J.DIFFERENTIAL GEOMETRY
Vol. 43, No.1 January, 1996

HOLOMORPHIC MAPS OF A RIEMANN SURFACE INTO A FLAG MANIFOLD

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Abstract

We prove that there exist constants $c_0 > 0, c_1$ such that the inclusion

$$\mathrm{Hol}_k^0(\Sigma, G/P) \to \mathrm{Map}_k^0(\Sigma, G/P)$$

of the space of holomorphic maps of degree $k = (k_1, k_2, ..., k_r), k_i \geq 0$ of a Riemann surface Σ to a flag manifold G/P, into the corresponding space of continuous maps, induces isomorphisms in homology groups H_i for $i < c_0|k| - c_1$, with $|k| = \min(k_i)$.

1. Introduction

The space of based smooth maps $\mathrm{Map}^0(\Sigma, X)$ from a compact surface Σ into a manifold X can be considered from many points of view, in terms of its topology, its geometry, and, thanks to string theory, even its physics. When both Σ and X are Riemannian, there is a natural energy functional

$$E(f) = \int_\Sigma | df |^2 \tag{1.1}$$

defined on $\mathrm{Map}^0(\Sigma, X)$, whose critical points are harmonic maps. One class of manifolds X that has been considered quite extensively is that of the generalised complex flag manifolds G/P, G a semi-simple complex Lie group, and P a parabolic subgroup. In this case, the minima of the functional correspond to holomorphic or anti-holomorphic maps, and in general, there are other, non-minimal, critical points.

Received March 30, 1994. During the preparation of this work the author was supported by NSERC and FCAR grants.

In the spirit of Morse theory, it is natural to ask what the relation is between the space of criticial points of the functional and the total space. Unfortunately, Morse theory is not applicable in this context (condition "C" does not hold) and, indeed, one has the example of $\Sigma = X = \mathbb{P}_1(\mathbb{C})$, for which there are no critical points apart from minima, and the space of the latter (holomorphic or anti-holomorphic maps) is quite different from $\mathrm{Map}^0(\mathbb{P}_1, \mathbb{P}_1) \simeq \Omega^2 S^2$.

Nevertheless, in certain cases, an interesting stability phenomenon seems to occur, in that the components of the space of minima begin to mimic the components of the larger space $\mathrm{Map}^0(\Sigma, X)$ as one increases a degree k classifying the components of $\mathrm{Map}^0(\Sigma, X)$. More precisely, the inclusion induces isomorphisms in homotopy groups π_i and homology group H_i for $i \leq f(k)$, where f is some increasing function of k. This was first brought to light by Segal [16], who proved:

Theorem 1.1. *Let Σ be a Riemann surface of genus g. The inclusion in the degree k component*

$$I : \mathrm{Hol}_k^0\ (\Sigma, \mathbb{P}_n) \hookrightarrow \mathrm{Map}_k^0\ (\Sigma, \mathbb{P}_n)$$

induces isomorphisms

$$I_* : H_i(\mathrm{Hol}_k^0\ (\Sigma, \mathbb{P}_n) \simeq H_i\left(\mathrm{Map}_k^0\ (\Sigma, \mathbb{P}_n)\right)$$

for $i < (k - 2g)(2n - 1)$. When $\Sigma = \mathbb{P}_1$, the result holds for homotopy groups as well.

Segal's result was then extended to the case of X a Grassmannian by Kirwan [12], to $\Sigma = \mathbb{P}_1(\mathbb{C})$ and X certain flag manifolds, in homology, by Guest [7], to $\Sigma = \mathbb{P}_1(\mathbb{C})$ and X an $Sl(n, \mathbb{C})$ flag manifold, in homology, by Mann and Milgram [14], and to $\Sigma = \mathbb{P}_1(\mathbb{C})$ and G/P an arbitrary flag manifold, in both homology and homotopy in [3]. For $X = G/P$, the components $\mathrm{Map}_k^0(\mathbb{P}_1, G/P)$ of $\mathrm{Map}^0(\mathbb{P}_1, G/P)$ are indexed by $k = (k_1, \ldots, k_r) \in \mathbb{Z}^r$. Setting

$$|\, k \,| = \min(k_i) \tag{1.2}$$

one has:

Theorem 1.2 ([3]). *Let $k_i \geq 0$, for all i. There exist constants $c_0 = c_0(G/P) > 0$, $c_1 = c_1(G/P)$ such that the inclusion*

$$\mathrm{Hol}_k^0\ (\mathbb{P}_1, G/P) \to \mathrm{Map}_k^0\ (\mathbb{P}_1, G/P)$$

induces isomorphism in homology groups H_i *and homotopy groups* π_i *for* $i < c_0(| k |) - c_1$.

There is a corresponding stability result for instantons over a four-manifold X, known as the Atiyah-Jones conjecture. Proofs, for $X = S^4$, can be found in [4], as well as [13], [17]; for $X = \mathbb{P}^2(\mathbb{C})$ or a ruled surface, see [10].

Roughly, Theorem (1.2) depends on a description of holomorphic maps in terms of their principal parts, a generalisation due to Segal-Gravesen of the principal parts of a map $\mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C})$. The space of holomorphic maps then becomes a space of labelled configurations of points in $\mathbb{C} \subset \mathbb{P}_1(\mathbb{C})$, the points being the poles of the map and the labels the principal parts associated to each pole. These spaces then admit natural stabilisation maps

$$\mathrm{Hol}_k^0\,(\mathbb{P}_1, G/P) \to \mathrm{Hol}_{k'}^0(\mathbb{P}_1, G/P) \tag{1.3}$$

which increase the degree, and which are defined by adding in extra principal parts. The theorem is then obtained by combining two results:

- The first, due to Gravesen [6], shows that the limit space which one obtains from (1.3) is weakly homotopic to $\mathrm{Map}^0(\mathbb{P}_1, G/P)$.
- The second relies on an analysis of the homology of labelled configurations as in [4] to prove that the stabilisation maps induce homology isomorphisms through the range.

The homotopy result is obtained by analysing the universal covers of these spaces, and applying Whitehead's theorem.

Our purpose in this note is to extend this result in homology to arbitrary Riemann surfaces; due to the difficulties in analysing the relevant covers, the homotopy result seems for the moment to be inaccessible.

Theorem 1.3. *Let* $k_i \geq 0$, *for all* i. *There exist constants* $c_0 = c_0(G/P) > 0$, $c_1 = c_1(G/P)$ *such that the inclusion*

$$\mathrm{Hol}_k^0(\Sigma, G/P) \to \mathrm{Map}_k^0(\Sigma, G/P)$$

induces isomorphism in homology groups H_i *for* $i < c_0(| k |) - c_1$.

As for the case $\Sigma = \mathbb{P}_1$, holomorphic maps can be described as configurations of principal parts, and these configurations will exhibit the same stability phenomena. The problem, however, is that an arbitrary configuration of principal parts does not necessarily represent a holomorphic map, rather in the same way that an arbitrary collection of zeroes and poles on Σ does not necessarily represent a meromorphic function. There is an obstruction which must be analysed.

It seems that this stability phenomenon is not restricted to flag manifolds. Indeed Guest [8] has already proven a similar result for toric varieties. More generally, they seem to hold for certain manifolds admitting actions of a solvable group [2].

In section 2, we introduce our objects of study and establish some notation. Section 3 is devoted to describing the space $\mathrm{Conf}_k(\Sigma, G/P)$ of configurations of principal parts on Σ, and to establishing the stability theorem for these configurations. Section 4 is devoted to the relative topology of the inclusion $\mathrm{Hol}_k^0(G/P) \hookrightarrow \mathrm{Conf}_k(\Sigma, G/P)$ and to establishing our principal result.

2. Preliminaries

In this section we begin with a presentation of some facts about G/P. We refer to the references [9], [1] for more details.

Let G denote any complex semi-simple Lie group, and $\mathfrak{g}$ its Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let $\mathfrak{u}^+$ and $\mathfrak{u}^-$ denote the positive and negative root spaces, respectively, with respect to $\mathfrak{h}$. One then has Borel subalgebras

$$\begin{aligned} \mathfrak{b}^+ &= \mathfrak{h} + \mathfrak{u}^+, \\ \mathfrak{b}^- &= \mathfrak{h} + \mathfrak{u}^-. \end{aligned} \tag{2.1}$$

We consider *parabolic* subalgebras $\mathfrak{p}$ obtained by adjoining to $\mathfrak{b}^+$ a certain number of negative root spaces. Then, $\mathfrak{p}$ has a natural complement $\mathfrak{n}^-$ in $\mathfrak{g}$ with $\mathfrak{n}^- \subset \mathfrak{u}^-$ and $\mathfrak{b}^+ \subset \mathfrak{p}$, so that

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{n}^-. \tag{2.2}$$

We will denote by $B^\pm$, P, $U^\pm$, and N^- the Borel, parabolic, and unipotent subgroups of G corresponding to the Lie algebras $\mathfrak{b}^\pm$, $\mathfrak{p}$, $\mathfrak{u}^\pm$, and $\mathfrak{n}^-$, respectively. Let $N = N^-$.

The group U^- acts on G/P, and its orbits give a cell decomposition of G/P:

$$G/P = \bigsqcup_{w \in W^P} U^-(wP),$$

where W^P is a suitable subset of the Weyl group of G; we refer the reader to [1]. There is one open dense orbit, $U^-(P)$, as well as complex codimension-one orbits (cells) X_α, which are in bijective correspondence with the simple root spaces R_α, $\alpha = 1, \cdots, r$, which are in $\mathfrak{n}$. Let Z_α be the closure of X_α in G/P; one then has:

$$G/P = (U^-(P)) \sqcup \left(\overset{r}{\underset{\alpha=1}{\cup}} Z_\alpha \right).$$

The group $N \subset U^-$ also acts on G/P, and preserves the cell structure. N acts transitively and freely on the big cell $U^-(P)$, and G/P can be thought of as a compactification of N by r varieties Z_α at infinity. The Z_α freely generate $H^2(G/P, \mathbb{Z})$ via the intersection pairing.

Let Σ be a Riemann surface with base point p_0. Maps f from a Riemann surface into G/P are classified topologically by their degree, an r-tuple of integers $(k_1, \ldots, k_r)$ defined by

$$\deg (f) = (k_1, \cdots k_r) \Leftrightarrow f^*(Z_\alpha) = k_\alpha \text{ points}$$

counted with multiplicity. For holomorphic maps the k_α are non-negative.

The group N is nilpotent, and has a central series

$$0 \to N_{\mu+1} \to N_\mu \overset{\pi_\mu}{\to} \mathbb{C}^{\ell_\mu} \to 0 \tag{2.3}$$

with $N_1 = N$, $N_s = 0$ and $[N, N_\mu] \subset N_{\mu+1}$.

We can also first define what is in some sense a refinement of the central series (2.3). Recall (e.g. [9]) that:

1) The Lie algebra u^- is spanned by the negative root spaces R_α; any negative root α is a (positive) sum of simple negative roots.
2) The algebra $n \subset u^-$ is obtained by fixing r simple roots $\alpha_1, \cdots \alpha_r$; one then sets n to be the span of the root spaces $R_\alpha \subset U^-$ such that the decomposition of α into simple roots has at least one non-zero coefficient for one of the α_i.
3) From the fact that $[R_\alpha, R_\beta] \subset R_{\alpha+\beta}$ if $[R_\alpha, R_\beta] \neq 0$, one has that U^- normalises n. One can filter n by ideals m_μ, such that m_μ is spanned by the root spaces R_α of n for which α is the sum of μ or more simple roots. In other words, there is a natural length function $\ell(\alpha)$ on the roots, given by the sum of the multiplicies of the simple roots in the decomposition, and m_μ is the span of the root spaces R_α with $\ell(\alpha) \geq \mu$. Similarly, one can define ideals u_μ^- of u^-.

More concretely, when $G = S\ell(n, \mathbb{C})$, N will be (conjugate to) a subalgebra of strictly bloch lower triangular matrices, while U^- consists of all the strictly lower triangular matrices. The root spaces correspond to the matrix entries, and the length function is simply the distance from the diagonal ($\ell(a_{ij}) = i - j$).

Corresponding to the m_μ and the u_μ^-, one has filtrations of N and U^- by normal subgroups.

$$\begin{aligned} 0 \to M_{\mu+1} &\to M_\mu \stackrel{\pi_\mu}{\to} Q_\mu = \mathbb{C}^{s_\mu} \to \quad 0, \\ 0 \to U_{\mu+1}^- &\to U_\mu^- \to T_\mu \;= \mathbb{C}^{t_\mu} \to \quad 0, \end{aligned} \tag{2.4}$$

with $[U_\nu^-, M_\mu] \subset M_{\mu+\nu}$. Note that $s_1 = r$, the number of simple root spaces in N.

Finally, one has:

Proposition 2.1. *The exponential map*

$$\exp : \mathfrak{n} \longrightarrow N$$

is a biholomorphic diffeomorphism.

3. Principal parts and the space $\mathrm{Conf}_k(\Sigma, G/P)$

Let D be an open set of Σ, and consider $M(D, G/P)$, the subset of the holomorphic maps $D \to G/P$ which only meet the Z_α in a discrete set. This is acted on in a natural way by the group $\mathrm{Hol}(D, N)$, using the action of N on G/P. We can define the quotient set

$$\mathcal{PP}(D) = M(D, G/P)/\,\mathrm{Hol}(D, N). \tag{3.1}$$

This quotienting is compatible with restriction and so one can define a sheaf of sets $\mathcal{PP}$, the sheaf of principal parts of maps into G/P. The example to bear in mind is that of $G = Sl(2, \mathbb{C})$, P the upper triangular subgroup, so that $G/P = \mathbb{P}_1(\mathbb{C}), N = \mathbb{C}$, and one is considering ordinary meromorphic maps. In this example, by representing elements of $M(D, G/P)$ as meromorphic maps into N, the action of $\mathrm{Hol}(D, N)$ is simply by addition, and one obtains the classical notion of the principal part of a map $D \to \mathbb{P}_1(\mathbb{C})$, i.e., of meromorphic maps modulo holomorphic maps. More details and examples can be found in [3]. The more general principal parts defined above share many of the properties of the classical ones. For example:

- Any map $D \to N \cdot P \subset G/P$ is equivalent in $\mathcal{PP}(D)$ to the trivial map $D \to eP$. In other words, in the principal part space, any map which does not meet "infinity" is trivial.

- If $f : D \longrightarrow G/P$ is a map with $f(z_0) \in Z_\alpha$, then $n(z_0)f(z_0) \in Z_\alpha$ for all $n : D \longrightarrow N$. Thus, the property of belonging to Z_α is invariant, and the points mapped to Z_α are to be thought of as the poles of the map. There are r different Z_α's for the map to hit, and, correspondingly, r different types or colours of poles. A local section of the principal part space is then given by a discrete set of points, along with some extra data "concentrated" at these points.

- A map with poles can be represented, using the action of N on the big cell, as a meromorphic map into N, with the notion of what constitutes a meromorphic map being defined via the exponential diffeomorphism of Proposition 2.1

- The action of N preserves the local intersection number with Z_α, and so the principal parts have a natural degree at a point $p \in \Sigma$:

$$k^p(f) = (k_1^p(f), \cdots, k_r^p(f)) \in (\mathbb{Z}_+)^r. \tag{3.2}$$

If Z_α is cut out locally by $g_\alpha = 0$, then $k_\alpha^p(f)$ is the order of vanishing of $g_\alpha \circ f$ at p. The local degrees add up to give a global degree for a section $s \in H^0(\Sigma, \mathcal{PP})$:

$$k(s) = (k_1(s), \cdots, k_r(s)) = \sum_{p \in \Sigma} k^p(s). \tag{3.3}$$

Let $H_k^0(\Sigma, \mathcal{PP})$ be the subspace of sections of degree k. As our basing condition $(f(p_0) = P)$ will preclude poles over p_0, we will be interested in the space

$$\mathrm{Conf}_k(\Sigma, G/P) = \{s \in H_k^0(\Sigma, \mathcal{PP}) | k^{p_0}(s) = 0\}. \tag{3.4}$$

This space has the structure of a smooth complex variety. To see this, note that if one has coordinate patches, $\rho : U \simeq D,\ U \subset \mathbb{P}_1, D \subset \Sigma$, one can pull back principal parts to obtain $\mathcal{PP}(D) \simeq \mathcal{PP}(U)$. However, configurations on $\mathbb{P}_1$ uniquely determine uniquely based rational maps $\mathbb{P}_1 \to G/P$ ([6]). The space of such maps is a smooth complex variety, [3], and so one can use this to give a similar structure to $\mathcal{PP}(D)$. More generally, if $D_i,\ i = 1, \cdots, n$ are disjoint coordinate patches in Σ, then

$$\mathcal{PP}(\cup_i D_i) = \mathcal{PP}(D_1) \times \cdots \times \mathcal{PP}(D_n),$$

and so can be thought of as a product of pieces of the space of rational maps. We then obtain, from the analoguous result for $\mathrm{Hol}(\mathbb{P}_1, G/P)$:

Proposition 3.1. *$\mathrm{Conf}_k(\Sigma, G/P)$ is a smooth complex manifold of dimension $\sum_{i=1}^r k_\alpha \cdot (\dim s_\alpha + 1)$ where s_α is the Lie subalgebra of* $\mathfrak{n}$ *stabilising a point of the codimension one cell X_α in G/P.*

Let $k' \geq k \Leftrightarrow k'_i \geq k_i$ for all i. If $k' \geq k$, $k'_i - k_i = c_i, \Sigma c_i = c$, one can define a stabilisation map

$$i(k,k') : \mathrm{Conf}_k(\Sigma, G/P) \to \mathrm{Conf}_{k'}(\Sigma, G/P) \tag{3.5}$$

as follows. One chooses fixed principal parts f_i of multiplicity $k_j(f_i) = \delta_{ij}$. Let $\rho : D_1 \to \Sigma$, $\rho(0) = p_0$ be a parametrisation of a neighbourhood of p_0 by the unit disk D_1. If $\tau \in \mathrm{Conf}_k(\Sigma, G/P)$, let

$$d(\tau) = \sup\{s \in (0,1] | \rho(D_s) \quad \text{contains no pole of } \tau.\},$$

where D_s is the disk of radius s. Then $i(k,k')(\tau)$ is the configuration of principal parts obtained from τ by adding to it the principal part f_1 at the points $\rho\left(\frac{d(\tau)}{c+1}\right)$, $\rho\left(2\frac{d(\tau)}{c+1}\right), \cdots \rho\left(c_1\frac{d(\tau)}{c+1}\right)$, the principal part f_2 at the points $\rho\left((c_1+1)\frac{d(\tau)}{c+1}\right), \cdots \rho\left((c_1+c_2)\frac{d(\tau)}{c+1}\right)$, etc, finishing with the principal part f_r at $\rho\left((c-c_r+1)\frac{d(\tau)}{c+1}\right), \cdots, \rho\left(c\frac{d(\tau)}{c+1}\right)$. This stabilisation has the following important property:

Theorem 3.2. *There exist constants c_0, c_1 such that the map $i(k,k')$ induces isomorphisms in homology*

$$i(k,k')_* : H_j(\mathrm{Conf}_k(\Sigma, G/P)) \to H_j(\mathrm{Conf}_{k'}(\Sigma, G/P))$$

for $j < c_0 \mid k \mid -c_1$.

Proof. The proof for the case $\Sigma = \mathbb{P}_1(\mathbb{C})$ is given in [3]. Once one has this case, one can exploit the cofibering

$$\vee_1^{2g} S^1 \to \Sigma \to \mathbb{P}_1(\mathbb{C})$$

to obtain the result for arbitrary Σ.

This extension, in a different context, is given in [10, Section 3D]: [10] considers moduli spaces of holomorphic vector bundles over a ruled surface X; X is a holomorphic fibering, with $\mathbb{P}_1$ fibers, over a Riemann surface Σ. In the end, the analysis boils down to studying strata of labelled configurations of points over Σ. This is the same situation that one is considering here.

As in [6], the maps $i(k, k')$ can be used to define an infinite mapping telecope $\widetilde{\mathrm{Conf}}(\Sigma, G/P)$. One can build a similar telescope $\widetilde{Map}^0(\Sigma, G/P)$ for the mapping spaces, and one has

Theorem 3.3 (Gravesen [6]). *For all i, there is an isomorphism*

$$j_* : H_i(\widetilde{\mathrm{Conf}}(\Sigma, G/P)) \to H_i(\widetilde{\mathrm{Map}}^0(\Sigma, G/P)).$$

Combining this with Theorem 3.2 and the fact that the components $\mathrm{Map}^0_k(\Sigma, G/P)$ of $\mathrm{Map}^0(\Sigma, G/P)$ are all homotopy equivalent , we obtain as in [3]:

Theorem 3.4. *There are isomorphisms*

$$j_* : H_i(\mathrm{Conf}_k(\Sigma, G/P)) \to H_i(\mathrm{Map}^0_k(\Sigma, G/P))$$

for $i < c_0 \mid k \mid -c_1$.

Furthermore, Gravesen's isomorphism is mediated by the holomorphic maps, in the sense that if one considers the maps:

$$\mathrm{Conf}_k(\Sigma, G/P) \overset{K}{\hookleftarrow} \mathrm{Hol}^0_k(\Sigma, G/P) \overset{I}{\hookrightarrow} \mathrm{Map}^0_k(\Sigma, G/P),$$

then $I_* = j_* \circ K_*$ in homology. To prove that I_* is an isomorphism through some range, we are thus reduced to understanding K_*.

4. The obstruction space $H^1(\Sigma, N(-p_0))$

An arbitrary choice of principal parts does not necessarily define a holomorphic map $f : \Sigma \to G/P$, $f(p_0) = P$ (p_0, P are our base-points). Indeed, the map will exist only if a certain obstruction vanishes. To see this, we consider $\tau \in \mathrm{Conf}_k(\Sigma, G/P)$ with poles at p_i, $i = 1, \cdots, n$. Let D_i be disks, disjoint and open, centred at p_i with $p_0 \notin D_i$, and set $D_0 = \Sigma \backslash \{p_1, \cdots, p_n\}$. The configuration τ is then described by holomorphic functions:

$$n^{i0} : D_i \backslash \{p_i\} \longrightarrow N \tag{4.1}$$

with poles at p_i . If $n^i : D_i \to N$ is holomorphic, we note that $n^i n^{i0}$ and n^{i0} define the same principal part at p_i. The question of existence of a map $\Sigma \to G/P$ with principal parts τ is then equivalent to the existence of holomorphic maps

$$n^i : D_i \longrightarrow N, i = 0, \dots, n \tag{4.2}$$

with

$$n^0 = n^i n^{i0}\ ,\ i = 1, \dots, n \tag{4.3}$$

on $D_i \backslash \{p_i\}$. We see that this is tantamount to asking that the cocycle n^{i0} be a coboundary (i.e., trivial) in the non-abelian cohomology $H^1(\Sigma, N)$ or, equivalently, that the principal N-bundle with transition functions $n^{i0}(z)$ be trivial. (We note that, as bundles on open Riemann surfaces are all trivial, the open sets D_0, D_i constitute a Leray cover). If one now adds in the base point condition that $n^0(p_0) = 1$, one obtains the more restrictive constraint that the class $[n^{i0}]$ vanish in $H^1(\Sigma, N(-p_0))$, where $N(-p_0)$ is the sheaf of holomorphic maps in N which are the identity at p_0. The space $H^1(\Sigma, N(-p_0))$ is a pointed set, with origin O corresponding to the trivial bundle.

There is thus an obstruction map

$$\Omega : \mathrm{Conf}_k(\Sigma, G/P) \to H^1(\Sigma, N(-p_0))$$

with

$$\mathrm{Hol}_k^0(\Sigma, G/P) = \Omega^{-1}(O).$$

The structure of $H^1(\Sigma, N(-p_0))$ is given by the following:

Proposition 4.1. *Let δ be the complex dimension of N. Then $H^1(\Sigma, N(-p_0))$ is isomorphic as a complex manifold to $\mathbb{C}^{g\delta}$, where g is the genus of Σ.*

Proof. We consider the successive quotients $\mathbb{C}^{s_\mu}$ of (2.3). The corresponding vector spaces $V_\mu = H^1(\Sigma, \mathcal{O}(-p_0)^{\oplus s_\mu})$ are of dimension $s_\mu g$, by the Riemann Roch theorem. We will prove the proposition by building a biholomorphic map

$$H^1(\Sigma, N(-p_0)) \to \oplus_\mu V_\mu. \tag{4.4}$$

Let W_μ be subspaces of the space of cocycles $Z^1(\Sigma, \mathfrak{m}_\mu(-p_0))$, $\mathfrak{m}_\mu$ the Lie algebra of M_μ, such that

$$(\Pi_\mu)^* \circ \exp : W_\mu \to V_\mu \tag{4.5}$$

is an isomorphism (we note that on $\mathbb{C}^{s_\mu}$, the exponential map is just the identity). We will establish (4.4) by successive normalisation of cocycles. Let n_1^{i0} represent an element of $H^1(\Sigma, M_1(-p_0))$ $(N = M_1)$. The map π_1 of (2.4) induces a surjection:

$$\pi_1^* : H^1(\Sigma, M_1(-p_0)) \to V_1,$$

and so one can choose $w_1^{i0} \in W_1$ with $\exp(w_1^{i0}), n^{i0}$ having the same image in V_1. Therefore, one can find $n_1^i \in H^0(D_i, M_1(-p_0))$, $i = 0, \ldots, n$ and $n_2^{i0} \in H^0(D_i \cap D_0,\ M_2(-p_0))$, with

$$n_1^i n_1^{i0} (n_1^0)^{-1} = n_2^{i0}\ \exp(w_1^{i0}). \tag{4.6}$$

One can ask how unambiguously n_2^{i0} is defined. A different choice of n_1^i, n_1^0 would give a different $\hat{n}_2^{i0}$, with

$$\hat{n}_2^{i0} = \rho^i n_2^{i0}\ Ad_{\exp(w_1^{i0})}(\rho^0) \tag{4.7}$$

for $\rho^i \in H^0(D_i, N(-p_0))$, $i = 0, \ldots, n$. Applying π_1 gives $\pi_1(\rho^i) = \pi_1(\rho^0)$ on $D_0 \cap D_i$, and so $\pi_1(\rho^i)$ defines a section of $H^0(\Sigma, \mathcal{O}(-p_0)^{\oplus s_\mu}) = 0$. The ρ^i thus take values in M_2, and (4.7) tells us that invariantly, the n_2^{i0} define unambiguous elements of

$$H^1(\Sigma, Ad_{(1)}(M_2)(-p_0)),$$

where $Ad_{(1)}(M_2)$ is the M_2-bundle with transition functions given by the adjoint action of $\exp(w_1^{i0})$. As $[M_1, M_2] \subset M_3$, taking π_2 of (4.7) gives:

$$\pi_2(\hat{n}_2^{i0}) = \pi_2(\rho^i)\pi_2(n_2^{i0})\pi_2(\rho^0),$$

and so $\pi_2(n_2^{i0})$ is invariantly an element of $V_2 = H^1(\Sigma, \mathcal{O}(-p_0)^{\oplus s_2})$. Pursuing the normalisation, we write:

$$n_2^i n_2^{i0}\ (n_2^0)^{-1} = n_3^{i0} \exp(w_2^{i0}),$$

with $w_2^{i0} \in W_2$, $n_2^i \in H^0(D_i, M_2(-p_0))$, $n_3^{i0} \in H^0(D_i \cap D_0, M_3(-p_0))$. Again, invariantly, n_3^{i0} represents an element in $H^1(\Sigma, Ad_{(2)}(M_3)(-p_0))$ for a suitable bundle $Ad_{(2)}(M_3)$, defined by the adjoint action of $\exp(w_2^{i0})\exp(w_1^{i0})$; n_3^{i0} projects under π_3 to an element of V_3. Continuing in this way, we obtain a unique representative

$$\exp(w_s^{i0}) \exp\ (w_{s-1}^{i0}) \cdots\ \exp(w_1^{i0})$$

for our cocycle, as well as a corresponding element

$$(\pi_1(n_1^{i0}), \pi_2(n_2^{i0}), ..., \pi_d(n_d^{i0})) \in \oplus_\mu V_\mu,$$

establishing the isomorphism (4.4).

This proposition gives us, associated to any filtration of N with abelian quotients, a somewhat artificial vector space structure on $H^1(\Sigma, N(-p_0))$ which is however, natural on the projection to the first quotient. We want to show that for some positive constants c_0', c_1', the relative homotopy groups π_n $(\mathrm{Conf}_k(\Sigma, G/P), \mathrm{Hol}_k(\Sigma, G/P))$ vanish for $n < c_0' \mid k \mid -c_1'$. Thus, given a family of elements of $\mathrm{Conf}_k(\Sigma, G/P)$ parametrized by a disk, we want to deform it so that its image in $H^1(\Sigma, N(-p_0))$ is zero. We will do this by using two mechanisms:

- The first, which works well when N (or a quotient) is an abelian group A, uses the torus of G to "rescale" individual principal parts so that they balance out in $H^1(\Sigma, A(-p_0))$.

- The second uses the coadjoint action of the group U^- on N to ensure, roughly, that the portion of the obstruction in H^1 due to the commutator $[U^-, N] \subset N$ can be made to vanish.

A configuration $\tau \in \mathrm{Conf}_k(\Sigma, G/P)$ can be described, as we saw, by meromorphic functions n^{i0} into N near its poles p_i, and these define an element $\Omega(\tau)$ of $H^1(\Sigma, N(-p_0))$. Projecting from $N = M_1$ to the quotient Q_1 of (2.4), we obtain $\pi_1(\Omega(\tau)) \in V_1 \equiv H^1(\Sigma, Q_1(-p_0)) \simeq \mathbb{C}^{gr}$. The latter is an abelian group, and indeed $\pi_1(\Omega(\tau))$ is the sum of individual contributions from each principal part located at the p_i.

We first show that given a family $\tau(s)$ of elements of $\mathrm{Conf}_k(\Sigma, G/P)$, we can deform $\tau(s)$ so that the contributions of sufficiently many of the p_i span V_1. If the dimension of the family is small compared to $\mid k \mid$, this follows from the codimension estimate:

Proposition 4.2. *Let $2g < d <\mid k \mid$. Let Ξ be the subset of $Conf_k(\Sigma, G/P)$ consisting of elements τ such that:*

1) *for each $i \in \{1, \ldots, r\}$, there are at least d principal parts of multiplicity*
   $(k_1, \ldots, k_i, \ldots k_r) = (0, \ldots, 0, 1, 0, \ldots, 0)$, *i.e.,* $k_j = \delta_{ij}$,
2) *any choice of d of these "simple poles" for each degree i spans V_1 as a real vector space.*

Then there are constants $c_0, c_1 > 0$ depending only on G/P, g such that the codimension of $(Conf_k(\Sigma, G/P)\backslash\Xi)$ in $Conf_k(\Sigma, G/P)$ is greater than $min(d - 2g, 2c_0(\mid k \mid -d) - c_1)$.

Proof. From [3, Proposition (6.5)] one has that the complex codimension of those configurations not satisfying 1) is bounded below by $c_0(|\ k\ | - d) - c_1$. As for constraint 2), we first remark that when $N = \mathbb{C}$, the result is quite classical: the generic configuration of $2g$ simple poles spans $H^1(\Sigma, \mathcal{O}(-p_0))$ as a real vector space, and, for each additional principal part, the constraint of belonging to a hyperplane in $H^1(\Sigma, \mathcal{O}(-p_0))$ containing the images of the first $(2g-1)$ principal parts imposes one extra constraint, giving a codimension of $d - 2g + 1$. For arbitrary N, we note that V_1 decomposes naturally as $Q_1 \otimes H^1(\Sigma, \mathcal{O}(-p_0))$, so that all one needs is some generic behaviour for the residues of the poles in Q_1. One must therefore examine the nature of these poles.

Principal parts of total degree-one were considered in some detail in [3, Section 4]. On $\Sigma = \mathbb{P}_1(\mathbb{C})$, each principal part determines a unique map $\mathbb{P}_1 \to G/P$, and one can study the principal parts in terms of their corresponding maps. Let us then consider a map of degree $k_j = \delta_{ij}$. There is a larger parabolic subgroup $P' \supset P$, whose Lie algebra contains that of P as well as the i-th simple root space. Correspondingly, the opposite unipotent algebra n' contains all the simple root spaces of n, except for the i-th. There is a projection

$$G/P \to G/P'$$

whose fiber is again a flag manifold $\hat{G}/\hat{P}$, with $\hat{P}$ maximal parabolic, so that the Lie algebra of the corresponding $\hat{N}$ contains one simple root space of $\hat{G}$, which can be identified with the i-th root space α_i of N. The degree-one maps $\mathbb{P}_1 \to G/P$ have constant image under the projection to G/P', and so are in essence maps into $\hat{G}/\hat{P}$. Furthermore, there is a line bundle $\theta(\lambda_i)$ on $\hat{G}/\hat{P}$ whose sections embed $\hat{G}/\hat{P}$ into $\mathbb{P}_n$, in such a way that the degree one maps $\mathbb{P}_1 \to \hat{G}/\hat{P}$ give lines $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$, with the pole of the map being cut out by the $\mathbb{P}_{n-1}$ at infinity. The sections of $\mathcal{O}(\lambda_i)$ are acted on linearly by $\hat{G}$, with the highest weight vector corresponding to the section S_{λ_i} vanishing at infinity. There is also a "next highest" weight vector $S_{\lambda_i+\alpha_i}$. Under the representation of $\varphi : \mathbb{P}_1 \to \hat{G}/\hat{P}$ as a meromorphic map $\Phi : \mathbb{P}_1 \to \hat{N}$, the component of Φ corresponding to the simple root space R_{α_i} is essentially $S_{\lambda_i+\alpha_i}(x)/S_{\alpha_i}(x)$. (This calculation can be found in a different form in [15], [11].) For the generic map $\mathbb{P}_1 \to \hat{G}/\hat{P}$, this function has a nontrivial residue at the pole; one can use, for example, the transitivity of the action of $\hat{G}$ on $\hat{G}/\hat{P}$ to prove this.

Returning to the description of maps $\mathbb{P}_1 \to G/P$ in terms of maps $\mathbb{P}_1 \to N$, this means that for maps of degree $k_j = \delta_{ij}$, the entry in N corresponding to the i-th simple root has, generically, one simple pole with non-zero residue. The entries corresponding to the other simple roots are constant, as the composition $\mathbb{P}_1 \to G/P \to G/P'$ is constant. One can thus choose the polar behaviour of the principal parts corresponding to each simple root space in Q_1 independently. As one has at least d simple poles for each degree i, $Q_1 \otimes H^1(\Sigma, \mathcal{O}(-p_0))$ exhibits the same genericity behaviour as $H^1(\Sigma, \mathcal{O}(-p_0))$, giving the result.

Corollary 4.3. *The real codimension of $Hol_k(\Sigma, G/P) \setminus (Hol_k(\Sigma, G/P) \cap \Xi)$ in $Hol_k(\Sigma, G/P)$ is greater than $[min(d-2g, 2c_0(\mid k \mid -d) - c_1)] - 2g\delta$.*

Proof. $\mathrm{Hol}_k(\Sigma, G/P)$ is cut out of $\mathrm{Conf}_k(\Sigma, G/P)$ by the vanishing of $g\delta$ holomorphic constraints, by Proposition (4.1).

Theorem 4.4. *There exist constants $c_0' > 0$, c_1' such that*

$$\pi_n(\mathrm{Conf}_k(\Sigma, G/P),\ \mathrm{Hol}_k(\Sigma, G/P)) = 0$$

for $n < c_0' \mid k \mid -c_1'$.

Proof. Let σ be a map (simplex)

$$\sigma : (B^n, \partial B^n) \to (\mathrm{Conf}_k(\Sigma, G/P),\ \mathrm{Hol}_k(\Sigma, G/P)).$$

We want to deform it so that $\sigma(B^n)$ lies in $\mathrm{Hol}_k(\Sigma, G/P)$, keeping the boundary in $\mathrm{Hol}_k(\Sigma, G/P))$. First note that by Proposition (4.2) and Corollary (4.3) one can choose c_0' and c_1' such that for $n < c_0' \mid k \mid -c_1', \sigma$ can be supposed to lie in the generic set Ξ. Next, we would like to choose (continuously) once and for all our d simple poles of each degree over all of σ. This may not be possible, as, for example any pair of simple poles at one point of σ may coalesce into a double pole somewhere else on σ. The codimension estimates of Proposition (4.2) and Corollary (4.3) guarantee that if one subdivides σ into sufficiently small simplices σ_i, this choice of simple poles can be performed over the σ_i. We will then deform inductively over the skeleta of this subdivision, starting with the 0-skeleton. Our deformations will fix the points already in $\mathrm{Hol}_k(\Sigma, G/P)$, and so in particular the edges of the simplex, allowing the induction to proceed.

In any case, we are left with the problem of deforming a simplex of configurations, over which d simple poles have been "marked".

We will deform this simplex in successive steps, killing the obstruction corresponding to the successive quotients Q_μ of (2.4). We begin with the obstruction $(\pi_1)_* \cdot \Omega \circ \sigma(x) \in H^1(\Sigma, Q_1(-p_0)) \simeq \mathbb{C}^{rg}$ corresponding to the simple root spaces. To deform the principal parts, we will use the left action of the torus of G on principal parts. This gets translated into conjugation when one represents the principal part by a meromorphic function into N; so that if t is an element of the torus, and $n : U \subset \mathbb{C} \to N, then\ t \cdot (n(x)) = tn(x)t^{-1},\ x \in U$. For simplicity, we will choose a subgroup $H = \mathbb{C}^*$ of the torus which acts with the same weight on all of the simple root spaces, and just use this subgroup. We will need:

Lemma 4.5. *Let*

$$\begin{aligned} V : B^n &\to (\mathbb{C}^\ell)^{M+K+1} \\ x &\mapsto (v_1(x), \cdots, v_{M+K+1}(x)) \end{aligned}$$

be such that

1) $\sum_{i=1}^{M+K+1} v_i(x) = 0$ *on* ∂B^n,
2) $\{v_1(x), \cdots, v_M(x)\}$ *and* $\{v_{M+1}(x), \cdots, v_{M+K}(x)\}$ *both span* $\mathbb{C}^\ell$ *as a real vector space for all* x.

Then there exists a map C

$$\begin{aligned} C : B^n &\to (\mathbb{C}^*)^{M+K} \\ x &\mapsto (c_1(x), \dots, c_{M+K}(x)) \end{aligned}$$

with:

1) $C|_{\partial B^n} = (1, \cdots, 1)$,
2) $\sum_{i=1}^{M+K} c_i v_i(x) + v_{M+K+1}(x) = 0$ *for all* $x \in B^n$.

Furthermore, C *is homotopic, by a homotopy which is constant on* ∂B^n*, to the constant map* $x \mapsto (1, 1, ..., 1)$.

Proof. Let F be the affine subbundle of the trivial $\mathbb{R}^M$-bundle over $(\mathbb{C}^\ell)^{M+K+1}$: $\mathbb{R}^M \times (\mathbb{C}^\ell)^{M+K+1} \to (\mathbb{C}^\ell)^{M+K+1}$, defined by:

$$F = \Big\{(c_1, \cdots, c_M, v_1, \cdots, v_{M+K+1}) \Big| \sum_{i=1}^{M} c_i v_i + \sum_{i=M+1}^{M+K+1} v_i = 0\Big\}.$$

Let $\pi : \mathbb{R}^M \times \mathbb{C}^{M+K+1} \to F$ be a projection of bundles, that is a fiber preserving map which is the identity when restricted to F, defined for example by an orthogonal projection. The map $V : B^n \to (\mathbb{C}^\ell)^{M+K+1}$ lifts to a constant section $(1, V) : B^n \to \mathbb{R}^M \times (\mathbb{C}^\ell)^{M+K+1}$, taking values in F on ∂B^n. Let us set $\hat{c}(x) = \pi(1, V)(x)$. One then has $\sum_{i=1}^M \hat{c}_i(x)v_i(x) + \sum_{i=M+1}^{M+K+1} v_i(x) = 0$, which is what one wants; the problem is that some of the $\hat{c}_i$ could vanish. Let $0 < \epsilon < 1$, and set

$$c_i(x) = \begin{cases} \hat{c}_i(x) & \text{if } \hat{c}_i(x) \notin (-\epsilon, \epsilon), \\ \left(\exp \pi i \left(\frac{\epsilon - \hat{c}_i(x)}{2\epsilon}\right) \cdot \epsilon\right) & \text{if } \hat{c}_i(x) \in (-\epsilon, \epsilon). \end{cases}$$

The idea is to deform the real axis into the upper half plane in a small neighbourhood of the origin so that it avoids the origin. Then:

$$\sum_{i=1}^{M} c_i(x)v_i(x) + \sum_{i=M+1}^{M+K+1} v_i(x) = \sum_{i=1}^{M} (c_i - \hat{c}_i)(x)v_i(x).$$

We note that $\mid c_i - \hat{c}_i \mid \leq \epsilon$. Now let us set $v_j(x) = \sum_{i=M+1}^{M+K} d_{ji}(x)v_i(x)$, for $j = 1, \ldots, M$. Substituting, we have

$$\sum_{i=1}^{M} c_i(x)v_i(x) + \sum_{i=M+1}^{M+K} \left(1 - \sum_{j=1}^{M} (c_j - \hat{c}_j)d_{ji}\right) v_i(x) + v_{M+K+1}(x) = 0.$$

Taking ϵ sufficiently small, and setting $c_i = 1 - \sum_{j=1}^{M} (c_j - \hat{c}_j)d_{ji}$, $i = M+1, \ldots, M+K$ gives the result. We note that as the c_i never encounter, say, the negative imaginary axis, the map $(c_1, \cdots, c_{M+K})$ is automatically homotopic in $(\mathbb{C}^*)^{M+K}$ to the constant map.

We apply this lemma to our situation, with $\ell = rg$, $M + K = d$, as follows: one sets $v_i(x)$, $i = 1, \cdots, d$, to be the individual contributions in $H^1(\Sigma, Q_1(-p_0)) \simeq \mathbb{C}^{rd}$ of our marked single poles, and lump the contributions of the other poles together into $v_{M+K+1}(x)$. The lemma then tells us that we can rescale $v_i(x)$, $i = 1, \cdots, M+K$, so that the total obstruction $\Sigma v_i(x)$ vanishes. This rescaling is performed by using the adjoint action of the group H.

What this then means is that, once this rescaling is done, if one represents the principal parts by holomorphic N-valued functions n^{i0} over $D_i \cap D_0$, one can find holomorphic N-valued functions n^i on the D^i such that $n^i n^{i0}(n^0)^{-1}$ takes values in the subgroup M_2.

More generally, let us suppose that we have $n^i n^{i0}(n^0)^{-1} \in M_\mu$ for all i. We would like to modify n^{i0} to $\hat{n}^{i0}$, say, so that one can solve $\hat{n}^i\hat{n}^{i0}(\hat{n}^0)^{-1} \in M_{\mu+1}$, for suitable $\hat{n}^i, \hat{n}^0$. To begin, we note that if q^i is some element of $U^-_{\mu-1}$, the $(\mu-1)$-th subgroup of the filtration (2.4), then $Ad_{q^i}(n^{i0})$ and n^{i0} have the same image in N/M_μ, so that $n^i Ad_{q^i}(n^{i0})(n^0)^{-1} \in M_\mu$. Writing

$$Ad_{q^i}(n^{i0}) = \rho^{i0}n^{i0}, \quad \rho^{i0} \in M_\mu,$$

and projecting to $Q_\mu = M_\mu/M_{\mu+1}$, we have:

$$\begin{aligned}\pi_\mu(n^i Ad_{q^i}(n^{i0})(n^0)^{-1}) &= \pi_\mu(n^i\rho^{i0}n^{i0}(n^0)^{-1}) \\ &= \pi_\mu(n^i\rho^{i0}(n^i)^{-1}n^i n^{i0}(n^0)^{-1}) \\ &= \pi_\mu(\rho^{i0}) + \pi_\mu(n^i n^{i0}(n^0)^{-1}).\end{aligned}$$

This scheme allows us to use the action, not of the torus, but of U^- to modify the cocyle in $H^1(\Sigma, Q_\mu)$, adding to it $\pi_\mu(\rho^{i0})$. To fix our ideas, let us consider a simple example, taking $U^- = N$ in $S\ell(4,\mathbb{C})$. Let us write n^{i0} (fixing i) as:

$$n^{i0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n_1 & 1 & 0 & 0 \\ n_2 & n_3 & 1 & 0 \\ n_4 & n_5 & n_6 & 1 \end{pmatrix} \tag{4.8}$$

for some functions n_j, and let us suppose that there exist n^i, n^0 so that $n^i n^{i0}(n^0)^{-1}$ is of form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m & 0 & 0 & 1 \end{pmatrix} \tag{4.9}$$

for some function m, so that one is in M_3. Conjugating n^{i0} by q^i in M_2, where

$$q^i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ r & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.10}$$

gives:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ n_1 & 1 & 0 & 0 \\ n_2 & n_3 & 1 & 0 \\ n_4 - rn_6 & n_5 & n_6 & 1 \end{pmatrix} \tag{4.11}$$

and so

$$\rho^{i0} = \begin{pmatrix} 1 & 0\,0\,0 \\ 0 & 1\,0\,0 \\ 0 & 0\,1\,0 \\ -rn_6 & 0\,0\,1 \end{pmatrix}. \tag{4.12}$$

We can thus modify the element m in (4.10) by adding to it $(-rn_6)$.

We now remark that our genericity assumptions tell us that the residues of the simple root terms (n_1, n_3 and n_6 in the example) of the simple poles span $Q_1 \otimes H^1(\Sigma, \mathcal{O}(-p_0))$ and that the commutator gives a surjective map $Q_1 \otimes T_{\mu-1} \to Q_\mu$. The presence of sufficiently many simple poles in generic position will allow us to kill the μ-th obstruction, choosing q^i's so that $\pi_\mu(n^i Ad_{q^i}(n^{i0})(n^0)^{-1})$ vanishes in $H^1(\Sigma, Q_\mu(-p_0))$. We can then modify n^i, n^0 into $\hat{n}^i, \hat{n}^0$ so that

$$\hat{n}^i Ad_{qi}(n^{i0})\hat{n}^{0-1} \in M_{\mu+1},$$

giving us our inductive step, at least for one configuration. We now have to see how this should be done for a family, by a transformation which is a deformation.

Given a family $n^{i0}(s), s \in B^n$ of these cocyles, we will proceed in a fashion similar to that of Lemma (4.5). The part of the adjoint action of q^i that we are interested in only depends on its projection to $T_{\mu-1} \simeq \mathbb{C}^{t_\mu}$ (recall (2.4)). We consider the subbundle F of $(T_{\mu-1})^{rd} \times B^n$ defined by

$$F = \Big\{(q^i, x)\Big| \sum_{i=1}^{rd} \pi_\mu(n^i Ad_{q^i}(n^{i0}(x))(n^0)^{-1}) \\ + \sum_{i>rd} \pi_\mu(n^i n^{i0}(x)(n^0)^{-1}) = 0 \text{ in } Q_\mu\Big\},$$

where the first (rd) points of our configuration are taken to be our marked simple poles and $\mathcal{Q}_\mu$ denotes the space $H^1(\Sigma, Q_\mu(-p_0))$. By our genericity assumption, this is an affine subbundle, and we choose a projection $\pi : (T_{\mu-1})^d \times B^n \to F$. The natural 0-section $0 : B^n \to (T_{\mu-1})^d \times B^n$ lies in F over the boundary of B^n, and one defines the $q^i(x)$ by $\pi \circ 0$, which is naturally homotopic to 0, and so gives us our deformation.

This completes the inductive step; as $M_\mu = 0$ for μ sufficiently large, one can, after completing this deformation process, solve $n^i(x)n^{i0}(x) = n^0(x)$ over the simplex and so obtain holomorphic maps.

Combining Theorems (3.2) and (4.4), proves the stability result, Theorem (1.3)

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