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THE GROMOV-LAWSON-ROSENBERG CONJECTURE FOR GROUPS WITH PERIODIC COHOMOLOGY

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1. Introduction

By a well-known result of Lichnerowicz [18], there are manifolds which do not admit positive scalar curvature metrics. Lichnerowicz observes that the existence of such a metric on a closed spin manifold M of dimension n congruent to 0 mod 4 implies that the (chiral) Dirac operator $D^+(M)$ (cf. §4) is invertible (cf. [17, Ch. II, Thm. 8.8]). In particular,

$$\operatorname{index}(D^+(M)) = \dim\ker D^+(M) - \dim\operatorname{coker} D^+(M)$$

vanishes. Note that $\operatorname{index}(D^+(M))$, unlike the dimension of the kernel and the dimension of the cokernel of $D^+(M)$, is *independent of the metric* used in the construction of $D^+(M)$. In fact, according to the Atiyah-Singer Index Theorem, it is equal to a topological invariant $\hat{A}(M)$, the $\hat{A}$-genus of M (cf. [17, Ch. III, Thm. 13.10]). We recall that $\hat{A}(M)$ is a characteristic number defined by evaluating a certain polynomial in the Pontrjagin classes of the tangent bundle on the fundamental class of M.

Lichnerowicz' result was generalized by Hitchin [12], who constructs a version of the Dirac operator $D(M)$ which is selfadjoint and commutes with an action of the Clifford algebra $C\ell_n$, where n is the dimension of

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M. In particular, the kernel of $D(M)$ is a $\mathbb{Z}/2$-graded module over $C\ell_n$. The element represented by $\ker D(M)$ in the Grothendieck group $\mathcal{Z}(C\ell_n)$ of finitely generated $\mathbb{Z}/2$-graded modules over $C\ell_n$ in general will depend on the metric. However, it can be shown that the class $[\ker D(M)] \in \mathcal{Z}(C\ell_n)/i^*\mathcal{Z}(C\ell_{n+1})$ is *independent of the metric*, where i^* is the map induced by the inclusion $i\colon C\ell_n \to C\ell_{n+1}$ (cf. [17, Ch. III, Prop. 10.6]). This class is known as the *Clifford index* of $D(M)$. We note that the cokernel of i^* is isomorphic to $KO(S^n)$, the real K-theory of the n-sphere by a result of Atiyah-Bott-Shapiro (cf. [17, Ch. I, Thm. 9.27]).

If M has finite fundamental group π, the manifold M in the discussion above can be replaced by its universal covering $\widetilde{M}$. The π-action on $\widetilde{M}$ by deck transformations induces a π-action on the kernel of the Dirac operator $D(\widetilde{M})$. This action commutes with the $C\ell_n$-module structure, and hence $\ker D(\widetilde{M})$ is a module over $C\ell_n \otimes \mathbb{R}\pi$, where $\mathbb{R}\pi$ is the real group ring of π. As in the case $\pi = \{1\}$ discussed above, the element

$$\alpha(M) := [\ker D(\widetilde{M})] \in \mathcal{Z}(C\ell_n \otimes \mathbb{R}\pi)/i^*\mathcal{Z}(C\ell_{n+1} \otimes \mathbb{R}\pi) \tag{1.1}$$

is independent of the metric. The quotient of these Grothendieck groups is isomorphic to $KO_n(\mathbb{R}\pi)$, the real K-theory of $\mathbb{R}\pi$ (more or less by the definition of these K-theory groups).

Let π be an arbitrary fundamental group. Rosenberg constructs an invariant $\alpha(M)$ in $KO_n(C_r^*\pi)$, where $C_r^*\pi$ is a suitable completion of $\mathbb{R}\pi$, known as the (reduced) group C^*-algebra $C_r^*\pi$. This invariant can be thought of as the 'equivariant Clifford index' of the Dirac operator on the universal covering of M and agrees with (1.1) for finite groups π. We refer to [21], [22], [23] for details.

We hope the discussion above makes it clear that the existence of a positive scalar curvature metric on M implies the vanishing of $\alpha(M)$. Modifying a conjecture of Gromov and Lawson, Rosenberg conjectures that the converse holds as well:

Conjecture 1.1 (Gromov-Lawson-Rosenberg Conjecture). *A closed connected smooth spin manifold M of dimension $n \geq 5$ with fundamental group π admits a positive scalar curvature metric if and only if $\alpha(M)$ vanishes.*

This conjecture has been proved for simply connected manifolds [28]. It has been proved for manifolds whose fundamental group is odd order cyclic [23], [15], is of order 2 [24], or which is one of a (short) list of infinite groups, including free groups, free abelian groups, and fundamental groups of orientable surfaces [25].

The main result of this paper is the following theorem.

Theorem 1.2. *The Gromov-Lawson-Rosenberg Conjecture is true for finite groups π with periodic cohomology.*

We remark that by a theorem of Kwasik-Schultz [15] the Gromov-Lawson-Rosenberg Conjecture is true for a finite group π if and only if it is true for all Sylow subgroups of π. It is well known that finite groups with periodic cohomology are precisely those finite groups whose p-Sylow subgroups are cyclic for odd p and cyclic or (generalized) quaternion groups for $p = 2$. So the new result is that the conjecture is true for cyclic groups of order 2^k for $k \geq 2$ and for generalized quaternion groups.

We recall that a *smooth space form* is a manifold whose universal covering is diffeomorphic to the sphere S^n. The fundamental group of a space form has periodic cohomology, and space forms are spin manifolds for $n \equiv 3 \bmod 4$. Using the fact that $KO_n(C_r^*\pi)$ vanishes for finite groups π and $n \equiv 3 \bmod 8$, Theorem 1.2 implies in particular that every smooth spherical space form of dimension $n \equiv 3 \bmod 4$ admits a positive scalar curvature metric, a result due to Kwasik and Schultz [16].

In Section 2, we outline the proof of Theorem 1.2 and explain the role of the remaining sections.

2. Outline of the proof

Let M be a spin manifold of dimension n (all manifolds considered in this paper are smooth and compact; their boundary is empty unless mentioned otherwise). In the introduction, we described the invariant $\alpha(M)$, the 'equivariant Clifford index' of the Dirac operator on the universal covering of M, which lives in $KO_n(C_r^*\pi_1(M))$. Generalizing slightly, if $f\colon M \to B\pi$ is a map to the classifying space of a discrete group π, we define $\alpha(M, f) \in KO_n(C_r^*\pi)$ to be the π-equivariant Clifford index of the Dirac operator on $\widetilde{M}$, where $\widetilde{M} \to M$ is the pull back via f of the covering $E\pi \to B\pi$. It turns out that $\alpha(M, f)$ depends only on the bordism class (M, f) represents in the spin bordism group $\Omega_n^{spin}(B\pi)$. Hence the α-invariant induces a homomorphism $\alpha\colon \Omega_n^{spin}(B\pi) \to KO_n(C_r^*\pi)$. This homomorphism can be factored as follows [30];

$$\alpha = A \circ p \circ D \colon \Omega_n^{spin}(B\pi) \to ko_n(B\pi) \to KO_n(B\pi) \to KO_n(C_r^*\pi). \tag{2.1}$$

Here $ko_n(B\pi)$ (resp. $KO_n(B\pi)$) is the connective (resp. periodic) real K-homology of $B\pi$, D and p are natural transformations between these generalized homology theories, and A is known as the *assembly map.*

The proof of Theorem 1.2 will be based on the following result. Let $\Omega_n^{spin,+}(B\pi)$ be the subgroup consisting of bordism classes represented by pairs (N, f) for which N admits a positive scalar curvature metric, and let $ko_n^+(B\pi) = D(\Omega_n^{spin,+}(B\pi))$. See Jung [13] and Stolz [29], cf. [30, Thm 4.5] for the following

Theorem 2.1. *Let M be a spin manifold of dimension $n \geq 5$ with fundamental group π, and let $u\colon M \to B\pi$ be the classifying map of the universal covering. Then M has a positive scalar curvature metric if and only if $D[M, u]$ is in $ko_n^+(B\pi)$.*

Corollary 2.2. *The Gromov-Lawson-Rosenberg conjecture is true for a group π if and only if we have $ko_n^+(B\pi) = \mathcal{Y}_n(B\pi)$ for $n \geq 5$, where $\mathcal{Y}_n(B\pi)$ denotes the kernel of the map $A \circ p\colon ko_n(B\pi) \to KO_n(C_r^*\pi)$.*

We note that the vanishing of the α invariant for manifolds with positive scalar curvature metrics implies the inclusion $ko_n^+(B\pi) \subseteq \mathcal{Y}_n(B\pi)$. To prove the converse inclusion when π is a cyclic group C_l or a quaternion group Q_l of order $l = 2^k$, we consider lens spaces and lens space bundles over S^2 (in the case $\pi = C_l$), respectively lens spaces and quaternionic lens spaces (for $\pi = Q_l$). We furnish these manifolds with spin structures and maps to $B\pi$, and consider the Ω_*^{spin}-submodule of $\mathcal{M}_*(B\pi) \subseteq \widetilde{\Omega}_*^{spin}(B\pi)$ generated by their bordism classes. These manifolds admit metrics of positive scalar curvature, and hence we have the following chain of inclusions:

$$D(\mathcal{M}_n(B\pi)) \subseteq \widetilde{ko}_n^+(B\pi) \subseteq \widetilde{\mathcal{Y}}_n(B\pi), \tag{2.2}$$

where $\widetilde{\mathcal{Y}}_n(B\pi)$ denotes the kernel of $A \circ p$ restricted to the reduced ko-homology group $\widetilde{ko}_n(B\pi)$, and $\widetilde{ko}_n^+(B\pi)$ is the intersection of $\widetilde{ko}_n(B\pi)$ with $ko_n^+(B\pi)$.

Theorem 2.3. *Let π be a cyclic or (generalized) quaternion group whose order is a power of 2. Then $D(\mathcal{M}_*(B\pi)) = \widetilde{\mathcal{Y}}_*(B\pi)$.*

To prove this result, we basically compare the order of the group $\widetilde{ko}_n(B\pi)$ and the order of the subgroup $D(\mathcal{M}_n(B\pi))$. For this calculation, we need invariants that allow us to identify the elements represented by lens spaces, resp. lens space bundles, resp. quaternionic lens spaces in $\widetilde{ko}_*(B\pi)$. It turns out that the eta invariant provides the desired invariant. We recall that the eta invariant $\eta(D)$ of a selfadjoint elliptic operator D is a real number which measures the asymmetry of the spectrum of D with respect to the origin (naïvely, it is the number of non-negative eigenvalues of D minus the number of negative eigenvalues).

We recall that a finite dimensional complex representation ρ of a discrete group π determines a vector bundle V_ρ over the classifying space $B\pi$. Given a spin manifold M of odd dimension n, and a map $f\colon M \to B\pi$, we denote by $D(M,f,\rho)$ the Dirac operator on M twisted by the flat vector bundle f^*V_ρ, and by $\eta(M,f)(\rho) \in \mathbb{R}$ the eta invariant of $D(M,f,\rho)$. Since the eta invariant is additive with respect to the direct sum of representations, we may extend $\eta(M,f)(\rho)$ to virtual representations ρ.

In Section 4 we show that the eta invariant induces a homomorphism

$$\eta(\rho)\colon KO_n(B\pi) \to \mathbb{R}/\mathbb{Z}. \tag{2.3}$$

If $n \equiv 3 \bmod 8$ and ρ is of *real type* (*i.e.*, if ρ is the complexification of a real representation), or if $n \equiv 7 \bmod 8$ and ρ is of *quaternionic type* (*i.e.*, if ρ is obtained from a representation of π on a quaternionic vector space by restricting the scalars), we still get a well defined map if we replace the range of (2.3) by $\mathbb{R}/2\mathbb{Z}$.

Choosing a basis for the free $\mathbb{Z}$-module of π-representations of virtual dimension 0 appropriately (keeping track of the type of these representations), and combining the eta invariants corresponding to these representations in a single map, we get a homomorphism

$$\check{\eta}\colon KO_n(B\pi) \to \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/2\mathbb{Z}. \tag{2.4}$$

The proof Theorem 2.3 is based on comparing the order of the group $\widetilde{ko}_n(B\pi)$ with the order of $\check{\eta} \circ p(\widetilde{\mathcal{Y}}(B\pi))$:

Theorem 2.4. *Let π be the cyclic group C_l or the (generalized) quaternion group Q_l. Let n be a positive integer, and assume $n \equiv 3$ mod 4 for $\pi = Q_l$. Then the order of $\widetilde{ko}_n(B\pi)$ is $\epsilon(n,\pi)$ where we define*

n	$8j+1$	$8j+2$	$8j+3$	$8j+5$	$8j+7$
$\epsilon(n,C_l)$	$2(l/2)^{2j+1}$	2	$2(2l)^{2j+1}$	$(l/2)^{2j+2}$	$(2l)^{2j+2}$
$\epsilon(n,Q_l)$	$\star$	$\star$	$2^{4j+4}l^{2j+1}$	$\star$	$2^{4j+4}l^{2j+2}$

and we set $\epsilon(n,C_l)=1$ *and* $\epsilon(n,Q_l)=\star$ *otherwise.*

We prove this result in Section 3 using the Atiyah-Hirzebruch spectral sequence. For the quaternion group, we only determine the order of $\widetilde{ko}_n(BQ_l)$ for $n \equiv 3 \bmod 4$, since the calculation in the other cases would involve determining some higher differentials. This can be done, but is not done in this paper, since the above statement turns out to suffice for our purposes.

Theorem 2.5. *If* π *is the cyclic group* C_l *and* n *odd,* $n \not\equiv 1$ *mod* 8*, or if* π *is the (generalized) quaternion group* Q_l *and* $n \equiv 3$ *mod* 4*, then the order of* $\check{\eta} \circ p(D(\mathcal{M}_n(B\pi)))$ *is greater or equal to* $\epsilon(n,\pi)$*. For* $n \equiv 1$ *mod* 8*, the order of* $\check{\eta} \circ p(D(\mathcal{M}_n(C_l)))$ *is greater or equal to* $\frac{1}{2}\epsilon(n,\pi)$*, and the order of* $\check{\eta} \circ p(\widetilde{ko}_n(BC_l))$ *is greater or equal to* $\epsilon(n,\pi)$*.*

This Theorem is proved by explicit *eta*-calculations in Section 5 (for $\pi = C_l$) and in Section 6 (for $\pi = Q_l$), respectively. We note that taken together, Theorems 2.4 and 2.5 imply the following result, which might be of independent interest.

Corollary 2.6. *The composition*

$$\check{\eta} \circ p : \widetilde{ko}_n(B\pi) \to \widetilde{KO}_n(B\pi) \to \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/2\mathbb{Z}$$

is injective if π *is the cyclic group and* n *is odd, or if* π *is the (generalized) quaternion group and* $n \equiv 3$ *mod* 4*.*

Also, Theorems 2.4 and 2.5 imply Theorem 2.3 for $\pi = C_l$, $n \not\equiv 1,2$ mod 8, and for $\pi = Q_l$, $n \equiv 3$ mod 4. In fact, they yield the stronger statement $D(\mathcal{M}_n(B\pi)) = \widetilde{ko}_n(B\pi)$ in these cases. In particular, the map

$$A \circ p : \widetilde{ko}_n(B\pi) \to \widetilde{KO}_n(B\pi) \to KO_n(C_r^*\pi)$$

is trivial in these cases.

To settle the remaining cases, we need some general facts about the assembly map $A\colon KO_*(B\pi) \to KO_*(\mathbb{R}\pi)$ for finite 2-groups π which are proved in [26] by dualizing the well-known result of Atiyah-Segal concerning the real K-cohomology of $B\pi$. The group $\widetilde{KO}_n(B\pi)$ is a direct sum of copies of $\mathbb{Z}/2$ and $\mathbb{Z}/2^\infty$. The number of summands in each dimension n is determined by the irreducible complex representations of π as follows. Each irreducible representation of complex type (*i.e.*, a representation not isomorphic to its complex conjugate) contributes a copy of $\mathbb{Z}/2^\infty$ for n odd. A non-trivial irreducible representation of real type contributes a $\mathbb{Z}/2^\infty$ in dimensions $n \equiv 3 \bmod 4$ and a $\mathbb{Z}/2$ in dimensions $n \equiv 1, 2 \bmod 8$. Each irreducible representation of quaternionic type contributes a $\mathbb{Z}/2^\infty$ in dimensions $n \equiv 3 \bmod 4$ and a $\mathbb{Z}/2$ in dimensions $n \equiv 5, 6 \bmod 8$.

We will need the following fact; see [26].

Proposition 2.7. *The assembly map $A\colon \widetilde{KO}_*(B\pi) \to KO_*(C_r^*\pi)$ for a finite 2-group π is injective on the $\mathbb{Z}/2$-summands.*

Lemma 2.8. *For $n \equiv 1, 2$ mod 8, $n \geq 1$, there are elements in $\widetilde{ko}_n(BC_l)$, which map to indivisible elements in $\widetilde{KO}_n(BC_l)$.*

This shows that for $n \equiv 1, 2 \bmod 8$, $n \geq 1$ the kernel of $A \circ p$ from $\widetilde{ko}_n(BC_l)$ to $KO_n(C_r^*C_l)$ is a subgroup of index at least 2. This implies Theorem 2.3 in these cases. To prove the result for $\pi = Q_l$, $n \not\equiv 3 \bmod 4$, we recall that all the irreducible complex representations of the quaternion group Q_l, $l = 2^k$, are all of real or quaternionic type. Hence the assembly map for Q_l is injective in degrees $n \not\equiv 3 \bmod 4$, which implies that $\widetilde{\mathcal{Y}}_n(BQ_l) = \ker\left(p\colon \widetilde{ko}_n(BQ_l) \to \widetilde{KO}_n(BQ_l)\right)$ for $n \not\equiv 3 \bmod 4$. Theorem 2.3 for $\pi = Q_l$ in degrees $n \not\equiv 3 \bmod 4$ follows from the next lemma.

Lemma 2.9. *The kernel of $p\colon \widetilde{ko}_n(BQ_l) \to \widetilde{KO}_n(BQ_l)$ is trivial for $n \equiv 2$ mod 4; for $n \equiv 0$ mod 4, $\ker p$ is equal to the image of the map*

$$\eta\colon \widetilde{ko}_{n-1}(BQ_l) \to \widetilde{ko}_n(BQ_l)$$

given by multiplication by the generator $\eta \in ko_1 = \mathbb{Z}/2$; for $n \equiv 1$ mod 4, $\ker p$ is equal to the image of the map $\widetilde{ko}_{n-2}(BQ_l) \to \widetilde{ko}_n(BQ_l)$ given by multiplication by $\eta^2 \in ko_2 = \mathbb{Z}/2$.

3. *ko*-homology calculations

The goal of this section is to provide the necessary information about the connective KO-homology of the classifying spaces of cyclic and quaternion groups. We do not attempt to calculate these groups, but rather content ourselves with extracting all the information necessary for the proof of Theorem 2.3. The method we use is the Atiyah-Hirzebruch spectral sequence, and for the convenience of the reader we recall its basic properties.

Atiyah-Hirzebruch spectral sequence. Let X be a CW-complex. The Atiyah-Hirzebruch spectral sequence converging to $\widetilde{ko}_*(X)$ has the following form:

$$E^2_{p,q}(X) = \widetilde{H}_p(X; ko_q) \Longrightarrow \widetilde{ko}_{p+q}(X). \tag{3.1}$$

We recall, for $q \geq 0$, that $ko_q = \mathbb{Z}$ if $q \equiv 0 \bmod 4$, that $ko_q = \mathbb{Z}/2$ if $q \equiv 1, 2 \bmod 8$, and that $ko_q = 0$ otherwise.

It is well-known (cf. [1, Proof of Lemma 5.6] for the dual statement concerning KO-cohomology) that for $q \equiv 1 \bmod 8$ the differential

$$d^2 \colon E^2_{p,q}(X) = \widetilde{H}_p(X; \mathbb{Z}/2) \to E^2_{p-2,q+1}(X) = \widetilde{H}_{p-2}(X; \mathbb{Z}/2) \tag{3.2}$$

is the map dual to the cohomology operation

$$Sq^2 \colon H^{p-2}(X; \mathbb{Z}/2) \to H^p(X; \mathbb{Z}/2).$$

Similarly, for $q \equiv 0 \bmod 8$, the differential

$$d^2 \colon E^2_{p,q}(X) = \widetilde{H}_p(X; \mathbb{Z}) \to E^2_{p-2,q+1}(X) = \widetilde{H}_{p-2}(X; \mathbb{Z}/2) \tag{3.3}$$

is mod 2 reduction composed with the homomorphism dual to Sq^2.

Multiplicative Structure. We remark that

$$E^r_{*,*}(X) = \bigoplus_{p,q} E^r_{p,q}(X)$$

is a graded module over the graded ring

$$ko_* = \bigoplus_n ko_n = \mathbb{Z}[\eta, \omega, \mu]/(\eta^3, 2\eta, \eta\omega, \omega^2 - 4\mu),$$

where η, ω, μ are elements of degree 1, 4, 8 respectively. On

$$E^2_{*,*}(X) = \widetilde{H}_*(X; ko_*)$$

this module structure comes from the action of ko_* on the coefficients. On the E^∞-term, the module structure is induced from the ko_*-action on $\widetilde{ko}_*(X)$. These module structures are compatible with the differentials (*i.e.*, $d^r(xy) = xd^r(y)$ for $x \in ko_*$ and $y \in E^r$), and the module structure on E^r induces the module structure on E^{r+1}.

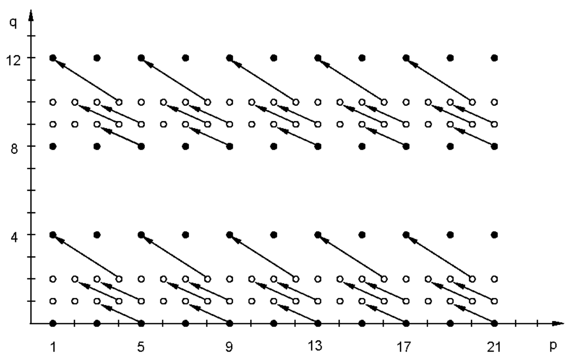

FIGURE 1. $\widetilde{H}_p(BC_l; ko_q) \Longrightarrow \widetilde{ko}_{p+q}(BC_l)$

Homology of BC_l. Let l be a power of 2. Then $\widetilde{H}_p(BC_l;\mathbb{Z}) = \mathbb{Z}/l$ for p odd, $\widetilde{H}_p(BC_l;\mathbb{Z}) = 0$ for p even, and $\widetilde{H}_p(BC_l;\mathbb{Z}/2) = \mathbb{Z}/2$ for all $p \geq 1$. We claim that the homomorphism

$$\widetilde{H}_p(BC_l;\mathbb{Z}/2) \to \widetilde{H}_{p-2}(BC_l;\mathbb{Z}/2) \tag{3.4}$$

dual to Sq^2 is non-trivial if and only if $p \equiv 0, 1 \bmod 4$, $p \geq 4$.

To see this we recall that $H^*(BC_l;\mathbb{Z}/2)$ is the polynomial algebra $\mathbb{Z}/2[x]$ for $l = 2$, and the tensor product $\Lambda(x) \otimes \mathbb{Z}/2[y]$ of an exterior algebra $\Lambda(x)$ and a polynomial algebra for $l = 2^k$, $k > 1$. Here x is a 1-dimensional generator, and y is a 2-dimensional generator. Using $Sq^1(y) = 0$ and the Cartan formula, we conclude that $Sq^2\colon H^{p-2}(BC_l;\mathbb{Z}/2) \to H^p(BC_l;\mathbb{Z}/2)$ is non-trivial if and only if $p \equiv 0, 1 \bmod 4$, $p \geq 4$.

This homology information leads to the following picture of the Atiyah-Hirzebruch spectral sequence converging to $\widetilde{ko}_*(BC_l)$; see Figure 1.

Here a black dot at the location (p,q) means that $E^2_{p,q}$ is isomorphic to $\mathbb{Z}/l$, while a white dot means $E^2_{p,q} \cong \mathbb{Z}/2$. The arrows in the picture represent non-trivial differentials originating and terminating in the range shown. The d^2-differentials are determined as explained above, for the d^3-differentials we use the following argument.

d^3-differentials. We observe that for dimensional reasons the only possibly non-trivial d^3-differentials are

$$d^3\colon E^3_{p,q}(BC_{2^k}) \to E^3_{p-3,q+2}(BC_{2^k}) \tag{3.5}$$

for $p \equiv 0 \bmod 4$, $p \geq 4$, $q \equiv 2 \bmod 8$, and we claim that these are in fact non-trivial. For $k = 1$, $p = 4$, $q = 2$, the differential (3.5) has to be non-zero, since otherwise the non-trivial element in $E^3_{4,2}$ would survive to the E^∞-term contradicting $\widetilde{ko}_6(BC_2) = 0$ (cf. [19, Lemma 7.3]).

To settle the differentials for $k = 1$, $p = 4m+4$, and $q = 2$, we replace $BC_2 = \mathbb{RP}^\infty$ by the quotient $\mathbb{RP}^\infty/\mathbb{RP}^{4m-1}$, which does not affect the differential (3.5). We note that this quotient is homeomorphic to the Thom space of the Whitney sum of $4m$ copies of the Hopf line bundle over $\mathbb{RP}^\infty$. This is a spin bundle which implies a homotopy equivalence between $ko \wedge \mathbb{RP}^\infty/\mathbb{RP}^{4m-1}$ and $ko \wedge \Sigma^{4m}\mathbb{RP}^\infty_+$. It follows that the differential (3.5) can be identified with the differential $d^3\colon E^3_{4,2}(BC_2) \to E^3_{1,4}(BC_2)$, and hence is non-trivial.

Since the differentials in the Atiyah-Hirzebruch spectral sequence are compatible with multiplication by elements in ko_*, this implies that the differential (3.5) is non-trivial for $k = 1$, $p \equiv 0 \bmod 4$, $q \equiv 2 \bmod 4$.

Finally, to prove that the differential (3.5) is non-trivial for any k, we consider the map $f\colon BC_2 \to BC_{2^k}$ induced by the inclusion $C_2 \to C_{2^k}$. In even dimensions, the induced map $f_*\colon H_*(BC_2;\mathbb{Z}/2) \to H_*(BC_{2^k};\mathbb{Z}/2)$ is an isomorphism; in odd dimensions $f_*\colon H_*(BC_2;\mathbb{Z}) \to H_*(BC_{2^k};\mathbb{Z})$ is non-trivial. This yields by naturality of the spectral sequence that the differential (3.5) is non-trivial for all k.

Proof of Theorem 2.4 for cyclic groups. Inspection of the picture shows $E^4_{p,q}$ is trivial for $p+q$ even, except for the term $E^4_{1,q} = \mathbb{Z}/2$ with $q \equiv 1 \bmod 8$. Hence all differentials d_r are trivial for $r > 3$ for dimensional reasons except differentials with range $E^r_{1,q}$, $q \equiv 1 \bmod 8$. However, multiplication by η acts trivially on $E^r_{p,q}$, $r > 3$, $p > 1$, but non-trivially on $E^r_{1,q}$. Hence the multiplicative structure as defined in §3 forces these differentials to be zero.

Hence the order of the group $\widetilde{ko}_n(BC_l)$ can be read off from the picture above by multiplying the orders of the groups $E^\infty_{p,q} = E^3_{p,q}$ on the line $p+q = n$. Inspection leads to us to see $|\widetilde{ko}_n(BC_l)| = \epsilon(n, C_l)$ and proves Theorem 2.4 for the cyclic group. q.e.d.

Proof of Lemma 2.8. Let $x \in \widetilde{ko}_1(BC_l)$ be the generator. We claim that the elements $\mu^j\eta^i x \in \widetilde{ko}_{8j+i+1}(B\pi)$, $i = 0, 1$, $j = 0, 1, \ldots$ map to indivisible elements in $\widetilde{KO}_n(BC_l)$. Clearly these elements are non-zero, since in the E^∞-term of the spectral sequence, they represent the generators of $E^\infty_{1,8j+i}$. The same is true in the Atiyah-Hirzebruch spectral sequence converging to $\widetilde{KO}_*(BC_l)$ (which is obtained by extending the picture of the spectral sequence above by periodicity to negative q-values), and hence these elements map non-trivially to $\widetilde{KO}_*(BC_l)$. Furthermore, $p(\mu^j\eta x) \in \widetilde{KO}_{8j+2}(B\pi)$ is indivisible, since $\widetilde{KO}_{8j+2}(B\pi) \cong \mathbb{Z}/2$, and $p(\mu^j x) \in \widetilde{KO}_{8j+1}(B\pi)$ is indivisible since otherwise its product with η would be zero. q.e.d.

A stable splitting of BQ_l. Let $SL_2(\mathbb{F}_q)$ be the group of 2 by 2 matrices with determinant 1 over the field $\mathbb{F}_q$ of order q. It turns out that for q odd the 2-Sylow subgroup of $SL_2(\mathbb{F}_q)$ is Q_{2^k}, where k the exponent of 2 in the prime factor decomposition of q^2-1. A standard transfer argument shows that localized at the prime 2, $BSL_2(\mathbb{F}_q)$ is a stable summand of BQ_{2^k}. Mitchell and Priddy [20] identify the other summands; they show that there is a stable, 2-local splitting

$$BQ_{2^k} \cong BSL_2(\mathbb{F}_q) \vee \Sigma^{-1}BS^3/BN \vee \Sigma^{-1}BS^3/BN, \tag{3.6}$$

where $N \subset S^3$ is the normalizer of $S^1 \subset S^3$.

To calculate the order of $\widetilde{ko}_n(BQ_l)$, we make use of the splitting (3.6) and deal with each summand separately. The connective K-theory of

$$\Sigma^{-1}BS^3/BN$$

is known. We refer to Bayen and Bruner [5] for the proof of the following result.

Theorem 3.1.
1) $ko_n(\Sigma^{-1}BS^3/BN) = \mathbb{Z}/2$ *if* $n = 8j+1$ *or* $n = 8j+2$.
2) $ko_n(\Sigma^{-1}BS^3/BN) = \mathbb{Z}/2^{2j+2}$ *if* $n = 8j+3$ *or* $n = 8j+7$.
3) $ko_n(\Sigma^{-1}BS^3/BN) = \mathbb{Z}/2$ *otherwise.*

Proof of Theorem 2.4 for quaternion groups. Without further mention, all groups in the proof are localized at 2. We have

$$\widetilde{H}_n(BSL_2(\mathbb{F}_q);\mathbb{Z}) = \mathbb{Z}/l$$

if $n \equiv 3 \bmod 4$ and is 0 otherwise. Hence the E^2-term of the Atiyah-Hirzebruch spectral sequence converging to $\widetilde{ko}_*(BSL_2(\mathbb{F}_q))$ looks as follows, where as above a black dot at the location (p,q) means that $E^2_{p,q}$ is isomorphic to $\mathbb{Z}/l$, while a white dot means $E^2_{p,q} \cong \mathbb{Z}/2$.

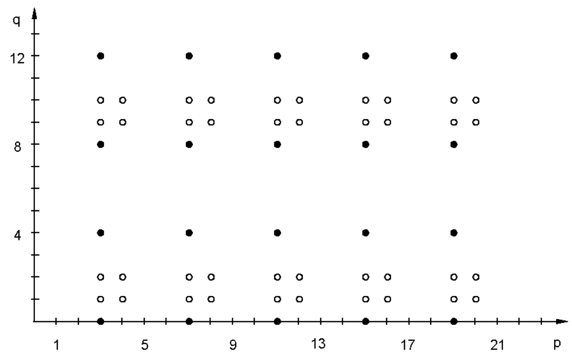

FIGURE 2. The E^2-term of the spectral sequence
$\widetilde{H}_p(BSL_2(\mathbb{F}_q); ko_q) \Longrightarrow \widetilde{ko}_{p+q}(BSL_2(\mathbb{F}_q))$

We note that for dimensional reasons, there are no non-trivial d_r differentials originating in or mapping to a group $E^r_{p,q}$ for $p+q \equiv 3 \bmod 4$ (there are non-trivial differentials in other total degrees!). Hence by multiplying the orders of the groups $E^\infty_{p,q} = E^3_{p,q}$ on the line $p+q = 4i+3$ we see that $\widetilde{ko}_{4i+3}(BQ_l)$ has order l^{i+1}. Together with Theorem 3.1 this implies Theorem 2.4 for the quaternion group. q.e.d.

Proof of Lemma 2.9. Let x be a non-trivial element in the kernel of $p\colon \widetilde{ko}_n(BQ_l) \to \widetilde{KO}_n(BQ_l)$ for $n \not\equiv 3 \bmod 4$. Assume that x has skeletal filtration p, which means $x \in F_p$, but $x \notin F_{p-1}$, where F_p is the image of the map $\widetilde{ko}_n(BQ_l^{(p)}) \to \widetilde{ko}_n(BQ_l)$ induced by the inclusion of the p-skeleton of BQ_l. Then x projects to a non-zero element $\bar{x} \in E^\infty_{p,n-p} = F_p/F_{p-1}$. We note that this implies $p \equiv 3 \bmod 4$, since if $\bar{x}$ were an element in the only other non-zero group $E^\infty_{p,n-p}$, $p \equiv 0 \bmod 4$, then $\bar{x}$ would map to a non-zero element in the Atiyah-Hirzebruch spectral sequence converging to periodic K-theory $\widetilde{KO}_n(BQ_l)$ (for dimensional reasons, elements in $E^r_{p,n-p}$, $p \equiv 0 \bmod 4$ of that spectral sequence can never be hit by a differential). For $n \equiv 2 \bmod 4$, $p \equiv 3 \bmod 4$ the vanishing of $E^\infty_{p,n-p}$ implies $x \in F_{p-1}$ contradicting our assumption. For $n \equiv 0 \bmod 4$ (resp. $n \equiv 1 \bmod 4$) the multiplicative structure of the spectral sequence implies that modulo elements in F_{p-1} the element x is in the image of η (resp. η^2). Inductively it follows that the kernel of

p is equal to the image of η (resp. η^2) for $n \equiv 0 \mod 4$ (resp. $n \equiv 1 \mod 4$). q.e.d.

4. Analytic results concerning the eta invariant

The goal of this section is twofold: on one hand we want to show that the eta invariant leads to a well defined homomorphism

$$\eta(\rho)\colon KO_n(B\pi) \to \mathbb{R}/\mathbb{Z} \quad \text{resp. } \mathbb{R}/2\mathbb{Z} \tag{4.1}$$

for n odd (cf. 2.3). Here ρ is a virtual representation of a discrete group π of virtual dimension zero, and the range of $\eta(\rho)$ depends on $n \bmod 8$, and on the 'type' of the representation; the eta invariant takes values in $\mathbb{R}/2\mathbb{Z}$ if $n \equiv 3 \bmod 8$ and ρ is of real type or if $n \equiv 7 \bmod 8$ and ρ is of quaternion type, and it takes values in $\mathbb{R}/\mathbb{Z}$ otherwise. On the other hand, we want to show how the eta invariant of a pair (M, f) consisting of a spin manifold M and a map $f\colon M \to B\pi$ can be computed in terms of the fixed point data of a π-manifold W whose boundary is $\widetilde{M}$, the π-covering of M determined by f. We begin by reviewing Dirac operators and the Atiyah-Singer Index Theorem for manifolds with boundary.

Dirac operator. Let M be an n-dimensional spin manifold and let S be the (complex) spinor bundle over M. The *Dirac operator* is a first order differential operator $D(M)\colon \Gamma(S) \to \Gamma(S)$ acting on the space $\Gamma(S)$ of sections of the spinor bundle (a general reference for the construction of D is the book of Lawson and Michelsohn [17], where this operator is called the "Atiyah-Singer operator"). More generally, given a vector bundle E over M with a unitary connection, one can 'twist' the Dirac operator by E to get a self-adjoint elliptic operator $D(M) \otimes E\colon \Gamma(S \otimes E) \to \Gamma(S \otimes E)$. If the dimension of M is even, the spinor bundle decomposes $S = S^+ \oplus S^-$. With this decomposition, the twisted Dirac operator $D(M) \otimes E$ can be written in the form $D = D^+ + D^-$, where $D^\pm(M) \otimes E\colon \Gamma(S^\pm \otimes E) \to \Gamma(S^\mp \otimes E)$.

Index Theorem. Assume M is the boundary of an even dimensional spin manifold W over which the bundle E extends. Atiyah, Patodi and Singer showed that $D^+(W) \otimes E$ is a Fredholm operator if one imposes a certain (global) boundary condition. In particular, the index is well defined:

$$\operatorname{index}(D^+(W) \otimes E) := \dim\ker D^+(W) \otimes E - \dim\operatorname{coker} D^+(W) \otimes E.$$

We use the Index Theorem [2]:

$$\operatorname{index}(D^+(W)\otimes E) = \textstyle\int_W \hat{A}(W)ch(E) - \eta(D(M)\otimes E).$$

Here the integrand is a certain polynomial in the Pontrjagin forms of W and the Chern forms of E, and $\eta(D(M)\otimes E)$ is the *eta-invariant* which is a measure for the asymmetry of the spectrum of the operator $D(M)\otimes E$ with respect to the origin.

Definition of $\eta(M,f)(\rho)$. Let ρ be a finite dimensional complex representation of a discrete group π. We assume that ρ is unitary henceforth to ensure the resulting operators are self-adjoint; this assumption is automatic for finite groups of course. Let M be a Riemannian spin manifold of odd dimension n, and let $f\colon M \to B\pi$ be a map. The map f and the representation ρ determine a flat vector bundle $V_\rho := \widetilde{M}\times_\pi \rho$, where $\widetilde{M}\to M$ is the π-covering of M classified by f (*i.e.*, the pull back of the universal covering $E\pi \to B\pi$ via f). We denote by $D(M,f,\rho)$ the Dirac operator on M twisted by V_ρ, and by $\eta(M,f)(\rho)$ its eta invariant. Often, the map f is understood and we will just write $\eta(M)(\rho)$. It is clear from the definition that $\eta(M,f)(\rho)$ is additive in ρ, and hence its definition can be extended to virtual representations.

We would like to show that the eta-invariant gives rise to a homomorphisms from $KO_*(B\pi)$ to $\mathbb{R}/\mathbb{Z}$. To do this, we need the following.

Geometric interpretation of $KO_n(X)$. The homomorphism pD discussed in §2 induces an isomorphism [14]

$$\left(\Omega_*^{spin}(X)/T_*(X)\right)[B^{-1}] \cong KO_*(X);$$

here $T_n(X)$ is the subgroup of $\Omega_n^{spin}(X)$ represented by pairs $(M, f\,q)$, $q\colon M\to N$ is an $\mathbb{HP}^2$-bundle over a $(n-8)$-dimensional spin manifold N, and $f\colon N\to X$ is a map. By $\mathbb{HP}^2$-bundle we mean a fiber bundle whose fiber is the quaternionic projective plane $\mathbb{HP}^2$ and whose structure group is the isometry group of $\mathbb{HP}^2$ with its standard metric. The Bott manifold $B = B^8$ is any simply connected spin manifold of dimension 8 with $\hat{A}(B)=1$. By definition, $\left(\Omega_*^{spin}(X)/T_*(X)\right)[B^{-1}]$ is the direct limit of the sequence given by Cartesian product with the Bott manifold B:

$$\Omega_*^{spin}(X)/T_*(X) \to \Omega_*^{spin}(X)/T_*(X) \to \Omega_*^{spin}(X)/T_*(X) \to \dots .$$

Theorem 4.1. *Let ρ be a virtual representation of π of virtual dimension* 0*. Then the homomorphisms*

$$\eta(\rho)\colon \Omega_n^{spin}(B\pi) \to \mathbb{R}/\mathbb{Z} \text{ and } \eta(\rho)\colon KO_n(B\pi) \to \mathbb{R}/\mathbb{Z}$$

which send a class represented by a spin manifold M of dimension n (resp. $n + 8k$), and by a map $f\colon M \to B\pi$ to $\eta(M,f)(\rho)$, are well defined. Moreover, if ρ is of real type and $n \equiv 3$ mod 8, or if ρ is of quaternionic type and $n \equiv 7$ mod 8, then we can replace the range of $\eta(\rho)$ by $\mathbb{R}/2\mathbb{Z}$.

For the proof of this result, we need to show:
1) $\eta(M,f)(\rho)$ depends only on the spin bordism class of (M,f),
2) $\eta(M,f)(\rho) = 0$ for $[M,f] \in T_*(B\pi)$, and
3) $\eta(M \times B, f)(\rho) = \eta(M,f)(\rho)$.
This follows from the next three lemmas.

Lemma 4.2. *If (M,f) is the zero element in $\Omega_n^{spin}(B\pi)$, then $\eta(M,f)(\rho) \in \mathbb{Z}$. Moreover, if ρ is of real type and $n \equiv 3$ mod 8, or if ρ is of quaternionic type and $n \equiv 7$ mod 8, then $\eta(M,f)(\rho) \in 2\mathbb{Z}$.*

Proof. Let (W,F) be a zero bordism for (M,f). Then the flat bundle V_ρ extends to a flat bundle V_ρ over W. We note that the Chern character of this bundle vanishes identically, and hence the Index Theorem discussed above implies index $(D^+(W) \otimes V_\rho) = -\eta(M,f)(\rho)$. This proves the first part of the assertion of the lemma. To prove the second part, we show that under those assumptions, the space of sections $\Gamma(S_W^\pm \otimes V_\rho)$ has a quaternionic structure and that the Dirac operator $D^+(W) \otimes V_\rho$ is $\mathbb{H}$-linear, which implies in particular that the (complex) dimension of its kernel and cokernel is even. Let $J_1\colon \rho \to \rho$ be complex conjugation (resp. multiplication by $j \in \mathbb{H}$) if ρ is a representation of real type (resp. quaternionic type). We note that J_1 is $\mathbb{C}$-antilinear, it commutes with the action of the real group ring $\mathbb{R}\pi$, and $J_1^2 = 1$ (resp. $J_1^2 = -1$) if ρ is a real (resp. quaternionic) type. We call J_1 a real (resp. quaternionic) structure on ρ. We claim that the $\mathbb{C}C\ell_n$-module Δ has a real (resp. quaternionic) structure $J_2\colon \Delta \to \Delta$ for $n \equiv 7$ mod 8 and a real structure for $n \equiv 3$ mod 8 (*i.e.*, J_2 is a $\mathbb{C}$-antilinear homomorphism which commutes with the action of the *real* Clifford algebra $C\ell_n$ such that $J_2^2 = \pm 1$). This follows from the fact that $C\ell_n$ is the sum of two matrix algebras over $\mathbb{R}$ (resp. $\mathbb{H}$) for $n \equiv 7$ mod 8 (resp. $n \equiv 3$ mod 8). It follows that $J_1 \otimes J_2\colon \Delta \otimes_C \rho \to \Delta \otimes_C \rho$ is a quaternionic structure if either ρ is of real type and $n \equiv 3$ mod 8, or ρ is of quaternionic type and $n \equiv 7 \bmod 8$. This quaternionic structure induces a quaternionic structure on the space of sections $\Gamma(S_W^{\pm} \otimes V_\rho)$, which makes the Dirac operator $\mathbb{H}$-linear. q.e.d.

Lemma 4.3. *Let $p\colon E \to B$ be a Riemannian submersion with totally geodesic fibers such that the induced metric on the fibers has positive scalar curvature. Let g be the metric on E, and for $t > 0$ let g_t be the canonical variation of g (cf. [6, Ch. 9 G]), i.e., the new metric obtained from g by rescaling in fiber direction by multiplication by t. Let $f\colon B \to B\pi$ be a map and let $\eta(E_t, f \circ p)(\rho)$ be the eta invariant with respect to the metric g_t. Then*

$$\lim_{t\to 0} \eta(E_t, f \circ p)(\rho) = 0.$$

Proof. Let $\rho = \sigma_1 - \sigma_0$ where σ_0 and σ_1 are actual representations of the group π of the same dimension. Let ξ_0^B and ξ_1^B be the vector bundles over B corresponding to the representations σ_0 and σ_1. Let ξ_0^E and ξ_1^E be the pull back of these bundles to E. We give these bundles the natural flat unitary connections.

The restriction of ξ_0^E and ξ_1^E to any fiber F_x is trivial since these bundles arise from a π structure on the base. Since the scalar curvature of F_x is positive, the kernel of the Dirac operator on F_x with coefficients in ξ_ν^E is trivial. Let D_ν^E be the Dirac operator on the total space E with coefficients in ξ_ν^E for $\nu = 0, 1$. We apply the Adiabatic limit theorem of Bismut and Cheeger [7, (0.5)]. (Note that Bismut and Cheeger use a somewhat different convention for rescaling the metric than does Besse). We use this result to conclude that $\lim_{t\to 0} \eta(D_\nu^E) = \int_B \hat{A}(iR^B)\tilde{\eta}_\nu$, where $\hat{A}(iR^B)$ is the differential form representing the $\hat{A}$ genus of B obtained from the Chern-Weil homomorphism, and $\tilde{\eta}_\nu$ is an explicit differential form on B whose value at $x \in B$ only depends on global information on F_x and on the horizontal and vertical distributions of the fibration. However, since the restriction of ξ_0^E and ξ_1^E to F_x are trivial bundles of the same dimension, $\tilde{\eta}_0 = \tilde{\eta}_1$. Since the eta invariant is independent of the metric chosen, we complete the proof by checking

$$\lim_{t\to 0} \eta(E_t, f \circ p)(\rho) = \lim_{t\to 0} \{\eta(D_0^E) - \eta(D_1^E)\} = \int_B \hat{A}(iR^B)(\tilde{\eta}_0 - \tilde{\eta}_1) = 0.$$

We note that the standard metric on $\mathbb{HP}^2$ has positive scalar curvature, and that we can equip the total space of an $\mathbb{HP}^2$-bundle $p\colon M \to N$ with a Riemannian metric which makes p a Riemannian submersion with totally geodesic fibers such that the induced metric on the fibers is the standard metric. Hence the previous lemma implies that the η-invariant vanishes on $T_*(B\pi)$. We refer to [4, p.84] for the proof of the following result.

Lemma 4.4. *Let N be an even dimensional spin manifold and let M be an odd dimensional spin manifold with a given map $f\colon M^n \to B\pi$. Then for any virtual representation ρ of virtual dimension 0, we have that*

$$\eta(M \times N)(\rho) = \widehat{A}(N)\eta(M)(\rho).$$

Remark 4.5. If the π structure on M is trivial, then the eta invariant vanishes. Thus the eta invariant is carried by the reduced theories in Theorem 4.1.

There is a result of H. Donnelly [9] which permits us to compute the eta invariant in terms of Dedekind sums. Let π be a finite group and let ϕ be an action of π on an odd dimensional spin manifold M without boundary by fixed point free isometries. Let ρ be a representation of π. Let $\bar{M} = M/\pi$ with the induced structures. Let $E(\lambda, M)$ be the eigenspaces of the Dirac operator on M and $E(\lambda, \bar{M})$ be the eigenspaces of the Dirac operator on $\bar{M}$. The eigenspaces $E(\lambda, M)$ inherit a natural π action and we may identify $E(\lambda, \bar{M})$ with $E(\lambda, M)^\pi$. Consequently

$$\dim(E(\lambda, \bar{M})) = |\pi|^{-1} \sum_{g \in \pi} \mathrm{Tr}(\phi(g)) \text{ on } E(\lambda, M),$$

so $\eta(\bar{M}) = |\pi|^{-1} \sum_{g\in\pi} \eta(\phi(g))$ where we define

$$\begin{aligned} 2\eta(\phi(g)) = & \sum_{\lambda \neq 0} (\mathrm{Tr}(\phi(g)) \text{ on } E(\lambda, M)) sign(\lambda) |\lambda|^{-s}|_{s=0} \\ & + \quad (\mathrm{Tr}(\phi(g)) \text{ on } E(0, M)). \end{aligned}$$

A similar argument shows that if we twist by representation of π then

$$\eta(\bar{M})(\rho) = |\pi|^{-1} \sum_{g \in \pi} \mathrm{Tr}(\rho(g))\eta(\phi(g)). \tag{4.2}$$

Let $R(\pi)$ be the group representation ring of a finite group π and let $R_0(\pi)$ be the augmentation ideal of all virtual representations of virtual dimension 0. If we restrict to $\rho \in R_0(\pi)$, then we can assume that $g \neq 1$ in equation (4.2). We shall assume henceforth that the metric on M

has positive scalar curvature so that there are no harmonic spinors and $E(0, M) = \{0\}$; this will be crucial for our analysis.

We now assume that M is the boundary of a compact spin manifold N and that the action ϕ of the group π extends to an action Φ of π by spin isometries on N. We do not, however, assume that $\Phi(g)$ is fixed point free for $g \neq 1$. We assume that the metric on N has positive scalar curvature and that M is totally geodesic. We wish to compute $\eta(\phi(g))$ in terms of the fixed points of the action on N.

Suppose for $g \neq 1$ that $\Phi(g)$ has isolated fixed points in the interior of N. At such a fixed point x, $\det(I - d\Phi(g))(x) \neq 0$. We denote the rotation angles of $d\Phi(g)(x)$ by $\theta_\nu \in (0, 2\pi)$ and define the defect relative to the spin complex by

$$\operatorname{def}(\Phi(g), \operatorname{spin})(x) = (2i)^{-1} \prod_\nu \operatorname{cosec}(\theta_\nu/2);$$

since cosec is not periodic mod π, $\operatorname{def}(\Phi(g), \operatorname{spin})(x)$ depends on the lift to spin. If $\Phi(g)$ in fact arises from a complex matrix with complex eigenvalues $\{\lambda_\nu\}$, then

$$\operatorname{def}(\Phi(g), \operatorname{spin})(x) = \frac{\{\prod_\nu \lambda_\nu\}^{1/2}}{\prod_\nu (1 - \lambda_\nu)}; \tag{4.3}$$

the choice of a square root in equation (4.3) in this instance is equivalent to a lift from the unitary to the spin group; see Hitchin [12] for details.

Donnelly [9] has generalized the index formula of Atiyah, Patodi, and Singer [2], [3], [4] to the equivariant setting. Since the metric on the boundary $\partial N = M$ has positive scalar curvature, the Lichnerowicz-Weitzenböck formula [18], shows there are no harmonic spinors on M. Since the metric on N also has positive scalar curvature, a similar similar argument shows there are no harmonic spinors on N; see [8] for details. Consequently, the Lefschetz fixed point formula for manifolds with boundary becomes in this setting for $g \neq 1$ (see [3, Section 2]):

$$\eta(\phi(g)) = \sum_i \operatorname{def}(\Phi(g), \operatorname{spin})(x_i). \tag{4.4}$$

Let $\tau : \pi \to U(m)$ be a fixed point free representation. Let M be the quotient manifold $S^{2m-1}/\tau(\pi)$. The standard metric on S^{2m-1} descends to define a metric of constant sectional curvature $+1$ on M, and every compact odd dimensional manifold admitting such a metric arises in this way. These are the odd dimensional spherical space forms; the sphere and real projective spaces are the only even dimensional spherical space forms and will play no role in our discussion.

The stable tangent bundle $T(M) \oplus 1$ is naturally isomorphic to the underlying real vector bundle of the complex vector bundle defined by τ over M, and thus $T(M)$ admits a stable almost complex structure. The lift described by Hitchin of $U(m)$ to the group $\mathrm{spin}^c(2m)$ provides M with a natural spin^c structure. This structure can be reduced to a spin structure if and only if one can take the square root of the associated determinant line bundle or equivalently if and only if we can take the square root of the linear representation $\det(\tau)$. This can also be expressed in topological terms. The second Stiefel-Whitney class of M is the mod 2 reduction of the first Chern class of the determinant line bundle. This vanishes if and only if $c_1(\det(\tau))$ is divisible by 2 or equivalently if it is possible to take the square root of the determinant line bundle. Since every line bundle over a spherical space form arises from a linear representation, this is possible if and only if we can take the square root of the representation $\det(\tau)$.

If $|\pi|$ is odd, this is always possible; if $|\pi|$ is even, this is possible if and only if m is even, *i.e.*, the dimension of M is congruent to 3 mod 4. In this case, there are inequivalent spin structures on M, and the choice of a spin structure is equivalent to the choice of the square root of $\det(\tau)$. This square root plays an important role in the formula for the eta invariant in the following theorem.

Theorem 4.6. *Let $\rho \in R_0(\pi)$ and let $\tau : \pi \to U(m)$ be a fixed fixed point free representation. Assume that there exists a representation of π, which we denote by $\det(\tau)^{1/2}$, whose tensor square is the representation $\det(\tau)$. Let $M = S^{2m-1}/\tau(\pi)$ with the inherited structures. Then*

$$\eta(M)(\rho) = |\pi|^{-1} \sum_{g\in\pi, g\neq 1} \frac{Tr(\rho(g))\det(\tau(g))^{1/2}}{\det(I-\tau(g))}.$$

Proof. We regard S^{2m-1} as the boundary of the upper hemisphere S^{2m}_+ of the sphere S^{2m} and extend the orthogonal action to S^{2m}_+ with an isolated fixed point at the north pole. Theorem 4.6 then follows from the equations (4.2), (4.3), and (4.4). q.e.d.

If the fixed submanifold is higher dimensional, the defect formula is a bit more complicated. We assume that the fixed point set of $\Phi(g)$ in N is a Riemann surface X and that the normal bundle ν_X decomposes as a direct sum of complex line bundles $\nu_X = H_1 \oplus ... \oplus H_k$. We further assume that the action of $d\Phi(g)$ on ν_X respects this decomposition so $d\Phi(g)_{\nu_X}$ is the diagonal matrix $\text{diag}(\lambda_1, ..., \lambda_k)$; the eigenvalues are constant and independent of the point of X which is chosen.

The spin complex is multiplicative with respect to products. If H_1 is non-trivial, and all the other line bundles are trivial, then the defect in the Atiyah-Singer index theorem is given by:

$$\text{def}(\Phi(g), \text{spin})_{\nu_X} = \frac{\{\prod_i \lambda_i\}^{1/2}}{\prod_i (1-\lambda_i)} \cdot \mathcal{F}(\lambda_1) c_1(H_1)[X];$$

see equation (4.3), where $\mathcal{F}$ is some universal function from $S^1 - \{1\}$ to $\mathbb{C}$ which is independent of all the choices made. This is easily seen from standard heat equation proofs of the Lefschetz fixed point formulas or also directly from the equivariant index theorem.

On the other hand, since X is 2-dimensional, the Chern classes do not interact and thus the formula in general can be written in the form

$$\text{def}(\Phi(g), \text{spin})_{\nu_X} = \frac{\{\prod_i \lambda_i\}^{1/2}}{\prod_i (1-\lambda_i)} \cdot \sum_i \mathcal{F}(\lambda_i) c_1(H_i)[X]. \tag{4.5}$$

We apply this observation as follows. Let $\vec{a} = (a_1, ..., a_k)$ be a collection of integers coprime to l. Let M be the total space of the sphere bundle $S(H_1 \oplus \cdots \oplus H_k)$ where H_i are complex line bundles over S^2; M is spin if and only if $c_1(H_1) + ... + c_1(H_k)$ is even; assume this condition henceforth. Let

$$\tau(\vec{a})(\lambda) = \text{diag}(\lambda_1^{a_1}, ..., \lambda_k^{a_k})$$

define a fixed point free action of the cyclic group C_l on M, and let

$$M(l; \vec{a}; H_1 \oplus \cdots \oplus H_k) := S/\tau(\vec{a})(C_l).$$

We shall be interested in the case l even. The same discussion as that given for spherical space forms shows that these lens space bundles are spin if and only if k is even. Again, the spin structure is determined by the square root of the defining representation.

Theorem 4.7. *Let $\rho \in R_0(\pi)$. The following formula holds:*

$$\begin{aligned}\eta(M(l; \vec{a}; H_1 \oplus \cdots \oplus H_{2i}))(\rho) = & \frac{1}{l} \sum_{\lambda^l = 1, \lambda \neq 1} \text{Tr}(\rho) \frac{\lambda^{(a_1 + \cdots + a_{2i})/2}}{(1-\lambda^{a_1}) \cdots (1-\lambda^{a_{2i}})} \\ & \cdot \sum_j \frac{1}{2} c_1(H_j)[\mathbb{CP}^1] \frac{1+\lambda^{a_j}}{1-\lambda^{a_j}}.\end{aligned}$$

Proof. We take D to be the disk bundle of $H_1 \oplus \cdots \oplus H_{2i}$ with a suitable metric. We stated the Lefschetz fixed point formula in equation (4.4) for isolated fixed points, however it holds in a more general context. We use equations (4.2), (4.4), and (4.5) to see that

$$\eta(M(l;\vec{a};H_1 \oplus \cdots \oplus H_{2i}))(\rho) = \frac{1}{l} \sum_{\lambda^l=1,\lambda\neq 1} \mathrm{Tr}(\rho) \frac{\lambda^{(a_1+\cdots+a_{2i})/2}}{(1-\lambda^{a_1})\cdots(1-\lambda^{a_{2i}})}$$

$$\cdot \sum_j c_1(H_j)[\mathbb{CP}^1]\mathcal{F}(\lambda_j).$$

We apply this to the special case $k=2$, $a_1 = a_2 = 1$ to see

$$\eta(M(l;1,1;H \oplus H)) = \frac{1}{l} \sum_{\lambda^l=1,\lambda\neq 1} \mathrm{Tr}(\rho) \frac{\lambda}{(1-\lambda)^2} 2\tilde{\mathcal{F}}(\lambda)$$

where H is the Hopf bundle. On the other hand, we computed (see equation (2.12) of [8]) that

$$\eta(M(n;1,1;H \oplus H)) = \frac{1}{l} \sum_{\lambda^l=1,\lambda\neq 1} \mathrm{Tr}(\rho) \frac{\lambda(1+\lambda)}{(1-\lambda)^3}.$$

We compare these two equations to see $\tilde{\mathcal{F}}(\lambda) = (1+\lambda)/2(1-\lambda)$. q.e.d.

5. The range of the eta invariant for cyclic groups

In this section, we prove Theorem 2.5 for cyclic groups, see Proposition 5.1 below. It then follows that the eta invariant completely detects the connective K-theory of cyclic groups in odd dimensions. We adopt the following notational conventions. Let C_l be the cyclic group of order $l = 2^k$, which we will identify with the subgroup of S^1 consisting of l-th roots of unity, *i.e.*,

$$C_l = \{\lambda \in S^1 | \lambda^l = 1\}.$$

Given an integer a, let ρ_a be the complex 1-dimensional representation of S^1, where $\lambda \in S^1$ acts by multiplication by λ^a. Given a tuple of integers $\vec{a} = (a_1, \ldots, a_t)$, we note that the representation $\rho_{a_1} \oplus \ldots \oplus \rho_{a_t}$ restricts to a free C_l-action on the unit sphere S^{2t-1} of $\mathbb{C}_t$ if and only if all the a_i's are odd. Let $t = 2i$ be even. The quotient manifold

$$X^{4i-1}(l;\vec{a}) := S^{4i-1}/(\rho_{a_1} \oplus \ldots \oplus \rho_{a_{2i}})(C_l)$$

is a *lens space* of dimension $4i-1$. The discussion of Section 4 shows it inherits a natural spin structure, and we use Theorem 4.6 to compute its eta invariant.

The corresponding lens spaces of dimension $4i+1$ do not admit spin structures. To get spin manifolds of dimension $n \equiv 1 \bmod 4$, we consider lens space bundles; these manifolds also play an important role in [8]. Let H be the (complex) Hopf line bundle over S^2, and let $H^{\otimes 2} \oplus (2i-1)\underline{\mathbb{C}}$ be the Whitney sum of $H \otimes H$ and $2i-1$ copies of the trivial complex line bundle $\underline{\mathbb{C}}$. Given a tuple of integers $\vec{a} = (a_1, \dots, a_{2i})$, we let $\lambda \in S^1$ act on this sum by multiplication by λ^{a_ν} in the ν-th summand. This action restricts to a free action of the cyclic group C_l on the sphere bundle $S(H^{\otimes 2} \oplus (2i-1)\underline{\mathbb{C}})$ if and only if all the a_ν's are odd. The quotient

$$X^{4i+1}(l; \vec{a}) := S\left(H^{\otimes 2} \oplus (2i-1)\underline{\mathbb{C}}\right)/C_l$$

is a manifold of dimension $4i+1$, which fibers over S^2 with fiber $X^{4i-1}(l;\vec{a})$. It is spin. The corresponding lens space bundles in dimensions $4i-1$ are not spin.

There is a natural identification $\pi_1(X^{4i\pm 1}(l;\vec{a})) = C_l$ that gives these manifolds natural C_l structures. Since l is even, these manifolds admit two inequivalent spin structures. We fix the spin structure by defining the square root of the canonical (or determinant) line bundle to be

$$\det(\rho_{a_1} \oplus \dots \oplus \rho_{a_{2i}})^{1/2} := \rho_{(a_1+\dots+a_{2i})/2}.$$

We may then use Theorems 4.6 and 4.7 to compute the eta invariant. In taking the square root of the canonical bundle, we needed the a_ν to be integer valued and not just defined mod l. We give these manifolds the natural metrics of positive scalar curvature.

Let X_0^n be X^n with the trivial C_l structure, and let $\tilde{X}^n := X^n - X_0^n$ define a bordism class $[\tilde{X}^n]$ in $\widetilde{\Omega}_n^{spin}(BC_l)$. Let $\mathcal{M}_*(BC_l) \subseteq \Omega_*^{spin}(BC_l)$ be the Ω_*^{spin}-submodule generated by the elements $[\tilde{X}^n(l;\vec{a})]$; see (2.2). Define the eta invariant

$$\vec{\eta}(M) := (\eta(M)(\rho_1 - \rho_0), \eta(M)(\rho_2 - \rho_0), \dots, \eta(M)(\rho_{l-1} - \rho_0)).$$

Since η extends to maps in bordism and in K-theory, let $\vec{\eta}([M]) = \vec{\eta}(M)$ take values in $\mathbb{R}^{l-1}/\mathbb{Z}^{l-1}$. Let $q = l/2$ and let $\eta_q(M) = \eta(M)(\rho_q - \rho_0)$. If $n \equiv 3 \bmod 8$, the virtual representation $\rho_q - \rho_0$ is real, so by Theorem 4.1, η_q extends to a map in bordism taking values in $\mathbb{R}/2\mathbb{Z}$. It is convenient to separate out the eta invariant with values in $\mathbb{R}/2\mathbb{Z}$; $\check{\eta} = \vec{\eta} + \eta_q$ if

$n = 8j + 3$, and $\check{\eta} = \vec{\eta}$ otherwise (see §2). Let $\mathcal{L}_n(BC_l)$ and $\mathcal{S}_n(BC_l)$ be the range of the eta invariant on $\mathcal{M}_n(BC_l)$ and $\widetilde{\Omega}_n^{spin}(BC_l)$ respectively. If $n \equiv 3 \mod 8$, let $\mathcal{K}_n(BC_l)$ be the range of the extended eta invariant $\vec{\eta} \oplus \eta_q$. That means

$$\mathcal{L}_n(BC_l) := \operatorname{span}\{\vec{\eta}([M]) \;:\; [M] \in \mathcal{M}_n(BC_l)\} \subset (\mathbb{R}/\mathbb{Z})^{l-1},$$

$$\mathcal{S}_n(BC_l) := \operatorname{span}\left\{\vec{\eta}([M]) \;:\; [M] \in \widetilde{\Omega}_n^{spin}(BC_l)\right\} \subset (\mathbb{R}/\mathbb{Z})^{l-1},$$

$$\begin{aligned}\mathcal{K}_{8j+3}(BC_l) := \operatorname{span}\{\vec{\eta}([M]) \oplus \eta_q([M]) \;:\; [M] \in \mathcal{M}_{8j+3}(BC_l)\}\\ \subset (\mathbb{R}/\mathbb{Z})^{l-1} \oplus (\mathbb{R}/2\mathbb{Z}).\end{aligned}$$

Theorem 2.5 and Corollary 2.6 for cyclic 2-groups are immediate consequences of the following result and Theorem 4.1.

Proposition 5.1. *Let $l = 2^k$.*
a) *If $i \geq 1$, then $|\mathcal{L}_{4i+1}(BC_l)| \geq 2^{(i+1)(k-1)}$.*
b) *If $j \geq 0$, then $|\mathcal{S}_{8j+1}(BC_l)| \geq 2^{1+(2j+1)(k-1)}$.*
c) *If $i \geq 2$, then $|\mathcal{L}_{4i-1}(BC_l)| \geq 2^{i(k+1)}$.*
d) *If $j \geq 0$, then $|\mathcal{K}_{8j+3}(BC_l)| \geq 2^{1+(2j+1)(k+1)}$.*

We shall need some technical results to prove Proposition 5.1. Let B^8 be the Bott manifold; B^8 is a simply connected spin manifold with $\hat{A}(B^8) = 1$ which plays a central role in our discussion. If f is a complex function on C_l, let

$$\Sigma_\lambda f(\lambda) := \sum_{\lambda^l=1} f(\lambda) \quad \text{and} \quad \tilde{\Sigma}_\lambda f(\lambda) := \sum_{\lambda^l=1,\lambda\neq 1} f(\lambda)$$

be the sum of f over C_l and the reduced sum of f over $C_l - \{1\}$ respectively. If $f(1) = 0$, we may replace $\tilde{\Sigma}_\lambda$ by Σ_λ. If $\rho \in R(C_l)$, the orthogonality relations imply $l^{-1}\Sigma_\lambda Tr(\rho(\lambda)) \in \mathbb{Z}$. Let $f_n(1) = 0$, and for $\lambda \neq 1$ let

$$f_{4i-1}(\vec{a})(\lambda) := \lambda^{(a_1+\ldots+a_{2i})/2}(1-\lambda^{a_1})^{-1}\cdots(1-\lambda^{a_{2i}})^{-1},$$

$$f_{4i+1}(\vec{a})(\lambda) = f_{4i-1}(\vec{a})(1+\lambda^{a_1})(1-\lambda^{a_1})^{-1}.$$

Lemma 5.2. *Let $n = 4i \pm 1 = 2j - 1 \geq 3$, $\vec{b}_j = (a_1, ..., a_j)$, $\rho \in R_0(C_l)$, and $\sigma = \rho_{-3}(\rho_0 - \rho_3)^2$.*
a) *If $s \geq 0$, then $\eta(\tilde{X}^n(l;\vec{b}_{2i}) \times (B^8)^s)(\rho) = l^{-1}\tilde{\Sigma}_\lambda f_n(\vec{b}_{2i})(\lambda)Tr(\rho(\lambda))$.*
b) *We have the identity $l^{-1}\tilde{\Sigma}_\lambda(1-\lambda)^{-1} = (l-1)/(2l)$.*
c) *If $\gamma \in R_0(C_l)^{j+1}$, then $\eta(\tilde{X}^n(l;\vec{b}_{2i}))(\gamma) \in \mathbb{Z}$.*

d) *If* $n \geq 7$, *then* $\eta(\tilde{X}^n(l;\vec{b}_{2i-2},3,3))(\sigma\rho) = \eta(\tilde{X}^{n-4}(l;\vec{b}_{2i-2}))(\rho)$.
e) *If* $n \geq 3$, *then*

$$\begin{aligned}\eta(\tilde{X}^n(l;\vec{b}_{2i-1},1) - 3\tilde{X}^n(l;\vec{b}_{2i-1},3))(\rho) \\ = \eta(\tilde{X}^n(l;\vec{b}_{2i-1},3))(\rho(\rho_1 - 2\rho_0 + \rho_{-1})).\end{aligned}$$

Proof. The eta invariant vanishes if the C_l structure is trivial, so we may replace $\tilde{X}^n$ by X^n in computing the eta invariant. Assertion (a) now follows from Lemma 4.4, Theorem 4.6, and Theorem 4.7. We note that the lack of symmetry between the first and the remaining indices for f_{4i+1} is caused by taking $H^{\otimes 2}$ as the first line bundle, and the trivial line bundle as the remaining line bundles in the definition of X^{4i+1}. Since

$$\begin{aligned}l^{-1}\tilde{\Sigma}_\lambda(1-\lambda)^{-1} =& l^{-1}\tilde{\Sigma}_\lambda\{(1-\lambda)^{-1} + (1-\bar{\lambda})^{-1}\}/2 \\ =& l^{-1}\tilde{\Sigma}_\lambda(2-\lambda-\bar{\lambda})(1-\lambda)^{-1}(1-\bar{\lambda})^{-1}/2 \\ =& (l-1)/(2l),\end{aligned}$$

assertion (b) follows. If a is odd, then

$$R_0(C_l) = (\rho_a - \rho_0)R(C_l).$$

Suppose $\gamma \in R_0(C_l)^{2i+1}$. Then there exists $\rho \in R_0(C_l)$ such that

$$\gamma = (\rho_0 - \rho_{a_1})...(\rho_0 - \rho_{a_{2i}})\rho.$$

This shows that $f_{4i-1}(\vec{b}_{2i})\gamma \in R_0(C_l)$. Similarly if $\gamma \in R_0(C_l)^{2i+2}$ then $f_{4i+1}(\vec{b}_{2i})\gamma \in R_0(C_l)$. Let $4i \pm 1 = 2j-1$. If $\gamma \in R_0(C_I)^{j+1}$, then

$$f_{4i\pm 1}(\vec{b}_{2i})\gamma = \rho \in R_0(C_l).$$

Since $Tr(\rho)(1) = 0$,

$$\eta(\tilde{X}^{4i\pm 1}(l;\vec{b}_{2i}))(\gamma) = l^{-1}\tilde{\Sigma}_\lambda Tr(\rho(\lambda)) = l^{-1}\Sigma_\lambda Tr(\rho(\lambda)) \in \mathbb{Z},$$

and assertion (c) follows. Assertion (d) follows from the identity

$$Tr(\sigma)f_n(\vec{b}_{2i-2},3,3) = f_{n-4}(\vec{b}_{2i-2})$$

and assertion (a). As $(1-\lambda)^{-1} = (\lambda^2+\lambda+1)(1-\lambda^3)^{-1}$,

$$f_n(\vec{b}_{2i-1},1) = (\rho_1 + \rho_0 + \rho_{-1})f_n(\vec{b}_{2i-1},3),$$

and assertion (e) follows. q.e.d.

We define

$$\begin{aligned}
&Y^3 = \tilde{X}^3(l;1,1) - 3\tilde{X}^3(l;1,3),\\
&Y^{8j+3} = Y^3 \times (B^8)^j \text{ for } j > 0,\\
&Z^3 = \tilde{X}^3(l;1,1),\\
&Z^5 = \tilde{X}^5(l;1,1) - 3\tilde{X}^5(l;1,3),\\
&Z^7 = \tilde{X}^7(l;1,1,1,1) - 3\tilde{X}^7(l;1,1,1,3),\\
&Z^9 = \tilde{X}^9(l;1,1,1,1) - 3\tilde{X}^9(l;1,1,1,3)\\
&\qquad\quad - 3(\tilde{X}^9(l;1,1,3,1) - 3\tilde{X}^9(l;1,1,3,3)),\\
&Z^n = Z^{n-8} \times B^8 \text{for } n > 9,\\
&\sigma = \rho_{-3}(\rho_0 - \rho_3)^2,\\
&\delta(M) = (\eta(M)(\sigma(\rho_1 - \rho_0)), ..., \eta(M)(\sigma(\rho_{l-1} - \rho_0))).
\end{aligned}$$

Lemma 5.3.
a)*If $j \geq 0$, then $\eta_q(Y^{8j+3}) = \pm 1$ and $\vec{\eta}(Y^{8j+3}) = 0$.*
b) *If $n \geq 3$, then $\delta(Z^n) = 0$.*
c) *If $i \geq 1$, then $\vec{\eta}(Z^{4i-1})$ has order at least 2^{k+1} in $\mathbb{R}^{l-1}/\mathbb{Z}^{l-1}$.*
d) *If $i \geq 1$, then $\vec{\eta}(Z^{4i+1})$ has order at least 2^{k-1} in $\mathbb{R}^{l-1}/\mathbb{Z}^{l-1}$.*
e) *The vector $\delta(\tilde{X}^5(l;3,3))$ has order at least 2^{k-1} in $\mathbb{R}^{l-1}/\mathbb{Z}^{l-1}$.*

Proof. We use Lemma 5.2 in the proof throughout. We prove assertions (a)-(d) in dimensions 3, 5, 7, and 9; the remaining cases then follow from Lemma 4.4. To prove assertion (a), we compute that

$$\begin{aligned}
\eta_q(Y^3) &= \eta(\tilde{X}^3(l;1,3))((\rho_q - \rho_0)(\rho_1 + \rho_{-1} - 2\rho_0))\\
&= l^{-1}\tilde{\Sigma}_\lambda(\lambda^q - 1)\lambda^2(\lambda + \bar{\lambda} - 2)(1-\lambda)^{-1}(1-\lambda^3)^{-1}\\
&= l^{-1}\tilde{\Sigma}_\lambda(\lambda^{3q} - 1)\lambda(1-\lambda)(1-\lambda^3)^{-1}\\
&= -l^{-1}\tilde{\Sigma}_\lambda(1 + \lambda^3 + ... + \lambda^{3q-3})\lambda(1-\lambda).
\end{aligned}$$

We replace $l^{-1}\tilde{\Sigma}_\lambda$ by $l^{-1}\Sigma_\lambda$ since $(1-\lambda) = 0$ if $\lambda = 1$. Since $l^{-1}\Sigma_\lambda\lambda^j = \delta_{j,l}$ for $0 < j \leq 3q - 3 + 2 < 2l$, $\eta_q(Y^3) = \pm 1$ as claimed. If $\rho \in R_0(C_l)$, then $\rho(\rho_1 + \rho_{-1} - 2\rho_0) \in R_0(C_l)^3$ so $\eta(Y^3)(\rho) = 0$. This proves assertion (a). Since $\sigma \in R_0(C_l)^2$ and $(\rho_1 + \rho_{-1} - 2\rho_0) \in R_0(C_l)^2$, assertion (b) follows from Lemma 5.2 (c,e). We use Lemma 5.2 (a,b) to prove assertion (c). We compute that

$$\eta(\tilde{X}^3(l;1,1))(\rho_s(\rho_0 - \rho_1)) = l^{-1}\tilde{\Sigma}_\lambda\lambda^{s+1}(1-\lambda)^{-1}$$

has order $2l$ in $\mathbb{R}/\mathbb{Z}$ for $s=-1$. Similarly, we compute that

$$\begin{aligned}\eta(Z^7)(\rho_s(\rho_0-\rho_3)) =&\eta(\tilde{X}^7(l;1,1,1,3))(\rho_s(\rho_0-\rho_3)(\rho_1-2\rho_0+\rho_{-1}))\\ =&\eta(\tilde{X}^7(l;1,1,1,3))(\rho_{s-1}(\rho_0-\rho_3)(\rho_0-\rho_1)^2)\\ =&l^{-1}\tilde{\Sigma}_\lambda\lambda^{s+2}(1-\lambda)^{-1}\end{aligned}$$

has order $2l$ in $\mathbb{R}/\mathbb{Z}$ for $s=-2$. To check $\delta(\tilde{X}^5(l;3,3))$ has order at least 2^{k-1}, we compute that

$$\begin{aligned}&\eta(\tilde{X}^5(l;3,3))(\sigma(\rho_0-\rho_3)\\ &\quad= l^{-1}\tilde{\Sigma}_\lambda(1+\lambda^3)=-2/l+l^{-1}\Sigma_\lambda(1+\lambda^3)\equiv -2/l \bmod \mathbb{Z}.\end{aligned}$$

To check that $\vec{\eta}(Z^5)$ has order at least 2^{k-1}, we let $\rho=\rho_s(\rho_0-\rho_3)$ for suitably chosen s and compute

$$\begin{aligned}\eta(Z^5)(\rho) =&\eta(\tilde{X}^5(l;1,3))(\rho(\rho_1-2\rho_0+\rho_{-1}))\\ =&l^{-1}\tilde{\Sigma}_\lambda(1+\lambda)\equiv -2/l \bmod \mathbb{Z}.\end{aligned}$$

To check that $\vec{\eta}(Z^9)$ has order at least 2^{k-1}, we let $\rho=\rho_s(\rho_0-\rho_3)$ for suitably chosen s. Choose a so $3a\equiv 1 \bmod l$. Then

$$(1-\lambda)/(1-\lambda^3)=1+\lambda^3+\ldots+\lambda^{3a-3}.$$

Consequently we may complete the proof by computing

$$\begin{aligned}\eta(Z^9)(\rho) =&\eta(\tilde{X}^9(l;1,1,3,3))(\rho(\rho_1-2\rho_0+\rho_{-1})^2)\\ =&l^{-1}\tilde{\Sigma}_\lambda(1+\lambda)(1-\lambda)/(1-\lambda^3)\\ =&l^{-1}\tilde{\Sigma}_\lambda(1+\lambda)(1+\lambda^3+\ldots+\lambda^{3a-3})\\ \equiv&-2a/l \bmod \mathbb{Z}.\end{aligned}$$

Proof of Proposition 5.1. By Lemma 4.4 and Lemma 5.2 (d), δ induces a surjective map from $\mathcal{L}_n(BC_l)$ to $\mathcal{L}_{n-4}(BC_l)$ if $n\geq 7$. Let $n=5$. By Lemma 5.3 (b,d,e), the range of δ_5 and the kernel of δ_5 both have order at least 2^{k-1} and thus $|\mathcal{L}_5(BC_l)|\geq 2^{2k-2}$. We now proceed by induction. Since δ_{4i+1} is surjective, the order of the range of δ_{4i+1} is at least $|\mathcal{L}_{4i-3}(BC_l)|$. By Lemma 5.3 (b,d), the kernel of δ_{4i+1} has order at least 2^{k-1}; Proposition 5.1 (a) now follows. If $n=8j+1$, we prove Proposition 5.1 (b) by picking up an extra factor of 2 in $\ker(\delta_n)$. Let $M=S^1\times(B^8)^j$ with the trivial (*i.e.*, bounding) spin structure and canonical C_l structure. It is then an easy calculation to show $[M] \in \ker(\delta_n)$ and that $\vec{\eta}(M)$ is an element of order 2^k.

Let $n = 3$. By Lemma 5.3 (e), $\vec{\eta}(\tilde{X}^3(l; 3, 3))$ has order at least 2^{k+1}. Since δ_{4i-1} is surjective, the order of the range of δ_{4i-1} is at least $|\mathcal{L}_{4i-5}(BC_l)|$. By Lemma 5.3 (b,c), the kernel of δ_{4i-1} has order at least 2^{k+1}; Proposition 5.1 (c) now follows by induction. To prove Proposition 5.1 (d), we must pick up an extra factor of 2 using the refined eta invariant; again, it suffices to perform this calculation for $n = 3$ using the periodicity mod 8. This follows since $\vec{\eta}(Y^3) = 0$ and $\eta_q(Y^3) \neq 0$ by Lemma 5.3 (a). q.e.d.

6. The range of the eta invariant for generalized quaternion groups

In this section, we complete the proof of Theorem 2.5 by dealing with the generalized quaternion groups; see Proposition 6.1 below. We also show that the eta invariant completely detects the connective K-theory of these groups in dimensions congruent to 3 mod 4. We adopt the following notational conventions.

Let i, j, and k be the standard elements of the quaternions $\mathbb{H}$. For $q \geq 3$, identify the generalized quaternion group Q_l of order $l = 2^q$ with the subgroup of $S^3 \subset \mathbb{H}$ generated by the elements $\xi = e^{\pi i/2^{q-2}}$ and j; these are the generalized quaternion groups we shall be studying. This defines a natural representation $\tau\colon Q_l \to SU(2)$. We note that $i = \xi^{2^{q-3}}$. Any subgroup of order 4 in Q_l is conjugate to one of the following groups:

$$H_1 := \langle i \rangle, \quad H_2 := \langle j \rangle, \quad H_3 := \langle \xi j \rangle.$$

There are 4 inequivalent real representations ρ_0, κ_1, κ_2 and κ_3 of Q_l which are given on the generators by the formula:

$$\rho_0(j) := 1, \rho_0(\xi) := 1, \qquad \kappa_1(j) := 1, \kappa_1(\xi) := -1,$$

$$\kappa_2(j) := -1, \kappa_2(\xi) := 1, \qquad \kappa_3(j) := -1, \kappa_3(\xi) := -1.$$

We notice that the representations κ_t, $t = 1, 2, 3$ are nontrivial on the subgroup H_1 if and only if $q = 3$, so this case is slightly exceptional. To have a uniform notation that covers this case as well, we define the virtual representations

$\epsilon_2 := \rho_0 - \kappa_1$ if $q = 3$, $\quad \epsilon_2 := \kappa_3 - \rho_0$ if $q > 3$,

$\epsilon_3 := \rho_0 - \kappa_3$ if $q = 3$, $\epsilon_3 := \kappa_1 - \rho_0$ if $q > 3$.

Let S^{4m-1} be the unit sphere about the origin in $\mathbb{H}^m$. Let $t = 1, 2, 3$. Consider the following spherical space forms

$$M_Q^{4m-1} := S^{4m-1}/Q_l, \quad \text{and} \quad M_t^{4m-1} := S^{4m-1}/H_t.$$

These manifolds admits natural spin structures and natural metrics of positive scalar curvature. We identify $\pi_1(M_Q) = Q_l$ and $\pi_1(M_t) = H_t$. This gives M_Q a natural Q_l structure. The inclusion of H_t in Q_l gives the H_t natural Q_l structures as well. Let $\mathcal{M}_*(BQ_l) \subseteq \Omega_*^{spin}(BC_l)$ be the Ω_*^{spin}-submodule generated by the elements

$$\{[M_Q^{4m-1}],\ [M_1^{4m-1}],\ [M_2^{4m-1}],\ [M_3^{4m-1}]\};$$

see (2.2). Choose r large; we shall see presently $r = m$ will suffice. If $[M^{4m-1}]$ belongs to $\mathcal{M}_{4m-1}(BQ_l)$, define

$$\check{\eta}(M) := (\eta(M)(\epsilon_2), \eta(M)(\epsilon_3), \eta(M)((2-\tau)),$$

$$\eta(M)((2-\tau)^2), \dots, \eta(M)((2-\tau)^r).$$

Since η extends to a map in bordism, we may define $\check{\eta}([M]) = \check{\eta}(M)$. Since the representations ϵ_2, ϵ_3, $(2-\tau)^{2a}$ are real and the representations $(2-\tau)^{2a+1}$ are quaternion, Theorem 4.1 implies that if $n \equiv 3 \bmod 8$,

$$\check{\eta}([M]) \in \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \dots$$

while if $n \equiv 7 \bmod 8$,

$$\check{\eta}([M]) \in \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \dots.$$

Let $\mathcal{L}_n(BQ_l)$ be the range of the eta invariant on $\mathcal{M}_n(BQ_l)$;

$$\mathcal{L}_n(BQ_l) := \operatorname{span}\{\check{\eta}([M]) \ :\ [M] \in \mathcal{M}_n(BQ_l)\}.$$

Theorem 2.5 and Corollary 2.6 for the quaternion groups follow from the following result and Theorem 4.1.

Proposition 6.1. *We have that* $|\mathcal{L}_{8m+3}(BQ_l)| \geq 2^{4m+4} l^{2m+1}$ *and that*

$$|\mathcal{L}_{8m+7}(BQ_l)| \geq 2^{4m+4} l^{2m+2}.$$

Proof. We use Theorem 4.6 to compute some eta invariants. Let δ_{ts} be the Kronecker symbol and let $d = |Q_l| - 1 = 2^q - 1$. Since $\eta(M_t^n)((2-\tau)^\nu)$ is independent of t, we see that $\eta(M_1^n - M_t^n)((2-\tau)^\nu) = 0$. Suppose $q > 3$. Then the κ_t are trivial on H_1, so $\eta(M_1^m)(\epsilon_t) = 0$ for $t = 2, 3$. Note that

$$\epsilon_2(\pm 1) = 0, \quad \epsilon_2(\pm j) = -2, \quad \epsilon_2(\pm \xi j) = 0,$$

$$\epsilon_3(\pm 1) = 0, \quad \epsilon_3(\pm j) = 0, \quad \epsilon_3(\pm \xi j) = -2,$$
$$(2-\tau)^m(\pm j) = (2-\tau)^m(\pm \xi j) = 2^m.$$

It now follows that $\eta(M_1^n - M_t^n)(\epsilon_s) = 2^{-m}\delta_{ts}$ for $t, s = 2, 3$; a similar calculation establishes this result if $q = 3$. Finally, since

$$\det(I - m \cdot \tau) = (2-\tau)^m,$$

$$\eta(M_Q^{4m-1})((2-\tau)^m) = \eta(M_Q^3)((2-\tau)) = l^{-1} \sum_{\lambda \in Q_l, \lambda \neq 1} 1 = d/l.$$

To study $\eta(M_Q^{4m-1})((2-\tau)^{m+s})$ for $s > 0$ requires a slightly different argument and uses the orthogonality relations. If γ_t for $t = 1, 2$ are real class functions on the group Q_l, let

$$(\gamma_1, \gamma_2)_Q = l^{-1} \sum_{\lambda \in Q_l} \gamma_1(\lambda)\gamma_2(\lambda)$$

be the usual real L^2 inner product. We identify representations with their characters; the character of any representation of Q_l is real. If γ_t are irreducible representations of Q_l, then $(\gamma_1, \gamma_2)_Q = 0$ if the γ_i are not equivalent, and $(\gamma_1, \gamma_2)_Q = 1$ if the γ_i are equivalent. Let $s > 0$. We use Theorem 4.6 to see that

$$\eta(M_Q^{4m-1})((2-\tau)^{m+s}) = ((2-\tau)^s, \rho_0)_Q = b_s \in \mathbb{Z}.$$

If s is odd, then $(\tau^s, \rho_0)_Q = 0$, so that

$$b_s = ((2-\tau)^s, \rho_0)_Q \equiv (\tau^s, \rho_0)_Q \equiv 0 \mod 2\mathbb{Z}.$$

This shows that we may express $\check{\eta}(M^3) = (*, *, l^{-1}d, b_1, b_2, b_3, \dots)$ where b_s belongs to $\mathbb{Z}$ for s even, and to $2\mathbb{Z}$ for s odd. Thus

$$\check{\eta}(M^3) = (*, *, l^{-1}d, 0, 0, 0, \dots) \in \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \dots.$$

Let K^4 and B^8 be the Kummer surface and the Bott manifold respectively. These are simply connected spin manifolds with $\hat{A}(K^4) = 2$ and $\hat{A}(B^8) = 1$. We use Lemma 4.4 to study $\check{\eta}(M^7 \times K^4)$. Since $\hat{A}(K^4) = 2$, the numbers b_s play no role and we see that:

$$\check{\eta}(M^7 \times K^4) = (*, *, *, 2d/l, 0, 0, \dots) \in \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \dots .$$

This argument shows that if $n = 8m + 3$, then for $r = 2m + 1$,

$\check{\eta}(M_1^n - M_2^n) = (2^{-2m-1}, 0, 0, 0, 0, \dots, 0, 0, 0)$,
$\check{\eta}(M_1^n - M_3^n) = (*, 2^{-2m-1}, 0, 0, 0, \dots, 0, 0, 0)$,
$\check{\eta}(M_Q^3 \times (B^8)^m) = (*, *, d/l, 0, 0, \dots, 0, 0, 0)$,
$\check{\eta}(M_Q^7 \times K^4 \times (B^8)^{m-1}) = (*, *, *, 2d/l, 0, \dots, 0, 0, 0)$,
$\check{\eta}(M_Q^{11} \times (B^8)^{m-1}) = (*, *, *, *, d/l, \dots, 0, 0, 0)$,
....
$\check{\eta}(M_Q^{8m-1} \times K^4) = (*, *, *, *, *, \dots, 0, 2d/l, 0)$,
$\check{\eta}(M_Q^{8m+3}) = (*, *, *, *, *, \dots, 0, 0, d/l)$,

where $*$ indicates a term we are not interested in. The above matrix is lower triangular; the diagonal entries $2d/l \in \mathbb{R}/2\mathbb{Z}$ and $d/l \in \mathbb{R}/\mathbb{Z}$ all have order l. The first two entries $2^{-2m-1} \in \mathbb{R}/2\mathbb{Z}$ have order 2^{2m+1}. Thus the product of the orders on the diagonal furnishes the desired lower bound and completes the proof of the first inequality.

The situation is the same if $n = 8n + 7$. Let $r = 2m + 2$. We express

$$\check{\eta}(M_Q^7) = (*, *, *, d/l, b_1, b_2, \dots) \in \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \oplus \dots$$

and again, the numbers b_ν for ν even belong to $\mathbb{R}/\mathbb{Z}$ and vanish, and b_ν for ν odd belong to $\mathbb{R}/2\mathbb{Z}$ and vanish as well. We have

$$\check{\eta}(M_1^n - M_2^n) = (2^{-2m-2}, 0, 0, 0, 0, \dots, 0, 0, 0),$$
$$\check{\eta}(M_1^n - M_3^n) = (*, 2^{-2m-2}, 0, 0, 0, \dots, 0, 0, 0),$$
$$\check{\eta}(M_Q^3 \times K^4 \times (B^8)^m) = (*, *, 2d/l, 0, 0, \dots, 0, 0, 0),$$
$$\check{\eta}(M_Q^7 \times (B^8)^{m-1}) = (*, *, *, d/l, 0, \dots, 0, 0, 0),$$
$$\check{\eta}(M_Q^{11} \times K^4 \times (B^8)^{m-1}) = (*, *, *, *, 2d/l, \dots, 0, 0, 0),$$
....
$$\check{\eta}(M_Q^{8m+3} \times K^4) = (*, *, *, *, *, \dots, 0, 2d/l, 0),$$
$$\check{\eta}(M_Q^{8m+7}) = (*, *, *, *, *, \dots, 0, 0, d/l).$$

Again, the matrix is lower diagonal. The entries $2^{-2m-2} \in \mathbb{R}/\mathbb{Z}$ have orders 2^{2m+2}. Each entry $2d/l$ in $\mathbb{R}/\mathbb{Z}$ is followed by an entry d/l in $\mathbb{R}/2\mathbb{Z}$, and the product of the orders is l^2. q.e.d..

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