5.1. Here we use a combinatorial argument to deal with the cases in which the essential annulus in $M(\gamma_1)$ intersects the Dehn filling solid torus at most twice, then we use β-taut sutured manifold hierarchy and a generalized version of Gabai disk argument to prove the theorem in the general case.

Some problems arise in this research. A challenging problem is to establish the lower limit of $\Delta(\gamma_1, \gamma_2)$ for the remaining cases in Table 0.2. Some other problems can be found in the paper.

Notation and Convensions. All 3-manifolds and surfaces below are assumed orientable. Surfaces in 3-manifolds are assumed properly embedded unless otherwise stated. 3-manifolds are compact and connected unless it is the exterior of some essential lamination in a compact manifold. A non-sphere surface F in a 3-manifold M, then it is *essential* if it is incompressible, ∂-incompressible, and not parallel to a surface on ∂M. A sphere in M is essential if it is a reducing sphere. We refer the reader to the books of Hempel [20] and Jaco [21] for standard definitions and basic results on 3-manifold topology, to the paper of Gabai-Oertel [9] for those of essential lamination and essential branched surface, and to the papers of Gabai [5]–[7] and Scharlemann [27]–[28] for sutured manifold theory.

We use $N(B)$ to denote a regular neighborhood of a set B in M. If F is a surface in M, we use $M|F$ to denote the manifold obtained by cutting M along F, i.e. $M|F = M - \text{Int}N(F)$. A train track τ is a branched compact 1-manifold on a surface. A small neighborhood of a branched point is cut into 3 pieces, one of which contains a cusp at that branched point. A branched surface is a 2-dimensional generalization of train tracks, see [9] for more details. If B is a branched surface in a 3-manifold M, then its exterior is defined as $E(B) = M - \text{Int}N(B)$.

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1. Cusped manifolds

Definition 1.1. A *cusped manifold* is a compact orientable 3-manifold M with a specified (possibly empty) collection of annuli and tori on ∂M, denoted by $\partial_v M$, called *cusps* or *vertical surfaces*. The surface $\partial_h M = \partial M - \text{Int}\partial_v M$ is called the *horizontal surface* of M.

Example 1.2. If B is a branched surface in $\text{Int}M$, then $E(B)$ has a natural cusped manifold structure. The horizontal surface of $E(B)$ is $\partial_h E(B) = \partial_h N(B) \cup \partial M$, and the vertical surfaces of $E(B)$ is $\partial_v E(B) = \partial_v N(B)$, where $\partial_h N(B)$ and $\partial_v N(B)$ are the horizontal and vertical boundaries of $N(B)$; see [9] for definitions. Sutured manifolds are also cusped manifolds; see Section 2.

Definition 1.3. Let F be a surface in M such that $F \cap \partial_v M$ is a set of essential arcs and circles in $\partial_v M$. We consider the arcs of $F \cap \partial_v M$ as cusps on ∂F. The cusped Euler characteristic of F is defined as $\chi_c(F) = \chi(F) - \frac{1}{2}C(F)$, where $\chi(F)$ is the Euler characteristic of F, and $C(F)$ is the number of cusps on ∂F. This number has an additivity property: If τ is a train track on F, then $\chi(F) = \sum \chi_c(F_i)$, where F_i are the components of $F - \text{Int}N(\tau)$. In particular, the formula holds if B is a branched surface in $\text{Int}M$, F a surface transverse to B, and F_i the components of $F \cap E(B)$.

Suppose M is a cusped manifold. A *monogon* in M is a disk D properly embedded in M, such that $\partial D \cap \partial_v M$ is a single essential arc in $\partial_v M$. In this case D is also called a monogon of the horizontal surface $\partial_h M$. If $D \cap \partial_v M$ consists of two essential arcs in $\partial_v M$, then D is called a *bigon*. An annulus A properly embedded in M is *cuspless* if ∂A is disjoint from $\partial_v M$.

Definition 1.4. A cusped manifold M is χ_c-*irreducible* if it contains no essential surface F with $\chi_c(F) > 0$. This is equivalent to that (i) M is irreducible, (ii) $\partial_h M$ is incompressible, and (iii) M has no monogons.

Note that if M is connected then (i) and (ii) imply that either (iii) is true or M is a solid torus with $\partial_v M$ a longitudinal annulus. The proof is simple: If D is a monogon, then the frontier of $N(\partial_v M \cup D)$ is a disk with boundary on $\partial_h M$, and hence is parallel to a disk on $\partial_h M$, so M is homeomorphic to the solid torus $N(\partial_v M \cup D)$.

We say that a branched surface B is *intrinsically essential* if (i) it has no disk of contact, (ii) it has no Reeb branched subsurface, (iii) no component of $\partial_h N(B)$ is a sphere, and (iv) it fully carries a lamination.

A branched surface B embedded in M is essential if it is intrinsically essential, and $E(B)$ is χ_c-irreducible. The intrinsic part is independent of the embedding of B. Thus to show an essential branched surface B in M remains essential after Dehn filling on a torus T disjoint from B, it would suffice to show that $X(\gamma)$ is a χ_c-irreducible cusped manifold, where X is the component of $E(B)$ containing T.

An essential annulus A in M is called an *accidental annulus* if it has one boundary component in each of T and $\partial_h M$. The curve $A \cap T$ is the *slope* of A on T. Two such annuli A_1, A_2 are *parallel* if they are disjoint, and cut off a product region $A_1 \times I$ containing no cusps of M. We need the following lemma.

Lemma 1.5. *If $A_1, \ldots, A_n$ are mutually nonparallel, mutually disjoint, accidental annuli in a χ_c-irreducible cusped manifold M which is not a $T^2 \times I$ with no cusps, then the frontier of $X = N(T \cup A_1 \cup \ldots \cup A_n)$ are essential annuli in $M - \partial_v M$.*

Proof. Denote by $A' = A'_1 \cup \ldots \cup A'_m$ the frontier of X. Since A_i are essential, A'_j are incompressible. Clearly there is no ∂-compressing disk of A'_j in X. If there is a ∂-compressing disk D of A'_j in $X' = M - \text{Int}X$ disjoint from $\partial_v M$, then the frontier of $N(D \cup A'_j)$ contains a disk Δ. Since M is χ_c-irreducible, Δ must be parallel to a disk on $\partial_h M$, which implies that A'_j is parallel to an annulus on $\partial_h M$. If $n \geq 2$, then the two annuli among the A_i which are adjacent to A'_j would be parallel. If $n = 1$, then M is a $T^2 \times I$ without cusp. Both cases contradict the assumptions of the lemma. q.e.d.

Theorem 1.6. *Let M be a χ_c-irreducible cusped manifold, and T a torus component of $\partial_h M$. Suppose M is not a $T^2 \times I$ without cusp, and there is an accidental annulus A with slope l on T.*

(1) If $\Delta(l, \gamma) > 1$, then $M(\gamma)$ is χ_c-irreducible.

(2) If $\Delta(l, \gamma) > 2$, then $M(\gamma)$ is not an I-bundle over a surface S with $\partial_v M(\gamma)$ the I-bundle over ∂S.

(3) If $\Delta(l, \gamma) > 2$, any collection of bigons, tori and cuspless annuli in $M(\gamma)$ can be rel ∂ isotoped into M.

Proof. (1) Put $X = N(T \cup A)$, and let A' be the frontier of X. By Lemma 1.5, A' is essential in M. After Dehn filling, $X(\gamma)$ is a solid torus with A' running $\Delta(l, \gamma)$ times along the longitude. So if $\Delta(l, \gamma) > 1$ then A' remains an essential annulus in $M(\gamma)$. By an innermost circle outermost arc argument one can show that $M(\gamma)$ is χ_c-irreducible, hence (1) holds.

(2) Suppose $\Delta(l, \gamma) > 2$, and suppose $M(\gamma)$ is an I-bundle over a surface S such that $\partial_v M(\gamma)$ is the I-bundle over ∂S. Since the annulus A' above is incompressible, with $\partial A' \subset \partial_h M(\gamma)$, it is isotopic to a vertical annulus, so after cutting along A', the manifold is still an I-bundle over some surface S', with the two copies of A' and $\partial_h M(\gamma)$ as the I-bundle over $\partial S'$. But this is impossible because $X(\gamma)$ is a solid torus with A' running along the longitude at least 3 times.

(3) Suppose $\Delta(l, \gamma) > 2$, and suppose F is a collection of bigons and cuspless annuli in $M(\gamma)$. Since A' is essential, by an isotopy we may assume that $A' \cap F$ consists of essential arcs and circles in F. Here an arc in a bigon B is essential if each component of $B - A'$ intersects a cusp of M. Thus each component C of $F \cap X(\gamma)$ is either an annulus with boundary disjoint from $\partial A'$ or a disk intersecting A' twice. If C is an annulus, it can be pushed off the Dehn filling solid torus to lie in M. Since $\Delta(l, \gamma) > 2$, a meridian disk of $X(\gamma)$ intersects A' at least three times, so if C is a disk, it cannot be an essential disk of $X(\gamma)$, hence again it can be isotoped into M.

By restricting the above isotopies to $M - N(\partial M)$ and extending continuously over M, we may assume that the restriction of the above isotopies to ∂M are identity isotopies. q.e.d.

A stronger result than Theorem 1.6(1) holds if there is no accidental annulus in M.

Theorem 1.7. *Let M be a χ_c-irreducible cusped manifold, and T a torus component of $\partial_h M$. Suppose there is no accidental annulus A in M. If $M(\gamma_1)$ and $M(\gamma_2)$ are χ_c-reducible, then $\Delta(\gamma_1, \gamma_2) \leq 1$. In particular, $M(\gamma)$ are χ_c-irreducible for all but at most three γ.*

Proof. If each $M(\gamma_i)$ is reducible or has a compressing disk for $\partial_h M$, the result follows from the main theorems of [31], [28], [13]. If $M(\gamma_1)$ has a monogon but no compressing disk for $\partial_h M$ or reducing sphere, then $M(\gamma_1)$ is a solid torus with $\partial_h M$ a longitude, in which case it is also easy to show that $M(\gamma_2)$ is χ_c-irreducible when $\Delta(\gamma_1, \gamma_2) \geq 2$. See the last paragraph of the proof of Theorem 1 in [33]. q.e.d.

If in Theorem 1.6 we can find two different annuli in M, then a stronger result holds:

Theorem 1.8. *Let M be a χ_c-irreducible cusped manifold, and T a torus component of $\partial_h M$. Suppose there are two disjoint, nonparallel, accidental annuli A_1 and A_2 in M with slope l on T.*

(1) If $\gamma \neq l$, then $M(\gamma)$ is χ_c-irreducible.

(2) If $\Delta(l, \gamma) > 1$*, then* $M(\gamma)$ *is not an* I*-bundle over a surface* S *with* $\partial_v M(\gamma)$ *the* I*-bundle over* ∂S*.*

(3) If $\Delta(l, \gamma) > 1$*, then any collection of bigons, tori and cuspless annuli in* $M(\gamma)$ *can be rel* ∂ *isotoped into* M*.*

Proof. The proof is similar to that of Theorem 1.6, only that we take $X = N(T \cup A_1 \cup A_2)$. The frontier of X now consists of two annuli A' and A'', which are essential in M by Lemma 1.5. If $\gamma \neq l$ then $A' \cup A''$ is essential in $X(\gamma)$, and if $\Delta(l, \gamma) > 1$ then a meridian disk of $X(\gamma)$ intersects $A' \cup A''$ more than two times, hence the argument in the proof of Theorem 1.6 applies here. q.e.d.

The "two annuli" condition was first applied by Menasco to show that essential surfaces remain essential after Dehn surgery on certain knots [23]. If there are three such annuli in M, then we can get the strongest possible conclusion. The proof of the following theorem is similar to that of Theorem 1.8, and is omitted.

Theorem 1.9. *If in Theorem 1.8 there are three disjoint, nonparallel, accidental annuli* A_i*, then the conclusions of (1), (2) and (3) in Theorem 1.8 hold for all* $\gamma \neq l$*.*

2. Intersection between essential surface and essential lamination

Given two essential surfaces F_1, F_2, it is always possible to isotope one of them so that they intersect in circles essential on both surfaces. This is not possible in general if one of them is an essential branched surface B. However, Theorem 2.2 shows that after some splitting of B, it is possible to make them intersect essentially. As a corollary, it is shown that we can isotope an essential surface F so that its intersection with an essential lamination in M^3 is an essential lamination on F. The results will be applied in Section 3 to prove Theorems 3.3 and 3.4. As a by-product, we will prove Theorem 2.5, which, when combined with a result of Gabai-Mosher, shows that all but 5 lines of surgeries on a hyperbolic knot produce hyperbolike manifolds.

Lemma 2.1. *Suppose* λ *is an essential lamination fully carried by a branched surface* B*. Then* λ *is fully carried by an essential branched surface* B' *which is a* λ*-splitting of* B*.*

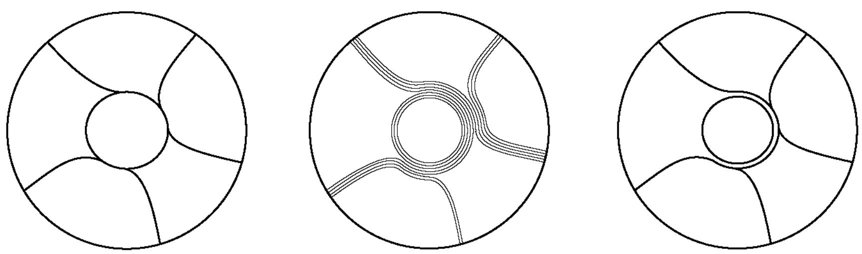
Proof. This follows from the proofs of Lemma 4.3 and Proposition 4.5 in [9]. The argument goes as follows. By thickening λ if necessary we may assume that $\partial_h N(B) \subset \lambda$. By a λ-splitting we may assume that B has no compact surface of contact. Since λ is essential, $E(B)$ is irreducible, and $\partial_h E(B)$ is incompressible in $E(B)$. Also, a monogon of $E(B)$ could be extended to a monogon for λ via some half-infinite vertical strip in $N(B) - \lambda$, which would contradict the essentiality of λ. It follows that the branched surface B satisfies all conditions of an essential branched surface except possibly the condition that it has no Reeb branched surfaces. Since λ is essential, it has no vanishing cycle, so [9, Lemma 4.3], says that λ is also fully carried by an essential branched surface B'. By examining the proof of that lemma, one can see that B' is actually obtained by a λ-splitting of B. q.e.d.

Theorem 2.2. *Suppose B is an essential branched surface which fully carries a lamination λ, and suppose F is an essential surface in M. Then there is an essential branched surface B' which is a λ-splitting of B, and a surface F' isotopic to F, such that $F' \cap B'$ is an essential train track on F'.*

Proof. By an isotopy we may assume that F is transverse to B. Thus $F \cap B$ is a train track τ on F. We may further assume that $F \cap N(B) = N(\tau)$, and the I-fibers of $N(\tau)$ are also I-fibers of $N(B)$; see [9, Lemma 2.6]. The train track τ cannot have any monogon, because a monogon bounded by τ would also be a monogon for the exterior of B, which is impossible since B is essential. We need to modify B and F so that τ has no 0-gons either.

Suppose D is a 0-gon of τ in F. Let D_1 be a small neighborhood of D in F. Then the train track τ in D_1 consists of a circle C bounding the 0-gon D, and some arcs from some branch points on C to the boundary of D_1, so it looks like that in Figure 2.1(a). The foliation $\mathcal{F}$ of $N(B)$ induces a foliation $\mathcal{F}_\infty = \mathcal{F} \cap \mathcal{F}$ on $N(\tau)$; see Figure 2.2(b) for an example. We claim that $\mathcal{F}_\infty$ has no noncompact leaf, otherwise there would be a circle γ which is the limit of noncompact leaves. Let l be the leaf of $\mathcal{F}$ that contains γ. Since the lamination λ is essential, l is π_1-injective, so γ is a trivial loop on l, which contradicts the Reeb stability of λ, (see [9, Lemma 2.2].)

It follows that there is an annulus $\gamma \times I$ in D_1, such that each leaf of $\mathcal{F}_\infty$ in $\gamma \times I$ is a circle $\gamma \times t$ for some t, $\gamma \times 0 = \partial D$, and $\gamma \times 1$ intersects $\partial_h N(\tau)$. Since λ is essential, $\gamma \times 1$ bounds a disk D' in a leaf l which intersects $\partial_h N(B)$, so there is an interstitial I-bundle J in M which contains the part of D' that is not on $\partial_h N(B)$. By splitting along a compact subbundle of J that contains $J \cap D'$, we get a new branched surface B', such that $\tau' = B' \cap F$ is a train track obtained by splitting τ, and $\tau' \cap D_1$ is obtained by splitting $\tau \cap D_1$ along $\gamma \times 1$, as shown in Figure 2.1(c). Note that splitting a train track with no monogon will not increase the number of 0-gons. Since M is irreducible (because it contains an essential lamination), $D \cup D'$ bounds a 3-ball V. By pushing F off the ball V to the side of D', we get a surface F' such that $F' \cap B'$ has at least one less 0-gons than τ' on F. Repeating this process eliminates all 0-gons, so eventually $F' \cap B'$ becomes an essential train track on F', as requires.



(a) (b) (c)

FIGURE 2.1

By Lemma 2.1, B' can be further λ-split to an essential branched surface B''. The train track $B'' \cap F'$ is a splitting of $B' \cap F'$, so it is still essential. q.e.d.

Corollary 2.3. *If λ is an essential lamination, and F is an essential surface in M, then F can be isotoped so that $\lambda \cap F$ is an essential lamination on F. In particular, for any leaf l in λ, each circle component of $l \cap F$ is essential on both F and l.*

Proof. By the theorem there is an essential branched surface B' fully carrying λ, such that $B' \cap F$ is an essential train track on F. Since $\lambda \cap F$ is fully carried by the essential train track $B \cap F$, it is an essential lamination on F. If C is a circle component of $\lambda \cap F$, then it is essential on F. Since F is incompressible, C must also be essential on the leaf of λ that contains C. q.e.d.

Question 2.4. Can Corollary 2.3 be generalized to the intersection of two essential laminations?

Following Gordon [11], we say that a closed manifold M is *hyperbolike* if it is irreducible, atoroidal, and is not a small Seifert fiber space.

Thurston's geometrization conjecture [29] asserts that hyperbolike manifolds are hyperbolic. Gabai and Mosher [8] showed that a branched surface B as in the following theorem always exists. The slope γ_0 is called a degenerate slope.

Theorem 2.5. *Suppose M is a hyperbolic manifold with torus boundary T. Let B be an essential branched surface in M such that $M - \text{Int}N(B)$ contains an essential annulus A with one boundary on $\partial_h N(B)$ and the other a curve on T of slope γ_0. Then $M(\gamma)$ is hyperbolike for all γ with $\Delta(\gamma_0, \gamma) > 2$.*

Proof. Let $X = M - \text{Int}N(B)$, and assume $\Delta(\gamma_0, \gamma) > 2$. By Theorem 1.6 we have (i) $X(\gamma)$ is a χ_c-irreducible cusped manifold, and (ii) $X(\gamma)$ is not an I-bundle over a surface S with $\partial_v X(\gamma)$ the I-bundle over ∂S. Now (i) implies that B remains essential in $M(\gamma)$, so $M(\gamma)$ is irreducible; and by [2], (ii) implies that $M(\gamma)$ is not a small Seifert fiber space. To prove the theorem, it remains to show that $M(\gamma)$ is atoroidal.

Assuming the contrary, let F be a torus in $M(\gamma)$. Let λ be an essential lamination fully carried by B. By Theorem 2.2, there is an essential branched surface B' which λ-splits B, and a surface isotopic to F (still denoted by F), such that $B' \cap F$ is an essential train track on F. Notice that the component of $X' = M - \text{Int}N(B')$ containing T cannot be a $T \times I$ without cusp, otherwise $T \times 1$ would be a leaf of λ, so λ would be inessential after all Dehn fillings on M; but since λ is fully carried by B, which is essential in $M(\gamma)$, this is impossible.

Since B' is a splitting of B, the essential annulus A also lives in X', with ∂_1 on $\partial_h X' - T$. Since $B' \cap F$ is an essential train track on F, each component of $F \cap X'(\gamma)$ is either a bigon or a cuspless annulus in $X'(\gamma)$. According to Theorem 1.6(3), the surface $F \cap X'(\gamma)$ can be rel ∂ isotoped into X', which implies that F is isotopic to a torus in M. This contradicts the assumption that M is atoroidal, completing the proof.

q.e.d.

Corollary 2.6. *Let M and B be as in Theorem 2.5. Then $M(\gamma)$ is hyperbolike for all but at most 20 slopes γ.*

Proof. This follows from Theorem 2.5, the 2π-theorem of Gromov and Thurston [1], and the proof of [3]. Brittenham showed that there are at most 20 slopes γ which have $\Delta(\gamma, l) \leq 2$ and have length at most 2π on T, and they contain all the reducible or small Seifert fibered slopes. Theorem 2.5 says that this set also contains all the toroidal slopes. q.e.d.

3. Sutured manifold decomposition and essential branched surfaces

A *sutured manifold* is a triple (M, γ, β), where M is a compact orientable 3-manifold, γ a set of annuli or tori on ∂M, and β a proper 1-complex in M. The pair (M, γ) is a cusped manifold M such that each component of $\partial_h M$ is oriented $+$ or $-$, and each annulus cusp is adjacent to two components of $\partial_h M$ with different orientation. In this case the cusps γ are called *sutures* in [5], [27]. Denote by $\partial_+ M$ (resp. $\partial_- M$) the union of all components of $\partial_h M$ with $+$ (resp. $-$) orientation. We use $\partial_\pm M$ to denote "$\partial_+ M$ or $\partial_- M$". They are denoted by $R_\pm$ in [5], [27]. In this section we assume $\beta = \emptyset$.

A sutured manifold M is *taut* if it is χ_c-irreducible, and both $\partial_\pm M$ minimize the Thurston norm in $H_2(M, \partial_v M)$. Note that since each annular cusp is adjacent to two different component of $\partial_h M$, it is automatically true that M has no monogon.

If F is an oriented surface properly embedded in M, such that ∂F intersects each torus component of $\partial_v M$ in coherently oriented circles, then when cutting along F, the manifold $M_1 = M|F = M - \mathrm{Int}N(F)$ has a natural sutured manifold structure (M_1, γ_1); see [5], [27]. Such process of obtaining a new sutured manifold from the old one by cutting along an oriented surface is called a *sutured manifold decomposition*, and is denoted by $(M, \gamma) \stackrel{F}{\rightsquigarrow} (M_1, \gamma_1)$. The decomposition is *taut* if both M and M_1 are taut sutured manifolds.

Theorem 3.1 (Gabai). *Let M be a Haken 3-manifold with toroidal boundary. Let P be a specified component of ∂M. Suppose M is atoroidal, and $H_2(M, \partial M - P) \neq 0$. Then there exists a sequence*

$$(M, P) = (M_0, \gamma_0) \stackrel{S_1}{\rightsquigarrow} (M_1, \gamma_1) \stackrel{S_2}{\rightsquigarrow} \cdots \stackrel{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

of sutured manifold decompositions with the following properties:

(1) Each (M_i, γ_i) is taut and each separating component of S_{i+1} is a product disk.

(2) Some component of γ_n is the torus P.

(3) (M_n, γ_n) is a union of a product sutured manifold and a sutured manifold $(H, \delta) = T^2 \times I$ where $P = T^2 \times 0$, and $\delta \cap (T^2 \times 1) \neq \emptyset$.

Proof. When $\partial M = P$, this is exactly Step 1 in the proof of [6, Theorem 1.7]. Since ∂M is incompressible, the sutured manifold (M, P) is taut, with $\partial_+ M = \partial M - P$. Hence (M, P) satisfies the assumption of [6, Theorem 1.8]. As remarked in the paragraph before [6, Theorem 1.8], the proof of Theorem 1.7 there applies verbatim to this more general setting. q.e.d.

Given any sequence of sutured manifold decomposition of M, there is an associated branched surface B. The construction is obvious: B is the union of $\partial M - P$ and all the S_i in the sequence, smoothed at ∂S_i according to its orientation. See [7, Construction 4.16] for details. The following theorem is also due to Gabai.

Theorem 3.2. *The branched surface B associated to the sequence in Theorem 3.1 fully carries an essential lamination λ.*

Proof. The construction of λ was described in [7, Construction 4.17]. The lamination extends to taut foliations after all but one Dehn filling on P, so it is essential in all but one $M(\gamma)$. It follows that λ is also essential in M.

One can also prove the essentiality of λ directly from the construction. From Description 2 of [7, Construction 4.17], we see that the only compact leaves of λ are $\partial M - P$, which are incompressible by our assumption. As usual, let $M_\lambda = M - \text{Int}\lambda_1$, where λ_1 is a thickening of λ. Then M_λ is the union of $E(B)$ and a noncompact product sutured manifold (W, β) along the annular sutures β of B. Since $E(B)$ is a taut sutured manifold, $E(B)$ is χ_c-irreducible, so M_λ is irreducible, and ∂M_λ is incompressible and end-incompressible. By definition λ is essential. q.e.d.

Theorem 3.3. *Let M be an atoroidal, irreducible, ∂-irreducible, compact 3-manifold with ∂M a set of tori. Let T be a specified component of ∂M. Suppose $H_2(M, \partial M - T) \neq 0$. Let γ_1 and γ_2 be slopes on T such that $M(\gamma_1)$ is toroidal, and $M(\gamma_2)$ is reducible or ∂-reducible. Then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Proof. Let

$$(M, \partial M) = (M_0, \gamma_0) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$$

be a sequence of sutured manifold decomposition given by Theorem 3.1. Let B and λ be the branched surface and essential lamination given by the Theorem. By Lemma 2.1, B can be λ-split into an essential branched surface B'. Let X (resp. X') be the component of $E(B)$ (resp. $E(B')$) containing T. Note that since B' is a splitting of B, we have $X \subset X'$.

According to Theorem 3.1(3), X is a sutured manifold $T^2 \times I$ with $T = T^2 \times 0$, and $\partial_v X \cap (T^2 \times 1) \neq \emptyset$. Thus $\partial_v X$ consists of (at least two) annuli, cutting $T^2 \times 1$ into $\partial_+ X$ and $\partial_- X$. Hence there are two essential annuli $A_\pm$ in X, each having one boundary component on T, and the other on different components of $\partial_\pm X$. After splitting the component X' may no longer be a $T^2 \times I$, but since $X \subset X'$, the above annuli $A_\pm$ are also essential annuli in X'. Let γ' be the slope $A_+ \cap T$ on T. Applying Theorem 1.8 to X', we see that if $\gamma \neq \gamma'$ then $X'(\gamma)$ is χ_c-irreducible, hence B' remains an essential branched surface in $X'(\gamma)$ for all $\gamma \neq \gamma'$. Since we assumed that $M(\gamma_2)$ is reducible or ∂-reducible, we must have $\gamma' = \gamma_2$.

Now consider an essential torus F in $M(\gamma_1)$. By Lemma 2.1, after an isotopy of F we may assume that there is a branched surface B'' which is a λ-splitting of B', such that $F \cap B''$ is an essential train track τ on F. Since the Euler number of F is zero, all components of $F - \text{Int}N(\tau)$ are bigons or cuspless annuli. Let X'' be the component of $E(B'')$ that contains T. Then in particular $Y = F \cap X''(\gamma)$ is a union of bigons and cuspless annuli. By Theorem 1.8, if $\Delta(\gamma_1, \gamma_2) \geq 2$, then Y can be rel ∂ isotoped into M. But then F would be isotopic to an essential torus in M, contradicting the assumption that M is atoroidal. q.e.d.

Theorem 3.4. *Let M be a simple compact 3-manifold with ∂M a set of tori. Let T be a specified component of ∂M. Suppose $H_2(M, \partial M - T) \neq 0$. Let γ_1 and γ_2 be slopes on T such that $M(\gamma_1)$ is annular, and $M(\gamma_2)$ is reducible or ∂-reducible. Then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Proof. The proof is the same as that of Theorem 3.3, with the essential torus F in $M(\gamma_1)$ replaced by an essential annulus A. Since $\partial M(\gamma_1)$ is contained in the branched surface B'' in that proof, we can assume that $A \cap B''$ is an essential train track τ in A containing ∂A. Hence $A - \text{Int}N(\tau)$ consists of bigons and cuspless annuli. So if $\Delta(\gamma_1, \gamma_2) \geq 2$, then A can be rel ∂ isotoped into M, which would contradict the assumption that M is simple. q.e.d.

4. Surgery on manifolds with large boundary

In [32] and [24] it was proved that if M is a hyperbolic manifold with T a torus boundary component, and if $M(\gamma_1)$ is reducible and $M(\gamma_2)$ toroidal, then $\Delta(\gamma_1, \gamma_2) \leq 3$. Theorem 3.3 says that if ∂M consists of tori, and $H_2(M, \partial M - T) \neq 0$, then we actually have a much stronger conclusion that $\Delta(\gamma_1, \gamma_2) \leq 1$. The following theorem shows that the first condition can be removed. Note that if ∂M is not one or two tori, then M automatically satisfies the condition that $H_2(M, \partial M - T) \neq 0$.

Theorem 4.1. *Let M be an irreducible, ∂-irreducible, atoroidal 3-manifold with torus T as a boundary component, such that $H_2(M, \partial M - T) \neq 0$. Let γ_1 and γ_2 be slopes on T such that $M(\gamma_1)$ is toroidal, and $M(\gamma_2)$ is reducible or ∂-reducible. Then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Proof. First assume that $M(\gamma_2)$ is reducible. Since M is atoroidal, by a theorem of Scharlemann [28], we see that $M(\gamma)$ remains ∂-irreducible if $\gamma \neq \gamma_2$.

For each component F of ∂M with genus ≥ 2, choose a simple manifold M_F with $\partial M_F = F$, and $H_2(M_F) \neq 0$. Such a manifold can be constructed as follows: Let g =genus(F). Choose a compact manifold X such that ∂X is a surface of genus $g-1$, and $H_2(X)$ has rank ≥ 2. (For example, let $X = V \#(T^2 \times S^1)$, where V is a handlebody of genus $g-1$.) According to Myers [26], there is an arc α in X, such that $Y = X - \text{Int}N(\alpha)$ is a simple manifold. Clearly, $\partial Y = F$. The Meyer-Vietoris sequence of the pair $(Y, N(\alpha))$ gives the following exact sequence:

$$0 \to H_2(Y) \to H_2(X) \to H_1(Y \cap N(\alpha)) \to \cdots$$

Since $Y \cap N(\alpha)$ is an annulus, it follows that $H_2(Y) \neq 0$. Hence we can take $M_F = Y$.

Gluing an M_F to M along F for each nontorus boundary component F of M, we get a manifold $\widehat{M}$, which is atoroidal and Haken, with $\partial\widehat{M}$ a set of tori. Since M_F has only one boundary component, $H_2(M_F) \neq 0$ implies that M_F contains a closed nonseparating surface, which is then nonseparating in $\widehat{M}$; hence $H_2(\widehat{M}, \partial\widehat{M} - T) \neq 0$. If S is a reducing sphere in $M(\gamma_2)$, then since M_F contains a nonseparating surface, gluing M_F to $M(\gamma_2) - \text{Int}N(S)$ will not produce a 3-ball bounded by S. Hence S remains a reducing sphere in $\widehat{M}(\gamma_2)$. Since ∂M is incompressible in $\widehat{M}(\gamma_1)$, an essential torus P in $M(\gamma_1)$ remains essential in $\widehat{M}(\gamma_1)$, so $M(\gamma_1)$ is toroidal. We have thus shown that $\widehat{M}$ satisfies all conditions of Theorem 3.3, hence by that theorem we have $\Delta(\gamma_1, \gamma_2) \leq 1$.

Now assume that $M(\gamma_2)$ is ∂-reducible. Let F be a component of $\partial M - T$ which is compressible in $M(\gamma_2)$, and let C be a curve on F bounding a compressing disk D in $M(\gamma_2)$. If F is a torus, then the frontier of $N(F \cup D)$ is a 2-sphere S. If S is an essential sphere, then $M(\gamma_2)$ is reducible, and the result has been proved above. If S is inessential, then ∂M consists of two tori, so the result follows from Theorem 3.3. Hence we can assume that F has genus ≥ 2.

Let P be a planar surface having at least three boundary components. Let $\varphi : \partial P \times I \to F$ be a map such that ∂P is sent to curves parallel to C. Denote by $Y = M \cup_\varphi (P \times I)$, the manifold obtained by gluing $P \times I$ to M using φ as gluing map. By a standard innermost circle outermost arc argument it can be shown that Y is irreducible, ∂-irreducible and atoroidal. Since $\partial Y - T$ is not a torus, we also have $H_2(Y, \partial Y - T) \neq 0$. The surface P extends to a sphere in $Y(\gamma_2)$ having some boundary components on each side, hence $Y(\gamma_2)$ is reducible. If the annuli $\partial P \times I$ are incompressible in $M(\gamma_1)$, then an essential torus in $M(\gamma_1)$ remains essential in $Y(\gamma_1)$. Hence the result follows from the first case proved above, with M replaced by Y. If $\partial P \times I$ is compressible in $M(\gamma_1)$, then $Y(\gamma_1)$ is also reducible. By the Reducible Surgery Theorem of Gordon-Luecke [13], we also have $\Delta(\gamma_1, \gamma_2) \leq 1$. q.e.d.

Remark 4.2. The idea of gluing a large simple manifold to M to get the result is due to John Luecke. If M is an irreducible atoroidal manifold with torus boundaries, such that $H_2(M, \partial M - T) \neq 0$, then a theorem of Gabai [6, Corollary 2.4] says that at most one Dehn filling on T could be reducible. Using the above trick, Luecke showed that this is true even if ∂M has some higher genus components.

Example 4.3. (1) If W is a solid torus, and K is a hyperbolic knot in W with winding number 0, then $M = W - \text{Int}N(K)$ satisfies the conditions of Theorems 3.3 and 3.4. Hence if γ surgery on K produces toroidal or annular manifold, then γ is a longitudinal slope, i.e., $\Delta(\gamma, m) = 1$, where m is the meridional slope of K. Together with results of [31] and [28], it shows that non-integral surgeries on such knots always produce hyperbolic manifolds.

(2) As noticed above, the condition $H_2(M, \partial M - T)$ is true unless $\partial M - T$ is empty or a single torus. This condition cannot be removed. Hayashi and Motegi [19] have an example of a simple manifold M, such that ∂M is a union of two tori, $M(\gamma_1)$ is reducible and ∂-reducible, $M(\gamma_2)$ is toroidal and annular, and $\Delta(\gamma_1, \gamma_2) = 2$.

(3) If W is a handlebody of genus ≥ 2, and K is a knot in W such that $W - \text{Int}N(K)$ is irreducible, ∂-irreducible and atoroidal, then only integral surgeries on K can yield toroidal manifolds.

Question 4.4. Are there any hyperbolic knots in a solid torus which admits some non-integral toroidal or annular surgeries?

This has recently been answered in positive by Miyazaki and Motegi [25].

Corollary 4.5. *Let K be a hyperbolic knot in S^3. Suppose there is an incompressible surface F in $E(K)$, cutting $E(K)$ into anannular manifolds X and Y. Then $K(\gamma)$ is hyperbolic for all non-integral slopes γ.*

Proof. Let X be the component of $E(K) - \text{Int}N(F)$ containing $T = \partial N(K)$. Let γ be a non-integral slope on $N(K)$, and let m be the meridional slope. Clearly, F is compressible in $X(m)$. Therefore, by [31] and [28], $X(\gamma)$ is irreducible and ∂-irreducible. Hence $K(\gamma) = Y \cup X(\gamma)$ is a Haken manifold. Since Y is anannular, any essential torus S in $K(\gamma)$ can be isotoped to be disjoint from F. Since K is hyperbolic, S cannot be in Y, otherwise it would be an essential torus in $E(K)$. By Theorem 4.1, $X(\gamma)$ is atoroidal, so S cannot be in $X(\gamma)$ either. Thus $K(\gamma)$ is also atoroidal. If $K(\gamma)$ were a Seifert fiber space, then F would be isotopic to a surface transverse to the fibers, hence $K(\gamma) - \text{Int}N(F)$ would be an I-bundle over surface, which is impossible because X is anannular. It now follows from Thurston's Geometrization Theorem [29] that $K(\gamma)$ is hyperbolic. q.e.d.

By a theorem of Gordon [10, Theorem 1.1], if $Y(\gamma_1)$ and $Y(\gamma_2)$ are nonsimple, then $\Delta(\gamma_1, \gamma_2) \leq 5$. Since γ_i are integral, there are at most 6 such slopes. Hence those knots K in Corollary 4.5 admit at most 6 nontrivial, nonhyperbolic surgeries.

Theorem 4.6. *Let M be a compact simple 3-manifold with torus T as a boundary component, such that $H_2(M, \partial M - T) \neq 0$. Let γ_1 and γ_2 be slopes on T such that $M(\gamma_1)$ is annular, and $M(\gamma_2)$ is reducible. Then $\Delta(\gamma_1, \gamma_2) \leq 1$.*

Proof. Let A be an essential annulus in $M(\gamma_1)$. If ∂M consists of tori, the theorem follows from Theorem 3.4. If ∂A lies on torus boundary components of M, we can glue M_F to M to get a simple manifold Y with toroidal boundary, as in the proof of Theorem 4.1, then apply Theorem 3.4. Hence we may assume that A has at least one boundary on a nontorus component of ∂M.

Suppose both components of ∂A lie on nontorus boundary components of M. Let $X = P \times I$, where P is an annulus. Gluing $(\partial P) \times I$ to a neighborhood of ∂A on ∂M, we get a manifold Y. One can check that Y is still a simple manifold, $H_2(Y, \partial Y - T) \neq 0$, the manifold $Y(\gamma_1)$ is toroidal, and $Y(\gamma_2)$ is reducible. Hence the result follows from Theorem 4.1.

Now suppose that ∂A has one component on a non-torus boundary component F of M, and the other on a torus G on ∂M. Notice that the above construction fails, because then G pushed into M would be an essential torus in Y. We proceed as follows: Let A_1, A_2 be two parallel copies of A. Glue the above manifold $X = P \times I$ to M with $\partial P \times I$ identified to a neighborhood of the two curves of ∂A_i on the nontorus component F of ∂M. Then the resulting manifold Y again satisfy all the conditions of the theorem, and $A_1 \cup A_2 \cup P$ is an essential annulus in $Y(\gamma_1)$ with both boundary on the torus G. Hence the result follows from the first case proved above. q.e.d.

The following examples show that the results of Theorems 4.1 and 4.6 are the best possible.

Example 4.7. Let M be the exterior of the Borromean ring L shown in Figure 4.1, and let T be a specified component of ∂M. It is well known that M is hyperbolic. The trivial surgery $M(m)$ is reducible and ∂-reducible, and the longitudinal surgery $M(l)$ is toroidal because a component of L bounds a punctured torus disjoint from the other components, which extends to a torus F in $M(l)$. Notice that F is nonseparating, so if it were compressible in $M(l)$, then $M(l)$ would be reducible, which would contradict the theorem of Gabai that M admits at most one reducible Dehn filling [6, Corollary 2.4].

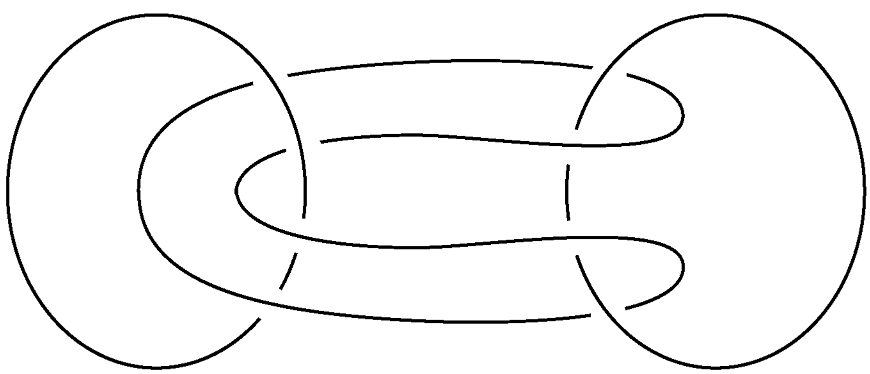
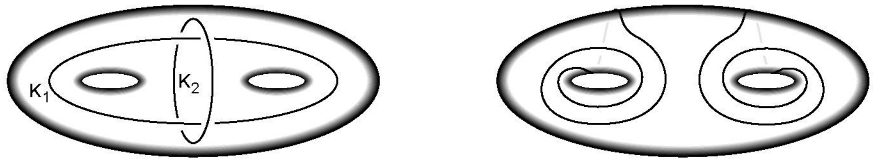


FIGURE 4.1

Example 4.8. It is more difficult to construct an example of large manifold M with $M(\gamma_1)$ annular and $M(\gamma_2)$ reducible, and $\Delta(\gamma_1, \gamma_2) = 1$. Here is a sketch of such an example.

Let $L = K_1 \cup K_2$ be the link in a handlebody H as shown in Figure 4.2(a). Let $M_1 = H - \text{Int}N(L)$, with $T_i = \partial N(K_i)$.



(a) (b)

FIGURE 4.2

It is easy to show that M_1 is irreducible, ∂-irreducible, and atoroidal. There is an essential annulus, however, from T_2 to ∂H. We need to modify the manifold to make it anannular.

Let C_1, C_2 be the curves on ∂H as shown in Figure 4.2(b). Then $P = \partial H - \text{Int}N(C_1 \cup C_2)$ is a sphere with 4 punctures. Choosing a simple manifold X with ∂X a genus 2 surface, and gluing it to M_1 along P, we get a M. One can show that M is a simple manifold.

Let m be the meridian of K_1 on T_1, and let l be the longitude, i.e., the blackboard framing slope. Clearly, $M(m)$ is reducible. We claim that $C_1 \cup C_2$ bounds an essential annulus in $M(l)$. To see the annulus, choose a Möbius band F_i on each handle of H, with $\partial F_i = C_i$. Tubing F_1 and F_2 together, we get a twice punctured Klein bottle F. Isotope F so that it contains K_1 and is disjoint from K_2. Then $F \cap M$ is a twice punctured annulus, which can be capped off in $M(l)$ to become an annulus A bounded by $C_1 \cup C_2$. Since C_1 and C_2 are on different components of ∂M, A is ∂-incompressible. It must also be incompressible, otherwise $M(l)$ would be ∂-reducible; but since $M(m)$ is reducible, this would contradict Scharlemann's Theorem [28].

Question 4.9. (1) If $M(\gamma_2)$ in Theorem 4.6 is ∂-reducible instead of reducible, is the theorem still true? It is true if the boundary of a ∂-reducing disk lies on a torus.

(2) If both $M(\gamma_i)$ are toroidal or annular, what is the upper bound of $\Delta(\gamma_1, \gamma_2)$? For general M, Gordon [10] proved that $\Delta \leq 8$, and $\Delta \leq 5$ if $\partial M - T \neq \emptyset$. With the extra assumption that $H_2(M, \partial M - T) \neq 0$, the upper bound could be much smaller.

5. Annular surgery and toroidal surgery

Theorem 5.1. *Suppose M is a simple manifold with torus T as a boundary component. If $M(\gamma_1)$ is annular, and $M(\gamma_2)$ is reducible, then*

$\Delta(\gamma_1, \gamma_2) \leq 2$.

This whole section is devoted to the proof of this theorem. By Theorem 4.6, we may assume that $\partial M - T$ is a torus. We may also assume that $M(\gamma_1)$ is irreducible and ∂-irreducible, otherwise the result follows from the Reducible Surgery Theorem of Gordon and Luecke [13] or Scharlemann's theorem [28]. We will further assume $\Delta(\gamma_1, \gamma_2) \geq 3$. The theorem will follow from the contradiction in the conclusions of Lemma 5.6 and 5.7.

Let F_1 be an essential annulus in $M(\gamma_1)$. Let F_2 be either a reducing sphere in $M(\gamma_2)$, or a disk embedded in $\text{Int}M(\gamma_2)$. Denote by J_i the attached solid torus in $M(\gamma_i)$. Let $P_i = F_i \cap M$. Let $u_1, \dots, u_{n_1}$ (resp. $v_1, \dots, v_{n_2}$) be the disks of $F_1 \cap J_1$ (resp. $F_2 \cap J_2$), labeled successively when traveling along J_i. Let Γ_1 be the graph in F_1 with u_i as (fat) vertices, and the arc components of $P_1 \cap P_2$ as edges. Similarly, Γ_2 is a graph in F_2 with v_j as vertices and the arcs of $P_1 \cap P_2$ as edges. Notice that if F_2 is a disk, and e is an arc component of $P_1 \cap P_2$ with an end on ∂F_2, then that end of e is not attached to any fat vertices. We say that e is a *ghost edge*, so Γ_i are actually graphs with ghost edges. The end of e which is not on a vertex is called a *ghost end*. On F_2 all ghost ends are on ∂F_2, while on F_1 the ghost ends are in the interior of P_1.

Each vertex of Γ_i is given a sign according to whether the J_i passes F_i from the positive side or negative side at this vertex. Two vertices of Γ_i are parallel if they have the same sign, otherwise they are antiparallel. An edge of Γ_i is a *positive edge* if it connects parallel vertices, otherwise it is a *negative edge*. The *parity rule* of [4] says that an edge of $P_1 \cap P_2$ is a positive edge in Γ_1 if and only if it is a negative edge in Γ_2.

A trivial loop in Γ_i is an edge cutting off a disk in P_i with interior disjoint from Γ_i. Such a disk can be used to ∂-compress the surface F_j, $j \neq i$. We choose F_1 so that n_1 is minimal, which guarantees that Γ_2 has no trivial loops. In the following, F_2 is either a reducing sphere of $M(\gamma_2)$, or a disk in the interior of $M(\gamma_2)$ such that all vertices of Γ_2 are parallel. In the first case, we choose n_2 to be minimal among all reducing spheres. In the second case, by the parity rule Γ_1 cannot have any loops. In any case, we have that neither Γ_i has any trivial loop. Doing some disk swappings if necessary, we may also assume that all circle components of $P_1 \cap P_2$ are essential in both P_i. In particular, each disk face of Γ_i has interior disjoint from P_j, $j \neq i$.

We may assume that each circle ∂u_i intersects each ∂v_j exactly Δ times. If e is an edge of Γ_1 with an end x on a fat vertex u_i, then x

is labeled j if x is in $u_i \cap v_j$. The labels in Γ_1 are considered mod n_2 integers. In particular, $n_2 + 1 = 1$. When going around ∂u_i, the labels of the ends of edges appear as $1, 2, \ldots, n_2$ repeated Δ times. Label the ends of edges in Γ_2 similarly. Each label in Γ_2 is a mod n_1 integer. Ghost ends are not labeled.

A set of positive edges $\{e_1, \ldots, e_k\}$ on Γ_i is called a *Scharlemann cycle* if (1) they bound a disk on P_i with interior disjoint from Γ_i, (2) all the vertices on the ends of e_j are parallel, and (3) the two labels at the ends of e_j are the same as that of e_1. The two labels of e_i must be $\{j, j+1\}$ for some j. We call $\{j, j+1\}$ the label pair of the Scharlemann cycle.

If $\{e_1, e_2, e_3, e_4\}$ are four parallel positive edges with e_i adjacent to e_{i+1} for $i = 1, 2, 3$, and if the two middle edges $\{e_2, e_3\}$ form a Scharlemann cycle, then the set of these four edges is called an *extended Scharlemann cycle.* This is enough for our purpose. We refer the reader to [13] for more general definition.

Lemma 5.2. *Suppose F_2 is a reducing sphere. Then the following hold:*

(1) Γ_1 cannot have n_2 parallel edges.

(2) Γ_1 cannot have an extended Scharlemann cycles.

(3) Any two Scharlemann cycles on Γ_1 have the same label pair.

(4) Γ_1 cannot have more than $(n_2/2) + 1$ parallel positive edges.

(5) If Γ_1 has a Scharlemann cycle, then F_2 bounds a punctured lens space. In particular it is separating.

Proof. (1) is proved by Gordon and Litherland in [12, Proposition 1.3]. (2)–(4) follow from the proof of [30, Lemma 2.2–2.4].

(5) is well known: If Γ_1 has a Scharlemann cycle, then we can find another sphere F_2' which has two fewer intersections with Dehn filling solid torus, and cobounds with F_2 a punctured lens space. By the minimality of n_2 the surface F_2' must bounds a 3-ball. See the proof of [4, Lemma 2.5.2(a)] for more details. q.e.d.

Note that since M is a simple manifold, we have $n_2 > 2$.

Lemma 5.3. *Theorem 5.1 is true if $n_1 \leq 2$.*

Proof. If $n_1 = 1$, all edges of Γ_1 are parallel. Since $\Delta \geq 3$, this contradicts Lemma 5.2(1).

Suppose $n_1 = 2$. Consider the reduced graph $\widehat{\Gamma}_1$ obtained by replacing a family of parallel edges in Γ_1 with a single edge. By calculating the Euler number, one can see that $\widehat{\Gamma}_1$ has at most 4 edges. Since by

Lemma 5.2(1) there are no n_2 parallel edges, each of the two vertices in $\widehat{\Gamma}_1$ must have valency 4, so the graph looks like that in Figure 5.1(a). The edges a and d are positive edges, hence each represents at most $(n_2/2)+1$ edges in Γ_1. Since each of b and c represents at most $n_2 - 1$ edges, and the valency of each vertex in Γ_1 is $3n_2$, it follows that each of a and d represents exactly $(n_2/2)+1$ edges, and each of b and c represents $n_2 - 1$ edges. See Figure 5.1(b) for the case that $n_2 = 6$. We separate two cases.

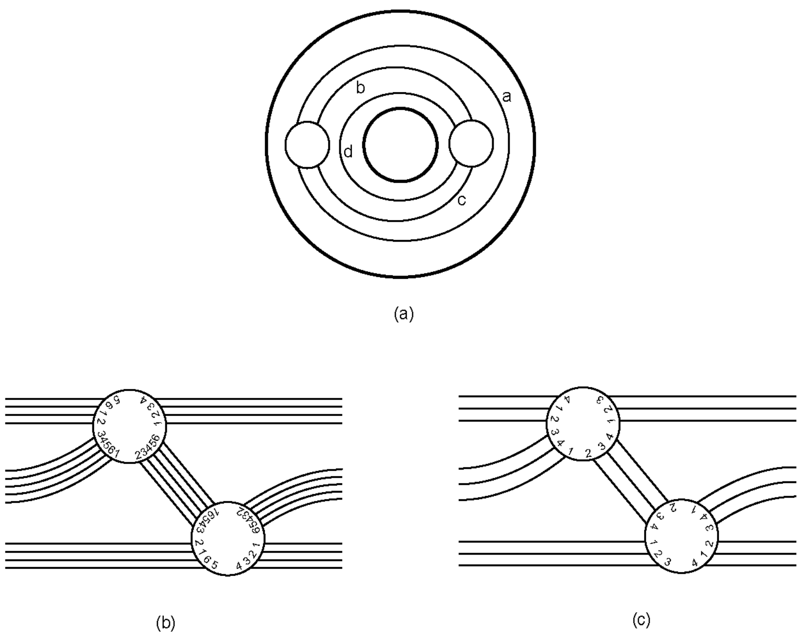


FIGURE 5.1

Case 1. *The two vertices of* Γ_1 *are not parallel.*

It is clear that any family of $(n_2/2)+1$ parallel positive edges contain a Scharlemann cycle $\{e_1, e_2\}$. Moreover, they must lie on one side of the family, for otherwise there would be an extended Scharlemann cycle. Since $n_2 > 2$, the Scharlemann cycles cannot be on the side near the boundary of P_1. Also, by Lemma 5.2(3) we may assume that both Scharlemann cycles have the same label pair, say $\{1, 2\}$. It is now clear that the labeling of the graph looks like that in Figure 5.1(c). In particular, for each label j there is a negative edge in Γ_1 with ends labeled j and $j+1$. By the parity rule, v_j and v_{j+1} are parallel. Thus all vertices of Γ_2 are parallel. But then there cannot be any positive edges in Γ_1, contradicting the existence of family a and d.

Case 2. *The two vertices of* Γ_1 *are parallel.*

Since the families of edges in a and d contain Scharlemann cycles, by Lemma 5.2(5) F_2 is a separating sphere. Hence n_2 is even. In this case all edges on Γ_1 are positive, so b and c also represents at most $(n_2/2)+1$ edges, and we have $4[(n_2/2)+1] \geq 3n_2$. Since n_2 is even, and $n_2 > 2$, this holds only if $n_2 = 4$, and each family contains 3 edges. Thus the graph looks like that in Figure 5.1(c). One can see that there is an edge with both ends labeled 4, which is impossible by the parity rule. q.e.d.

Lemma 5.4. *(1) If* Γ_2 *has a Scharlemann cycle, then* F_1 *is a separating annulus.*

(2) Γ_2 *cannot have two Scharlemann cycles with different label pairs.*

(3) Γ_2 *cannot have an extended Scharlemann cycle.*

Proof. (1) Let $\{e_1, \ldots, e_n\}$ be a Scharlemann cycle in Γ_2 with label pair $\{1,2\}$, bounding a Scharlemann disk D on the surface F_2. Let V be the part of the Dehn filling solid torus J_1 in $M(\gamma_1)$ lying between the two meridian disks u_1, u_2, and containing no other u_j. If F_1 is a nonseparating annulus, then the frontier of $N(F_1 \cup V \cup D)$ in $M(\gamma_1)$ consists of two nonseparating annuli F_1 and F_1'. The annulus F_1' has 2 less intersection with J_1 than F_1. Since $M(\gamma_1)$ is irreducible and ∂-irreducible, any nonseparating annulus is essential. Hence F_1' is essential in $M(\gamma_1)$, contradicting the minimality of n_1.

(2) Use the notation above. On the annulus F_1 consider the subgraph $G = e_1 \cup \ldots \cup e_k \cup u_1 \cup u_2$ of Γ_1. If G is contained in a disk D_1, then as in the proof of [4, Lemma 2.5.2] it is easy to see that a regular neighborhood of $D_1 \cup V \cup D$ is a once punctured lens space. Since we have assumed that $M(\gamma_1)$ is irreducible, this is impossible. Hence we may assume that G cuts F_1 into two annuli A_1, A_2 and some disk components. Let A_i' be the closure of $F_1 - A_i$. Consider the manifold $Y = N(A_1' \cup V \cup D)$. Clearly, ∂Y is a torus, so the frontier of Y in $M(\gamma_1)$ is an annulus Q.

Claim. Q is an essential annulus in $M(\gamma_1)$.

Proof. The central curve C of Q is isotopic to the central curve C' of F_1. Since F_1 is incompressible, C' (and hence C) is not null homotopic in $M(\gamma_1)$, so Q is also incompressible. If Q is not essential, it has a boundary compressing disk D'. Put $X = M(\gamma_1) - \text{Int}Y$. We have assumed that $M(\gamma_1)$ is irreducible and ∂-irreducible, so if X contains

D', then it is a solid torus with Q as a longitudinal annulus. Similarly for Y.

First assume D' is in X. Notice that F_1 can be isotoped into X. Choose D' so that $D' \cap F_1$ is minimal. If $D' \cap F_1 = \emptyset$, then F_1 would lie in a 3-ball. If $D' \cap F_1 \neq \emptyset$, an outermost component of $D' - F_1$ disjoint from Q would be a boundary compressing disk of F_1. Both cases contradict the essentiality of F_1.

Now assume D' is in Y. Then Y is a solid torus with Q a longitudinal annulus. Let $\widehat{Y}$ be the manifold obtained by attaching a 2-handle H to Y along the longitudinal annulus $\partial Y - Q$. Then $\widehat{Y}$ is a 3-ball. Recall that $Y = N(A_1' \cup V \cup D)$, so it can be considered as obtained by attaching a 2-handle $H' = N(D)$ to the handlebody $W = N(A_1' \cup V)$ along the curve ∂D. Switch the order of the two 2-handle additions. It is easy to see that after attaching H to W, the manifold is a solid torus, with ∂D intersecting a meridian exactly k times, where $k > 1$ is the length of the Scharlemann cycle. Therefore $\widehat{Y}$ is a punctured lens space, which is a contradiction. q.e.d.

We continue with the proof of Lemma 5.4. Notice that if A_1' contains m vertices of Γ_1 (including u_1 and u_2), then the new essential annulus Q above intersects J_1 exactly $2m - 2$ times. By the minimality of n_1 we must have $2m - 2 \geq n_1$.

Suppose $\{e_1', \ldots, e_t'\}$ is another Scharlemann cycle of Γ_2 with label pair $\{p, p+1\}$. Since the label pairs are different, they can have at most one label in common, say $p = 2$. Let $G' = e_1' \cup \ldots \cup e_t' \cup u_p \cup u_{p+1}$ be the corresponding graph on F_1. The two graphs G' and G are disjoint except possibly intersecting at their common vertex $u_p = u_2$. Like before, G' cannot be contained in a disk, hence we may assume that G' is contained in the annulus $A_1 \cup u_2$.

By the above, the annulus A_1 contains at most $n_1/2 - 1$ vertices, so the annulus $A_1 \cup u_2$ contains at most $n_1/2$ vertices. Applying the Claim to G', we see that the frontier of $Y' = N((A_1 \cup u_2) \cup V' \cup D')$ is an essential annulus in $M(\gamma_1)$ intersecting J_1 at most $2(n_1/2) - 2 = n_1 - 2$ times. This contradicts the minimality of n_1, completing the proof of (2).

(3) Let $\{e_1, e_2, e_3, e_4\}$ be an extended Scharlemann cycle with label pair $\{2, 3\}$, say. Then as above, the set $C_1 = e_2 \cup e_3 \cup u_2 \cup u_3$ cuts F_1 into two annuli, each containing exactly $n_1/2 - 1$ vertices of Γ_1 in its interior. The cycle $C_2 = e_1 \cup e_4 \cup u_1 \cup u_4$ must lie on one of these two annuli. Like C_1, the cycle C_2 cannot be contained in a disk, so $F_1 - C_2$ consists

of two annuli. Let A_1 be the one which does not contain C_1. Let D be the disk on F_2 bounded by e_1, e_4 and two arcs on the boundary of fat vertices. Let V be the part of the Dehn filling solid torus J_1 between u_1 and u_4 and which contains u_2 and u_3. By the same proof as in (2), one can show that the frontier of $N(A_1 \cup u_1 \cup u_4 \cup V \cup D)$ is an essential annulus in $M(\gamma_1)$ which intersects J_1 less than n_1 times, leading to a contradiction to the minimality of n_1. q.e.d.

Consider a disk F_2 in the interior of $M(\gamma_2)$. We assume that ∂F_2 is disjoint from J_2. Recall that $F_1 \cap F_2$ form the graph Γ_2 in F_2 which may have some ghost edges connecting the fat vertices of Γ_2 to ∂F_2.

Definition 5.5. (1) A disk F_2 in $M(\gamma_2)$ is a *generalized Gabai disk* if all the fat vertices on F_2 are parallel, and the number of ghost edges is less than Δn_1, the valency of fat vertices in Γ_2.

Lemma 5.6. *If $\Delta = \Delta(\gamma_1, \gamma_2) \geq 3$, then $M(\gamma_2)$ cannot have a generalized Gabai disk.*

Proof. Recall that a non-ghost edge of Γ_2 is an i-edge if it has i as a label on one of its ends. An i-edge cycle in Γ_2 is a cycle consisting of i-edges.

Since there are less than Δn_1 ghost edges, at least one of the labels, say i, has the property that there are at most $\Delta - 1$ ghost edges with label i on their non-ghost ends, so every fat vertex v_j has a non-ghost edge with label i at v_j. One can then find a cycle of edges in Γ_2, each starting with the label i. Such a cycle is called a great i-cycle in [4]. By [4, Lemma 2.6.2] this implies that Γ_2 has a Scharlemann cycle, so by Lemma 5.4(1) F_1 is a separating surface. According to Lemma 2.2 of [13], any disk D on F_2 bounded by an i-edge cycle in Γ_2 contains a Scharlemann cycle if Int$D$ contains no vertices of Γ_2. We are done by Lemma 5.4(2) unless all of these Scharlemann cycles have the same label pair $\{1, 2\}$, say.

Consider the subgraph Γ_2' of Γ_2 consisting of all i-edges. We may assume that Γ_2' is connected, otherwise consider a smaller disk and follow the argument here. There are at least $\Delta n_2 - (\Delta - 1)$ i-edges, with n_2 vertices. By calculating the Euler characteristic we see that there are at least $1 + \Delta n_2 - (\Delta - 1) - n_2$ faces. Since each of them contains at least one Scharlemann cycle, which contains at least 2 edges, there are at least $2(\Delta n_2 - \Delta - n_2 + 2)$ edges in Γ_1 connecting u_1 to u_2. Since the valency of u_i is Δn_2, we have

$$2(\Delta n_2 - \Delta - n_2 + 2) \leq \Delta n_2,$$

i.e., $(\Delta - 2)(n_2 - 2) \leq 0$. Since $\Delta > 2$, this holds only if $n_2 \leq 2$. Recall that there is no trivial loop in Γ_2, so $n_2 \neq 1$, otherwise all edges would be ghost edges, contradicting the assumption that F_2 is a generalized Gabai disk. If $n_2 = 2$, all non-ghost edges must be parallel, and there are more than $3n_1/2$ such. This number is greater than $n_1/2 + 2$, so there exists an extended Scharlemann cycle by the proof of [30, Lemma 2.2], which contradicts Lemma 5.4(3). q.e.d.

Lemma 5.7. *If $\Delta \geq 3$, then $M(\gamma_2)$ contains a generalized gabai disk.*

Proof. We will use sutured manifold theory to prove this lemma. One is referred to [5]–[7] and [27] for definitions and theorems about sutured manifolds. In particular, we will use the planar surface $P_1 = F_1 \cap M$ as a parametrizing surface, and use Theorem 7.8 of [27].

Consider the sutured manifold (X, γ, β), where $X = M(\gamma_2)$, the suture set γ is empty, and β is the knot which is the center of the Dehn filling solid torus J_2 in X. Let $\partial X = \partial_+ X$, which is denoted by R_+ in [27]. Since $X - \beta$ is irreducible and ∂-irreducible, and the norm of $\partial_+ X$ is 0, by definition X is a β-taut sutured manifold.

Recall that a proper surface Q in $M = X - \text{Int}N(\beta)$ is a *parametrizing surface* if no component of Q is a disk with boundary in $\partial_\pm X$. By extending over $N(\beta)$, Q can also be considered as an immersed surface in X with interior embedded in X, and with boundary on $\partial X \cup \beta$. Isotop Q so that ∂Q intersects γ and $\partial N(\beta)$ in essential arcs or circles. The index of Q is defined as

$$I(Q) = \mu + \nu - 2\chi(Q),$$

where μ and ν are the numbers of essential arcs of ∂Q in γ and $\partial N(\beta)$ respectively. The index is additive over components of Q, and Q being a parametrizing surface means that the index of each component is nonnegative. If we view γ and $\partial N(\beta)$ as cusps, then Q would be a surface with some cusp points on its boundary, and $\mu + \nu$ is exactly the number of cusps on ∂Q. Hence $I(Q) = -2\chi_c(Q)$, where χ_c is the cusped Euler characteristic defined in Section 1.

Take $Q = F_1 \cap M$, the punctured essential annuli in M. Since X has no sutures, and since β is a circle, $\mu = \nu = 0$, so the index of Q is $I(Q) = -2\chi(Q) = 2n_1$, where as before n_1 denotes the number of times F_1 intersects the Dehn filling solid torus J_1. We refer the reader to [27] for the definition of taut sutured manifold decomposition, decomposition that respect a parametrizing surface, and sutured manifold hierarchy. An important fact about parametrizing surface is that if a decomposition respect a parametrizing surface, then $I(Q') \leq I(Q)$, where Q' is the parametrizing surface after the decomposition. The following is one of the fundamental theorems in sutured manifold theory.

Theorem [27, 7.8]. *If Q is a parametrizing surface for the β-taut sutured manifold (X, γ), then there is a β-taut sutured manifold hierarchy for (X, γ) which respects Q.*

Applying the theorem to (X, γ, β), with $Q = F_1 \cap M$ as a parametrizing surface, we get a sutured manifold hierarchy as in the theorem. By definition, each component of ∂M_n is a sphere. Denote by $Q_n = Q \cap M_n$ the parametrizing surface in M_n. Since the hierarchy respects Q, we have $I(Q_n) \leq I(Q) = 2n_1$.

There is a process called cancellation, see [27, Definition 4.1]. If D is a disk component of Q_n which passes each of γ_n and β_n exactly once, then we can cut along D to reduce the number of components in β_n, the resulting sutured manifold is still β-taut [27, Lemma 4.3], and the index of the parametrizing surface unchanged. After canceling all possible components in β_n, we get a new set β'_n, for which Q_n has no cancellation disks.

If $\beta'_n = \emptyset$, then (M_n, γ_n) would be $\emptyset$-taut, so by Corollary 3.9 of [27], the original manifold X would also be $\emptyset$-taut, (see *Proof of 9.1 from 9.7* on [27, p. 608] for more details.) This would be a contradiction because $X = M(\gamma_2)$ was assumed reducible. Therefore, there must be a component P of $\partial_+ M_n$ which contains some points of β'_n.

There is a graph $\Gamma(P)$ on P constructed in the obvious way: The vertices are $P \cap J_2 = P \cap N(\beta'_n)$, where $J_2 = N(\beta)$ is the Dehn filling solid torus, and the edges are the arcs in $P \cap Q = P \cap Q_n$. The orientation of β'_n comes from that of the knot β, and from the definition of sutured manifold decomposition we know that β'_n always intersect $\partial_+ M_n$ in the same direction. In particular, all the fat vertices of $\Gamma(P)$ are parallel. Thus if P is a sphere, then by removing a small disk from P, the resulting surface is a generalized Gabai disk for F_2 with no ghost edges, and we are done.

Therefore we assume that P lies on a sphere S of ∂M_n which contains some sutures. If P is not a disk, consider a disk component P' of $S \cap \partial_\pm M_n$. If P' has some intersection with β'_n, we can take P' instead of P. If P' does not intersect β'_n, then the existence of a nondisk component in $S \cap \partial_\pm M_n$ implies that some components in $S \cap \partial_\pm M_n$ is compressible in $M_n - \beta'_n$, which contradicts the β-tautness of M_n. Therefore we may assume that P is a disk in S. Since all fat vertices of $\Gamma(P)$ are parallel, we need only show that $\Gamma(P)$ has less than Δn_1 ghost edges. P would then be the required generalized Gabai disk for F_1.

Let W be the component of M_n containing S. Let $D_1, \dots, D_k$ be the components of Q_n in W which intersect ∂P. Thus all ghost edges of $\Gamma(P)$ are contained in $\cup \partial D_i$. Recall that the index of D_i is $I(D_i) = \mu_i + \nu_i - 2\chi(D_i)$. Since each suture and each arc of β'_n connect two components of $\partial_\pm M_n$ with different orientation, $\mu_i + \nu_i$, the number of cusps on ∂D_i, is always even. Since there is no cancellation disk anymore, either $\mu_i + \nu_i \geq 4$, or $\chi(D_i) \leq 0$. In any case we have $I(D_i) \geq (\mu_i + \nu_i)/2$. Summing over all such disks we have

$$\sum(\mu_i + \nu_i) \leq 2\sum I(D_i) \leq 2I(Q_n) \leq 4n_1.$$

The left-hand side is equal to the total number of arc components of $\cup \partial D_i$ on $\partial_\pm M_n$, exactly half of which are on $\partial_+ M_n$. It follows that the number of ghost edges on $\Gamma(P)$ is at most $2n_1$. Hence P is a generalized Gabai disk for F_1, completing the proof of Lemma 5.7. q.e.d.

The contradiction in the conclusion of Lemmas 5.6 and 5.7 completes the proof of Theorem 5.1.

In [14] Gordon and Luecke showed that if $M(\gamma_1) = S^3$ and $M(\gamma_2)$ is toroidal, then $\Delta = \Delta(\gamma_1, \gamma_2) \leq 2$. Moreover, if $\Delta = 2$ then the essential torus in $M(\gamma_2)$ intersects the Dehn filling solid torus exactly twice.

Question 5.8. If in Theorem 5.1 we have $\Delta(\gamma_1, \gamma_2) = 2$, is it true that $n_1 = 2$?

References

[1] S. Bleiler & C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996) 809–833.

[2] M. Brittenham, *Essential laminations in Seifert-fibered spaces*, Topology **32** (1993) 61–85.

[3] ———, *Essential laminations, exceptional Seifert-fibered spaces, and Dehn filling*, Preprint.

[4] M. Culler, C. Gordon, J. Luecke & P. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987) 237–300.

[5] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. **18** (1983) 445–503.

[6] ______, *Foliations and the topology of 3-manifolds. II*, J. Differential Geom. **26** (1987) 461–478.

[7] ______, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. **26** (1987) 479–536.

[8] D. Gabai & Lee Mosher, In preparation.

[9] D. Gabai & U. Oertel, *Essential laminations in 3-manifolds,*, Ann. of Math. **130** (1989) 41–73.

[10] C. Gordon, *Boundary slopes of punctured tori in 3-manifolds*, Trans. Amer. Math. Soc., to appear.

[11] ______, *Dehn filling: A survey*, Preprint.

[12] C. Gordon & R. Litherland, *Incompressible planar surfaces in 3-manifolds*, Topology Appl. **18** (1984) 121-144.

[13] C. Gordon & J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996) 385–409.

[14] ______, *Dehn surgery on knots creating essential tori I*, Comm. Anal. Geom. **3** (1995) 597–644.

[15] ______, *Toroidal and boundary-reducing Dehn fillings*, Topology Appl., to appear.

[16] C. Gordon & Y-Q. Wu, *Toroidal and annular Dehn fillings*, Proc. London Math. Soc., to appear.

[17] ______, *Annular and boundary reducing Dehn fillings*, In preparation.

[18] M. Gromov & W. Thurston, *Pinching constants for hyperbolic manifolds*, Invent. Math. **89** (1987) 1–12.

[19] C. Hayashi & K. Motegi, *Dehn surgery on knots in solid tori creating essential annuli*, Trans. Amer. Math. Soc. **349** (1997) 4897–4930.

[20] J. Hempel, *3-manifolds*, Ann. of Math. Stud. No. 86, Princeton University Press, 1976.

[21] W. Jaco, *Lectures on three-manifold topology*, Regional Conference Series in Mathematics, Amer. Math. Soc. Vol. 43, 1977.

[22] J. Luecke, Private communication.

[23] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984) 37–44.

[24] S. Oh, *Reducible and toroidal manifolds obtained by Dehn filling*, Topology Appl. **75** (1997) 93–104.

[25] K. Miyazaki & K. Motegi, *Toroidal and annular Dehn surgeries of solid tori*, Preprint.

[26] R. Myers , *Simple knots in compact orientable 3-manifolds*, Trans. Amer. Math. Soc. **273** (1982)75–91.

[27] M. Scharlemann, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom. **29** (1989) 557–614.

[28] ______, *Producing reducible 3-manifolds by surgery on a knot*, Topology **29** (1990) 481–500.

[29] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982) 357–381.

[30] Y-Q. Wu , *The reducibility of surgered 3-manifolds*, Topology Appl. **43** (1992) 213–218.

[31] ______, *Incompressibility of surfaces in surgered 3-manifolds*, Topology **31** (1992) 271–279.

[32] ______, *Dehn surgeries producing reducible manifolds and toroidal manifolds*, Topology **37** (1998) 95–108.

[33] ______, *Essential laminations in surgered 3-manifolds*, Proc. Amer. Math. Soc. **115** (1992) 245–249.

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