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METRIC STRUCTURE OF CUT LOCI IN SURFACES AND AMBROSE'S PROBLEM

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Abstract

It is shown that every compact subset of the cut locus of a point in a complete, two-dimensional Riemannian manifold has finite one-dimensional Hausdorff measure. When combined with a result from an earlier paper, this completes the solution of the two-dimensional case of a long-standing problem of Ambrose.

1. Introduction

This paper is devoted to the proof of the following theorem.

Theorem 1.1. *Let M be a complete, 2-dimensional Riemannian manifold, and let p be a point in M. Then every compact subset of the cut locus of p in M has finite 1-dimensional Hausdorff measure.*

Consequently, the cut locus of a point in a complete, two-dimensional Riemannian manifold must satisfy stringent conditions relative to the differentiable structure of the manifold. This means that the problem of determining what subsets of a given surface can be realized as the cut locus of some point for some Riemannian metric depends not only on the topology, but also on the metric structure (or more precisely on the local Lipschitz structure) that they inherit as subsets of the surface. To illustrate this, recall that Gluck and Singer [5] show how a certain subset of the sphere, consisting of countably many great circle arcs radiating out from a common endpoint, can be realized as a cut locus. Altering this set by extending the arcs slightly, so that their lengths form a divergent series, results in a set homeomorphic to the original, but possessing infinite Hausdorff locus. (1-measure, and so no longer realizable as a cut locus (c.f. Example 6.2).

Along the way to proving Theorem 1.1, it is shown that embedded arcs in the cut locus of a point in a surface have finite length (Theorem 4.7). Thus there is an intrinsic metric on the cut locus in which the distance between two cut points is defined to be the length of the shortest arc in

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the cut locus that joins them. We will use Theorem 1.1 to show that this intrinsic metric induces the same topology on the cut locus as that which it inherits as a topological subspace of the surface (Theorem 6.1).

Recall that W. Ambrose proved that a complete, connected, simply connected, Riemannian manifold is determined up to a global isometry by the behavior of the curvature tensor under parallel translation along all the broken geodesics emanating from a given point [1]. This result is sometimes called the Cartan-Ambrose-Hicks Theorem (e.g., [2, pp. 37–40]). However, Ambrose thought the hypothesis might be too strong and posed the problem to decide whether or not the behavior of curvature along the *unbroken* geodesics emanating from the point would yet be enough to determine the manifold.

In an earlier paper [8], the author solved Ambrose's problem in the case of a surface for which every compact subset of the cut locus of the given point has finite one-dimensional Hausdorff measure. Thus combining Theorem 1.1 with Theorem 1 in [8] results in the next theorem which completes the solution of Ambrose's problem in dimension 2. For higher dimensions, Ambrose's problem remains unsolved (*cf.* [6], [8]).

Theorem 1.2. *Suppose that M and $\overline{M}$ are both complete, connected Riemannian manifolds of dimension 2, possessing respective Gaussian curvature functions $\kappa : M \to \mathbf{R}$ and $\bar{\kappa} : \overline{M} \to \mathbf{R}$. Given points $p \in M$ and $\bar{p} \in \overline{M}$, let $I : T_pM \to T_{\bar{p}}\overline{M}$ be a linear isometry between the tangent spaces at p and $\bar{p}$ respectively. Assume that M is simply connected, and that*

$$\kappa \circ \exp_p(X) = \bar{\kappa} \circ \exp_{\bar{p}}(I(X))$$

for all $X \in T_pM$ where $\exp_p : T_pM \to M$ and $\exp_{\bar{p}} : T_{\bar{p}}\overline{M} \to \overline{M}$ are the exponential maps at p and $\bar{p}$ respectively. Then there exists an isometric immersion $F : M \to \overline{M}$ such that $F(p) = \bar{p}$ and $dF = I$ at p.

An isometric immersion between two complete Riemannian manifolds of the same dimension is a covering map. Thus in Theorem 1.2 if $\overline{M}$ is also simply connected, then F must be an isometry.

In contrast to higher dimensions, parallel translation of curvature plays no part in the hypothesis of the Cartan-Ambrose-Hicks Theorem in dimension 2. This accounts for the simple formulation of the curvature assumption in Theorem 1.2. This assumption can be rephrased in terms of unbroken geodesics emanating from p. Every geodesic segment $\gamma : [0, 1] \to M$ emanating from p takes the form $\gamma(t) = \exp_p(tX)$ for $0 \leq t \leq 1$ for some $X \in T_pM$. By means of the linear isometry I, a corresponding geodesic $\bar{\gamma} : [0, 1] \to \overline{M}$ emanating from $\bar{p}$ is defined by $\bar{\gamma}(t) = \exp_{\bar{p}}(tI(X))$

for $0 \leq t \leq 1$. Obviously the curvature assumption in Theorem 1.2 is equivalent to the assumption that $\bar{\kappa}(\bar{\gamma}(1)) = \kappa(\gamma(1))$ for every geodesic segment γ emanating from p, where $\bar{\gamma}$ is the corresponding geodesic in $\overline{M}$ emanating from $\bar{p}$.

Throughout this paper, $\mathscr{H}^1$ denotes 1-dimensional Hausdorff measure [3], [4], [11], and $A \backslash B$ denotes the relative complement of the set B in the set A. All Riemannian manifolds are smooth, i.e., C^∞.

2. Topological structure of cut loci

Let M be a complete n-dimensional Riemannian manifold, let p be a point in M, and let $C(p)$ be the cut locus of p in M. A cut point $q \in C(p)$ is said to be a *conjugate cut point* if it is conjugate to p along at least one minimizing geodesic joining p to q, and is said to be a *nonconjugate cut point* otherwise. The *order* of a nonconjugate cut point $q \in C(p)$ is the number of minimizing geodesics joining p to q. The order is always finite and at least 2. A cut point is said to be a *cleave point* if it is a nonconjugate cut point of order two; it is a *noncleave point* otherwise. The topological structure of $C(p)$ is summarized in the following proposition which was proved in [8, Proposition 1.2].

Proposition 2.1. *Let M be a complete n-dimensional Riemannian manifold, and let $p \in M$. The cut locus, $C(p)$, is a closed subset of M. The set of cleave points is a relatively open subset of $C(p)$ forming a smooth $(n-1)$-dimensional submanifold of M. The set of noncleave points in $C(p)$ is a closed subset whose $(n-1)$-dimensional Hausdorff measure is zero.*

Remark 2.2. By applying the Morse-Sard-Federer Theorem [4, 3.4.3], [11, p. 113] in place of the version of Sard's Theorem used in the proof of the above proposition, one obtains the stronger conclusion that the s-dimensional Hausdorff measure of the set of noncleave points in $C(p)$ is 0 for all $s > n-2$. Thus the Hausdorff dimension of the set of noncleave points is at most $n-2$.

It is also known that at a cleave point q, the two minimizing geodesics joining p to q make the same angle with the tangent space at q to the submanifold of cleave points, but from opposite sides [14, Remark 2.5]. Further, if M is compact, then $C(p)$ is connected and nonempty. If M is not compact, then $C(p)$ may be empty, or may have countably many connected components, none of which is compact. Only finitely many components meet a given compact subset of M.

When M is a surface, *i.e.*, when $n = 2$, the topological structure of $C(p)$ was investigated in more detail by S. B. Myers [12], [13]. Three definitions are needed in order to concisely state his work. An *arc* is a topological space homeomorphic to the unit interval [0,1]. A *tree* is a topological space with the property that every pair of points, q_1, q_2, is contained in a unique arc with endpoints q_1, q_2. A *local tree* is a topological space in which every point is contained in arbitrarily small closed neighborhoods which are themselves trees.

Theorem 2.3 (Myers). *Let M be a complete two-dimensional Riemannian manifold, and let $C(p)$ be the cut locus of a point $p \in M$. Then $C(p)$ is a local tree. Furthermore, if M is simply connected, then each connected component of $C(p)$ is a tree.*

Myers' results are even stronger when the Riemannian metric on M is real analytic. Then he showed the cut locus is triangulable. Gluck and Singer [5] have produced examples of smooth metrics with nontriangulable cut loci.

Each point of a local tree is either an endpoint, ordinary point, or branch point depending upon the number (one, two, or more respectively) of connected components possessed by a deleted neighborhood of the point. The relation between Proposition 2.1 and Theorem 2.2 is that (1) every endpoint is a conjugate cut point, (2) every cleave point is ordinary, and (3) every nonconjugate, noncleave point is a branch point. These three statements follow easily from Myers' paper, especially §3 of [13].

Remark 2.4. There are countably many branch points in the cut locus in a surface since the cut locus will be an increasing union of countably many compact subtrees, and compact trees have countably many branch points [10, p. 309].

The set of endpoints of a local tree embedded in a surface may be uncountable, although it must be a totally disconnected set [10, p. 309]. However, if it is a cut locus, then Remark 2.2 implies that the Hausdorff dimension of the set of ends be 0. This is a restriction upon which local trees can be cut loci.

Example 2.5. Here is an example of a compact tree in the Euclidean plane $\mathbf{R}^2$ whose set of endpoints is the Cantor set. Since the Cantor set has Hausdorff dimension $(log 2)/(log 3)$, the dimension of the set of endpoints of this tree is not zero [3, p. 14]. By regarding $\mathbf{R}^2$ as a local coordinate system on the two-dimensional sphere, were it not for Remark 2.2, this tree might have been a candidate to be the cut locus of a point for some Riemannian metric on the sphere. Note that this example does have finite Hausdorff 1-measure. Thus its failure to be a cut locus is not on account of Theorem 1.1.

This example is constructed in parallel with the usual Cantor set construction. Following [10, pp. 310–311], for any closed bounded interval $[a, b]$, let the union of the two line segments in $\mathbf{R}^2$ which join the point $((a+b)/2, (a-b)/2)$ to the two points $(a, 0)$ and $(b, 0)$ be called the *roof* on $[a, b]$. Start the construction by forming the roof on $[0, 1]$. Then form the roofs on $[0, \frac{1}{3}]$ and on $[\frac{2}{3}, 1]$, and, in general, form the roofs on each of the 2^n intervals of length 3^{-n} that arise in the nth stage of the construction of the Cantor set. The closure of the union of all these roofs is a tree in $\mathbf{R}^2$ whose set of endpoints is the Cantor set contained in the x-axis. Since the one-dimensional Hausdorff measure of this tree is just the sum of the lengths of all the line segments constructed to form the roofs, it is routine to verify that its Hausdorff 1-measure has the exact value $2\sqrt{2}$.

3. An arclength estimate

Lemma 3.1. *Let A be an oriented arc in $\mathbf{R}^2$ starting at $q_0 = (x_0, y_0)$. Assume $A = V \cup S$ where V, relatively open in A, is a smooth one-dimensional submanifold of $\mathbf{R}^2$, and $\mathscr{H}^1(S) = 0$. Assume that the forward pointing tangent vectors*

$$a(q)\left.\frac{\partial}{\partial x}\right|_q + b(q)\left.\frac{\partial}{\partial y}\right|_q$$

satisfy (i) $a(q) > 0$ *and* (ii) $|b(q)/a(q)| < 1$ *at every point q of V. Then*

(1) *the projection mapping $x\colon A \to \mathbf{R}$ on the first coordinate is a strictly increasing function of the parameter on A, and*

(2) *for all $q_1, q_2 \in A$,*

$$\mathscr{H}^1(A') \leq \sqrt{2}\,|x(q_2) - x(q_1)|,$$

where A' is the subarc of A from q_1 to q_2.

Proof. Since A is an arc, it is possible to parameterize A by the homeomorphism $h(t) = (x(t), y(t))$ for $t \in [0, 1]$, where $h(0) = (x(0), y(0)) = (x_0, y_0)$. Set $G = h^{-1}(V)$, $E = h^{-1}(S)$. The hypothesis (i) implies that for all $t \in G$, there exists arbitrarily small $\Delta t > 0$ such that $x(t + \Delta t) \geq x(t)$.

We first show that x is monotone increasing. For if not there would exist $t_0 < t_1$ such that $x(t_0) > x(t_1)$. Since the projection map is Lipschitz, and since $\mathscr{H}^1(S) = 0$, it follows that the image of the projection of

S has $\mathscr{H}^1$-measure zero. Thus there exists an $x^* \in (x(t_1), x(t_0))$ that is not in the image of S.

Set

$$t^* = \sup\{t \in [t_0, t_1] : x(t) \geq x^*\}.$$

Then by continuity of x, $x(t^*) = x^*$. Thus $t^* \in G$ and $t^* < t_1$. By construction, $x(t) < x^* = x(t^*)$ for all $t \in (t^*, t_1]$ which contradicts the assertion that $x(t^* + \Delta t) \geq x(t^*)$ for some arbitrarily small Δt. Hence x is monotone increasing.

It is strictly increasing, for if not there would exist a nondegenerate interval $I \subset [0, 1]$ on which x is constant, say c. Thus $h(I)$ is an arc contained in $A \cap (\{c\} \times \mathbf{R})$. But $V \cap (\{c\} \times \mathbf{R})$ consists of at most countably many points. (Assumption (i) implies each component of V is transverse to all vertical lines, hence meets each vertical line in at most one point. There are only countably many components of V.) And since $\mathscr{H}^1(S) = 0$, $S \cap (\{c\} \times \mathbf{R})$ has $\mathscr{H}^1$-measure zero. Thus $\mathscr{H}^1(A \cap (\{c\} \times \mathbf{R})) = 0$, which contradicts that $A \cap (\{c\} \times \mathbf{R})$ contains an arc. Therefore x is strictly increasing. This proves (1). Thus A is the graph of a function $y = y(x)$.

Given $q_1, q_2 \in A$, let A' be the subarc of A from q_1 to q_2, and let V_i be the countably many components of $A' \cap V$. Set $(a_i, b_i) = x(V_i)$. These intervals are mutually disjoint since the projection map x is one-to-one on A. The function $y = y(x)$ is smooth on (a_i, b_i) because its graph V_i is smooth. Since the Hausdorff 1-measure of a rectifiable curve is its arclength [3, p. 29], the calculus formula for arclength and (ii) gives

$$\mathscr{H}^1(V_i) = \int_{a_i}^{b_i} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \leq \sqrt{2}\,(b_i - a_i),$$

in consequence of $dy/dx|_{x(q)} = b(q)/a(q)$. Therefore,

$$\begin{aligned} \mathscr{H}^1(A') &= \mathscr{H}^1(A' \cap S) + \sum_i \mathscr{H}^1(V_i) \\ &\leq \sqrt{2} \sum_i (b_i - a_i) \leq \sqrt{2}\,|x(q_2) - x(q_1)|, \end{aligned}$$

because S has measure zero, and the intervals (a_i, b_i) are mutually disjoint subintervals of $x(A')$.

An arbitrary Riemannian metric on $\mathbf{R}^2$ induces a distance function ρ which is locally Lipschitz equivalent to the Euclidean distance d, i.e., for every compact set K, there exists $k > 0$ such that

$$k^{-1} d(q_1, q_2) \leq \rho(q_1, q_2) \leq k d(q_1, q_2)$$

for all $q_1, q_2 \in K$. Hence on all subsets of K we have $\mathscr{H}_d^1 \leq k\mathscr{H}_\rho^1$ and $\mathscr{H}_\rho^1 \leq k\mathscr{H}_d^1$, where the subscripts denote which distance function was used to define $\mathscr{H}^1$. Therefore the condition that a subset has zero Hausdorff 1-measure and the condition that a compact subset has finite Hausdorff 1-measure are independent of which of the two distance functions, d or ρ, is used to define the measure [4, 3.2.46].

Corollary 3.2. *Under the same hypothesis as Lemma 3.1, but now with $\mathscr{H}^1$ defined using the distance function ρ induced by an arbitrary Riemannian metric on $\mathbf{R}^2$, there exists a constant C such that for all $q_1, q_2 \in A$,*

$$\mathscr{H}^1(A') \leq C\rho(q_2, q_1),$$

where A' is the subarc of A from q_1 to q_2.

Proof. This follows immediately from Lemma 3.1. The value of C is $\sqrt{2}k^2$, where k is the Lipschitz constant relating d and ρ on some compact set K containing A.

4. Arcs in the cut locus

Throughout this section assume that M is a complete two-dimensional Riemannian manifold, and that $C(p)$ is the cut locus of a point $p \in M$. Let ρ denote the Riemannian distance function on M. For $q \in M$ and $r > 0$, let

$$D(q; r) = \{x \in M : \rho(x, q) < r\}$$

denote the metric disk in M centered at q with radius r.

If $q \in C(p)$, then the *link* at q is the set Λ_q consisting of the unit tangent vectors at q that are tangent to a minimizing geodesic joining q to p. Clearly, Λ_q is a closed nonempty subset of the circle $S_q = \{X \in T_qM : |X| = 1\}$ of unit tangent vectors at q. Obviously $\Lambda_q = S_q$ if and only if $C(p) = \{q\}$. (See [7].)

Proposition 4.1. *Let $q \in C(p)$, and let $r > 0$ be less than the injectivity radius of M at q. Let Γ be a closed neighborhood of q in $C(p)$ which is a tree and satisfies $\Gamma \subset D(q; r)$. (Such neighborhoods exist because $C(p)$ is a local tree by Theorem* 2.3.) Then the connected components of $S_q \backslash \Lambda_q$ are in one-to-one correspondence with the connected components of the deleted neighborhood $\Gamma \backslash \{q\}$.

Proof. The connected components of the set

$$E = \{\exp(tX) : X \in S_q \backslash \Lambda_q,\ 0 < t < r\}$$

are open wedges in the disk $D(q\,; r)$. Clearly, these wedge components W of E are in one-to-one correspondence with the components J of $S_q \backslash \Lambda_q$ where W corresponds to J if and only if

$$W = \{\exp(tX) : X \in J\,,\ 0 < t < r\}.$$

In turn, these wedges W are in one-to-one correspondence with the components K of $\Gamma \backslash \{q\}$ where W corresponds to K if and only if $K = \Gamma \cap W$.

Lemma 4.2. *Let $q \in C(p)$, and let Γ and $r > 0$ be as in Proposition 4.1. Suppose K is a connected component of $\Gamma \backslash \{q\}$ and J is a component interval of $S_q \backslash \Lambda_q$ that correspond to each other under Proposition 4.1. Let X^+ and X^- be the endpoints of J. Let q_n be a sequence of points in K converging to q.*

If X_n is a sequence of vectors that converges to $X_0 \in TM$ such that $X_n \in \Lambda_{q_n}$ for all n, then either $X_0 = X^+$ or $X_0 = X^-$.

Proof. Since the geodesics $\exp(tX_n)$, $0 \leq t \leq \rho(q_n\,, p)$, are minimizing geodesics joining q_n to p, the geodesic to which they converge, $\exp(tX_0)$, $0 \leq t \leq \rho(q_0\,, p)$, is minimizing joining q to p. Thus $X_0 \in \Lambda_q$. None of the geodesics $\exp(tX_n)$, $0 \leq t \leq \rho(q_n\,, p)$, meet either of the two geodesics $\exp(tX^+)$ or $\exp(tX^-)$, $0 \leq t \leq \rho(q_0\,, p)$, except at p, because the latter two are minimizing from q to p. Hence, when n is sufficiently large, i.e., when $\rho(q_n\,, q_0) < r/2$, then $\exp((r/2)X_n) \in W$, where

$$W = \{\exp(tX) : X \in J\,,\ 0 < t < r\}.$$

Therefore $\exp((r/2)X_0)$ is in the closure of the wedge W. Since $X_0 \in \Lambda_q$, either $X_0 = X^+$ or $X_0 = X^-$.

Let A be an oriented arc in $C(p)$ starting at q_0. Then A determines a unique component interval of $S_{q_0} \backslash \Lambda_{q_0}$ as follows: Choose $r > 0$ less than both the diameter of A and the injectivity radius of M at q_0. Let q^* be the first point on A such that $\rho(q_0\,, q^*) = r$. There is such a point because r is less than the diameter of A. Let A_0 be the subarc of A from q_0 to q^*. Choose a tree neighborhood Γ of q_0 in $C(p)$ such that $\Gamma \subset D(q_0\,; r)$. Then A_0 meets exactly one of the components K of $\Gamma \backslash \{q_0\}$. Clearly the component interval of $S_{q_0} \backslash \Lambda_{q_0}$ that corresponds to K under Proposition 4.1 is independent of the choice of both r and Γ. Hence it is determined only by the arc A.

Continuing, let J denote the component interval of $S_{q_0} \backslash \Lambda_{q_0}$ determined by A. The two endpoints $X^+, X^- \in \Lambda_{q_0}$ of J will be called respectively the *left* and *right link vectors determined by* A. Next set

$$W = \{\exp(tX) : X \in J,\ 0 < t < r\}.$$

Clearly $A_0 \backslash \{q_0, q^*\} \subset W$. Hence

$$W \backslash A_0 = W^+ \cup W^-,$$

where W^+ and W^- are two disjoint open topological disks in M. (This uses the Jordan Curve Theorem.) Here notation is chosen so that the geodesic segment $\exp(tX^+)$, $0 \le t \le r$, lies in the closure of W^+, and $\exp(tX^-)$, $0 \le t \le r$, lies in the closure of W^-. For any $q \in C(p) \cap W$ and $X \in \Lambda_q$, the geodesic $\exp(tX)$, $0 \le t \le \rho(q, p)$, will meet no point of $C(p)$ other than q, because it minimizes from q to p. In particular, it does not meet A unless $q \in A$. Hence there exists an $\epsilon > 0$ such that the geodesic segment $\exp(tX)$, $0 < t < \epsilon$, either lies in W^+ or in W^-. Depending upon which one of the two cases holds, we will say X is either on the *left side* or the *right side* of A respectively.

Lemma 4.3. *Let A be an oriented arc in $C(p)$ starting at q_0. Let J be the component interval of $S_{q_0} \backslash \Lambda_{q_0}$ determined by A, and let X^+ and X^- be the left and right link vectors determined by A. Let*

$$W = \{\exp(tX) : X \in J,\ 0 < t < r\},$$

where $r > 0$ is less than both the diameter of A and the injectivity radius of M at q_0. Suppose q_n is a sequence of points in $C(p) \cap W$ converging to q_0, and suppose $X_n \in \Lambda_{q_n}$ is on the left side (respectively right side) of A for all n. Then the sequence X_n converges to X^+ (respectively X^-).

Proof. We will consider only the left side case. The right side case is similar.

Let $\Gamma \subset D(q_0; r)$ be a tree neighborhood of q_0 in $C(p)$. We may assume $q_n \in K = \Gamma \cap W$ for all n because q_n converges to q_0. Since X_n is a unit vector at q_n, the X_n are contained in a compact subset of the tangent bundle. Hence it suffices to show every convergent subsequence of the X_n converges to X^+. Thus we may as well assume X_n converges to some $X_0 \in TM$. By Lemma 4.2, either $X_0 = X^+$ or $X_0 = X^-$. But as we saw in the proof of Lemma 4.2, none of the geodesics $\exp(tX_n)$, $0 \le t \le \rho(q_n, p)$, meet either of the two geodesics $\exp(tX^+)$ or $\exp(tX^-)$, $0 \le t \le \rho(q_0, p)$, except at p. Hence, when n is sufficiently large, i.e. when $\rho(q_n, q_0) < r/2$, then $\exp((r/2)X_n) \in W^+$, because X_n is on the left side of A for all n. Therefore $\exp((r/2)/X_0)$ is in the closure of W^+ which implies $X_0 = X^+$.

Lemma 4.4. *Let A be an arc in $C(p)$. Then $A = V \cup S$ where S is a closed set with $\mathscr{H}^1(S) = 0$, and V, which is relatively open in A, forms a smooth one-dimensional submanifold of M.*

Proof. This follows immediately from Proposition 2.1 upon setting S to be the intersection of A with the set of noncleave points of $C(p)$ and V to be the intersection of A with the set of cleave points in $C(p)$.

Let $q \in C(p)$. A midpoint of one of the component intervals of $S_q \backslash \Lambda_q$ will be called a *$C(p)$-tangent vector at q*. In the special case when q is a cleave point, Λ_q consists of two vectors, and by Proposition 2.1, there exists a tree neighborhood of q in $C(p)$ which is a smoothly embedded arc passing through q. Because the two vectors in Λ_q make the same angle with this arc but on opposite sides, the $C(p)$-tangent vectors at q are just the two unit tangent vectors to this smooth arc considered as a submanifold.

Now let A be an oriented arc in $C(p)$ starting at q_0 and ending at q_1. For each $q \in A$, $q \neq q_1$, let $A(q)$ denote the subarc of A starting at q and ending at q_1. Define $Y(q)$ to be the $C(p)$-tangent vector at q which is the midpoint of the component interval of $S_q \backslash \Lambda_q$ determined by $A(q)$. The resulting function $Y: A\backslash\{q_1\} \to TM$ will be called the *field of forward pointing $C(p)$-tangent vectors along A*.

Lemma 4.5. *Let A be an oriented arc in $C(p)$ starting at q_0 and ending at q_1, and let $Y: A\backslash\{q_1\} \to TM$ be the field of forward pointing $C(p)$-tangent vectors along A. Then whenever q is in the smooth set V of Lemma 4.4, $Y(q)$ is just the oriented unit tangent vector to V at q. Furthermore,*

$$\lim_{q \to q_0} Y(q) = Y(q_0).$$

Proof. The first statement follows immediately from the fact mentioned earlier that the two $C(p)$-tangent vectors at a cleave point are tangent to the submanifold of cleave points.

Let q_n be a sequence in A converging to q_0. We must show

$$\lim_{n \to \infty} Y(q_n) = Y(q_0).$$

Let $r > 0$ be less than both the diameter of A and the injectivity radius of M at q_0. Let J be the component interval of $S_{q_0} \backslash \Lambda_{q_0}$ determined by A, and let X^+ and X^- be the left and right link vectors at q_0 determined by A. Without loss of generality we may assume

$$q_n \in W = \{\exp(tX) : X \in J,\ 0 < t < r\}$$

for all n. The right and left link vectors X_n^+ and X_n^- at q_n of the subarc

$A(q_n)$ of A from q_n to q_1 can be chosen consistently so that X_n^+ is on the left and X_n^- is on the right side of A for all n. By Lemma 4.3, X_n^+ converges to X^+, and X_n^- converges to X^-.

If $\lim_{n\to\infty} Y(q_n) \neq Y(q_0)$, then we can find a subsequence of $Y(q_n)$ that converges to some tangent vector $Y_0 \neq Y(q_0)$. Hence we can assume that $\lim_{n\to\infty} Y(q_n) = Y_0$. Now by definition, $Y(q_n)$ is the midpoint of an interval in S_{q_n} whose endpoints are X_n^+ and X_n^-. Thus in the limit, Y_0 is a midpoint of an interval in S_{q_0} whose endpoints are X^+ and X^-. This implies that $Y_0 = \pm Y(q_0)$ because by definition $Y(q_0)$ is such a midpoint. But clearly, for all n sufficiently large, i.e. when $\rho(q_n, q_0) < r/2$, then $\exp((r/2)Y(q_n)) \in W$. On taking limits, $\exp((r/2)Y_0)$ lies in the closure of W. Thus $Y_0 = Y(q_0)$ which is a contradiction. This completes the proof.

Corollary 4.6. *Let A be an oriented arc in $C(p)$ starting at q_0. There is a subarc A_0 of A containing q_0 that has finite length.*

Proof. Let $Y\colon A\backslash\{q_1\} \to TM$ be the field of forward pointing $C(p)$-tangent vectors along A. Let $r > 0$ be less than both the diameter of A and the injectivity radius of M at q_0. Choose a normal coordinate system (x, y) centered at q_0 in the disk $D(q_0; r)$ such that

$$\left.\frac{\partial}{\partial x}\right|_{(0,0)} = Y(q_0).$$

Write

$$Y(q) = a(q)\left.\frac{\partial}{\partial x}\right|_q + b(q)\left.\frac{\partial}{\partial y}\right|_q$$

for $q \in A \cap D(q_0; r)$. By Lemma 4.5, $a(q) \to a(q_0) = 1$ and $b(q) \to b(q_0) = 0$ as $q \to q_0$, $q \in A \cap D(q_0; r)$. Therefore, we may assume $a(q) > 0$ and $|b(q)/a(q)| < 1$ for all $q \in A \cap D(q_0; r)$ by taking r to be sufficiently small.

Let A_0 be any subarc of A containing q_0 that is contained in $D(q_0; r)$. By Lemmas 4.4 and 4.5, Corollary 3.2 can be applied to the arc A_0 to prove $\mathscr{H}^1(A_0) \leq Cr$ for some constant C. Therefore A_0 has finite length.

Theorem 4.7. *Suppose that M is a complete two-dimensional Riemannian manifold and that $C(p)$ is the cut locus of a point $p \in M$. Then every arc in $C(p)$ has finite length.*

Proof. Let A be an arc in $C(p)$. Then every $q_0 \in A$ has a neighborhood in A which has finite length. If q_0 is an endpoint of A, this follows directly from Corollary 4.6, while if q_0 is not an endpoint, then q_0 separates A into two subarcs both having q_0 as an endpoint. Again

Corollary 4.6 produces a neighborhood of q_0 in each subarc having finite length. The union of these neighborhoods is a neighborhood of q_0 in A with finite length. Since A is compact, it follows that A is contained in a union of finitely many subarcs of finite length. Hence A itself has finite length.

5. The Hausdorff 1-measure of compacta in $C(p)$

As in the previous section, throughout this section assume that M is a complete two-dimensional Riemannian manifold, and that $C(p)$ is the cut locus of a point $p \in M$.

Define the function $\mu: S_p \to [0, \infty]$ by setting $\mu(X)$, for $X \in S_p$, equal to the distance to the cut point along the geodesic $\exp(tX)$, $t \geq 0$, if there is a cut point, and to ∞ otherwise. It is well known that μ is a continuous function [9, p. 98]. Thus the set

$$\mathscr{U} = \{X \in S_p : \mu(X) < \infty\}$$

is open, and the map $w : \mathscr{U} \to C(p)$ defined by $w(X) = \exp(\mu(X)X)$ is continuous. It is easy to see that both μ and w are proper maps, i.e., the inverse image of compact sets is compact.

Define $\phi : S_p \to [0, \infty]$ by setting $\phi(X)$ equal to the distance to the first conjugate point along the geodesic $\exp(tX)$, $t \geq 0$, if there is such a first conjugate point, and to ∞ otherwise. In general, $\mu(X) \leq \phi(X)$ for all $X \in S_p$ because the cut point along a geodesic occurs no later than the first conjugate point. Let

$$\mathscr{V} = \{X \in S_p : \phi(X) < \infty\}.$$

It is known that $\mathscr{V}$ is an open subset of S_p, and that ϕ is smooth on $\mathscr{V}$ [12, p. 381]. Thus $\mathscr{V} \subset \mathscr{U}$, and the map $v: \mathscr{V} \to M$ defined by $v(X) = \exp(\phi(X)X)$ is smooth. Finally, define the set

$$\mathscr{C} = \{X \in \mathscr{U} : \mu(X) = \phi(X)\}.$$

Then $\mathscr{C}$ is closed, and $w(\mathscr{C})$ is the set of conjugate cut points in $C(p)$.

Let (r, θ) denote polar coordinates on T_pM. We will employ the usual interval notation to denote open and closed subintervals of the circle S_p of unit tangent vectors at p. Thus $[\alpha, \beta] = \{(r, \theta) : r = 1, \alpha \leq \theta \leq \beta\}$ and $(\alpha, \beta) = \{(r, \theta) : r = 1, \alpha < \theta < \beta\}$.

Lemma 5.1. *Suppose $[\alpha, \beta]$ is a subinterval of $\mathscr{U}$ such that $w([\alpha, \beta])$ is contained in a subtree Γ of $C(p)$. Let A_0 be the unique arc in Γ joining $w(\alpha)$ to $w(\beta)$. Then the set*

$$\Delta = \{\exp(tX) : X \in [\alpha, \beta],\ 0 \leq t \leq \mu(X)\}$$

is a topological disk in M whose boundary consists of the arc A_0 together with the two geodesic segments

$$\gamma_\alpha(t) = \exp(t\alpha) \qquad (0 \le t \le \mu(\alpha))$$

and

$$\gamma_\beta(t) = \exp(t\beta) \qquad (0 \le t \le \mu(\beta)).$$

Furthermore,

$$w([\alpha, \beta]) = \Delta \cap C(p),$$

and the initial tangent vector of every minimizing geodesic joining p to a cut point in the interior of Δ lies in (α, β). Moreover, $(\alpha, \beta) \cap \mathscr{C} \neq \varnothing$, unless $w([\alpha, \beta]) = A_0$.

Proof. Note that $A_0 \subset w([\alpha, \beta])$ because $w([\alpha, \beta]) \subset \Gamma$ is connected. Since Γ is a tree, the arc A_0 and the restriction $w|[\alpha, \beta]$ are homotopic in Γ keeping endpoints fixed. The loop $\gamma_\alpha \cdot (w|[\alpha, \beta]) \cdot \gamma_\beta^{-1}$ is contractible in the set Δ since it is obviously the image under exp of a contractible loop in T_pM. It follows that the simple closed curve

$$\sigma = \gamma_\alpha \cdot A_0 \cdot \gamma_\beta^{-1}$$

is contractible. Therefore, σ bounds a disk in M, which is obviously the set Δ.

From the definitions of w and Δ, it follows immediately that $w([\alpha,\beta]) \subset \Delta \cap C(p)$. To see that $\Delta \cap C(p) \subset w([\alpha, \beta])$, let $q \in \Delta \cap C(p)$, and let $X \in S_p$ be the initial tangent vector of a minimizing geodesic segment γ joining p to q. Then either $X \in [\alpha, \beta]$, or γ must intersect the boundary σ of Δ at some point other than p. The latter case can occur only if γ meets σ at $q \in A_0$. Thus, in either case, $q \in w([\alpha, \beta])$. Note that this argument shows that the initial tangent vector of every minimizing geodesic from p to a cut point q in the interior of Δ lies in (α, β).

The simple connectivity of Δ implies that $\Delta \cap C(p)$ is a tree (*cf.* Theorem 2.3). Thus unless $w([\alpha, \beta]) = A_0$, $C(p)$ will have an endpoint q in the interior of Δ. The initial tangent vector to the minimizing geodesic segment from p to q will provide an element of $(\alpha, \beta) \cap \mathscr{C}$ since every endpoint of $C(p)$ is a conjugate cut point.

Lemma 5.2. *Let $[\alpha, \beta]$ be a subinterval of $\mathscr{U} \cap \mathscr{V}$ such that $w([\alpha, \beta])$ is contained in a subtree Γ of $C(p)$. Suppose that there exist an open set U in T_pM and a local coordinate system (x, y) defined throughout some open set D in M satisfying the conditions:*

(1) $\Gamma \subset D$.

(2) $\exp(U) \subset D$.

(3) *For all* $(r, \theta) \in U$, (i) $f(r, \theta) > 0$, *and (ii)* $|g(r, \theta)/f(r, \theta)| < 1$ *where* f *and* g *are the real-valued functions defined throughout* U *by*

$$\exp_*\left(\frac{\partial}{\partial r}\Big|_{(r,\theta)}\right) = f(r, \theta)\frac{\partial}{\partial x} + g(r, \theta)\frac{\partial}{\partial y}.$$

(4) $(r, \theta) \in U$ *whenever* $\theta \in [\alpha, \beta]$ *and* $\mu(\theta) \leq r \leq \phi(\theta)$.

Let Δ *be the topological disk constructed from* $[\alpha, \beta]$ *by Lemma* 5.1, *and let* $q \in C(p)$ *be a cut point in the interior of* Δ. *Then we conclude*:

(5) $x(v(\theta)) \geq x(w(\theta))$ *for all* $\theta \in [\alpha, \beta]$.

(6) *Every* $C(p)$*-tangent vector*

$$Y = a\frac{\partial}{\partial x}\Big|_q + b\frac{\partial}{\partial y}\Big|_q$$

at q *satisfies* $a \neq 0$ *and* $|b/a| < 1$.

(7) *There exists exactly one such* $C(p)$*-tangent vector at* q *satisfying* $a > 0$.

(8) *There exists a cut point* q' *in the interior of* Δ *such that* $x(q') > x(q)$.

Proof. The definitions of v and w with assumptions (3.i) and (4) immediately imply (5).

Let Y be a $C(p)$-tangent vector at q, and let $X^+, X^- \in \Lambda_q$ be the endpoints of the component interval of $S_q \backslash \Lambda_q$ of which Y is the midpoint. Thus X^+ and X^- lie on opposite sides of the line in T_qM that passes through the origin and is tangent to Y, unless $X^+ = X^-$, in which case $Y = -X^+ = -X^-$. (This may happen when q is an endpoint of $C(p)$.)

Let (u, v) denote coordinates in T_qM with respect to the ordered basis

$$\{\partial/\partial x|_q, \partial/\partial y|_q\},$$

and let Y, X^+, and X^- have respective coordinates (a, b), (f^+, g^+), and (f^-, g^-). Observe that for every $X \in \Lambda_q$, if θ is the initial tangent vector to the minimizing geodesic joining p to q to which X is tangent at q, then $\theta \in (\alpha, \beta)$ by Lemma 5.1, since q is in the interior of Δ. Thus

$$-X = \exp_*\left(\frac{\partial}{\partial r}\Big|_{(r,\theta)}\right),$$

where $r = \mu(\theta)$. By assumption (4), $(r, \theta) \in U$. Applying this fact to X^+ and X^-, gives $f^+, f^- < 0$ and $|g^+/f^+|, |g^-/f^-| < 1$ because of assumption (3).

Now the line in T_pM tangent to Y has the equation

$$bu - av = 0.$$

Thus, when X^+ and X^- lie on opposite sides of the line, after interchanging X^+ and X^- if necessary, we have the two inequalities

$$bf^+ - ag^+ > 0 \quad \text{and} \quad bf^- - ag^- < 0.$$

From these we easily deduce in this case, and even more easily in the case $Y = -X^+ = -X^-$, that $a \neq 0$ because $f^+, f^- < 0$, and that $|b/a| < 1$ because $|g^+/f^+|, |g^-/f^-| < 1$. This proves (6).

Let

$$H = \{(u, v) : u > 0\}$$

be the right half-plane of T_qM. Since for every $X \in \Lambda_q$, there exists $(r, \theta) \in U$ such that

$$-X = \exp_* \left(\left. \frac{\partial}{\partial r} \right|_{(r,\theta)} \right),$$

it follows $X \notin H$ by (3.i). Thus there exists a component interval J_0 of $S_q \backslash \Lambda_q$ such that $H \cap S_q \subset J_0$. The midpoint Y_0 of J_0 is a $C(p)$-tangent vector at q that lies in H. This proves (7) because obviously no other $C(p)$-tangent vector at q lies in H.

Let K be the component of $\Gamma \backslash \{q\}$ that corresponds to this J_0 under Proposition 4.1. Let A be an arc in Γ starting at q and ending at some point $q_1 \in K$. Let Y be the field of forward pointing $C(p)$-tangent vectors along A, and set

$$Y(t) = a(t) \left. \frac{\partial}{\partial x} \right|_t + b(t) \left. \frac{\partial}{\partial y} \right|_t$$

for $t \in A \backslash \{q_1\}$. By construction $Y(q) = Y_0$. Thus $a(q) > 0$. By Lemma 4.5 we have

$$\lim_{t \to q} a(t) = a(q),$$

which implies that there is a subarc A' of A joining q to some point $q' \in A$ such that $a(t) > 0$ for all $t \in A'$. We may further assume A' is contained in the interior of Δ. Thuen due to (7), $|b(t)/a(t)| < 1$ for all $t \in A'$, and by Lemmas 4.4 and 4.5, the hypothesis of Lemma 3.1 is satisfied by the arc A'. Therefore Lemma 3.1(1) applied to A' implies that $x(q') > x(q)$.

Remark 5.3. If in Lemma 5.2 we also assume $w(\alpha) = w(\beta)$, then the coordinate projection map x restricted to the compact set $C(p) \cap \Delta = w([\alpha, \beta])$ attains its maximum value only at the point $w(\alpha) = w(\beta)$.

In fact, by Lemma 5.2(8), x does not attain its maximum value at any $q \in C(p) \cap \Delta$ that lies in the interior of Δ, while by Lemma 5.1, the set A_0, which is the set of points in $C(p) \cap \Delta$ that lie in the boundary of Δ, consists of the one point $w(\alpha) = w(\beta)$.

Lemma 5.4. *Let $[\alpha, \beta]$ be a subinterval of $\mathcal{U} \cap \mathcal{V}$ such that $w([\alpha, \beta])$ is contained in a subtree Γ of $C(p)$. Suppose that there exist an open set U in T_pM and a local coordinate system (x, y) defined throughout some open set D in M satisfying conditions* (1) *through* (4) *of Lemma* 5.2. *Assume $w(\alpha) = w(\beta)$, and $(\alpha, \beta) \cap \mathcal{C} \neq \varnothing$. Define*

$$\theta^* = \inf([\alpha, \beta] \cap \mathcal{C}),$$

and suppose $\theta^* \neq \alpha$. Then the set $A = w([\alpha, \theta^*])$ is an arc in $C(p)$ with endpoints $w(\alpha)$ and $w(\theta^*)$, and

$$\mathcal{H}^1(A) \leq \sqrt{2}|x(v(\theta^*)) - x(v(\alpha))|,$$

where $\mathcal{H}^1$ is defined using the Euclidean coordinates (x, y) on D.

Proof. Because $(\alpha, \theta^*) \cap \mathcal{C} = \varnothing$, Lemma 5.1 applied to the interval $[\alpha, \theta^*]$ implies that A is an arc in $C(p)$ with endpoints $w(\alpha)$ and $w(\theta^*)$. Let A be oriented so that it starts at $q_0 = w(\theta^*)$ and ends at $q_1 = w(\alpha)$. Let

$$Y(q) = a(q)\frac{\partial}{\partial x}\bigg|_q + b(q)\frac{\partial}{\partial y}\bigg|_q$$

denote the forward pointing $C(p)$-tangent vector along A at $q \in A\backslash\{q_1\}$. Let Δ be the topological disk constructed from $[\alpha, \beta]$ by Lemma 5.1. As we saw in Remark 5.3, since $w(\alpha) = w(\beta) = q_1$, the point q_1 is the only point in $C(p) \cap \Delta$ that lies in the boundary of Δ. Hence every $q \in A\backslash\{q_1\}$ is contained in the interior of Δ. Therefore by Lemma 5.2(6), $a(q) \neq 0$ and $|b(q)|a(q)| < 1$ for all $q \in A\backslash\{q_1\}$. Because of Lemmas 4.4 and 4.5, the hypothesis of Lemma 3.1 will be satisfied by the arc A with respect to the Euclidean coordinate system (x, y) on D once we show $a(q) > 0$ for all $q \in A\backslash\{w(q_1\}$.

To show this, let $h : [0, 1] \to A$ be a parameterization of the oriented arc A. Assume that there exists $t_1 \in [0, 1)$ such that $a(h(t_1)) < 0$. Were it the case that $a(h(t)) < 0$ for all $t \in [t_1, 1)$, then, by Lemma 3.1(1) applied to the arc $h([t_1, 1])$, $x(h(t))$ would be strictly decreasing on $[t_1, 1]$. But then, in contradiction to Remark 5.3, one would have $x(h(t_1)) > x(h(1)) = x(w(\alpha))$. Thus, let

$$t_2 = \inf\{t : t_1 \leq t < 1,\ a(h(t)) > 0\}.$$

Then $t_2 < 1$, and $x(h(t))$ is strictly decreasing on $[t_1, t_2]$ by Lemma 3.1(1) applied to the arc $h([t_1, t_2])$. By Lemma 4.3 applied to the arc

$h([t_2, 1])$, we have the one-sided limit

$$\lim_{t \to t_2^+} a(h(t)) = a(h(t_2)).$$

From the definition of t_2 it follows that $a(h(t_2)) \geq 0$, so that $a(h(t_2)) > 0$ because $a(h(t_2)) \neq 0$. Consequently, there exists $t_3 \in (t_2, 1)$ such that $a(h(t)) > 0$ for all $t \in [t_2, t_3]$. Thus $x(h(t))$ is strictly increasing on $[t_2, t_3]$ by Lemma 3.1(1) applied to the arc $h([t_2, t_3])$. Let $q = h(t_2)$. By construction, the arcs $h([t_1, t_2])$ and $h([t_2, t_3])$ lie in distinct components of $\Gamma \backslash \{q\}$. Since $x(h(t))$ is strictly decreasing on $[t_1, t_2]$ and strictly increasing on $[t_2, t_3]$, both of these arcs are to the right of q in the local coordinate system. If both arcs are oriented to start at q, then their respective forward pointing $C(p)$-tangent vectors at q will be two distinct $C(p)$-tangent vectors at q which both point to the right, contradicting Lemma 5.2(7). Therefore $a(h(t)) > 0$ for all $t \in [0, 1)$.

Hence applying Lemma 3.1 to A yields $x(w(\alpha)) > x(w(\theta^*))$ and

$$\mathscr{H}^1(A) \leq \sqrt{2}(x(w(\alpha)) - x(w(\theta^*))) \leq \sqrt{2}(x(v(\alpha)) - x(v(\theta^*))).$$

The second inequality follows from the two facts: $x(v(\alpha)) \geq x(w(\alpha))$ by Lemma 5.2(5), and $v(\theta^*) = w(\theta^*)$ because $\theta^* \in \mathscr{Q}$ by the definition of θ^* since $\mathscr{Q}$ is a closed set.

Lemma 5.5. *If $X_0 \in \mathscr{U}$, then there exists an interval $I \subset \mathscr{U}$ containing X_0 in its interior such that $\mathscr{H}^1(w(I)) < \infty$.*

Proof. Set $q_0 = w(X_0)$ and $Y_0 = \mu(X_0)X_0$.

First suppose $X_0 \notin \mathscr{Q}$. Then there exists a neighborhood U of Y_0 in T_pM on which exp is a diffeomorphism onto an open subset of M. By continuity of μ there exists a closed interval I containing X_0 in its interior such that $\mu(X)X \in U$ for all $X \in I$. Since $w(X) = \exp(\mu(X)X)$, it is clear that w is one-to-one on I. Hence $w(I)$ is an arc, and $\mathscr{H}^1(w(I)) < \infty$ by Theorem 4.7.

Now suppose $X_0 \in \mathscr{Q}$. Take any metric disk D in M centered at q_0 with radius less than the injectivity radius at q_0, and let Γ be a tree neighborhood of q_0 in $C(p)$ with $\Gamma \subset D$. Let (x, y) be a normal coordinate system defined throughout D so that

$$\exp_*\left(\frac{\partial}{\partial r}\bigg|_{Y_0}\right) = \frac{\partial}{\partial x}\bigg|_{q_0}.$$

Since exp is smooth, there exists a sufficiently small open neighborhood U of Y_0 in T_pM such that $\exp(U) \subset D$, and such that (i) $f(r, \theta) > 0$, and (ii) $|g(r, \theta)/f(r, \theta)| < 1$ for all $(r, \theta) \in U$ where f and g are defined by

$$\exp_*\left(\frac{\partial}{\partial r}\Big|_{(r,\theta)}\right) = f(r,\theta)\frac{\partial}{\partial x} + g(r,\theta)\frac{\partial}{\partial y}.$$

Since $X_0 \in \mathscr{Q}$, we have $\mu(X_0)X_0 = \phi(X_0)X_0 = Y_0 \in U$. Thus by continuity of μ, ϕ, and w, and because $\mathscr{V} \cap \mathscr{U}$ is open, there exists a closed interval I containing X_0 in its interior such that $(r,\theta) \in U$ whenever $\theta \in I$ and $\mu(\theta) \le r \le \phi(\theta)$, $w(I) \subset \Gamma$, and $I \subset \mathscr{V} \cap \mathscr{U}$. It will be shown that $\mathscr{H}^1(w(I)) < \infty$.

Since v is smooth, $x \circ v$ is Lipschitz on I. Hence there exists a constant C such that

$$|x(v(\theta_1)) - x(v(\theta_2))| \le C|\theta_1 - \theta_2|$$

for all $\theta_1, \theta_2 \in I$.

Since $I \subset \mathscr{V} \cap \mathscr{U}$ and $w(I) \subset \Gamma$, by applying Lemma 5.1 to the interval I, we can construct a topological disk Δ_0 such that

$$w(I) = C(p) \cap \Delta_0.$$

Thus by Lemma 5.1, $w(I)$ is the disjoint union of an arc A_0 in the boundary of Δ_0 with the part of $C(p)$ in the interior of Δ_0. By Proposition 2.1, the part of $C(p)$ contained in the interior of Δ_0 is the union of a smooth 1-dimensional manifold V of cleave points together with a set of $\mathscr{H}^1$-measure zero. Since $\mathscr{H}^1(A_0) < \infty$ by Theorem 4.7, it suffices to show $\mathscr{H}^1(V) < \infty$.

If $q \in V$, then there are exactly two minimizing geodesics joining p to q since q is a cleave point. By Lemma 5.1, since q lies in the interior of Δ_0, the initial tangent vectors α and β of these two geodesics both lie in I. Thus $[\alpha, \beta] \subset I$ and $w(\alpha) = w(\beta) = q$. If q' is a second point in V, and α' and β' are the initial tangent vectors to the two minimizing geodesics joining p to q' so that $[\alpha', \beta'] \subset I$ and $w(\alpha') = w(\beta') = q'$, then either $[\alpha, \beta]$ and $[\alpha', \beta']$ are disjoint or one of these intervals is contained in the other. For let Δ and Δ' be the topological disks constructed by Lemma 5.1 from $[\alpha, \beta]$ and $[\alpha', \beta']$ respectively. Note that by Remark 5.3, the boundaries of Δ and Δ' meet $C(p)$ at only in q and q' respectively. Thus either q lies in the interior of Δ' in which case $[\alpha, \beta] \subset [\alpha', \beta']$ by Lemma 5.1, or q' lies in the interior of Δ in which case $[\alpha', \beta'] \subset [\alpha, \beta]$ by Lemma 5.1, or neither happens in which case $[\alpha, \beta]$ and $[\alpha', \beta']$ are disjoint.

Now V has at most countably many connected components V_n. Each V_n is a smooth open arc of cleave points. If $q \in V_n$, then the two minimizing geodesics joining q to p lie on opposite sides of V_n. Thus the arc V_n

cuts through the boundary at q of the disk Δ of the previous paragraph. If Q' is also in V_n, then either $q' \in \Delta$ or $q \in \Delta'$ which implies that either $[\alpha', \beta'] \subset [\alpha, \beta]$ or $[\alpha, \beta] \subset [\alpha', \beta']$, that is, these two intervals are not disjoint. For each n, let

$$\alpha_n = \inf\{\theta : \theta \in I, \ w(\theta) \in V_n\}$$

and

$$\beta_n = \sup\{\theta : \theta \in I, \ w(\theta) \in V_n\}.$$

Then (α_n, β_n) is the union of all the intervals $[\alpha, \beta]$ corresponding to the $q \in V_n$. It follows that $w(\alpha_n) = w(\beta_n)$. Also, by Lemma 5.1 applied to $[\alpha_n, \beta_n]$ with Remark 5.3, $(\alpha_n, \beta_n) \cap \mathscr{Q} \neq \varnothing$ since $V_n \subset w([\alpha_n, \beta_n])$ shows that $w([\alpha_n, \beta_n] \neq \{w(\alpha_n)\}$. Thus we may define

$$\theta_n^* = \inf([\alpha_n, \beta_n] \cap \mathscr{Q}).$$

Clearly, $\theta_n^* \neq \alpha_n$ because no $q \in V_n$ is a conjugate cut point.

Set $A_n = w([\alpha_n, \theta_n^*])$. Since the hypothesis of Lemma 5.4 is satisfied by $[\alpha_n, \beta_n]$ for every n, we have

$$\mathscr{H}^1(A_n) \leq \sqrt{2}|x(v(\theta_n^*)) - x(v(\alpha_n))| \leq \sqrt{2}C|\theta_n^* - \alpha_n|$$

where C is the Lipschitz constant for $x \circ v$ on I.

Clearly, $V_n \subset A_n$ for every n. Also, our previous discussion implies that if $m \neq n$, then the open intervals (α_m, β_m) and (α_n, β_n) are either disjoint, or one is contained in the other. Thus, by definition of θ_n^*, if $m \neq n$, then the intervals (α_m, θ_m^*) and (α_n, θ_n^*) are either disjoint, or one is contained in the other. Hence, if $m \neq n$, then the arcs A_m and A_n either meet in at most one point, or one is contained in the other. Therefore, given any finite collection $A_1, A_2, \cdots, A_N$, by discarding any arc contained within another, there is a set of indices $\mathscr{I}_N \subset \{1, 2, \cdots, N\}$ such that

$$\bigcup_{n=1}^{N} A_n = \bigcup_{n \in \mathscr{I}_N} A_n,$$

and, if $m, n \in \mathscr{I}_N$ with $m \neq n$, then (α_m, θ_m^*) and (α_n, θ_n^*) are disjoint, and A_m and A_n meet in at most one point.

Thus, given $V_1, V_2, \cdots, V_N$,

$$\begin{aligned}\sum_{n=1}^{N} \mathscr{H}^1(V_n) &\leq \sum_{n \in \mathscr{I}_N} \mathscr{H}^1(A_n) \\ &\leq \sum_{n \in \mathscr{I}_N} \sqrt{2}C|\theta_n^* - \alpha_n| \leq \sqrt{2}C|I|,\end{aligned}$$

where $|I|$ is the length of the interval I because the (α_n, θ_n^*), $n \in \mathcal{J}_N$, are disjoint subintervals of I. By letting N approach infinity (if necessary), we obtain

$$\mathcal{H}^1(V) = \sum_{n=1}^{\infty} \mathcal{H}^1(V_n) \leq \sqrt{2}C|I| < \infty.$$

Therefore $\mathcal{H}^1(w(I)) < \infty$.

Corollary 5.6. *If $E \subset \mathcal{U}$ is compact, then $\mathcal{H}^1(w(E)) < \infty$.*

Proof. By Lemma 5.5, every $X_0 \in E$ is contained in the interior of some interval I such that $w(I)$ has finite Hausdorff 1-measure. Compactness implies that E is contained in a finite union of such intervals I. Therefore $w(E)$ is contained in a finite union of sets $w(I)$ of finite Hausdorff 1-measure, and thus itself must have finite Hausdorff 1-measure.

Since $w : \mathcal{U} \to C(p)$ is a proper map, the inverse image under w of every compact subset of $C(p)$ is a compact subset of $\mathcal{U}$. Hence every compact subset K of $C(p)$ is the image $w(E)$ of the compact set $E = w^{-1}(K)$ of $\mathcal{U}$. Therefore by Corollary 5.6, every compact subset of $C(p)$ has finite Hausdorff 1-measure. This completes the proof of Theorem 1.1.

REMARK 5.7. Corollary 5.6, in conjunction with [8, Proposition 5.1], implies that μ is absolutely continuous on $[\alpha, \beta]$, and that $w|[\alpha, \beta]$ is an absolutely continuous curve in M for every $[\alpha, \beta] \subset \mathcal{U}$.

6. The intrinsic metric

Let M be a complete two-dimensional Riemannian manifold, and let $C(p)$ be the cut locus of a point $p \in M$.

Given two points $q_1, q_2 \in C(p)$ that are in the same connected component, there is at least one arc joining them contained in $C(p)$. (If M is compact, there are at most finitely many [14, p. 97], and otherwise at most finitely many in any compact subset of $C(p)$.) Define $\delta(q_1, q_2)$ to be the length of the shortest arc in $C(p)$ with endpoints q_1 and q_2. By Theorem 4.7, this is finite. If q_1 and q_2 are not in the same component of $C(p)$, set $\delta(q_1, q_2) = 1$. It is easily verified that δ defines a metric on $C(p)$. This metric δ will be called the *intrinsic metric* on $C(p)$.

Theorem 6.1. *Suppose that M is a complete two-dimensional Riemannian manifold and that $C(p)$ is the cut locus of a point $p \in M$. Then the intrinsic metric δ on $C(p)$ induces the subspace topology on $C(p)$.*

Proof. Clearly, $\delta(q_1, q_2) \geq \rho(q_1, q_2)$ for all $q_1, q_2 \in C(p)$. Hence it is enough to show that if q_n is a sequence in $C(p)$ converging to $q_0 \in C(p)$ with respect to the subspace topology, then it converges to q_0 with respect to δ, that is, $\delta(q_n, q_0) \to 0$ as $n \to \infty$.

The subspace topology on $C(p)$ is a locally compact metric topology since $C(p)$ is a closed subset of M by Proposition 2.1. Since $C(p)$ is also a local tree by Theorem 2.3, we can find a countable family Γ_m of compact trees in $C(p)$, which form a base for the system of neighborhoods of q_0 in $C(p)$. We may certainly arrange that the Γ_m form a decreasing sequence of sets. Since the Γ_m form a base for the system of neighborhoods of q_0, $\bigcap_{m=1}^{\infty} \Gamma_m = \{q_0\}$. Since $\mathscr{H}^1(\Gamma_m) < \infty$ by Theorem 1.1, the well-known theorem on the measure of the intersection of a decreasing sequence of measurable sets [3, p. 2] implies that

$$\lim_{m\to\infty} \mathscr{H}^1(\Gamma_m) = \mathscr{H}^1\left(\bigcap_{m=1}^{\infty} \Gamma_m\right) = \mathscr{H}^1(\{q_0\}) = 0.$$

Thus given $\epsilon > 0$, there exists an m_0 such that $\mathscr{H}^1(\Gamma_{m_0}) < \epsilon$. Because q_n converges to q_0 and Γ_{m_0} is a neighborhood of q_0 in $C(p)$, there exists an N such that $q_n \in \Gamma_{m_0}$ for all $n \geq N$. Since Γ_{m_0} is a tree containing q_0, for every $n \geq N$, there exists an arc A_n joining q_0 to q_n in Γ_{m_0}. Hence

$$\delta(q_n, q_0) \leq \mathscr{H}^1(A_n) \leq \mathscr{H}^1(\Gamma_{m_0}) < \epsilon$$

for all $n \geq N$, and $\delta(q_n, q_0) \to 0$ as $n \to \infty$. This completes the proof of the theorem.

Example 6.2. Let (r, θ) denote polar coordinates on $\mathbf{R}^2$. Then the set $\{(r, \theta) : 0 \leq r \leq 1, , \theta = 0 \text{ or } \theta = 2\pi/n, \text{ for } n = 1, 2, 3, \cdots\}$ is a compact tree in $\mathbf{R}^2$ in which every arc has finite length but for which the resulting intrinsic metric does not induce the subspace topology. This tree clearly has infinite Hausdorff 1-measure.

The set $\{(r, \theta) : 0 \leq r \leq 1/n, \ \ \theta = 2\pi/n, \text{ for } n = 1, 2, 3, \cdots\}$ is a compact tree with infinite Hausdorff 1-measure for which the resulting intrinsic metric does induce the subspace topology. A set homeomorphic to it is realizable as a cut locus [5].

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