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A KNOT INVARIANT VIA REPRESENTATION SPACES

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0. Introduction

A beautiful construction of A. Casson on the representation spaces corresponding to a Heegard splitting of an oriented homology 3-sphere M gives rise to an integer invariant $\lambda(M)$ of M. This invariant generalizes the Rohlin invariant and gives stricking corollaries in low-dimensional topology. Defined as an intersection number of appropriate subspaces, this invariant $\lambda(M)$ can be roughly thought of as the number of conjugacy classes of irreducible representations of $\pi_1(M)$ into SU_2 counted with signs. A detailed discussion of this invariant can be found in an exposé by S. Akbulut and J. McCarthy [1]. Further works on Casson's invariant include the generalizations by K. Walker as well as S. Boyer and A. Nicas to rational homology 3-spheres [14], [3] and by S. Cappell, R. Lee, and E. Miller to representations into SU_n [4]. The works of C. Taubes [13] and A. Flore [6] interpret Casson's invariant as the Euler number of the instanton homology (or Flore homology) of M.

In this paper, analogous to Casson's original construction, we will define an intersection number of the representation spaces corresponding to a braid representative of a knot K in S^3. This intersection number turns out to be an integer knot invariant (see Theorem 1.8). The representations of the knot group $\pi_1(S^3\backslash K)$ used in our construction seem to be mysterious. They are representations of $\pi_1(S^3\backslash K)$ into SU_2 such that all meridians of K are represented by trace-zero matrices. Call such a representation of the knot group a *trace-free representation.* Then, roughly speaking, our knot invariant $h(K)$ is the number of conjugacy classes of irreducible trace-free representations of $\pi_1(S^3\backslash K)$ counted with signs.

Our knot invariant $h(K)$ can be computed via knot diagrams. It turns out that our algorithm of computing $h(K)$ by using the skein model is the same as the algorithm of computing $\frac{1}{2}\mathrm{sgn}(K)$ given by J. H. Conway [5],

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where $\operatorname{sgn}(K)$ is the signature of K. Thus, we have

$$h(K) = \tfrac{1}{2}\operatorname{sgn}(K)$$

(see Theorem 2.9 and Corollary 2.10).

Notice that $\operatorname{sgn}(K)$ is an invariant of (unoriented) knots and is always an even integer. It also seems to be mysterious why these two quantities with apparently different algebraic-geometric contents should ever be the same. The significance of the equality of these two knot invariants is yet to be explored.

The original idea of considering the space of all trace-free representations of a knot group came from the study of a paper by W. Magnus [11] where he proved the faithfulness of a representation of braid groups in the automorphism groups of the rings generated by the character functions on free groups. In Magnus' proof, the generators of a free group were chosen to be represented by trace-zero matrices, and this simplified the outcome of some complicated trace computations. We do not know whether the trace-free representations of knot groups have any geometric meanings.

As pointed out to us first by D. Ruberman, the work presented here can be generalized to representations of knot groups with the trace of the meridians fixed (not necessarily zero). We will discuss this and some other generalizations at another time and are content at present with the argument here for trace-free representations.

We divide the body of this paper into two parts. The first part (§1) concerns the definition of our knot invariant $h(K)$, and the second part (§2) the computation of $h(K)$.

1. The definition of $h(K)$

Let $\mathscr{B}_n$ be the braid group of rank n with the standard generators $\sigma_1, \cdots, \sigma_{n-1}$. Let F_n be the free group of rank n generated by $x_1, \cdots, x_n$. Then $\mathscr{B}_n$ can be faithfully represented by a subgroup of the automorphism group of F_n. In particular, the automorphism of F_n representing σ_μ is given by (still denote it by σ_μ)

$$\begin{aligned} \sigma_\mu \colon x_\mu &\mapsto x_\mu x_{\mu+1} x_\mu^{-1}, \\ x_{\mu+1} &\mapsto x_\mu, \\ x_\nu &\mapsto x_\nu, \qquad \nu \neq \mu,\ \mu+1. \end{aligned}$$

If $\sigma \in \mathscr{B}_n$, then the automorphism of F_n representing σ (still denoted by σ) maps each x_μ to a conjugation of some x_ν and preserves the product $x_1 \cdots x_n$. Moreover, these two conditions are also sufficient for an automorphism of F_n to be a braid automorphism (see [2]).

Consider the Lie group

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} ; a, b \in \mathbb{C}, a\overline{a} + b\overline{b} = 1 \right\}$$

and its Lie algebra

$$\mathfrak{su}_2 = \left\{ \begin{pmatrix} is & z \\ -\overline{z} & -is \end{pmatrix} ; s \in \mathbb{R}, z \in \mathbb{C} \right\},$$

where $i = \sqrt{-1}$. As a vector space, $\mathfrak{su}_2$ can be decomposed as the direct sum of a real subspace consisting of matrices $\left(\begin{smallmatrix} is & 0 \\ 0 & -is \end{smallmatrix}\right)$, $s \in \mathbb{R}$, and a complex subspace consisting of matrices $\left(\begin{smallmatrix} 0 & z \\ -\overline{z} & 0 \end{smallmatrix}\right)$, $z \in \mathbb{C}$. The adjoint action of the diagonal element $\left(\begin{smallmatrix} a & 0 \\ 0 & \overline{a} \end{smallmatrix}\right) \in \mathrm{SU}_2$, $|a| = 1$, on $\mathfrak{su}_2$ leaves this decomposition invariant. Moreover, its restriction on the real subspace is the identity, and on the complex subspace is the scale multiplication by a^2.

Let $R_n = \mathrm{Hom}(F_n, \mathrm{SU}_2)$. Since F_n is a free group, R_n can be identified with the product of n copies of SU_2. For any $\sigma \in \mathscr{B}_n$, by applying the Hom functor, we get an induced diffeomorphism (still denoted by σ) of R_n. In terms of the identification $R_n = (\mathrm{SU}_2)^n$, the induced diffeomorphism, say σ_1, is given by

$$\sigma_1(X_1, X_2, X_3, \cdots, X_n) = (X_1 X_2 X_1^{-1}, X_1, X_3, \cdots, X_n)$$

with $X_\mu \in \mathrm{SU}_2$ for $\mu = 1, \cdots, n$. In general, we denote the action of the diffeomorphism of R_n induced by $\sigma \in \mathscr{B}_n$ by

$$\sigma(X_1, \cdots, X_n) = (\sigma(X_1), \cdots, \sigma(X_n))$$

with $X_\mu \in \mathrm{SU}_2$ for $\mu = 1, \cdots, n$.

Let

$$\mathbf{a} = \left(\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}, \cdots, \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \right) \in R_n, \qquad |a| = 1.$$

Then $\mathbf{a}$ is a fixed point for any $\sigma \in \mathscr{B}_n$ (thought of as a diffeomorphism of R_n). Let

$$\mathbf{e} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \in R_n.$$

We have a linear map

$$T_{\mathbf{a}}\sigma = T_{\mathbf{e}}(L_{\overline{\mathbf{a}}} \circ \sigma \circ L_{\mathbf{a}}) \colon (\mathfrak{su}_2)^n \to (\mathfrak{su}_2)^n.$$

Here $L_{\overline{\mathbf{a}}}$ and $L_{\mathbf{a}}$ are left multiplications on R_n by $\overline{\mathbf{a}}$ and $\mathbf{a}$, respectively, and we identify $T_{\mathbf{e}}R_n$ with $(\mathfrak{su}_2)^n$.

We can decompose $(\mathfrak{su}_2)^n$ into the direct sum of an n-dimensional real subspace and an n-dimensional complex subspace canonically. Then, the linear map $T_{\mathbf{a}}\sigma$ leaves this decomposition invariant. Moreover, under the standard basis, the matrix of $T_{\mathbf{a}}\sigma$ restricted on the real subspace is the permutation matrix of σ, and the matrix of $T_{\mathbf{a}}\sigma$ restricted on the complex subspace is the Burau matrix of σ with parameter a^2. Notice that σ fixes the whole diagonal

$$\{(X, \cdots, X) \in R_n;\ X \in \mathrm{SU}_2\}$$

of R_n. Consequently, $T_{\mathbf{a}}\sigma$ fixes the whole diagonal of $(\mathfrak{su}_2)^n$. Modulo this invariant subspace, the complex part of $T_{\mathbf{a}}\sigma$ gives us the reduced Burau matrix of σ with parameter a^2 (see [9]).

Let $\sigma \in \mathscr{B}_n$. The closure of σ, denoted by $\sigma^\wedge$, is a link in S^3. A result of J. Alexander asserts that any tame link is isotopic to the closure of a certain braid. On the other hand, two braids $\alpha \in B_n$ and $\beta \in B_m$ have isotopic closures iff one braid can be changed to another by a sequence of finitely many *Markov moves*. A Markov move of *type* I changes $\alpha \in B_n$ to $\xi^{-1}\alpha\xi \in B_n$ for any $\xi \in B_n$, and a Markov move of *type* II changes $\alpha \in B_n$ to $\sigma_n^{\pm 1}\alpha \in B_{n+1}$, or the inverse of this operation (see [2]).

For a braid $\sigma \in B_n$, an easy application of van Kampen's theorem gives us a presentation of $\pi_1(S^3\backslash\sigma^\wedge)$:

$$(1.1) \qquad \pi_1(S^3\backslash\sigma^\wedge) = \langle x_1, \cdots, x_n; x_\mu = \sigma(x_\mu),\ \mu = 1, \cdots, n\rangle.$$

Here each x_μ is represented by a meridian of $\sigma^\wedge$. So we have the following lemma.

Lemma 1.2. *Let $\sigma \in \mathscr{B}_n$. Then the fixed pint set of the diffeomorphism of R_n induced by σ can be identified with the representation space* $\mathrm{Hom}(\pi_1(S^3\backslash\sigma^\wedge), \mathrm{SU}_2)$.

Proof. Using the presentation (1.1) of $\pi_1(S^3\backslash\sigma^\wedge)$, we see that a representation of $\pi_1(S^3\backslash\sigma^\wedge)$ into SU_2 is determined by n matrices $X_1, \cdots, X_n$ in SU_2 or $(X_1, \cdots, X_n) \in R_n$ which satisfy the relation $X_\mu = \sigma(X_\mu)$, $\mu = 1, \cdots, n$, or $\sigma(X_1, \cdots, X_n) = (X_1, \cdots, X_n)$. So we can identify $\mathrm{Fix}(\sigma)$ with $\mathrm{Hom}(\pi_1(S^3\backslash\sigma^\wedge), \mathrm{SU}_2)$. q.e.d.

Let $\sigma \in \mathscr{B}_n$. Suppose $\sigma^\wedge = K$ is a knot (a link with one component). Let $R(K)$ be the space of conjugacy classes of representations of $\pi_1(S^3\backslash\sigma^\wedge)$ into SU_2, i.e., the quotient space of $\mathrm{Hom}(\pi_1(S^3\backslash\sigma^\wedge), \mathrm{SU}_2)$ by SU_2-conjugation. Notice that reducible representations of $\pi_1(S^3\backslash\sigma^\wedge)$ are conjugate to diagonal representations for which each generator X_μ of

$\pi(S^3\backslash\sigma^\wedge)$ is represented by $\left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)$ for a certain $a \in \mathbb{C}$ with $|a| = 1$. Moreover, $\left(\begin{smallmatrix} a_1 & 0 \\ 0 & \bar{a}_1 \end{smallmatrix}\right)$ is conjugate to $\left(\begin{smallmatrix} a_2 & 0 \\ 0 & \bar{a}_2 \end{smallmatrix}\right)$ iff $a_2 = \bar{a}_1$. Thus, the subset of $R(K)$ consisting of all conjugacy classes of reducible representations can be identified with the upper half unit circle $\{a = e^{i\theta}; 0 \le \theta \le \pi\}$ in the complex plane.

Let $\Delta_K(t)$ be the normalized Alexander polynomial of $K = \sigma^\wedge$, and let $\psi_t(\sigma)$ be the reduced Burau matrix of σ with parameter t. Then

$$\Delta_K(t) = \left(-\frac{1}{\sqrt{t}}\right)^{e-n+1}\left(\frac{1-t}{1-t^n}\right)\det(1-\psi_t(\sigma)), \tag{1.3}$$

where e is the exponent sum of σ (see [8]). In the case that $K = \sigma^\wedge$ is a knot, $e - n + 1$ is an even integer and $\Delta_K(t)$ is a Laurent polynomial in t with integer coefficients. Since $\Delta_K(1) = \pm 1$, we have $\Delta_K(-1) \ne 0$.

Lemma 1.4. *For $a_0 = e^{i\theta_0}$, $0 \le \theta_0 \le \pi$, if $\Delta_K(a_0^2) \ne 0$, then there is a neighborhood of a_0 in $R(K)$ consisting of only reducible representations.*

Proof. This is a more or less well-known fact (see, for example, [10]). Here we present an elementary argument.

Consider a subset $\mathscr{H}_n$ of $R_n \times R_n$ given by

$$\mathscr{H}_n = \{(X_1, \cdots, X_n, Y_1, \cdots, Y_n) \in R_n \times R_n; X_1 \cdots X_n = Y_1 \cdots Y_n\},$$

which is a manifold of dimension $6n - 3$ diffeomorphic to $(\mathrm{SU}_2)^{2n-1}$. Let

$$\Lambda_n = \{(X_1, \cdots, X_n, X_1, \cdots, X_n) \in R_n \times R_n\}$$

be the diagonal of $R_n \times R_n$, and let

$$\Gamma_\sigma = \{(X_1, \cdots, X_n, \sigma(X_1), \cdots, \sigma(X_n)) \in R_n \times R_n\}$$

be the graph of the induced diffeomorphism $\sigma: R_n \to R_n$. Then both Λ_n and Γ_n are $3n$-dimensional manifolds of $\mathscr{H}_n$.

Noticing that the conclusion of Lemma 1.4 is about the space $R(K)$, we are free to choose an appropriate n by changing σ by type II Markov moves so that $\Delta_K(a_0^2)$ implies $\det(1 - \psi_{a_0^2}(\sigma)) \ne 0$. Thus, the intersection of $T_{\mathbf{a}_0\times\mathbf{a}_0}\Lambda_n$ and $T_{\mathbf{a}_0\times\mathbf{a}_0}\Gamma_\sigma$ in $T_{\mathbf{a}_0\times\mathbf{a}_0}\mathscr{H}_n$ is of the minimal dimension 3, where

$$\mathbf{a}_0 = \left(\begin{pmatrix} a_0 & 0 \\ 0 & \bar{a}_0 \end{pmatrix}, \cdots, \begin{pmatrix} a_0 & 0 \\ 0 & \bar{a}_0 \end{pmatrix}\right) \in R_n.$$

So, in a neighborhood of $\mathbf{a}_0 \times \mathbf{a}_0$ in $\mathscr{H}_n$, $\Lambda_n \cap \Gamma_\sigma$ is a 3-dimensional manifold.

On the other hand, reducible representations of $\pi_1(S^3\backslash K)$ can be identified with the diagonal of R_n, which is diffeomorphic to $\mathrm{SU}_2 \cong S^3$.

Thus, there is a neighborhood of the representation corresponding to $\mathbf{a}_0$ in $\mathrm{Hom}(\pi_1(S^3\backslash K), \mathrm{SU}_2)$ consisting of only reducible representations. q.e.d.

The subset of SU_2 consisting of all trace-zero matrices is a manifold diffeomorphic to S^2. The tangent space of this manifold at $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ is the image of the complex subspace of $\mathfrak{su}_2$ under the left multiplication by $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. Define

$$Q_n = \{(X_1, \cdots, X_n) \in R_n;\ \mathrm{tr}(X_\mu) = 0\}.$$

Then Q_n is diffeomorphic to the product of n copies of S^2's.

Let $\sigma \in B_n$. Since σ maps each X_μ to a conjugation of some X_ν, it leaves Q_n invariant and gives rise to a diffeomorphism of Q_n. The fixed point set of $\sigma|Q_n$ can be identified with the set of all representations of $\pi_1(S^3\backslash\sigma^\wedge)$ into SU_2 such that each generator of $\pi_1(S^3\backslash\sigma^\wedge)$ is represented by a trace-zero matrix. We call such representations *trace-free* representations.

Using the notation in the proof of Lemma 1.4, we define

$$\begin{aligned} H_n &= \mathscr{H}_n \cap Q_n \times Q_n \\ &= \{(X_1, \cdots, X_n, Y_1, \cdots, Y_n) \in Q_n \times Q_n;\ X_1 \cdots X_n = Y_1 \cdots Y_n\}. \end{aligned}$$

Notice that H_n is no longer a manifold.

A point $(X_1, \cdots, X_n, Y_1, \cdots, Y_n) \in R_n \times R_n$ is called *reducible* if there is a matrix $A \in \mathrm{SU}_2$ such that $A^{-1}X_\mu A$ and $A^{-1}Y_\mu A$ are all diagonal matrices for $\mu = 1, \cdots, n$. Let S_n be the subset of H_n consisting of all reducible points in H_n.

Lemma 1.5. *$H_n\backslash S_n$ is an open manifold of dimension $4n-3$.*

Proof. Consider the map $f: Q_n \times Q_n \to \mathrm{SU}_2$ defined by

$$f(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = X_1 \cdots X_n Y_n^{-1} \cdots Y_1^{-1}.$$

We will show that the tangent map of f is onto at an irreducible point in $Q_n \times Q_n$ whose image in SU_2 is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since $H_n = f^{-1}(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$, this implies Lemma 1.5.

Without loss of generality, we only need to consider the map

$$\begin{aligned} &f: Q_m \to \mathrm{SU}_2, \\ &f(X_1, \cdots, X_m) = X_1 \cdots X_m. \end{aligned}$$

Suppose $\mathbf{X} = (X_1, \cdots, X_m)$ is an irreducible point such that $f(\mathbf{X}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Also without loss of generality, we may assume that

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad X_m = \begin{pmatrix} is & v \\ -\overline{v} & -is \end{pmatrix}$$

with $s^2 + v\overline{v} = 1$ and $v \neq 0$. Then

$$X_2 \cdots X_m = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and

$$X_1 \cdots X_{m-1} = \begin{pmatrix} -is & -v \\ \overline{v} & is \end{pmatrix}.$$

Let

$$X_1^t = \begin{pmatrix} i\sqrt{1-t^2} & -t \\ t & -i\sqrt{1-t^2} \end{pmatrix} \quad \text{and} \quad X_m^t = \begin{pmatrix} is & ve^{-it} \\ -\overline{v}e^{it} & -is \end{pmatrix}.$$

Then

$$\frac{d}{dt}(X_1^t X_2 \cdots X_m)|_{t=0} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and

$$\frac{d}{dt}(X_1 \cdots X_{m-1} X_m^t)|_{t=0} = \begin{pmatrix} iv\overline{v} & -sv \\ s\overline{v} & -iv\overline{v} \end{pmatrix}.$$

This shows that $T_{\mathbf{X}} f$ is onto. q.e.d.

We will still use Λ_n and Γ_σ to denote the diagonal of $Q_n \times Q_n$ and the graph of σ in $Q_n \times Q_n$, i.e.,

$$\Lambda_n = \{(X_1, \cdots, X_n, X_1, \cdots, X_n) \in Q_n \times Q_n\}$$

and

$$\Gamma_\sigma = \{(X_1, \cdots, X_n, \sigma(X_1), \cdots, \sigma(X_n)) \in Q_n \times Q_n\}.$$

Then we have $\Lambda_n \subset H_n$ and $\Gamma_\sigma \subset H_n$, and we can identify $\Lambda_n \cap \Gamma_\sigma$ with the set of trace-free representations of $\pi_1(S^3 \backslash \sigma^\wedge)$.

Lemma 1.6. *If $\sigma^\wedge$ is a knot, then*

$$(\Lambda_n \backslash \Lambda_n \cap S_n) \cap (\Gamma_\sigma \backslash \Gamma_\sigma \cap S_n)$$

is a compact subset of $H_n \backslash S_n$.

Proof. We have $\Delta_K(-1) \neq 0$ for any knot K. Since $i^2 = -1$, by Lemma 1.4, there is a neighborhood U of the point

$$\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \cdots, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in R_n$$

such that $U \cap \mathrm{Fix}(\sigma)$ consists of only reducible representations of $\pi_1(S^3 \backslash \sigma^\wedge)$. This proves Lemma 1.6. q.e.d.

Notice that $\Lambda_n \backslash \Lambda_n \cap S_n$ and $\Gamma_\sigma \backslash \Gamma_\sigma \cap S_n$ are $2n$-dimensional open submanifolds of $H_n \backslash S_n$, and also that SU_2 acts freely by conjugation on $H_n \backslash S_n$, $\Lambda_n \backslash \Lambda_n \cap S_n$, and $\Gamma_\sigma \backslash \Gamma_\sigma \cap S_n$. Let

$$\widehat{H}_n = \frac{H_n \backslash S_n}{\sim}, \quad \widehat{\Lambda}_n = \frac{\Lambda_n \backslash \Lambda_n \cap S_n}{\sim}, \quad \widehat{\Gamma}_\sigma = \frac{\Gamma_\sigma \backslash \Gamma_\sigma \cap S_n}{\sim},$$

where $\sim$ denotes the quotient by the SU_2-action. Then $\widehat{H}_n$ is an open manifold of dimension $4n-6$, $\widehat{\Lambda}_n$ and $\widehat{\Gamma}_\sigma$ are open submanifolds of dimension $2n-3$, and $\widehat{\Lambda}_n \cap \widehat{\Gamma}_\sigma$ is compact in $\widehat{H}_n$. To define an intersection number, we need to consider the orientations of these manifolds.

The standard basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

gives us an orientation of $\mathfrak{su}_2$, which induces an orientation on SU_2. We have an inner product on $\mathfrak{su}_2$ given by

$$\langle U, V \rangle = \tfrac{1}{2}\operatorname{trace}(U\overline{V}^{\mathrm{T}}), \qquad U, V \in \mathfrak{su}_2,$$

which yields a natural Riemannian metric on $\mathfrak{su}_2$. Let $S(0, \frac{\pi}{2})$ be the 2-sphere of radius $\frac{\pi}{2}$ centered at 0 in $\mathfrak{su}_2$. Then $S^2 = \exp(S(0, \frac{\pi}{2}))$ is the 2-sphere of trace-zero matrices in SU_2. By noticing that the injective radius of exp is π, the standard orientation of $S(0, \frac{\pi}{2})$ gives us an orientation of S^2.

Since $\Lambda_n \cong (S^2)^n$ and $\Gamma_\sigma \cong (S^2)^n$, Λ_n and Γ_σ are naturally oriented with separate product orientations and also $Q_n \times Q_n$ is naturally oriented. Recall $H_n = f^{-1}((\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}))$, where $f\colon Q_n \times Q_n \to SU_2$ is given by

$$f(\mathbf{X}, \mathbf{Y}) = X_1 \cdots X_n Y_n^{-1} \cdots Y_1^{-1}$$

for $(\mathbf{X}, \mathbf{Y}) = (X_1, \cdots, X_n, Y_1, \cdots, Y_n) \in Q_n \times Q_n$. We can orient $H_n \backslash S_n$ so that for each point $(\mathbf{X}, \mathbf{Y}) \in H_n \backslash S_n \subset Q_n \times Q_n$, we have

$$T_{(\mathbf{X},\mathbf{Y})}(Q_n \times Q_n) = T_{(\mathbf{X},\mathbf{Y})}(H_n \backslash S_n) \oplus f^*_{(\mathbf{X},\mathbf{Y})}(\mathfrak{su}_2)$$

as oriented vector spaces.

Noticing that the adjoint action of SU_2 on $\mathfrak{su}_2$ is orientation preserving, we get $\widehat{H}_n$, $\widehat{\Lambda}_n$, and $\widehat{\Gamma}_\sigma$ as oriented manifolds.

Definition 1.7. Isotopy $\widehat{\Gamma}_\sigma$ to $\widetilde{\Gamma}_\sigma$ with compact support so that $\widehat{\Lambda}_n \pitchfork \widetilde{\Gamma}_\sigma$. Define

$$h(\sigma) = \#_{\widehat{H}_n}(\widehat{\Lambda}_n \cap \widetilde{\Gamma}_\sigma).$$

Here $\#_{\widehat{H}_n}$ is the algebraic intersection number in $\widehat{H}_n$.

It is obvious that $h(\sigma)$ does not depend on the perturbation of $\widehat{\Gamma}_\sigma$. So we will simply denote

$$h(\sigma) = \langle \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma \rangle.$$

Theorem 1.8. *Let $\alpha \in \mathscr{B}_n$ and $\beta \in \mathscr{B}_m$ such that $\alpha^\wedge = \beta^\wedge$ as knots. Then $h(\alpha) = h(\beta)$.*

So $h(\cdot)$ is a knot invariant. We will denote $h(K) = h(\sigma)$ if $K = \sigma^\wedge$.

Proof. We only need to show that for $\sigma \in \mathscr{B}_n$ with $\sigma^\wedge$ being a knot, the Markov moves of type I and type II on σ do not effect the intersection number $h(\sigma)$.

Suppose we change σ to $\xi^{-1}\sigma\xi$ for some $\xi \in \mathscr{B}_n$. The induced diffeomorphism $\xi: Q_n \to Q_n$ is orientation preserving. Consider

$$\xi \times \xi: Q_n \times Q_n \to Q_n \times Q_n.$$

This diffeomorphism commutes with the SU_2-action and we have $(\xi \times \xi)(H_n \backslash S_n) = H_n \backslash S_n$, $(\xi \times \xi)(\Lambda_n) = \Lambda_n$, and $(\xi \times \xi)\Gamma_{\xi^{-1}\sigma\xi} = \Gamma_\sigma$ as oriented manifolds. Let $\Xi: \widehat{H}_n \to \widehat{H}_n$ be the quotient diffeomorphism of $\xi \times \xi$. Then it is orientation preserving and $\Xi(\widehat{\Lambda}_n) = \widehat{\Lambda}_n$ and $\Xi(\widehat{\Gamma}_{\xi^{-1}\sigma\xi}) = \widehat{\Gamma}_\sigma$ as oriented manifolds. So

$$h(\xi^{-1}\sigma\xi) = \langle \widehat{\Lambda}_n, \widehat{\Gamma}_{\xi^{-1}\sigma\xi} \rangle = \langle \Xi(\widehat{\Lambda}_n), \Xi(\widehat{\Gamma}_{\xi^{-1}\sigma\xi}) \rangle = \langle \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma \rangle = h(\sigma).$$

Next, let us change σ to $\sigma_n\sigma$, where $\sigma \in \mathscr{B}_{n+1}$ is given by

$$\sigma_n(x_\mu) = x_\mu, \qquad \mu = 1, \cdots, n-1,$$
$$\sigma_n(x_n) = x_n x_{n+1} x_n^{-1}, \qquad \sigma_n(x_{n+1}) = x_n.$$

Consider the imbedding $g: Q_n \times Q_n \to Q_{n+1} \times Q_{n+1}$ given by

$$g(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = (X_1, \cdots, X_n, Y_n, Y_1, \cdots, Y_n, Y_n).$$

It commutes with the SU_2-action and $g(H_n) \subset H_{n+1}$, and therefore reduces to an imbedding $\hat{g}: \widehat{H}_n \to \widehat{H}_{n+1}$. Thus we have

$$\hat{g}(\widehat{\Lambda}_n) \subset \widehat{\Lambda}_{n+1}, \quad \hat{g}(\widehat{\Gamma}_\sigma) \subset \widehat{\Gamma}_{\sigma_n\sigma}, \quad \hat{g}(\widehat{\Lambda}_n \cap \widehat{\Gamma}_\sigma) = \widehat{\Lambda}_n \cap \widehat{\Gamma}_{\sigma_n\sigma}.$$

We first perturb $\widehat{\Gamma}_\sigma$ to $\widetilde{\Gamma}_\sigma$ with compact support in $\widehat{H}_n$ so that $\widehat{\Lambda}_n \pitchfork \widetilde{\Gamma}_\sigma$. By the standard isotopy extension argument, we can subsequently perturb $\widehat{\Gamma}_{\sigma_n\sigma}$ to $\widetilde{\Gamma}_{\sigma_n\sigma}$ with compact support in $\widehat{H}_{n+1}$ so that $\widehat{\Lambda}_{n+1} \pitchfork \widetilde{\Gamma}_{\sigma_n\sigma}$ and

$$\widetilde{\Gamma}_{\sigma_n\sigma} \cap \hat{g}(H_n) = \hat{g}(\widetilde{\Gamma}_\sigma).$$

To conclude that the intersection numbers $\langle \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma \rangle$ and $\langle \widehat{\Lambda}_{n+1}, \widehat{\Gamma}_{\sigma_n\sigma} \rangle$ are the same, we need to consider the orientations of various manifolds involved.

Let $\mathbf{X} = (X_1, \cdots, X_n) \in Q_n$ be a fixed point of $\sigma | Q_n$. We have

$$T_{g(\mathbf{X},\mathbf{X})}\Lambda_{n+1} = dg(T_{(\mathbf{X},\mathbf{X})}\Lambda_n) \oplus \mathscr{U},$$
$$T_{g(\mathbf{X},\mathbf{X})}\Gamma_{\sigma_n\sigma} = dg(T_{(\mathbf{X},\mathbf{x})}\Gamma_\sigma) \oplus \mathscr{V}$$

as oriented vector spaces. Here $\mathscr{U}$ and $\mathscr{V}$ are oriented vector spaces of the forms

$$\mathscr{U} \cong \{(u, u, 0); u \in T_{X_n} S^2\}, \qquad \mathscr{V} \cong \{(v, 0, v); v \in T_{X_n} S^2\},$$

and also,

$$T_{g(\mathbf{X},\mathbf{X})}(Q_{n+1} \times Q_{n+1}) = dg(T_{(\mathbf{X},\mathbf{X})}(Q_n \times Q_n)) \oplus \mathscr{W}$$

as oriented vector spaces, where

$$\mathscr{W} \cong \{(u+v, u, v); u, v \in T_{X_n} S^2\} \cong \mathscr{U} \oplus \mathscr{V}$$

as oriented vector spaces.

These decompositions pass down to the tangent spaces of $\hat{g}(\widehat{H}_n) \subset \widehat{H}_{n+1}$, $\hat{g}(\widehat{\Lambda}_n) \subset \widehat{\Lambda}_{n+1}$, and $\hat{g}(\widehat{\Gamma}_\sigma) \subset \widehat{\Gamma}_{\sigma_n\sigma}$ at the corresponding points. Thus, assuming the intersection number of $\widehat{\Lambda}_{n+1}$ and $\widetilde{\Gamma}_{\sigma_n\sigma}$ at $\hat{g}(\mathbf{X}, \mathbf{X})$ is 1, we have

$$\begin{aligned}
T_{\hat{g}(\mathbf{X},\mathbf{X})}\widehat{H}_{n+1} &= T_{\hat{g}(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_{n+1} \oplus T_{\hat{g}(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_{\sigma_n\sigma} \\
&= d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n) \oplus \mathscr{U} \oplus d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_\sigma) \oplus \mathscr{V} \\
&= d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n) \oplus d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_\sigma) \oplus \mathscr{U} \oplus \mathscr{V} \\
&= d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_\sigma) \oplus \mathscr{U} \oplus \mathscr{V} \\
&= d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_\sigma) \oplus \mathscr{W}
\end{aligned}$$

as oriented vector spaces. On the other hand,

$$T_{\hat{g}(\mathbf{X},\mathbf{X})}\widehat{H}_{n+1} = d\hat{g}(T_{(\mathbf{X},\mathbf{X})}\widehat{H}_n) \oplus \mathscr{W}$$

as oriented vector spaces. Thus

$$T_{(\mathbf{X},\mathbf{X})}\widehat{H}_n = T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_\sigma$$

as oriented vector spaces, and this implies that the intersection number of $\widehat{\Lambda}_n$ and $\widetilde{\Gamma}_\sigma$ at $(\mathbf{X}, \mathbf{X})$ is also 1. Thus $h(\sigma) = h(\sigma_n\sigma)$. We can prove similarly that $h(\sigma) = h(\sigma_n^{-1}\sigma)$.

We have showed that $h(\sigma)$ is invariant under Markov moves. If $\alpha \in \mathscr{B}_n$ and $\beta \in \mathscr{B}_m$ have the same closure as knots, then β can be obtained from α by a sequence of Markov moves. So $h(\alpha) = h(\beta)$ and this proves Theorem 1.8.

2. The computation of $h(K)$

In this section, we derive a way to compute the knot invariant $h(K)$ via knot diagrams. This computation also identifies $h(K)$ with one half of the knot signature of K.

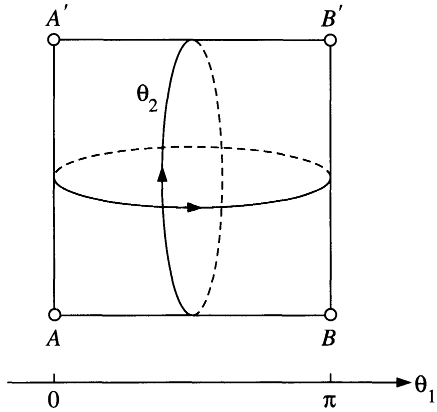

FIGURE 2.1

We first study the space $\widehat{H}_2$, which is a 2-dimensional open manifold.

Lemma 2.1. *$\widehat{H}_2$ is a 2-sphere with four cone points deleted.*

See Figure 2.1; it is called by some authors a "pillowcase."

Proof. Recall

$$H_2 = \{(X_1, X_2, Y_1, Y_2) \in Q_2 \times Q_2;\, X_1X_2 = Y_1Y_2\}.$$

Up to conjugation, we can assume that

$$X_1 = \begin{pmatrix} i\cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & -i\cos\theta_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad 0 \le \theta_1 \le \pi.$$

We have $Y_2 = Y_1^{-1}X_1X_2$. The condition that $Y_1^{-1}X_1X_2$ is of trace zero implies

$$Y_1 = \begin{pmatrix} i\cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & -i\cos\theta_2 \end{pmatrix}, \qquad -\pi \le \theta_2 \le \pi.$$

So, parameterized by θ_1 and θ_2, $\widehat{H}_2$ is a "pillowcase" as shown in Figure 2.1. The four deleted points are $A \sim (+,+,+,+)$, $A' \sim (+,+,-,-)$, $B \sim (+,-,-,+)$, and $B' \sim (+,-,+,-)$, where $+$ and $-$ stand for the matrices

$$\pm\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

respectively. q.e.d.

Notice that the cone point A corresponds to the origin of the (θ_1, θ_2)-plane modulo the involution $(\theta_1, \theta_2) \mapsto (-\theta_1, -\theta_2)$. The orientation of $\widehat{H}_2$ is given by the usual orientation of the (θ_1, θ_2)-plane.

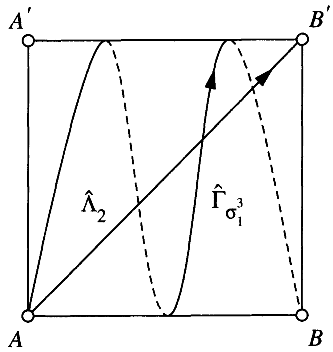

FIGURE 2.2

Consider the 1-dimensional submanifolds $\widehat{\Lambda}_2$ and $\widehat{\Gamma}_{\sigma_1^3}$ in $\widehat{H}_2$ (see Figure 2.2). These two submanifolds intersect transversally at a single point, and the intersection number at this point is 1. So $h(\sigma_1^3) = 1$, or our knot invariant for the right-handed trefoil knot is 1.

In general, let us consider the space

$$V_n = \{(X_1, \cdots, X_n, Y_1, \cdots, Y_n) \in H_n : X_\mu = Y_\mu,\ \mu = 3, \cdots, n\}.$$

Then

$$\widehat{V}_n = \frac{V_n \backslash V_n \cap S_n}{\sim}$$

is an oriented submanifold of $\widehat{H}_n$ whose dimension is $2n-2$.

Suppose $\sigma \in \mathscr{B}_n$ such that $\sigma^\wedge$ is a knot. Then $(\sigma_1^2\sigma)^\wedge$ is also a knot. We want to consider the difference $h(\sigma_1^2\sigma) - h(\sigma)$:

$$\begin{aligned} h(\sigma_1^2\sigma) - h(\sigma) &= \langle \widehat{\Lambda}_n, \widehat{\Gamma}_{\sigma_1^2\sigma}\rangle - \langle \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma\rangle \\ &= \langle \widehat{\Gamma}_{\sigma_1^{-2}}, \widehat{\Gamma}_\sigma\rangle - \langle \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma\rangle = \langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma\rangle. \end{aligned}$$

Noticing that the "difference cycle" $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n$ is carried by $\widehat{V}_n$, we can use the picture of $\widehat{H}_2$ to analyze the intersection of $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n$ and $\widehat{\Gamma}_\sigma$.

Let us first perturb $\widehat{\Gamma}_\sigma$ to $\widetilde{\Gamma}_\sigma$ with compact support so that

$$(\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n) \pitchfork \widetilde{\Gamma}_\sigma,$$

and then extend the isotopy so that $\widehat{V}_n \pitchfork \widetilde{\Gamma}_\sigma$. Thus $\widehat{V}_n \cap \widetilde{\Gamma}_\sigma$ is a 1-dimensional manifold. Notice that this is possible because $\widehat{V}_n$ and $\widehat{\Gamma}_\sigma$ are real algebraic sets.

We orient $\widehat{V}_n \cap \widetilde{\Gamma}_\sigma$ in the following way. Suppose $(\mathbf{X}, \mathbf{Y}) \in \widehat{V}_n \cap \widetilde{\Gamma}_\sigma$. Then there is an oriented subspace $\mathscr{P} \subset T_{(\mathbf{X},\mathbf{Y})}\widetilde{\Gamma}_\sigma$ of codimension 1 such that

$$T_{(\mathbf{X},\mathbf{Y})}\widehat{H}_n = T_{(\mathbf{X},\mathbf{Y})}\widehat{V}_n \oplus \mathscr{P}$$

as oriented vector spaces. We orient $\widehat{V}_n \cap \widetilde{\Gamma}_\sigma$ so that

$$T_{(\mathbf{X},\mathbf{Y})}\widetilde{\Gamma}_\sigma = \mathscr{P} \oplus T_{(\mathbf{X},\mathbf{Y})}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$$

as oriented vector spaces.

Lemma 2.2. $\langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma \rangle = \langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n, \widehat{V}_n \cap \widetilde{\Gamma}_\sigma \rangle_{\widehat{V}_n}$, *where* $\langle\ \rangle_{\widehat{V}_n}$ *is the intersection number in* $\widehat{V}_n$.

Proof. Let $(\mathbf{X}, \mathbf{X}) \in \widehat{\Lambda}_n \cap \widetilde{\Gamma}_\sigma$, and assume the intersection number of $\widehat{\Lambda}_n$ and $\widetilde{\Gamma}_\sigma$ at $(\mathbf{X}, \mathbf{X})$ is 1. Then

$$\begin{aligned} T_{(\mathbf{X},\mathbf{X})}\widehat{H}_n &= T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}\widetilde{\Gamma}_\sigma \\ &= T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus \mathscr{P} \oplus T_{(\mathbf{X},\mathbf{X})}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma) \\ &= T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma) \oplus \mathscr{P}, \end{aligned}$$

since $\dim \mathscr{P} = 2n - 4$ is even. Thus

$$T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma) = T_{(\mathbf{X},\mathbf{X})}\widehat{V}_n,$$

and therefore the intersection number of $\widehat{\Lambda}_n$ and $\widehat{V}_n \cap \widetilde{\Gamma}_\sigma$ at $(\mathbf{X}, \mathbf{X})$ in $\widehat{V}_n$ is also 1. This shows that

$$\langle \widehat{\Lambda}_n, \widehat{V}_n \cap \widetilde{\Gamma}_\sigma \rangle_{\widehat{V}_n} = \langle \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma \rangle.$$

Similarly, we have

$$\langle \widehat{\Gamma}_{\sigma_1^{-2}}, \widehat{V}_n \cap \widetilde{\Gamma}_\sigma \rangle_{\widehat{V}_n} = \langle \widehat{\Gamma}_{\sigma_1^{-2}}, \widehat{\Gamma}_\sigma \rangle,$$

which proves Lemma 2.2. q.e.d.

Let $\hat{p}: \widehat{V}_n \to \widehat{H}_2$ be the map induced by the projection

$$p(X_1, X_2, X_3, \cdots, X_n, Y_1, Y_2, X_3, \cdots, X_n) = (X_1, X_2, Y_1, Y_2).$$

Lemma 2.3. $\langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_n, \widehat{\Gamma}_\sigma \rangle = \langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_2, \hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma) \rangle_{\widehat{H}_2}$.

Proof. Letting $(\mathbf{X}, \mathbf{X}) \in \widehat{\Lambda}_n \cap \widetilde{\Gamma}_\sigma$, we have

$$T_{(\mathbf{X},\mathbf{X})}\widehat{V}_n = \hat{p}^*(T_{\hat{p}(\mathbf{X},\mathbf{X})}\widehat{H}_2) \oplus \mathscr{S}$$

and

$$T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n = \hat{p}^*(T_{\hat{p}(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_2) \oplus \mathscr{S}$$

as oriented vector spaces. Here $\dim \mathscr{S} = 2n-4$. Suppose the intersection number of $\widehat{\Lambda}_n$ and $\widehat{V}_n \cap \widetilde{\Gamma}_\sigma$ at $(\mathbf{X}, \mathbf{X})$ in $\widehat{V}_n$ is 1. Then

$$\begin{aligned} T_{(\mathbf{X},\mathbf{X})}\widehat{V}_n &= T_{(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_n \oplus T_{(\mathbf{X},\mathbf{X})}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma) \\ &= \hat{p}^*(T_{\hat{p}(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_2) \oplus \mathscr{S} \oplus \hat{p}^*(T_{\hat{p}(\mathbf{X},\mathbf{X})}\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)) \\ &= \hat{p}^*[(T_{\hat{p}(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_2) \oplus T_{\hat{p}(\mathbf{X},\mathbf{X})}\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)] \oplus \mathscr{S}. \end{aligned}$$

So

$$T_{\hat{p}(\mathbf{X},\mathbf{X})}\widehat{H}_2 = T_{\hat{p}(\mathbf{X},\mathbf{X})}\widehat{\Lambda}_2 \oplus T_{\hat{p}(\mathbf{X},\mathbf{X})}\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$$

or the intersection number of $\widehat{\Lambda}_2$ and $\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$ at $\hat{p}(\mathbf{X}, \mathbf{X})$ in $\widehat{H}_2$ is also 1. Similarly, we can get the same conclusion about the intersection of $\widehat{\Gamma}_{\sigma_1^{-1}}$ and $\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$ in $\widehat{H}_2$. This finishes the proof of Lemma 2.3. q.e.d.

Thus, we are led to study the intersection of $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_2$ with $\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$ in $\widehat{H}_2$. For this purpose, we need to understand the set $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$, especially its limiting behavior near the cone point A.

We picture $\widehat{H}_2$ in a different way. In Figure 2.3, we see a rectangle $\{0 \le \theta_1 \le \pi,\ -\pi \le \theta_2 \le \pi\}$ with six points deleted. Identifying three pairs of edges with the end points having the same labels, we get $\widehat{H}_2$. Also, we see the "difference cycle" $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Lambda}_2$ in Figure 2.3.

Suppose $c(t)$, $-\infty < t < \infty$, is a smooth curve in $\widehat{H}_2$ such that

$$\lim_{t \to -\infty} c(t) = A.$$

Let the $\theta_{1,2}$-parameter of $c(t)$ be $\theta_{1,2}(t)$, and further assume that

$$\theta_{1,2}^0 = \lim_{t \to -\infty} \theta'_{1,2}(t)$$

are finite numbers and $(\theta_1^0, \theta_2^0) \neq (0, 0)$. Then the *slope* of the curve c at A is defined to be

$$s = \begin{cases} \theta_2^0/\theta_1^0 & \text{if } \theta_1^0 \neq 0, \\ \infty & \text{if } \theta_1^0 = 0. \end{cases}$$

For example, the slope of $\widehat{\Lambda}_2$ at A is 1 and the slope of $\widehat{\Gamma}_{\sigma_1^{-2}}$ at A is -1.

Lemma 2.4. *Let n be odd. In a neighborhood of A on $\widehat{H}_2$, $\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$ is a curve approaching A. Moreover, the slope of $\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$ at A is not equal to ± 1.*

Proof. Consider the submanifold

$$\Lambda'_n = \{(X_1, X_2, X_3, \cdots, X_n, Y_1, Y_2, X_3, \cdots, X_n) \in Q_n \times Q_n\}$$

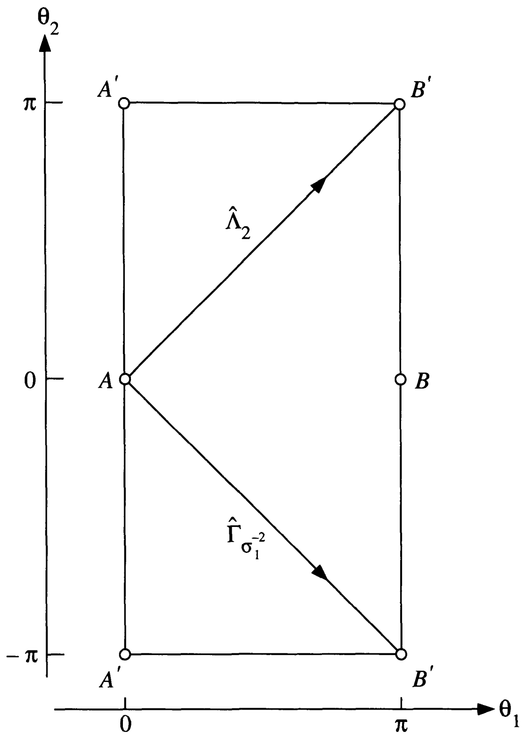

FIGURE 2.3

of $Q_n \times Q_n$, whose dimension is $2n+4$. So the minimal dimension of

$$T_{(\mathbf{i},\mathbf{i})}\Lambda'_n \cap T_{(\mathbf{i},\mathbf{i})}\Gamma_\sigma$$

is 4, where

$$\mathbf{i} = \left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \cdots, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \in Q_n.$$

Claim. $\dim T_{(\mathbf{i},\mathbf{i})}\Lambda'_n \cap T_{(\mathbf{i},\mathbf{i})}\Gamma_\sigma = 4$.

With this claim, we see that in a neighborhood of $\mathbf{i}$, $\Lambda'_n \cap \Gamma_\sigma$ is a manifold of dimension 4. Modulo the SU_2-action, we get the first statement of Lemma 2.4.

Proof of the Claim. At $\mathbf{i}$, the tangent map of $\sigma|Q_n$ under the standard basis is given by the Burau matrix of σ with parameter equal to -1. Denote this matrix by Ψ. It is a real matrix of rank n, so acts naturally on $\mathbb{R}^n$. Decompose Ψ as

$$\Psi = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix},$$

where $\mathbf{D}$ is an $(n-2)\times(n-2)$ matrix. Then, our claim amounts to saying that the real solutions of the equation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \tag{2.5}$$

are of dimension 2. To prove this, we need the following two properties of the matrix Ψ.

(1) The vectors fixed by Ψ are a subspace of dimension 1 spanned by $(1, \cdots, 1)^{\mathsf{T}}$.

(2) For a vector $\mathbf{v} = (v_1, \cdots, v_n)^{\mathsf{T}}$, let

$$\{\mathbf{v}\} = \sum_{\mu=1}^{n} (-1)^{\mu-1} v_\mu.$$

Then we have $\{\Psi \cdot \mathbf{v}\} = \{\mathbf{v}\}$ for any vector $\mathbf{v}$.

The first property comes from the facts that $\Delta_K(-1) \neq 0$ for any knot K and that $\sigma|Q_n$ leaves the whole diagonal of Q_n fixed. Notice that n has been chosen to be odd so that $\Delta_K(-1) \neq 0$ for $K = \sigma^\wedge$ implies $\det(1 - \psi_{-1}(\sigma)) \neq 0$ by (1.3), where $\psi_{-1}(\sigma)$ is the reduced Burau matrix of σ with parameter -1. The second property can be derived from the fact that σ preserves the product $X_1 \cdots X_n$ for any $(X_1, \cdots, X_n) \in Q_n$.

We now consider the following two cases.

Case 1. $\det(1 - \mathbf{D}) \neq 0$. In this case, solutions of (2.5) are given by

$$\begin{bmatrix} v_3 \\ \vdots \\ v_n \end{bmatrix} = (1 - \mathbf{D})^{-1} \mathbf{C} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Combined with property (1), it is easy to see that the real solutions of (2.5) are a subspace of dimension 2.

Notice that in this case, we can take

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and get

$$\Psi \cdot \begin{bmatrix} 1 \\ 0 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} s \\ s-1 \\ v_3 \\ \vdots \\ v_n \end{bmatrix},$$

where s is the slope of $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$ at A.

Case 2. $\det(1 - \mathbf{D}) = 0$. In this case, there is a nonzero vector $\mathbf{v} = (v_3, \cdots, v_n)^{\mathsf{T}}$ such that

$$\Psi \cdot \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} k \\ k \\ \mathbf{v} \end{bmatrix}$$

for some $k \neq 0$. If there are two such $\mathbf{v}$'s, say $\mathbf{v}$ and $\mathbf{v}'$, which are linearly independent, then

$$\Psi \cdot \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} - \frac{k}{k'}\mathbf{v}' \end{bmatrix} = \begin{bmatrix} k \\ k \\ \mathbf{v} \end{bmatrix} - \frac{k}{k'} \begin{bmatrix} k' \\ k' \\ \mathbf{v}' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} - \frac{k}{k'}\mathbf{v}' \end{bmatrix},$$

which contradicts property (1).

Now suppose for some vector $\mathbf{w} = (w_3, \cdots, w_n)^{\mathsf{T}}$, we have

$$\Psi \cdot \begin{bmatrix} 1 \\ 0 \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} s \\ s-1 \\ \mathbf{w} \end{bmatrix}.$$

Since $[{s \atop s-1}]$ and $[{k \atop k}]$ are linearly independent, there are some real numbers a and b such that

$$a \begin{bmatrix} s \\ s-1 \end{bmatrix} + b \begin{bmatrix} k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \Psi \cdot \begin{bmatrix} a \\ 0 \\ a\mathbf{w} + b\mathbf{v} \end{bmatrix} &= a \cdot \Psi \cdot \begin{bmatrix} 1 \\ 0 \\ \mathbf{w} \end{bmatrix} + b \cdot \Psi \cdot \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} \end{bmatrix} \\ &= a \cdot \begin{bmatrix} s \\ s-1 \\ \mathbf{w} \end{bmatrix} + b \cdot \begin{bmatrix} k \\ k \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ a\mathbf{w} + b\mathbf{v} \end{bmatrix}. \end{aligned}$$

By property (2) we must have $a = 1$, which contradicts property (1). This shows that the only nontrivial solution in this case is

$$\begin{bmatrix} 0 \\ 0 \\ \mathbf{v} \end{bmatrix}.$$

Thus, the real solutions of (3.5) in this case are also of dimension 2. Notice that the slope of $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$ at A is ∞ in this case.

We have finished the proof of our claim.

As for the second statement of Lemma 2.4, from the discussion of Case 1 above we first observe that if the slope of $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$ at A is 1, then property (1) will be contradicted. Next, since the Burau matrix of σ_1^2 with parameter -1 is $\left(\begin{smallmatrix}2 & -1\\ 1 & 0\end{smallmatrix}\right)$ and

$$\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}^2 \cdot \begin{bmatrix}-1\\ -2\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix},$$

we conclude that if the slope of $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$ at A is -1, then the slope of $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_{\sigma_1^2\sigma})$ at A is 1. This is impossible since $(\sigma_1^2\sigma)^\wedge$ is also a knot. This finishes the proof of Lemma 2.4. q.e.d.

Denote by s_σ the slope of $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$ at A. By Lemma 2.4, $s_\sigma \neq \pm 1$.

Lemma 2.6. *$h(\sigma_1^2\sigma) - h(\sigma) = \varepsilon$, where $\varepsilon = 1$ or 0 depending on whether $|s_\sigma|$ is greater or less than 1.*

Proof. Since $\sigma^\wedge$ is a knot, we claim that there is a neighborhood of $B'(A')$ on $\widehat{H}_2$ such that $\hat{p}(\widehat{V}_n \cap \widehat{\Gamma}_\sigma)$ is disjoint with that neighborhood. Suppose this is not true for B'. Then, by taking the limit, we will get a reducible fixed point $(X_1, \cdots, X_n) \in Q_n$ for $\sigma|Q_n$ such that $(X_1, X_2) \sim (+, -)$. This is impossible since the permutation induced by σ among the X_μ's has no nontrivial subcycle. As for A', our claim is even true without the assumption that $\sigma^\wedge$ is a knot.

By a small perturbation relative to a neighborhood of A on $\widehat{H}_2$ changing $\widehat{\Gamma}_\sigma$ to $\widetilde{\Gamma}_\sigma$, we get $\hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)$ as a 1-submanifold of $\widehat{H}_2$. In a neighborhood of A, it is a curve approaching A with the slope s_σ. The other end of that curve must approach B. Since

$$h(\sigma_1^2\sigma) - h(\sigma) = \langle \widehat{\Gamma}_{\sigma_1^{-2}\sigma} - \widehat{\Lambda}_2, \hat{p}(\widehat{V}_n \cap \widetilde{\Gamma}_\sigma)\rangle_{\widehat{H}_2},$$

by using Lemma 2.3, the topology of the space $\widehat{H}_2$, and the fact that the slopes of $\widehat{\Lambda}_2$ and $\widehat{\Gamma}_{\sigma_1^{-2}\sigma}$ are ± 1 respectively, we can easily derive the conclusion of Lemma 2.6. q.e.d.

The proof of Lemma 2.4 shows that the slope s_σ depends on the Burau matrix of σ with parameter -1. Let us make this fact more precise.

Lemma 2.7. *We have*

$$\frac{s_\sigma - 1}{s_\sigma + 1} = \frac{\det(1 - \psi_{-1}(\sigma))}{\det(1 - \psi_{-1}(\sigma_1^2\sigma))}. \tag{2.8}$$

Here, if $s_\sigma = \infty$, the left side of (2.8) *is defined to be* 1.

Proof. Use the notation in the proof of Lemma 2.4. The Burau matrices of σ and $\sigma_1^2\sigma$ with parameter -1 are

$$\Psi = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\mathbf{G}^2 & 0\\ 0 & \mathbf{I}\end{pmatrix}\Psi,$$

respectively, where $\mathbf{I}$ is the $(n-2) \times (n-2)$ identity matrix and $\mathbf{G} = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We first assume $\det(1 - \mathbf{D}) \neq 0$. In this case, similar to the proof of Lemma 2.4, let

$$\mathbf{v} = (1 - \mathbf{D})^{-1}\mathbf{C}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} s_\sigma \\ s_\sigma - 1 \end{bmatrix} = [\mathbf{A} + \mathbf{B}(1 - \mathbf{D})^{-1}\mathbf{C}]\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So we can denote

$$\mathbf{A} + \mathbf{B}(1 - \mathbf{D})^{-1}\mathbf{C} = \begin{pmatrix} s_\sigma & a \\ s_\sigma - 1 & b \end{pmatrix}$$

for some real numbers a and b. Let $\mathbf{T} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$. Then

$$\begin{aligned}(t-1)\det(1 - \psi_{-1}(\sigma)) &= \det\left(\begin{pmatrix} \mathbf{T} & 0 \\ 0 & \mathbf{I} \end{pmatrix} - \Psi\right) \\ &= \det(\mathbf{T} - [\mathbf{A} + \mathbf{B}(1 - \mathbf{D})^{-1}\mathbf{C}])\det(1 - \mathbf{D}) \\ &= \det\begin{pmatrix} 1 - s_\sigma & -a \\ 1 - s_\sigma & t - b \end{pmatrix}\det(1 - \mathbf{D}).\end{aligned}$$

For $t = 1$, we must have

$$\det\begin{pmatrix} 1 - s_\sigma & -a \\ 1 - s_\sigma & t - b \end{pmatrix} = 0.$$

Since $s_\sigma \neq 0$, we get $a = b - 1$ and

$$\det\begin{pmatrix} 1 - s_\sigma & -a \\ 1 - s_\sigma & t - b \end{pmatrix} = (t-1)(1 - s_\sigma).$$

Thus,

$$\det(1 - \psi_{-1}(\sigma)) = (1 - s_\sigma)\det(1 - \mathbf{D}).$$

Similarly,

$$\det(1 - \psi_{-1}(\sigma_1^2\sigma)) = (1 - s_{\sigma_1^2\sigma})\det(1 - \mathbf{D}).$$

Notice that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^2 \begin{bmatrix} s_\sigma \\ s_\sigma - 1 \end{bmatrix} = \begin{bmatrix} s_\sigma + 2 \\ s_\sigma + 1 \end{bmatrix}.$$

So $s_{\sigma_1^2\sigma} = s_\sigma + 2$. This proves Lemma 2.7 in the case that $\det(1 - \mathbf{D}) \neq 0$.

We can use a limiting argument to deal with the case $\det(1 - \mathbf{D}) = 0$. Since it is similar to the argument in the case $\det(1 - \mathbf{D}) \neq 0$, we omit the details here.

Theorem 2.9. *Let $K_+ = (\sigma_1^2\sigma)^\wedge$ and $K_- = \sigma^\wedge$ be two knots. Then*

$$h(K_+) - h(K_-) = \varepsilon,$$

where $\varepsilon = 0$ *or* 1 *depending on whether* $\Delta_{K_+}(-1)$ *and* $\Delta_{K_-}(-1)$ *have the same sign or not.*

Proof. Assume n is odd again. Since the difference of the exponent sums of $\sigma_1^2\sigma$ and σ is 2, by formulas (1.3) and (2.8), we get

$$\frac{s_\sigma - 1}{s_\sigma + 1} > 0 \quad \text{iff} \quad \Delta_{K_+}(-1)\Delta_{K_-}(-1) < 0$$

and

$$\frac{s_\sigma - 1}{s_\sigma + 1} < 0 \quad \text{iff} \quad \Delta_{K_+}(-1)\Delta_{K_-}(-1) > 0.$$

On the other hand,

$$\frac{s_\sigma - 1}{s_\sigma + 1} > 0 \quad \text{iff} \quad |s_\sigma| > 1$$

and

$$\frac{s_\sigma - 1}{s_\sigma + 1} < 0 \quad \text{iff} \quad |s_\sigma| < 1.$$

Thus, from Lemma 2.6 it follows our theorem. q.e.d.

Recall the signature of a knot is an integer invariant of (unoriented) knots. It is defined in the following way. Let K be an oriented knot. Suppose S is a compact, connected, and oriented surface imbedded in S^3 such that $\partial S = K$, i.e., S is a *Seifert surface* of K. Let l_1 and l_2 be two oriented simple loops on S. Define

$$q(l_1, l_2) = \mathrm{lk}(l_1, l_2^+),$$

where $\mathrm{lk}(\ ,\ )$ is the linking number in S^3, and l_2^+ is the push-off of l_2 away from S along its positive normal direction. Then $q(\ ,\ )$ induces a bilinear form on $H_1(S; \mathbb{Z})$. Suppose Q is the matrix of this bilinear form under some basis. Then, the signature $\mathrm{sgn}(K)$ of the knot K is defined to be the signature of the symmetric matrix $Q + Q^{\mathrm{T}}$ (see [12]).

The signature of a knot K is always an even integer. We have

$$\mathrm{sgn}(K) \equiv 0 \pmod 4 \quad \text{if } \Delta_K(-1) > 0$$

and

$$\mathrm{sgn}(K) \equiv 2 \pmod 4 \quad \text{if } \Delta_K(-1) < 0.$$

On the other hand, for the knots K_+ and K_- in Theorem 2.9,

$$0 \le \mathrm{sgn}(K_+) - \mathrm{sgn}(K_-) \le 2,$$

(see [5] as well as [7]). Thus, we get the following corollary of Theorem 2.9.

Corollary 2.10. *For a knot* K, *we have* $h(K) = \frac{1}{2}\mathrm{sgn}(K)$.

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