J. DIFFERENTIAL GEOMETRY
34 (1991) 769-834

STABLE PSEUDOISOTOPY SPACES OF COMPACT NONPOSITIVELY CURVED MANIFOLDS

F. T. FARRELL & L. E. JONES

0. Statement of results

In this section we formulate the main result of the paper (cf. Theorem 0.4), and derive from it a number of corollaries. At the end of this section we outline very briefly the proof of our main result. Our main result states that the space of stable pseudoisotopies $\mathscr{P}(M)$ of any closed Riemannian manifold M with sectional curvature $K \leq 0$ everywhere can be computed in a simple way from the stable pseudoisotopy spectrum $\mathscr{P}_*(S^1)$ of the circle. (More generally, a similar result is true for the stable pseudoisotopy spectrum $\mathscr{P}_*(M)$.) This is a new result even for the case when M is the flat 2-torus. All the results discussed in this section have been announced in [13], and have been proven in earlier papers of the authors' for the special case when M has $K < 0$ everywhere (cf. [7], [9], [12]). The reader is referred to [14] for an expository account of the authors' work to date on the stable pseudoisotopy spectrum for any compact aspherical manifold. The formula arrived at in this paper for computing the stable pseudoisotopy spectrum $\mathscr{P}_*(M)$ in terms of the stable pseudoisotopy spectrum $\mathscr{P}_*(S^1)$ involves the space of all closed geodesics in the compact nonpositively curved manifold M. There is an equivalent purely homotopic theoretic formulation of this result which involves the space of all continuous maps $S^1 \to M$ (cf. [14, §4]). This has motivated the authors to conjecture that the stable pseudoisotopy spectrum $\mathscr{P}_*(X)$ of any aspherical space X with torsion free fundamental group can be computed in a simple way from the stable pseudoisotopy spectrum $\mathscr{P}_*(S^1)$ (cf. [14, §4] for a precise statement of this conjecture).

Before stating our main theorem we review in §§0.1 and 0.2 the concept of homology theory with coefficients in an Ω-spectrum, and we outline in §0.3 the structure of the set of all closed geodesics in M.

Received April 16, 1990 and, in revised form, October 19, 1990. Both authors were supported in part by the National Science Foundation.

0.1. Ω-spectra. Recall that an *Ω-spectrum* consists of a bi-infinite sequence $\mathcal{S}_* = \{\mathcal{S}_j : j \in \mathbf{Z}\}$ of spaces with base point, together with weak homotopy equivalences $\{h_j : \mathcal{S}_j \to \Omega\mathcal{S}_{j+1}\}$, called the structure maps for the Ω-spectrum, from the jth space to the loop space of the $(j+1)$th space. (If the structure maps are not required to be weak homotopy equivalences, then the $\mathcal{S}_*$ together with its structure maps is called a *spectrum*.) For any integer k the *kth homotopy group of the Ω-spectrum* $\mathcal{S}_*$ is denoted by $\pi_k(\mathcal{S}_*)$ and is defined to be the homotopy group $\pi_{k+j}(\mathcal{S}_j)$ for any integer j satisfying $k + j > 0$. A *map of Ω-spectra* $r_* : \mathcal{S}_* \to \mathcal{S}'_*$ consists of a collection of maps $\{r_j : \mathcal{S}_j \to \mathcal{S}'_j\}$ such that $(\Omega r_{j+1}) \circ h_j$ is homotopic to $h'_j \circ r_j$ for all values of j. A map of Ω-spectra $r_* : \mathcal{S}_* \to \mathcal{S}'_*$ is called a *weak equivalence* if it induces an isomorphism on the homotopy groups of the Ω-spectra, and it is called an *equivalence* if there is a reverse map of Ω-spectra $r'_* : \mathcal{S}'_* \to \mathcal{S}_*$ such that each composite map $r_j \circ r'_j$, $r'_j \circ r_j$ is homotopic to the identity map.

Let X denote a manifold, possibly with nonempty boundary ∂X. Recall that a pseudoisotopy of X is a homeomorphism $h : X \times [0, 1] \to X \times [0, 1]$ such that the restricted map $h \mid X \times 0$ is the inclusion. We denote by $P(X)$ the space of all pseudoisotopies of X, equipped with the compact open topology. For each integer $n > 0$ let I^n denote the n-fold Cartesian product of the unit interval $I = [0, 1]$ with itself. Note that there is an "inclusion" map $P(X \times I^n) \to P(X \times I^{n+1})$ obtained by forming the Cartesian product of any pseudoisotopy $h : X \times I^n \times [0, 1] \to X \times I^n \times [0, 1]$ with the identity map $I \to I$. We denote by $\mathcal{P}(X)$ the direct limit space $\mathrm{limit}_{n\to\infty} P(X \times I^n)$, and call $\mathcal{P}(X)$ the *space of stable pseudoisotopies of* X. A result of A. Hatcher [17] states that $\mathcal{P}(X)$ is the zeroth space in a Ω-spectrum which is called the *stable pseudoisotopy spectrum* of X and is denoted by $\mathcal{P}_*(X) = \{\mathcal{P}_j(X) : j \in Z\}$ (cf. §1.3). By taking the direct limit of the Ω-spectra $\mathcal{P}_*(C)$ over all compact codimension zero submanifolds $C \subset X$ we get the Ω-spectrum $\mathcal{P}^c_*(X)$ of compactly supported stable pseudoisotopies of X. By appealing to semisimplicial constructions, $\mathcal{P}^c_*(X)$ may be defined for any topological space X; in fact $\mathcal{P}^c_*(\ )$ is a homotopy functor from the category of topological spaces and continuous maps to the category of Ω-spectra and Ω-spectra maps (cf. [25]).

0.2. Homotopy theory with coefficients in a spectrum. We remind the reader that Ω-spectra are the "coefficients" for generalized homology theories. Let X denote a topological space, let $\mathcal{S}_*$ denote an Ω-spectrum, and define for each integer j a space $\mathbb{H}_j(X, \mathcal{S}_*)$ to be the direct limit space $\mathrm{limit}_{i\to\infty} \Omega^i(X \times \mathcal{S}_{i+j}/X \times s_{i+j})$, where s_{i+j} is the base point for

$\mathscr{S}_{i+j}$. The collection of spaces $\mathbb{H}_*(X, \mathscr{S}_*) = \{\mathbb{H}_j(X, \mathscr{S}_*) : j \in \mathbf{Z}\}$ is an Ω-spectrum called the *homology spectrum* for X with coefficients in $\mathscr{S}_*$. The *jth homology group for* X with coefficients in $\mathscr{S}_*$ is denoted by $H_j(X, \mathscr{S}_*)$ and is defined to be $\pi_j(\mathbb{H}_*(X, \mathscr{S}_*))$.

There is a more complicated version of generalized homology theory where the "spectrum" of coefficients is both stratified and twisted over X (cf. [22], [25, Appendix], [14, §1]), which we will need in the formulation of our main theorem. The following version is taken from F. Quinn's paper [25]. Let $p: Y \to X$ denote a simplicially stratified fiber bundle over the space X (cf. Definition 1.1 for "simplicially stratified fiber bundle"). Let K denote the first barycentric subdivision for a triangulation of X for which $p: Y \to X$ satisfies 1.1(a), (b). For each integer j define a space $\mathbb{P}_j(p)$ to be the quotient space $(\bigcup_{\Delta \in K} \mathscr{P}_j(p^{-1}(\Delta)) \times \Delta)/\approx$, where the equivalence relation $\approx$ identifies $\mathscr{P}_j(p^{-1}(\Delta')) \times \Delta'$ with its image in $\mathscr{P}_j(p^{-1}(\Delta)) \times \Delta$ under the map induced by inclusion $\Delta' \to \Delta$ for every pair of simplices $\Delta', \Delta \in K$ which satisfy $\Delta' \subset \Delta$. By taking the union of the structure maps $\mathscr{P}_j(p^{-1}(\Delta)) \times \Delta \to \Omega\mathscr{P}_{j+1}(p^{-1}(\Delta)) \times \Delta$ we make the collection $\mathbb{P}_*(p) = \{\mathbb{P}_j(p) : j \in \mathbf{Z}\}$ into an ex-spectrum (cf. [24, Appendix]). Note that, in order to assure the union of structure maps $\mathscr{P}_j(p^{-1}(\Delta)) \times \Delta \to \Omega\mathscr{P}_{j+1}(p^{-1}(\Delta)) \times \Delta$ is well defined, we must know that they commute pointwise with the inclusion induced maps $\mathscr{P}_j(p^{-1}(\Delta')) \times \Delta' \to \mathscr{P}_j(p^{-1}(\Delta)) \times \Delta$. This can be arranged in various ways. For example if Y is a countable, locally finite simplicial complex of finite dimension and each $p^{-1}(\Delta)$ is a finite subcomplex of Y (this is the only situation which will occur in this paper), then we choose a PL embedding $Y \subset \mathbf{R}^n$, we choose a PL triangulation L for $\mathbf{R}^n$ which subdivides the given triangulation of Y, and for any finite subcomplex C of Y we define $\mathscr{P}_j(C)$ to be all stable pseudoisotopies in $\mathscr{P}_j(\mathbf{R}^n)$ whose support lies over $C' - \partial C'$, where C' denotes the union of all simplices in the second barycentric subdivision of L which intersect with C. Note also that X may be identified with the subspace $(\bigcup_{\Delta \in K} I_\Delta \times \Delta)/\approx$ of $\mathbb{P}_j(p)$, where I_Δ is the identity stable pseudoisotopy on $p^{-1}(\Delta) \times \mathbf{R}^j$. Thus we can define for each integer j a space $\mathbb{H}_j(X, \mathbb{P}_*(p))$ to be the direct limit space $\operatorname{limit}_{i \to \infty} \Omega^i(\mathbb{P}_{j+i}(p)/X)$. The collection of spaces $\mathbb{H}_*(X, \mathbb{P}_*(p)) = \{\mathbb{H}_j(X, \mathbb{P}_*(p)) : j \in \mathbf{Z}\}$ is an Ω-spectrum called the *homology spectrum* for X with coefficients in the (stratified and twisted) ex-spectrum $\mathbb{P}_*(p)$. The *jth homology group for* X with coefficients in $\mathbb{P}_*(p)$ is denoted by $H_j(X, \mathbb{P}_*(p))$ and is defined to be $\pi_j(\mathbb{H}_*(X, \mathbb{P}_*(p)))$. In the event that $p: Y \to X$ is a trivial fiber bundle over X having the compact manifold F for fiber we have the equalities $\mathbb{H}_*(X, \mathbb{P}_*(p)) = \mathbb{H}_*(X, \mathscr{P}_*(F))$ and $H_j(X, \mathbb{P}_*(p)) = H_j(X, \mathscr{P}_*(F))$ for all values of j.

The inclusion maps $p^{-1}(\Delta) \subset Y$, $\Delta \in K$, induce maps $\mathscr{P}_j(p^{-1}(\Delta)) \to \mathscr{P}_j^c(Y)$, $\Delta \in K$, the union of which yields a map $\psi_j \colon \mathbb{P}_j(p) \to \mathscr{P}_j^c(Y)$ for all values of j. Define $A_j \colon \mathbb{H}_j(X, \mathbb{P}_*(p)) \to \mathscr{P}_j^c(Y)$ to be the direct limit as $i \to \infty$ of the composite maps

$$\Omega^i(\mathbb{P}_{j+i}(p)/X) \xrightarrow{\Omega^i(\psi_{j+i})} \Omega^i(\mathscr{P}_{j+i}^c(Y)) = \mathscr{P}_j^c(Y).$$

The collection of all such maps, which is denoted by $A_* \colon \mathbb{H}_*(X, \mathbb{P}_*(p)) \to \mathscr{P}_*^c(Y)$ and is called the *assembly map*, is a map of Ω-spectra.

0.3. The structure of the set of closed geodesics in M. Let M denote a closed Riemannian manifold with sectional curvature $K \leq 0$ everywhere. Let SM and RPM denote respectively the unit sphere bundle and the real projective bundle associated to the tangent space for M. There is a geodesic flow $g^t \colon SM \to SM$, $t \in R$, on SM, and there is a smooth one-dimensional foliation $\mathscr{F}$ of RPM whose leaves are covered by the orbits of g^t under the canonical 2-fold covering projection $SM \to RPM$. Fix a Riemannian metric on RPM, and for any positive number s let E_s denote the union of all compact leaves of $\mathscr{F}$ which have length less than or equal to s. Let $p_s \colon E_s \to G_s$ denote the quotient map obtained by collapsing each closed leaf of $\mathscr{F}$ (contained in E_s) to a point. Finally let $p \colon E \to G$ denote the direct limit as $s \to \infty$ of the maps $p_s \colon E_s \to G_s$. Let $f \colon E \to M$ denote the direct limit as $s \to \infty$ of the composite maps $E_s \subset RPM \xrightarrow{\text{proj}} M$.

There are a couple of facts of which to take note. First we note that E is the collection of all closed (unparametrized) geodesics in M: a typical closed geodesic in M is the image of a fiber of $p \colon E \to G$ under the map $f \colon E \to M$. Thus G is the space which parametrizes the collection of all closed geodesics in M. The second fact to note is that $p \colon E \to G$ is a simplicially stratified fiber bundle having circles for fibers. This is not an obvious fact, but is an easy consequence of Theorem 2.4.

We can now state the main result of this paper.

0.4. Theorem. *Let M be a compact Riemannian manifold having sectional curvature $K \leq 0$ everywhere. Let $p \colon E \to G$ and $f \colon E \to M$ be as in* §0.3. *Then there is a weak equivalence of Ω-spectra*

$$e_* \colon \mathbb{H}_*(G, \mathbb{P}_*(p)) \to \mathscr{P}_*(M).$$

In fact we have that $e_* = \mathscr{P}_*^c(f) \circ A_*$, where $A_* \colon \mathbb{H}_*(G, \mathbb{P}_*(p)) \to \mathscr{P}_*^c(E)$

is the assembly map of §0.2, *and* $\mathscr{P}^c_*(f)\colon \mathscr{P}^c_*(E) \to \mathscr{P}_*(M)$ *is the image of* f *under the functor* $\mathscr{P}^c_*(\ )$.

Remark. For the special case when $K < 0$ holds everywhere for M, the authors have proven a fibered-orbifold version of Theorem 0.4 (cf. [8]). There is also such a generalized version of Theorem 0.4 when $K \leq 0$ holds everywhere. The case of compact local symmetric space orbifolds with $K \leq 0$ is discussed in [15].

Caveat. Theorem 1(ii) of [8] is incorrect in the generality stated. It is correct when the finite group action $F \times M \to M$ satisfies the following extra condition: Let ω be any closed geodesic of M, which is left invariant by some element $g \in F$, then $g|\omega$ is orientation preserving. Under the general assumption made in [8] there is always a map of spectra $X_{i=1}^{\infty} N\mathscr{P}(S^1; \rho_{\omega_i}) \to N\mathscr{P}(X; \rho)$ which induces an epimorphism on homotopy groups. A more precise conclusion is obtainable from the techniques of [15]. The remaining results in [8] are correct.

We can now state the corollaries of Theorem 0.4. The derivation of these corollaries from Theorem 0.4 will be obvious to the experts, so we only briefly indicate their proofs. The reader is referred to [14] for more detailed proofs of the corollaries.

The first of our corollaries was pointed out to us by Dieter Puppe. In this corollary we let J denote a countable collection of nonnegative integers (not necessarily distinct integers), and for each $j \in J$ we let $\mathscr{P}_{*+j}(S^1)$ denote the Ω-spectrum having for its kth space the space $\mathscr{P}_{k+j}(S^1)$. We also let $\bigoplus_{j\in J} \mathscr{P}_{*+j}(S^1)$ denote the direct limit as $i \to \infty$ of the finite Cartesian product spaces $\prod_{j\in J_i} \mathscr{P}_{*+j}(S^1)$, where J_i denotes the first i integers in J with respect to some fixed ordering of J.

0.5. Corollary. *Suppose that* M *is a flat* m*-dimensional torus where* $m > 1$. *Then there is a weak equivalence of* Ω*-spectra*

$$e'_*\colon \bigoplus_{j\in J} \mathscr{P}_{*+j}(S^1) \to \mathscr{P}_*(M).$$

Here J *is a countable collection of integers which satisfy the following two properties: if* $j \in J$ *then we must have* $0 \leq j \leq m-1$; *any integer* j *which satisfies* $0 \leq j \leq m-1$ *must occur an infinite number of times in* J.

Proof of Corollary 0.5. In this case $p\colon E \to G$ of §0.3 is just the disjoint union of a countable number of copies of the standard projection $T^{m-1} \times S^1 \to T^{m-1}$, where T^{m-1} denotes the $(m-1)$-dimensional torus. Thus $\mathbb{H}_*(G, \mathbb{P}_*(p)) = \bigoplus_{i=1}^{\infty} \mathbb{H}_*(T_i^{m-1}, \mathscr{P}_*(S^1))$. On the other hand it is well known that $\mathbb{H}_*(T^{m-1}, \mathcal{S}_*) = \prod_{k \in K} \mathcal{S}_{*+k}$ holds for any Ω-spectrum $\mathcal{S}_*$, where K is the finite collection of nonnegative integers which contains $\operatorname{rank}(H_r(T^{m-1}, \mathbf{Z}))$ many copies of any integer r satisfying $0 \leq r \leq m-1$, and contains no other integers. Now Corollary 0.5 follows from Theorem 0.4 and the preceding facts. This completes the proof of Corollary 0.5.

0.6. Corollary. *For M as in Theorem 0.4, we have that $\pi_j(\mathcal{P}_*(M)) = 0$ if $j < 0$, and $\pi_j(\mathcal{P}_*(M)) \otimes \mathbf{Z}(1/N) = 0$ if $j \geq 0$, where $N = [(j+4)/2]!$. Moreover $\pi_0(\mathcal{P}_*(M)) = \mathbf{Z}_2^\infty$.*

Remark. A. Nicas has proven Corollary 0.6 when $j \geq 0$ and $K = 0$ holds everywhere (cf. [22]). The authors have proven Corollary 0.6 for all values of j assuming that $K < 0$ holds everywhere (cf. [9; 12, Appendix]).

Proof of Corollary 0.6. It follows from results of Anderson and Hsiang [1], Waldhausen [28], [29], and Nicas [22], that the equalities in the first sentence of Corollary 0.6 are true if we replace M by the circle S^1. On the other hand Quinn has shown that there is a spectral sequence with $E^2_{i,j} = H_i(G, \pi_j(\mathbb{P}_*(p)))$ which abuts to $H_{i+j}(G, \mathbb{P}_*(p))$, where $\pi_j(\mathbb{P}_*(p))$ denotes the stratified system of groups $\{\pi_j(\mathcal{P}_*(p^{-1}(x))) : x \in G\}$ over G (cf. [25, Theorem 8.7]). Finally note that each $p^{-1}(x)$, $x \in G$, is a circle. This completes the verification of the equations in the first sentence of Corollary 0.6.

Now we will verify that $\pi_0(\mathcal{P}_*(M)) = \mathbf{Z}_2^\infty$. Note that $\pi_j(\mathcal{P}_*(S^1)) = 0$ (for all $j < 0$) implies that Quinn's spectral sequence is a first quadrant spectral sequence. From this we deduce:

0.6.1. $\pi_0(\mathcal{P}_*(M)) = H_0(G, \pi_0(\mathbb{P}_*(p)))$.

Results of Waldhausen [29] and Igusa [21, Theorem 13.1] show that $\pi_0(\mathcal{P}_*(S^1)) = \mathbf{Z}_2^\infty$. From this we deduce:

0.6.2. $\pi_0(\mathbb{P}_*(p))_x = \mathbf{Z}_2^\infty$ over each point $x \in G$.

Now the desired calculation follows from 0.6.1, 0.6.2 and from the properties of $p: E \to G$ stated in Theorem 2.4.

This completes the proof of Corollary 0.6.

0.7. Corollary. *For M as in Theorem* 0.4 *we have* $\mathrm{Wh}(\pi_1(M)) = 0$, $\tilde{K}_0(\mathbf{Z}(\pi_1(M))) = 0$, $K_i(\mathbf{Z}(\pi_1(M))) = 0$ *for all integers $i < 0$*, $\mathrm{Wh}_i(\pi_1(M)) \otimes \mathbf{Z}(1/N) = 0$ *for all integers $i > 2$, where $N = [(i+2)/2]!$, and* $\mathrm{Wh}_2(\pi_1(M)) = 0$.

Remark. Farrell and Hsiang verified most of the equalities in Corollary 0.7 (all those with $j \leq 1$) in the special case where $K = 0$ holds everywhere (or more generally when M is any compact aspherical manifold having a poly(finite or cyclic) fundamental group (cf. [6])). Nicas verified the higher Whitehead group equalities of 0.7 in the special case where

$K = 0$ holds everywhere (cf. [23]), and F. Quinn has announced a proof that the higher Whitehead groups vanish for any torsion free poly(finite or cyclic) group (cf. [26]). The authors verified all the equalities of 0.7 in the special case where $K < 0$ holds everywhere (cf. [7], [9]). Hu verified most of the equalities of 0.7 (all those with $j \leq 1$) for a special class of nonpositively curved manifolds (cf. [20]).

Proof of Corollary 0.7. Anderson and Hsiang [1] have shown that $\pi_j(\mathscr{P}_*(M))$ equals $K_{j+2}(\mathbf{Z}(\pi_1(M)))$ if $j \leq -3$, equals $\widetilde{K}_0(\mathbf{Z}(\pi_1(M)))$ if $j = -2$, and equals $\mathrm{Wh}(\pi_1(M))$ if $j = -1$. Waldhausen [29] and Nicas [23] have shown that $\pi_{i-2}(\mathscr{P}_*(M)) \otimes \mathbf{Z}(1/N) = \mathrm{Wh}_i(\pi_1(M)) \otimes \mathbf{Z}(1/N)$ for $i \geq 2$, where we must use that M is an aspherical manifold (cf. Lemma 2.1). Thus all of the equalities of 0.7, except the last, follow from 0.6 and the preceding facts.

To verify the last equality in 0.7 it will suffice to improve our calculation of the higher Whitehead groups to $\mathrm{Wh}_i(\pi_1(M)) \otimes \mathbf{Z}(1/N') = 0$, where $N' = [(i+1)/2]!$. Towards this end we let $\mathscr{Wh}_*(\ )$ denote the Ω-spectra valued functor which on the space X takes the value of the algebraic Whitehead groups spectrum for the fundamental group $\pi_1(X)$ if X is path connected (cf. [28]). If X is not path connected we let $\mathscr{Wh}_*(X)$ denote the direct limit of the finite product spaces $\prod_{j \in J} \mathscr{Wh}_*(X_j)$ taken over all finite collections $\{X_j : j \in J\}$ of distinct path components of X. Thus $\pi_i(\mathscr{Wh}_*(M)) = \mathrm{Wh}_i(\pi_1(M))$ for all $i > 0$. There is the homology theory $H_*(G, W\mathbb{H}_*(p))$ for G with coefficients in the ex-spectrum $W\mathbb{H}_*(p)$ defined as in 0.2, and there is a spectral sequence with $E^2_{k,j} = H_k(G, \pi_j(W\mathbb{H}_*(p)))$ which abuts to $H_{k+j}(G, W\mathbb{H}_*(p))$. Note that since each fiber of $p\colon E \to G$ is a circle, the stratified system of groups $\pi_j(W\mathbb{H}_*(p))$ is identically zero, and thus $H_*(G, W\mathbb{H}_*(p)) = 0$. On the other hand there is for each integer i a commutative diagram

$$\begin{tikzcd}
H_{i-2}(G, \mathbb{P}_*(p)) \arrow[r, "\pi_{i-2}(\mathscr{P}^c_*(f)\circ A_*)"] \arrow[d, "\psi_1"'] & \pi_{i-2}(\mathscr{P}_*(M)) \arrow[d, "\psi_2"] \\
H_{i-2}(G, W\mathbb{H}_{*-2}(p)) \arrow[r, "\pi_{i-2}(\mathscr{Wh}_{*-2}(f)\circ \overline{A}_*)"] & \pi_{i-2}(\mathscr{Wh}_{*-2}(M))
\end{tikzcd}$$

where $\mathscr{Wh}_*(f)\colon \mathscr{Wh}_*(E) \to \mathscr{Wh}_*(M)$ is induced by the map $f\colon E \to M$, $\overline{A}_*\colon \mathbb{H}_*(G, W\mathbb{H}_*(p)) \to \mathscr{Wh}_*(E)$ is an assembly map defined as in 0.2, and ψ_1, ψ_2 are the maps induced by the usual "forgetful map" $\mathscr{P}_*(\ ) \to \mathscr{Wh}_{*-2}(\ )$ between functors (cf. [17], [29]). According to [23, 2.4] the map ψ_2 is onto modulo N'-torsion, and by Theorem 0.4 the map

$\pi_{i-2}(\mathscr{P}^c_*(f)\circ A_*)$ is an isomorphism. It follows that $\mathrm{Wh}_i(\pi_1(M))\otimes\mathbf{Z}(1/N') = 0$, as desired.

This completes the proof of Corollary 0.7.

Since $\mathrm{Wh}_i(\pi_1(M))$ is defined in terms of $K_i(\mathbf{Z}(\pi_1(M)))$ (cf. [28]), Corollary 0.7 implies the following calculation for $K_i(\mathbf{Z}(\pi_1(M)))$.

0.8. Corollary. *For M as in Theorem* 0.4 *and all integer values of n we have that*

$$K_n(\mathbf{Z}(\pi_1(M)))\otimes\mathbf{Q} = H_n(M, \mathbf{Q}) \oplus \left(\bigoplus_{i=1}^{\infty} H_{n-1-4i}(M, \mathbf{Q})\right).$$

Remark. The authors have obtained Corollary 0.8 in the special case where M is a locally symmetric compact manifold by somewhat different arguments in [10].

Outline of the proof of Theorem 0.4. Our first step is to replace the assembly map $A_*: \mathbb{H}_*(G, \mathbb{P}_*(p)) \to \mathscr{P}^c_*(E)$ by a more geometric map $J_*: \mathscr{P}^c_*(p) \to \mathscr{P}^c_*(E)$ (cf. 1.3 and 1.4). Quinn has shown that there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathbb{H}_*(G, \mathbb{P}_*(p)) & \xrightarrow{\Psi_*} & \mathscr{P}^c_*(p) \\ {\scriptstyle A_*}\searrow & & \swarrow{\scriptstyle J_*} \\ & \mathscr{P}^c_*(E) & \end{array}$$

where the map Ψ_* is a weak equivalence of Ω-spectra (cf. [25, Appendix]). Thus to complete the proof of Theorem 0.4 it will suffice to show that $\mathscr{P}^c_*(f)\circ J_*: \mathscr{P}^c_*(p) \to \mathscr{P}_*(M)$ is a weak equivalence of Ω-spectra.

In §3 we show that $\mathscr{P}^c_*(f)\circ J_*$ induces a surjection on the homotopy groups of the Ω-spectra. We remark that this is enough to prove all the corollaries 0.6–0.8 discussed above. §§1 and 2 contain all the topological and geometric preliminaries that are needed to carry out the arguments in §3. Our proof that $\mathscr{P}^c_*(f)\circ J_*$ is surjective on homotopy groups of the Ω-spectra is similar in spirit to the argument used in [9] to show that $\mathscr{P}^c_0(f)\circ J_0$ is surjective if $K < 0$ holds everywhere. We set $N = M\times\mathbf{R}$ (equipped with the product metric) and let $S^+N \subset SN$ denote the subbundle of all vectors v in the unit sphere bundle of N which satisfy $\langle v, u\rangle_N \geq 0$, where $u: N \to SN$ is the unit length vector field pointing in the direction of the (increasing) $\mathbf{R}$-factor of N. Let $t_*: \mathscr{P}_*(M) \to \mathscr{P}^c_*(S^+N)$ denote the composite of the map $\mathscr{P}_*(M) \to \mathscr{P}^c_*(N)$, induced by the inclusion $M\times[0, 1] \subset N$, with the "special transfer" map $\mathscr{P}^c_*(N) \to \mathscr{P}^c_*(S^+N)$ (cf. §§3.9 and 3.10 for the "special transfer" map). Now apply the geodesic flow on S^+N to gain foliated control of $\mathrm{Image}(t_*)$, and next apply a foliated control theorem (for foliations having one-dimensional leaves, cf.

Theorem 1.5.3) to isolate the support of Image(t_*) over compact subsets of small tubular neighborhoods for a finite number of components of the space of all closed geodesic orbits for the geodesic flow on S^+N. The result of these constructions may be viewed as a deformation retraction of $\mathscr{P}_*(M)$ onto Image($\mathscr{P}_*^c(f) \circ J_*$). We note that the "special transfer" map used in this paper, which boils down to the method for lifting paths from N to S^+N discussed in §3.3, differs from the "asymptotic transfer" map used in [9].

In §6 we show that $\mathscr{P}_*^c(f) \circ J_*$ induces an injection on the homotopy groups of the Ω-spectra. §§4 and 5 contain further topological and geometric results which are needed to carry out the arguments of §6. To verify that $\mathscr{P}_*^c(f) \circ J_*$ is injective on homotopy groups we construct maps $r_*^i: \mathscr{P}_*(M) \to \widetilde{\mathscr{P}}_*^b(p_i^B)$ for each component E_i of E, and verify that the composite $r_*^i \circ \mathscr{P}_*^c(f_j) \circ J_*^j$ is null homotopic if $i \neq j$ and is a weak equivalence of Ω-spectra if $i = j$, where $f_j = f \mid E_j$ and $J_*^j = J \mid \mathscr{P}_*(p|E_j)$ (cf. Claim 6.2). The reader will find that this is just a parametrized version of the argument used in [9] to prove that $\mathscr{P}_0^c(f) \circ J_0$ is injective on the homotopy groups of the Ω-spectra.

This completes our outline of the proof for Theorem 0.4.

1. Spaces of pseudoisotopies

In this section we review two topological control theorems for spaces of stable pseudoisotopies. The first of these is a fibered control theorem (cf. §1.4) which was proven by F. Quinn [25]. The second is a foliated control theorem for foliations with one-dimensional leaves (cf. §1.5) which was proven by the authors [11].

Let E denote a Riemannian manifold (possibly with boundary), and let $d_E(\,,\,)$ denote the metric for E associated with its Riemannian structure. Let $p: E \to B$ be a continuous map to the space B, and let $d_B(\,,\,)$ denote a given metric on B. We assume that $p: E \to B$ is a simplicially stratified fiber bundle map in the sense of the following definition.

1.1. Definition. The mapping $p: E \to B$ is a *simplicially stratified fiber bundle* map if there is a triangulation K for B such that the following hold.

(a) For each simplex $\Delta \in K$ we have that $p: p^{-1}(\Delta - \partial\Delta) \to \Delta - \partial\Delta$ is a fiber bundle, and $p^{-1}(\Delta)$ is a finite polyhedron.

(b) For each $\Delta \in K$ there is a neighborhood U for $\partial\Delta$ in Δ, and there are deformation retractions $r_t: U \to U$, $t \in [0, 1]$, and $s_t: p^{-1}(U) \to$

$p^{-1}(U)$, $t \in [0, 1]$, of U and $p^{-1}(U)$ onto $\partial\Delta$ and $p^{-1}(\partial\Delta)$, such that $p(s_t(x)) = r_t(p(x))$ holds for all $x \in p^{-1}(U)$ and all $t \in [0, 1]$.

1.2. The spaces $\mathscr{P}(E)$, $\mathscr{P}(p)$, $\mathscr{P}^c(E)$, $\mathscr{P}^c(p)$. The *space of pseudoisotopies* of E, denoted by $P(E)$, consists of all homeomorphisms $h\colon E\times[0, 1] \to E\times[0, 1]$ which are the identity on $E\times 0$. Note that if I^k denotes the k-fold Cartesian product of $[0, 1]$ with itself, then there is an "inclusion" map $P(E\times I^k) \to P(E\times I^{k+1})$ obtained by forming the Cartesian product of each pseudoisotopy $h\colon E\times I^k\times[0, 1] \to E\times I^k\times[0, 1]$ with the identity map $I \to I$. Define $\mathscr{P}(E)$ to be the direct limit space $\operatorname{limit}_{k\to\infty} P(E\times I^k)$. The space $\mathscr{P}(E)$ is called the *space of stable pseudoisotopies of E*.

A path $g\colon [0, 1] \to E$ is ε-controlled over B, for some number $\varepsilon > 0$, if diameter(Image($p\circ g$)) $\leq \varepsilon$ holds in $(B, d_B(\ ,\ ))$. A *stable pseudoisotopy $h \in \mathscr{P}(E)$ is ε-controlled over B*, for some number $\varepsilon > 0$, if for every $y \in E\times I^k$ the composite function

$$[0, 1] = y\times[0, 1] \subset E\times I^k\times[0, 1] \xrightarrow{h} E\times I^k\times[0, 1] \xrightarrow{\text{proj}} E$$

is ε-controlled over B. Let $\mathscr{P}(p; \varepsilon)$ denote the subspace of all $h \in \mathscr{P}(E)$ which are ε-controlled over B, and define $\mathscr{P}(p)$ to be the space of all mappings $g\colon [0, \infty) \to \mathscr{P}(E)$ such that $g(t)$ is $(1+t)^{-1}$-controlled over B for all $t \geq 0$. The space $\mathscr{P}(p)$ is called the *space of stable pseudoisotopies of E controlled over B*.

We define $\mathscr{P}^c(E)$ to be the direct limit of the spaces $\{\mathscr{P}(C)\}$, where C is any compact codimension zero submanifold of E, and we define $\mathscr{P}^c(p)$ to be the direct limit of the spaces $\{\mathscr{P}(p_C)\}$, where C is any finite subcomplex of the triangulation K of B (cf. Definition 1.1) and $p_C = p \mid p^{-1}(C)$.

1.3. The Ω-spectra $\mathscr{P}_*(E)$, $\mathscr{P}_*(p)$, $\mathscr{P}^c_*(E)$, $\mathscr{P}^c_*(p)$. For any number $\alpha > 0$ and any integer $i > 0$, let $\mathscr{P}_i(E; \alpha)$ denote the subspace of all stable pseudoisotopies in $\mathscr{P}(E\times\mathbf{R}^i)$ which are α-controlled over $\mathbf{R}^i$ with respect to the projection $E\times\mathbf{R}^i \to \mathbf{R}^i$ and the Euclidean metric on $\mathbf{R}^i$. Define $\mathscr{P}_i(E)$ to be the direct limit space $\operatorname{limit}_{\alpha\to\infty}\mathscr{P}_i(E; \alpha)$.

Let $p^i\colon E^i \to B$ denote the composite map $E\times\mathbf{R}^i \xrightarrow{\text{proj}} E \xrightarrow{p} B$. Let $\mathscr{P}_i(p; \alpha)$ denote the subspace of all functions $g \in \mathscr{P}(p^i)$ such that $g(t) \in \mathscr{P}_i(E; \alpha)$ holds for all $t \geq 0$. Define $\mathscr{P}_i(p)$ to be the direct limit space $\operatorname{limit}_{\alpha\to\infty}\mathscr{P}_i(p; \alpha)$.

The following lemma is due to A. Hatcher [17] and F. Quinn [25].

1.3.1. Lemma. *For each integer $i \geq 1$ there are homotopy equivalences of spaces $\Omega\mathscr{P}_i(E) \cong \mathscr{P}_{i-1}(E)$ and $\Omega\mathscr{P}_i(p) \cong \mathscr{P}_{i-1}(p)$.*

For any integer $i < 0$ we define $\mathscr{P}_i(E)$, $\mathscr{P}_i(p)$ to be the $(-i)$-fold loop spaces $\Omega^{-i}\mathscr{P}_0(E)$, $\Omega^{-i}\mathscr{P}_0(p)$. Note that it follows from Lemma 1.3.1 that the collection of spaces $\mathscr{P}_*(E) = \{\mathscr{P}_i(E) : i \in \mathbf{Z}\}$ and $\mathscr{P}_*(p) = \{\mathscr{P}_i(p) : i \in \mathbf{Z}\}$ are Ω-spectra.

We define $\mathscr{P}_*^c(E)$ to be the direct limit of the Ω-spectra $\{\mathscr{P}_*(C)\}$, where C is any compact codimension zero submanifold of E, and we define $\mathscr{P}_*^c(p)$ to be the direct limit of the Ω-spectra $\{\mathscr{P}_*(p_C)\}$, where C is any finite subcomplex of the triangulation K for B and where $p_C = p \mid p^{-1}(C)$.

Remark. Hatcher proved that $\Omega(\mathscr{P}_i(E)) = \mathscr{P}_{i-1}(E)$ if E is compact (cf. [17, Appendix II]), but his proof works without the compactness assumption. Likewise Quinn proved an analogue of $\Omega(\mathscr{P}_i(p)) = \mathscr{P}_{i-1}(p)$ for spaces of stable embeddings (cf. [25, Theorem 5.9]), but his proof (being local in nature) also works for spaces of stable pseudoisotopies.

1.4. The fibered control theorem. Let $J_* : \mathscr{P}_*(p) \to \mathscr{P}_*(E)$ denote the map of Ω-spectra defined by $J_*(f) = f(0)$ for all functions $f : [0, \infty) \to \mathscr{P}_*(E)$ in $\mathscr{P}_*(p)$. The following theorem is due to F. Quinn [25, Theorem 5.6].

1.4.1. Theorem. *Suppose that E is compact. Then there is a number $\varepsilon > 0$ and a map $K_* : \mathscr{P}_*(p;\varepsilon) \to \mathscr{P}_*(p)$ of Ω-spectra such that $J_* \circ K_* =$ inclusion.*

1.5. The foliated control theorem. Let $p' : E' \to E$ denote a fiber bundle over E having a manifold for fiber, and let $\mathscr{F}$ denote a smooth foliation for the pair $(E, \partial E)$; that is, the local charts for $\mathscr{F}$ are all smooth, and if a leaf L of $\mathscr{F}$ intersects ∂E then $L \subset \partial E$. We assume that $\mathscr{F}$ satisfies the following properties.

1.5.1. (a) $\mathscr{F}$ is one-dimensional.

(b) $\mathscr{F}$ is of compact type (cf. Definition 1.5.2).

1.5.2. Definition. We say that E (or $(E, d_E(\ ,\ ))$ is of *compact type* if there is a collection $\{\phi_j : \widehat{E} \to \widehat{E}\}$ of isometries of the universal cover $\widehat{E}$ of E (where $\widehat{E}$ is equipped with the Riemannian structure pulled back from E) and a compact subset $C \subset \widehat{E}$ such that $\bigcup_j \phi_j(C) = \widehat{E}$. If in addition each $\phi_j : \widehat{E} \to \widehat{E}$ permutes the leaves of $\widehat{\mathscr{F}}$, where $\widehat{\mathscr{F}}$ is the foliation of $\widehat{E}$ which covers $\mathscr{F}$, then we say that the *pair $(E, \mathscr{F})$ is of compact type.*

We say that *a path $g : [0, 1] \to E'$ is (α, δ)-controlled over $(E, \mathscr{F})$*, for some numbers $\alpha, \delta > 0$, if there is another map $f : [0, 1] \to S$ into a leaf S of $\mathscr{F}$ such that Image(f) has diameter less than or equal to α in S, and such that $d_E(f(t), p' \circ g(t)) \leq \delta$ holds for all $t \in [0, 1]$. We say that *a stable pseudoisotopy* $h \in \mathscr{P}_i(E')$ *is* (α, δ)*-controlled over* $(E, \mathscr{F})$ if the following is true. For each $y \in E' \times \mathbf{R}^i \times I^k$ the composite map

$$[0, 1] = y \times [0, 1] \subset E' \times \mathbf{R}^i \times I^k \times [0, 1] \xrightarrow{h} E' \times \mathbf{R}^i \times I^k \times [0, 1] \xrightarrow{\text{proj}} E'$$

must be (α, δ)-controlled over $(E, \mathscr{F})$.

For any numbers $\alpha, \delta > 0$ we let $\mathscr{P}_i(p', \mathscr{F}; \alpha, \delta)$ denote the subspace of all $h \in \mathscr{P}_i(E')$ which are (α, δ)-controlled over $(E, \mathscr{F})$. For any subset $U \subset E$ and any numbers $\alpha, \delta > 0$ we let $U^{\alpha, \delta}$ denote the subset of all $z \in E$ for which there is a point $x \in U$, a point $y \in E$ with $d_E(x, y) \leq \delta$, and a smooth path $f\colon [0, 1] \to S$ of length less than or equal to α in a leaf S of $\mathscr{F}$ such that f starts at y and ends at z. The next theorem is due to the authors [11, Theorem 1.11].

1.5.3. Theorem. *Let* $(E, \mathscr{F})$ *be as in* 1.5.1, *and suppose that the fiber of* $p'\colon E' \to E$ *is a closed ball. There is a number* $\eta > 1$ *which depends only on* $\dim(E)$. *Given* $\alpha, \varepsilon > 0$ *there is a number* $\delta \in [0, \varepsilon]$, *where* δ *is independent of* p' *but does depend on the geometry of the pair* $(E, \mathscr{F})$. *Given any subsets* $U, V \subset E$ *such that* U *satisfies* (a) *below, there is a homotopy* $r_t\colon \mathscr{P}_i(p', \mathscr{F}; \alpha, \delta) \to \mathscr{P}_i(p', \mathscr{F}; \eta\alpha, \varepsilon)$, $t \in [0, 1]$, *which satisfies* (b) *and* (c).

(a) *If* $x \in U^{\eta\alpha, \varepsilon}$, *and* L_x *is the leaf of* $\mathscr{F}$ *which contains* x, *then we have that length* $(L_x) \geq \eta\alpha$.

(b) r_0 *is the inclusion* $\mathscr{P}_i(p', \mathscr{F}; \alpha, \delta) \subset \mathscr{P}_i(p', \mathscr{F}; \eta\alpha, \varepsilon)$; *for each* $h \in \mathscr{P}_i(p', \mathscr{F}; \alpha, \delta)$ *we have that* $r_1(h)$ *is the identity map on* $p'^{-1}(U) \times \mathbf{R}^i \times I^k \times [0, 1]$.

(c) *Suppose that* $h \in \mathscr{P}_i(p', \mathscr{F}; \alpha, \delta)$ *is the identity map on* $p'^{-1}(V^{\eta\alpha, \varepsilon}) \times \mathbf{R}^i \times I^k \times [0, 1]$, *then* $r_t(h)$ *is the identity map on* $p'^{-1}(V) \times \mathbf{R}^i \times I^k \times [0, 1]$ *for all* $t \in [0, 1]$.

Remark. The authors have proven 1.5.3 in [11] under the assumption that all relevant pseudoisotopies have compact support in the factor E. However the same proof (being local in nature) works when the compactness assumption is dropped.

2. Geometric preliminaries

In this section M will denote a complete Riemannian manifold having sectional curvature $K \leq 0$ everywhere. We let $d_M(\ ,\ )$ denote the metric on M associated to the Riemannian structure. In this section we state various geometric results concerning M which are needed in the rest of the paper. Most of these results are known to the experts (cf. [3], [5]).

A fundamental property of $d_M(\ ,\ )$ is that for any pair of geodesics $f(s)$, $g(t)$ in M the composite function $d_M(f(s), g(t))$ is a convex function in the two real variables s, t. It is a simple exercise to use this convexity property to prove the following three lemmas (cf. [3]).

2.0. Lemma. *Suppose that M is simply connected. Then for any $y \in M$ the exponential map $\exp_y \colon TM_y \to M$ is a diffeomorphism.*

2.1. Lemma. *Suppose that M is simply connected and that $f(s)$, $g(t)$ are two geodesics in M with $f(0) = g(0)$. Then, for any numbers $a \geq b > 0$, the following are satisfied.*

(a) $d_M(f(b), g(b)) \leq ba^{-1} d_M(f(a), g(a))$.

(b) *Suppose that $(M, d_M(\ ,\ ))$ is of compact type (cf. 1.5.2), and let $d_{SM}(\ ,\ )$ denote a Riemannian metric on the unit sphere bundle SM such that $(SM, d_{SM}(\ ,\ ))$ is also of compact type. Suppose also that $f(s)$, $g(t)$ have unit speed. Set $x = d_M(f(a), g(a))$ and $y = d_{SM}(df/ds(b), dg/dt(b))$. Then if $a \geq b+1$ we will have that $\operatorname{limit}_{x\to 0} y = 0$ holds uniformly in f, g, a, b.*

2.2. Lemma. *Suppose that M is simply connected. Then for any $y \in M$ and any number $r > 0$ the ball $B(y, r)$ of radius r centered at y in M is a convex subset of M (cf. §2.3).*

Before formulating the main result of this section it will be useful to collect some well-known facts about convex and locally convex subsets of M (cf. [3], [4]).

2.3. Convex and locally convex subsets of M. A subset $B \subset M$ is said to be *convex* if for any two points $x, y \in B$ there is (up to reparametrization) a unique shortest geodesic segment in M connecting x to y and this geodesic segment is contained in B. We say that B is *locally convex* subset of M if every $p \in B$ has a neighborhood U in B such that U is a convex subset of M. For any locally convex subset $B \subset M$ we denote by ∂B the subset of all $x \in B$ for which there is a geodesic $g \colon R \to M$ with $g(0) = x$, $g(\varepsilon) \notin B$, $g(-\varepsilon) \in B$ for all sufficiently small $\varepsilon > 0$. Note that $B - \partial B$ is always a smooth submanifold of M; and if $B \subset M$ is a closed subset (as well as locally convex) then the pair $(B, \partial B)$ is a topological submanifold pair of M (although, in general, not a smooth submanifold pair of M). Since $B - \partial B$ is a smooth submanifold of M, the tangent bundle $T(B - \partial B)$ is a subbundle of $TM \mid (B - \partial B)$. The closure of $T(B - \partial B)$ in $TM \mid B$ is also a subbundle of $TM \mid B$ (which we denote by TB) and satisfies $TB \mid (B - \partial B) = T(B - \partial B)$. We let SB

denote the unit sphere bundle for TB. A cross section $V: U \to SB$ on the open subset $U \subset B$ is said to be smooth if $V \mid (U - \partial B)$ is smooth, and for every $x \in \partial B$ there is an $\varepsilon > 0$ and a smooth extension of $V \mid (U \cap \exp(TB_{x,\varepsilon}))$ to $V_x: \exp(TB_{x,\varepsilon}) \to SM$, where $\exp: TM_x \to M$ is the exponential map at x, and $TB_{x,\varepsilon}$ is the ball of radius ε centered at the origin of TB_x. Any continuous cross section $V: B \to SB$ can be approximated by a smooth cross section $V': B \to SB$, provided that B is a closed subset of M (as well as locally convex).

Let SM denote the unit sphere bundle for M. There is a two-fold covering map $SM \to RPM$ onto the real projective bundle for M; let $\mathscr{F}$ denote the one-dimensional foliation of RPM which is covered by the orbits of the geodesic flow on SM. Define a subset $E \subset RPM$ to be the union of all compact leaves of $\mathscr{F}$, and let F denote the restriction of $\mathscr{F}$ to E. Let $F_1, F_2, \ldots$ denote the distinct equivalence classes of leaves of F, where the equivalence $\sim$ is defined as follows: for two closed leaves L, L' in F we have that $L \sim L'$ if and only if there are immersions $g: S^1 \to L$ and $g': S^1 \to L'$ such that the composite maps

$$S^1 \xrightarrow{g} L \subset RPM \xrightarrow{\text{proj}} M \quad \text{and} \quad S^1 \xrightarrow{g'} L' \subset RPM \xrightarrow{\text{proj}} M$$

are homotopic. Let $E_1, E_2, \ldots$ denote the union of leaves in $F_1, F_2, \ldots$. There are quotient mappings $p: E \to G$ and $p_i: E_i \to G_i$ obtained by collapsing each leaf of F and F_i to a point. The rest of this section will be spent proving the following theorem.

2.4. Theorem. *Suppose that M is compact. Then each E_i is a path component of E; the set $\{E_i\}$ is countable and nonempty. Moreover, given any $p_i: E_i \to G_i$ there are smooth connected compact manifolds A, B, and a smooth fiber bundle projection $q: A \to B$ having a circle for fiber. There are also smooth actions $G \times A \to A$ and $G \times B \to B$, by the finite group G which commute with the projection $q: A \to B$; the action $G \times A \to A$ is free, but the action $G \times B \to B$ need not be free. The mapping $p_i: E_i \to G_i$ is topologically conjugate to $q/G: A/G \to B/G$, where q/G is the quotient of q under the G-actions.*

Before beginning the proof of Theorem 2.4 we must introduce some terminology and notation, and state some lemmas, in the following subsections.

2.5. Flat bands. A *flat band* in M is any mapping $f: S^1 \times [0, 1] \to M$ (or $f: \mathbf{R} \times [0, 1] \to M$) such that for some numbers $a, b > 0$ the composite map

$$S_a^1 \times [0, b] \xrightarrow{I_a \times J_b} S^1 \times [0, 1] \xrightarrow{f} M$$

(or the composite map $\mathbf{R}\times[0, b] \xrightarrow{J_a\times J_b} \mathbf{R}\times[0, 1] \xrightarrow{f} M$) is a locally isometric immersion. (Thus these immersed surfaces satisfy locally a "totally geodesic subspace" condition.) Here S^1_a is the circle of radius a centered at the origin in $\mathbf{R}^2$, I_a is multiplication by a^{-1} in $\mathbf{R}^2$, and J_a, J_b are multiplication in $\mathbf{R}$ by a^{-1}, b^{-1}.

2.5.1. Lemma. *Let $f: S^1\times[0, 1] \to M$ be a map such that $f: S^1\times 0 \to M$ and $f: S^1 \times 1 \to M$ are both geodesics. Then, either $f: S^1 \times 0 \to M$ and $f: S^1 \times 1 \to M$ are equal up to reparametrization, or there is a homotopy $f_t: S^1 \times [0, 1] \to M$, $t \in [0, 1]$, of f which satisfies the following properties.*

(a) $f_t \mid S^1 \times 0 = f_0 \mid S^1 \times 0$ *for all* $t \in [0, 1]$.

(b) $f_t \mid S^1 \times 1 = (f_0 \mid S^1 \times 1) \circ r_t$ *for all* $t \in [0, 1]$, *where* $r_t: S^1 \times 1 \to S^1 \times 1$ *is a rotation.*

(c) $f_1: S^1 \times [0, 1] \to M$ *is a flat band.*

Proof of Lemma 2.5.1. We begin by paraphrasing a result of Eberlein and O'Neill [5, Proposition 5.1]. Let N denote any complete simply connected Riemannian manifold which has sectional curvature $K \leq 0$ everywhere. Two geodesics $g_0: \mathbf{R} \to N$, $g_1: \mathbf{R} \to N$ are said to be of *bounded distance apart* if there is a number $\alpha > 0$ such that $d_N(g_0(t), g_1(t)) < \alpha$ holds for all $t \in \mathbf{R}$.

2.5.1.1. Claim (Eberlein, O'Neill). If the geodesics g_0, $g_1: \mathbf{R} \to N$ are of bounded distance apart then either we have $\mathrm{Image}(g_0) = \mathrm{Image}(g_1)$, or there is a mapping $G: \mathbf{R}\times[0, 1] \to N$ which has the following properties.

(a) $G: \mathbf{R} \times [0, 1] \to N$ is an embedded flat band.

(b) $G \mid \mathbf{R} \times 0 = g_0$ and $G \mid \mathbf{R} \times 1 = g_1 \circ T_c$, where $T_c: \mathbf{R} \to \mathbf{R}$ is translation by some real number c.

Now we can complete the proof of 2.5.1. Let $g \in \pi_1(M, f(1, 0))$ denote the fundamental group element represented by $f \mid S^1 \times 0$, and let $g: \widehat{M} \to \widehat{M}$ also denote the corresponding deck transformation for the universal covering space $\widehat{M}$. Choose a covering map $\hat{f}: \mathbf{R} \times [0, 1] \to \widehat{M}$ for f such that the following holds.

2.5.1.2. $\hat{f}(x + 1, t) = g \circ \hat{f}(x, t)$ for all $x \in \mathbf{R}$ and all $t \in [0, 1]$.

It follows from 2.5.1.2 that $\hat{f} \mid \mathbf{R}\times 0$ and $\hat{f} \mid \mathbf{R}\times 1$ are geodesics in $\widehat{M}$ which are a bounded distance apart. Thus we may apply 2.5.1.1 (here we are considering the case where $f: S^1 \times 0 \to M$ and $f: S^1 \times 1 \to M$ are not equal up to reparametrization) to get a mapping $F: \mathbf{R} \times [0, 1] \to \widehat{M}$ which has the following properties.

2.5.1.3. F is an embedded flat band in $\widehat{M}$; $F \mid \mathbf{R}\times 0 = \hat{f} \mid \mathbf{R}\times 0$; $F \mid \mathbf{R}\times 1 = (\hat{f} \mid \mathbf{R}\times 1)\circ T_c$.

Note that it follows from 2.5.1.2 that F also has the following property.

2.5.1.4. $F(x+1, t) = g\circ F(x, t)$ for all $x\in\mathbf{R}$ and all $t\in[0, 1]$.

Now define a homotopy $\hat{f}_t\colon \mathbf{R}\times[0, 1]\to\widehat{M}$, $t\in[0, 1]$, of $\hat{f}$ by the following.

2.5.1.5. $\hat{f}_t(x, s) = r(x, s, t)$, where $r(x, s, t)$, $t\in[0, 1]$, is the geodesic segment in $\widehat{M}$ which starts at $\hat{f}(x, s)$ and ends at $F(x, s)$.

Note that it follows from 2.5.1.2–2.5.1.5 that the homotopy $\hat{f}_t\colon \mathbf{R}\times[0, 1]\to\widehat{M}$, $t\in[0, 1]$, covers a homotopy $f_t\colon S^1\times[0, 1]\to M$ which satisfies 2.5.1(a), (b), (c).

This completes the proof of Lemma 2.5.1.

2.6. The foliations $\overline{F}_i$, $\overline{F}_{i,j}$ and the sets $\overline{E}_{i,j}$. Recall that two unit speed geodesics $f, g\colon \mathbf{R}\to\widehat{M}$ are called *asymptotic* if $d_{\widehat{M}}(f(t), g(t))$ remains bounded as $t\to\infty$. A vector field $V\colon \widehat{M}\to S\widehat{M}$ is called an *asymptotic vector field* if for every $x, y\in\widehat{M}$ the vectors $V(x)$, $V(y)$ are tangent to asymptotic geodesics. Given $x\in\widehat{M}$ and any $v\in S\widehat{M}_x$ there is a unique asymptotic vector field $V\colon \widehat{M}\to S\widehat{M}$ with $V(x) = v$; moreover, V is a C^1 vector field on $\widehat{M}$ and it and its derivative are continuous in our choice of v (cf. [2], [18]). There is a foliation $\widehat{\mathscr{A}}$ of $S\widehat{M}$ whose leaves are just the subsets Image$(V)\subset S\widehat{M}$ for any asymptotic vector field $V\colon\widehat{M}\to S\widehat{M}$. The action of $\pi_1(M)$ on $S\widehat{M}$ (by the deck transformations for the covering $S\widehat{M}\to SM$) just permutes the leaves of $\widehat{\mathscr{A}}$, so the quotient of $\widehat{\mathscr{A}}$ by the action of $\pi_1(M)$ is a foliation of SM denoted by $\mathscr{A}$. We call $\widehat{\mathscr{A}}$, $\mathscr{A}$ the *asymptotic foliations* of $S\widehat{M}$, SM.

Let $\overline{F}_i$ denote the foliation in SM which covers F_i under the two-fold covering projection $SM\to RPM$. Let $\overline{F}_{i,1}, \overline{F}_{i,2}, \overline{F}_{i,3}, \ldots$ denote the equivalence classes for the equivalence relation $\sim$ defined on $\overline{F}_i$ as follows: $L\sim L'$ for two leaves $L, L'\in\overline{F}_i$ if there are orientation preserving immersions $g\colon S^1\to L$, $g'\colon S^1\to L'$ (the leaves of $\overline{F}_i$ are oriented in the direction of the geodesic flow on SM) such that the composite maps

$$S^1 \xrightarrow{g} L\subset SM \xrightarrow{\text{proj}} M \quad\text{and}\quad S^1 \xrightarrow{g'} L'\subset SM \xrightarrow{\text{proj}} M$$

are homotopic to one another. Let $\overline{E}_{i,j}$ denote the union of all the leaves in $\overline{F}_{i,j}$.

2.6.1. Lemma. *Each of the sets $\overline{E}_{i,j}$ is contained in a leaf $L_{i,j}$ of the asymptotic foliation $\mathscr{A}$. Moreover, the map $\overline{E}_{i,j}\to E_i$ (which is induced* *by the projection $SM \to RPM$) is either a two-fold covering map or a homeomorphism. Thus there are at most two distinct sets in the collection* $\{\overline{E}_{i,j} : j = 1, 2, \dots\}$.

Proof of Lemma 2.6.1. First we note that for any flat band $f: S^1 \times [0, 1] \to M$, if $f': S^1 \times [0, 1] \to SM$ is defined at each $(x, t) \in S^1 \times [0, 1]$ to be the unit tangent direction of $f \mid S^1 \times t$ at (x, t), then Image(f') will be contained in some leaf of the asymptotic foliation $\mathscr{A}$.

For any vectors $v_0, v_1 \in \overline{E}_{i,j}$ choose a map $g: S^1 \times [0, 1] \to M$ which has the following property.

2.6.1.1. $g \mid S^1 \times 0$ and $g \mid S^1 \times 1$ are both geodesics in M which have unit tangent directions v_0 and v_1 at $(1, 0)$ and $(1, 1)$ respectively.

Now apply Lemma 2.5.1 to g of 2.6.1.1 to get a homotopy $g_t: S^1 \times [0, 1] \to M$, $t \in [0, 1]$, of g which satisfies the following.

2.6.1.2. $g_1 \mid S^1 \times 0$, $g_1 \mid S^1 \times 1$ are both geodesics in M which have unit tangent directions v_0, v_1 at $(1, 0)$, $(x, 1)$ for some $x \in S^1$; $g_1 : S^1 \times [0, 1] \to M$ is a flat band in M.

It now follows from 2.6.1.2, and from the remark made at the beginning of this proof, that v_0, v_1 lie in the same leaf of $\mathscr{A}$. Hence $\overline{E}_{i,j}$ is contained in a leaf $L_{i,j}$ of $\mathscr{A}$.

To verify that $\overline{E}_{i,j} \to E_i$ is a covering projection, first note that it is a surjective map. Next let $\gamma: SM \to SM$ be the nontrivial deck transformation for the covering $SM \to RPM$, and note that either $\gamma(\overline{E}_{i,j}) = \overline{E}_{i,j}$ or $\gamma(\overline{E}_{i,j}) \cap \overline{E}_{i,j} = \varnothing$. Finally we note that each $\overline{E}_{i,j}$ is a closed subset of SM (cf. Lemma 2.7.3).

This completes the proof of Lemma 2.6.1.

2.7. The foliation $\widehat{F}_i$, the space $\widehat{E}_i$, and the action $\overline{\Gamma}_i \times \widehat{E}_i \to \widehat{E}_i$. Note that the composite map $L_{i,1} \subset SM \xrightarrow{\text{proj}} M$ is a covering projection for each leaf $L_{i,1}$ as in 2.6.1. We give to $L_{i,1}$ the geometry that is pulled back from M under the projection map $L_{i,1} \to M$. Note that the universal covering space projection $\widehat{M} \to M$ factors as $\widehat{M} \to L_{i,1} \to M$. Let $\widetilde{F}_i$ denote the one-dimensional foliation (of a region in $\widehat{M}$) which covers $\overline{F}_{i,1}$ under the covering projection $\widehat{M} \to L_{i,1}$. Let $\widetilde{F}_{i,1}, \widetilde{F}_{i,2}, \widetilde{F}_{i,3}, \dots$ denote the equivalence classes of leaves of $\widetilde{F}_i$ for the equivalence $\approx$ defined as follows: $\widetilde{L} \approx \widetilde{L}'$ for two leaves $\widetilde{L}, \widetilde{L}' \in \widetilde{F}_i$ if there are leaves $L, L' \in \overline{F}_{i,1}$, orientation preserving immersions $f: S^1 \to L$, $g: S^1 \to L'$, a homotopy $f_t: S^1 \to L_{i,1}$, $t \in [0, 1]$, from f to g, and a lifting of f_t,

$t \in [0, 1]$, to a homotopy $\hat{f}_t: R \to \widehat{M}$, $t \in [0, 1]$, such that $\text{Image}(\hat{f}_0) = \widetilde{L}$ and $\text{Image}(\hat{f}_1) = \widetilde{L}'$. Let $\hat{E}_i$ denote the union of all the leaves in $\widetilde{F}_{i,1}$, and let us use the shorter notation $\widehat{F}_i$ for $\widetilde{F}_{i,1}$. Let $\overline{\Gamma}_i \subset \pi_1(L_{i,1})$ denote the subgroup of $\pi_1(L_{i,1})$ consisting of all deck transformations on $\widehat{M}$ which leave $\widehat{E}_i$ invariant.

2.7.1. Lemma. (a) *$\widehat{E}_i$ is a convex subset of $\widehat{M}$.*

(b) *$\widehat{E}_i$ is isometric to the product $D_i \times \mathbf{R}$, where D_i is also a convex subset of $\widehat{M}$. Moreover, the foliation $\widehat{F}_i$ of $\widehat{E}_i$ is identified with the foliation of $D_i \times \mathbf{R}$ by the lines $\{y \times \mathbf{R} : y \in D_i\}$ under the isometry.*

(c) *The action $\psi_i: \overline{\Gamma}_i \times \widehat{E}_i \to \widehat{E}_i$ by deck transformations preserves the product structure $\widehat{E}_i = D_i \times \mathbf{R}$. That is, for each $\alpha \in \overline{\Gamma}_i$ there are isometries $\alpha_1: D_i \to D_i$ and $\alpha_2: \mathbf{R} \to \mathbf{R}$ such that $\psi_i(\alpha, (x, t)) = (\alpha_1(x), \alpha_2(t))$ holds for all $(x, t) \in D_i \times \mathbf{R}$.*

(d) *The map $\widehat{E}_i \to \overline{E}_{i,1}$, which is induced by $S\widehat{M} \to SM$, is a covering projection which has $\overline{\Gamma}_i$ as its group of deck transformations.*

Proof of Lemma 2.7.1. To complete the proof of 2.7.1(a)–(c) it will suffice to show (because the action $\overline{\Gamma}_1 \times \widehat{M} \to \widehat{M}$ permutes the leaves of $\widehat{F}_i$) that there is a continuous function $r: \widehat{M} \to \mathbf{R}$, and for any two leaves $\widehat{L}_0, \widehat{L}_1 \in \widehat{F}_i$ there is a mapping $h: \mathbf{R} \times [0, 1] \to \widehat{M}$, which has the following properties.

2.7.1.1. (a) h is an embedded flat band; $\text{Image}(h \mid \mathbf{R} \times 0) = \widehat{L}_0$ and $\text{Image}(h \mid \mathbf{R} \times 1) = \widehat{L}_1$; $\text{Image}(h) \subset \widehat{E}_i$.

(b) $r \circ h(s, t) = r \circ h(s, 0)$ for all $t \in [0, 1]$; $(r \circ h(s, 0) - r \circ h(s', 0)) = \text{length}(h([s', s] \times 0))$ holds for all $s', s \in \mathbf{R}$ satisfying $s' < s$.

Towards this end we choose (using the definition of $\overline{F}_{i,1}, \widehat{F}_i, \widehat{E}_i$) a mapping $f: S^1 \times [0, 1] \to L_{i,1}$ which has the following properties.

2.7.1.2. $f \mid S^1 \times 0$, $f \mid S^1 \times 1$ are orientation preserving immersions onto the leaves L_0, L_1 of $\overline{F}_{i,1}$ which are covered by the leaves $\widehat{L}_0, \widehat{L}_1$; there is a lifting of f to $\hat{f}: \mathbf{R} \times [0, 1] \to \widehat{M}$ such that $\text{Image}(\hat{f} \mid \mathbf{R} \times 0) = \widehat{L}_0$ and $\text{Image}(\hat{f} \mid \mathbf{R} \times 1) = \widehat{L}_1$.

There will be no loss of generality in 2.7.1.2 if we assume that each of $f \mid S^1 \times 0$, $f \mid S^1 \times 1$ have constant speed and are thus geodesics in $L_{i,1}$. So we can apply Lemma 2.5.1 to f to obtain a homotopy $f_t: S^1 \times [0, 1] \to L_{i,1}$, $t \in [0, 1]$, of f which satisfies the following.

2.7.1.3. $f_t \mid S^1 \times 0$, $f_t \mid S^1 \times 1$ are immersions of constant speed onto the leaves $L_0, L_1 \in \overline{F}_{i,1}$ for all $t \in [0, 1]$; $f_1: S^1 \times [0, 1] \to L_{i,1}$ is a flat band.

Now choose $h: \mathbf{R} \times [0, 1] \to M$ to be the correct lifting of $f_1: S^1 \times [0, 1] \to L_{i,1}$. We leave as an exercise the verification that this h satisfies 2.7.1.1(a).

Note that each leaf $\widehat{L} \in \widehat{F}_i$ is a directed geodesic in $\widehat{M}$, which ends at a point $u(\widehat{L})$ on the sphere at infinity (the direction of $\widehat{L}$ corresponds to the direction of the geodesic flow on $S(M)$). It follows from 2.7.1.1(a) that $u(\widehat{L}_0) = u(\widehat{L}_1)$ for any two leaves $\widehat{L}_0, \widehat{L}_1 \in \widehat{F}_i$, i.e., $\widehat{L}_0, \widehat{L}_1$ are asymptotic geodesics when parametrized by arc length. Thus if $r: \widehat{M} \to \mathbf{R}$ is defined to be one of the "horofunctions" associated to the point at infinity $u(\widehat{L})$ for any $\widehat{L} \in \widehat{F}_i$, then r and h will satisfy 2.7.1.1(b) (cf. [3, Lecture I, §3]).

To verify 2.7.1(d) we first note that for any $g \in \pi_1(L_{i,1})$ we have either $g(\widehat{E}_i) = \widehat{E}_i$ or $g(\widehat{E}_i) \cap \widehat{E}_i = \varnothing$. Next we wrote that if $g(\widehat{E}_i) \cap \widehat{E}_i = \varnothing$, then the distance from $g(\widehat{E}_i)$ to $\widehat{E}_i$ in $\widehat{M}$ is greater than ε, where $\varepsilon > 0$ is independent of g (cf. Lemma 2.7.3). Finally note that the composite map $\bigcup_{g \in \pi_1(L_{i,1})} g(\widehat{E}_i) \subset \widehat{M} \to L_{i,1} \supset \overline{E}_{i,1}$ is a covering space projection. This completes the proof of Lemma 2.7.1.

2.7.2. Lemma. *Let $\overline{\Gamma}_i \times (D_i \times \mathbf{R}) \to D_i \times \mathbf{R}$ be as in Lemma* 2.7.1. *Suppose that for $g \in \overline{\Gamma}_i$ and $y \in D_i$ we have $g(y \times \mathbf{R}) = y \times \mathbf{R}$. Then there is an integer $n > 0$ such that $g^n(z \times \mathbf{R}) = z \times \mathbf{R}$ for all $z \in D_i$.*

Proof of Lemma 2.7.2. By Lemma 2.7.1 the isometry $g: D_i \times \mathbf{R} \to D_i \times \mathbf{R}$ is a product of two isometries $g_1: D_i \to D_i$ and $g_2: \mathbf{R} \to \mathbf{R}$. For each positive integer m set $U_m = \{y \in D_i : g_1^m(y) = y\}$. To complete the proof of 2.7.2 we must show that $U_m = D_i$ for some m.

Note that each U_m is a closed convex subset of D_i satisfying $\partial U_m \subset \partial D_i$ (cf. §2.3). If for some m we have $\dim(U_m) = \dim(D_i)$, then U_m must contain an open subset of D_i. But since D_i is convex it follows that $U_m = D_i$.

If $\dim(U_m) < \dim(D_i)$, then choose $y \in (U_m - \partial U_m)$ and $z \in (D_i - U_m)$. Note that $y \times \mathbf{R}$, $z \times \mathbf{R}$ cover leaves of $\overline{F}_{i,1}$ which can be parametrized by orientation preserving immersions of constant speed $f_y: S^1 \to L_{i,1}$ and $f_z: S^1 \to L_{i,1}$. If z is chosen sufficiently close to y, then there will be a homotopy $h: S^1 \times [0, 1] \to L_{i,1}$ from f_z to $f_y \circ s$, for some covering map $s: S^1 \to S^1$ of constant speed $k \geq 0$. It follows that $z \in U_{km}$. Now, applying the argument of the preceding paragraph, we see that if $\dim(U_m) = \dim(U_{km})$, then we must have $U_m = U_{km}$. Now proceed by induction.

This completes the proof of Lemma 2.7.2.

2.7.3. Lemma. (a) *Each* $\overline{E}_{i,j}$ *is a closed subset of* SM.

(b) *If* $g(\widehat{E}_i) \cap \widehat{E}_i = \varnothing$ *for* $g \in \pi_1(L_{i,1})$, *then there is* $\varepsilon > 0$ *independent of* g *such that the distance in* $\widehat{M}$ *from* $g(\widehat{E}_i)$ *to* $\widehat{E}_i$ *is greater than* ε.

Proof of Lemma 2.7.3. We will prove part (a) by appealing to 2.7.2. The proof of part (b), which uses 2.7.2 in a manner similar to the proof of part (a), is left as an exercise.

Note that it follows from 2.7.2, 2.7.1(a)–(c), and the fact that the composite map $\widehat{E}_i \subset \widehat{M} \to L_{i,1} \supset \overline{E}_{i,1}$ is surjective (we cannot at this point use 2.7.1(d) because 2.7.3 is used in the proof of 2.7.1(d)), that there is an upper bound $b_{i,1} > 0$ to the lengths of leaves in $\overline{F}_{i,1}$. Let $\{x_k\}$ be a sequence of points in $\overline{E}_{i,1}$ which converge to $x \in SM$, and let $\{X_k\}$ denote the leaves of $\overline{F}_{i,1}$ that contain the points $\{x_k\}$. From the existence of the upper bound $b_{i,1}$ it follows that the $\{X_k\}$ converge to a closed orbit X of the geodesic flow on SM. Note that for sufficiently large k there will exist oriented immersions $f\colon S^1 \to X_k$ and $g\colon S^1 \to X$ which are homotopic in SM. Thus from the definition of $\overline{F}_{i,1}$ and $\overline{E}_{i,1}$ we have $X \in \overline{F}_{i,1}$ and $x \in \overline{E}_{i,1}$. A similar argument shows that $\overline{E}_{i,j}$ is a closed subset of SM for each $j \geq 2$.

This completes the proof of Lemma 2.7.3.

Proof of Theorem 2.4. We begin by verifying that E_i has the following properties.

2.8. (a) E_i is a compact connected submanifold of RPM; $E_i - \partial E_i$ is a C^∞-submanifold of RPM.

(b) $\pi_1(E_i)$ is a finitely presented group.

(c) The set $\{E_i\}$ is countable and nonempty.

Clearly 2.8(b) is a consequence of 2.8(a), and (2.8(a) follows from 2.3, 2.6.1, 2.7.1, and 2.7.3(a).

Towards verifying 2.8(c) we let, for each $\alpha > 0$, $\{E_i \in I_\alpha\}$ denote all the E_i such that each leaf of F_i has length less than or equal to α. Note that for any E_i there is $\alpha > 0$ such that $i \in I_\alpha$ (for by the proof of 2.7.3(a) there is an upper bound to the lengths of the leaves in $\overline{F}_{i,1}$, and $\overline{F}_{i,1}$ is by 2.6.1 a one- or two-fold covering of F_i). Thus to show that there are at most countably many of the $\{E_i\}$ it will suffice to show that each I_α is a finite set. If I_α is not finite, then there is a sequence $i_1, i_2, \ldots$ of distinct elements in I_α and a sequence of leaves $L_{i_1}, L_{i_2}, \ldots$ of foliations $F_{i_1}, F_{i_2}, \ldots$ such that the $\{L_{i_j}\}$ converge to a leaf L_k of some foliation F_k. Arguing as in the proof of 2.7.3 we can see that $F_k = F_{i_j}$ for any sufficiently large j, contradicting the assumption that $i_j \neq i_{j'}$ if $j \neq j'$.

To complete the verification of 2.8(c) we note that any closed Riemannian manifold with nonpositive curvature has a closed geodesic (cf. [3]). Thus there is at least one E_i, the one containing the closed unoriented geodesic.

It follows from 2.6.1, 2.7.1 that there are only the following possibilities.

2.9. (a) $\widehat{E}_i$ is the universal covering space for E_i, $\pi_1(E_i) = \overline{\Gamma}_i$, and $E_i = \widehat{E}_i/\overline{\Gamma}_i$; or

(b) $\widehat{E}_i$ is the universal covering space for E_i, $\overline{E}_{i,1} = \widehat{E}_i/\overline{\Gamma}_i$, and there is a short exact sequence $\overline{\Gamma}_i \xrightarrow{\text{incl}} \Gamma_i \xrightarrow{\psi} \mathbf{Z}_2$ (let Γ_i denote the group $\pi_1(E_i)$). Moreover there is $\alpha \in \pi_1(E_i)$ such that $\psi(\alpha)$ generates $\mathbf{Z}_2$; this α considered as a deck transformation $\alpha\colon \widehat{E}_i \to \widehat{E}_i$ is a product of two isometries $\alpha_1\colon D_i \to D_i$ and $\alpha_2\colon \mathbf{R} \to \mathbf{R}$ (recall that $\widehat{E}_i = D_i \times \mathbf{R}$ by 2.7.1), where $\alpha_2\colon \mathbf{R} \to \mathbf{R}$ reverses orientation.

Note that in either case 2.9(a) or case 2.9(b) we have that for each $\beta \in \Gamma_i$ deck transformation $\beta\colon \widehat{E}_i \to \widehat{E}_i$ is a product of two isometries $\beta_1\colon D_i \to D_i$ and $\beta_2\colon \mathbf{R} \to \mathbf{R}$ (cf. 2.7.1(c) and 2.9(b)). If $\beta \in \overline{\Gamma}_i$ then $\beta_2\colon \mathbf{R} \to \mathbf{R}$ preserves the orientation of $\mathbf{R}$, because the deck transformation $\beta\colon S\widehat{M} \to S\widehat{M}$ preserves the direction of the geodesic flow on $S\widehat{M}$. We define $\Gamma_{i,1}, \Gamma_{i,2}, \overline{\Gamma}_{i,1}, \overline{\Gamma}_{i,2}$ to be the following groups: $\{\beta_1 : \beta \in \Gamma_i\}, \{\beta_2 : \beta \in \Gamma_i\}, \{\beta_1 | \beta \in \overline{\Gamma}_i\}, \{\beta_2 : \beta \in \overline{\Gamma}_i\}$. Note that by sending each group element β to β_j, $j = 1, 2$, we obtain group homomorphisms $h_{i,j}\colon \Gamma_i \to \Gamma_{i,j}$ and $\bar{h}_{i,j}\colon \overline{\Gamma}_i \to \overline{\Gamma}_{i,j}$. We let $K_{i,j}, \overline{K}_{i,j}$ denote the kernels of the homomorphisms $h_{i,j}, \bar{h}_{i,j}$. The following conditions are satisfied by these new groups and group homomorphisms.

2.10. (a) All of the homomorphisms $h_{i,j}, \bar{h}_{i,j}$, $j = 1, 2$, are surjective; thus $\Gamma_i/K_{i,j} = \Gamma_{i,j}$ and $\overline{\Gamma}_i/\overline{K}_{i,j} = \overline{\Gamma}_{i,j}$.

(b) $\overline{\Gamma}_{i,2}$ acts by translations on $\mathbf{R}$.

Note that the following property is deduced from 2.8(b), 2.9, and 2.10(b).

2.11. $\overline{\Gamma}_{i,2}$ is a finitely generated free abelian group.

Now choose $\beta \in \overline{\Gamma}_i$ such that $\beta_1\colon D_i \to D_i$ has a fixed point but $\beta_2 \neq 0$. By Lemma 2.7.2 there is an integer $m > 0$ such that $\beta_1^m = \text{identity}$. Note that $\beta_2^m \neq 0$. For if $\beta_2^m = 0$ then we would have $\beta \neq \text{identity}$, $\beta^m = \text{identity}$. But Γ_i (and hence $\overline{\Gamma}_i$) is torsion free, because by 2.7.1(a) and 2.9, E_i has a contractible universal covering space. We summarize the results of this paragraph as follows.

2.12. There is $\gamma \in \overline{\Gamma}_i$ such that $\gamma_1 = \text{identity}$ but $\gamma_2 \neq 0$.

Write $\overline{\Gamma}_{i,2}$ as a direct sum of subgroups $\overline{\Gamma}_{i,2} = S \oplus S'$, where S is a cyclic subgroup containing γ_2 and where γ comes from 2.12 (cf. 2.11). Let C denote the subgroup of $\overline{\Gamma}_{i,2}$ which is generated by S' and γ_2. We claim that C has the following properties.

2.13. (a) The quotient group $\overline{\Gamma}_i/\bar{h}_{i,2}^{-1}(C)$ is a finite cyclic group.

(b) The subgroup $\bar{h}_{i,1}(\bar{h}_{i,2}^{-1}(C)) \subset \overline{\Gamma}_{i,1}$ acts freely and properly discontinuously on D_i.

Note that 2.13(a) follows immediately from the above description of C. To verify 2.13(b) we assume that contrary, i.e., there is a sequence $\{\beta_j\}$ in $\bar{h}_{i,2}^{-1}(C)$, a point $x \in D_i$, and a sequence $\{x_j\} \in D_i$, such that the following properties hold: $\text{limit}_{j\to\infty} \beta_{j,1}(x) = x$; $\beta_{j,1}(x_j) \neq x_j$ for all j. Applying 2.12 we see that there is a subsequence of $\{\beta_j\}$ (also denoted by $\{\beta_j\}$), a positive number b, and a sequence of integers $\{m_j\}$, such that the following hold: $\text{limit}_{j\to\infty}(\gamma^{m_j} \circ \beta_j^b)_1(x) = x$; $\text{limit}_{j\to\infty}(\gamma^{m_j} \circ \beta_j^b)_2(0) = 0$. (To get b we note that the isotropy subgroup for the action $\overline{\Gamma}_{i,1} \times D_i \to D_i$ at the point $x \in D_i$ is an infinite cyclic group $\overline{\Gamma}_{i,1,x}$ which contains γ_1; thus $\gamma_1 = g^b$, where g is a generator for $\overline{\Gamma}_{i,1,x}$.) Now since $\overline{\Gamma}_i$ acts freely and properly discontinuously on $\widehat{E}_i$, we deduce that $\gamma^{m_j} = \beta_j^{-b}$ for sufficiently large j. Thus $m_j T = -bT_j$, where the isometries $\gamma_2, \beta_{j,2}\colon \mathbf{R} \to \mathbf{R}$ are translations by numbers T, T_j respectively (cf. 2.10(b)). But then $T_j = a_j T$ holds for some integer a_j, because $\beta_{j,2} \in C$. So $m_j = -ba_j$ and $\gamma^{a_j} = \beta_j$ for sufficiently large j. (To see that $\gamma^{a_j} = \beta_j$, for sufficiently large j, we note that $(\gamma^{a_j} \circ \beta_j^{-1})_2(0) = 0$ and $\text{limit}_{j\to\infty}(\gamma^{a_j} \circ \beta_j^{-1})_1(x) = x$; since $\overline{\Gamma}_i$ acts freely and properly discontinuously on $\widehat{E}_i$ it follows that $\gamma^{a_j} = \beta_j$ for sufficiently large j.) This last equality and 2.12 contradict $\beta_{j,1}(x_j) \neq x_j$, and hence the verification of 2.13 is complete.

We can now complete the proof for Theorem 2.4 in the special case where the following hypothesis is satisfied.

2.14. Hypothesis. $\partial E_i = \varnothing$.

Choose $q\colon A \to B$ to be the quotient of the projection map $D_i \times \mathbf{R} \to D_i$ under the actions by $\bar{h}_{i,1}(\bar{h}_{i,2}^{-1}(C))$, $\bar{h}_{i,2}^{-1}(C)$ on D_i, $\widehat{E}_i = D_i \times \mathbf{R}$, respectively. Define finite group actions $G \times A \to A$ and $G \times B \to B$ to be the actions of $\Gamma_i/\bar{h}_{i,2}^{-1}(C)$ on $\widehat{E}_i/\bar{h}_{i,2}^{-1}(C)$ and $D_i/\bar{h}_{i,1}(\bar{h}_{i,2}^{-1}(C))$; these actions are well defined because $\bar{h}_{i,2}^{-1}(C)$ is a normal subgroup of Γ_i.

(Note that $\Gamma_i/\bar{h}_{i,2}^{-1}(C)$ acts on $D_i/\bar{h}_{i,1}(\bar{h}_{i,2}^{-1}(C))$ via the homomorphism $h_{i,1}$.) We leave as an exercise for the reader to deduce from 2.8, 2.9, 2.13, 2.14 that $q: A \to B$ and the actions $G \times A \to A$ and $G \times B \to B$ satisfy all the properties required of them in Theorem 2.4.

This completes the proof of Theorem 2.4 in the special case where Hypothesis 2.14 is satisfied.

If Hypothesis 2.14 is not satisfied, then the preceding argument yields the following weaker version of Theorem 2.4.

2.15. *There are compact manifolds A, B, and a fiber bundle projection $q: A \to B$ having a circle for fiber. There are topological group actions $G \times A \to A$ and $G \times B \to B$ by the finite group G which commute with q; the action $G \times A \to A$ is free, but the action $G \times B \to B$ need not be free. The projection $p_i: E_i \to G_i$ is topologically conjugate to $q/G: A/G \to B/G$.*

Note that the difference between 2.4 and 2.15 is that in 2.4 we require that A, B, $q: A \to B$, $G \times A \to A$, and $G \times B \to B$ all be C^∞, whereas in 2.15 they are only objects in the topological category. However it is clear from the preceding argument that the objects of 2.15 also have the following properties.

2.16. (a) A and B are locally convex, compact subsets of a complete Riemannian manifold having sectional curvature $K \leq 0$ everywhere.

(b) The finite group actions $G \times A \to A$ and $G \times B \to B$ are by isometries.

(c) $q: A - \partial A \to B - \partial B$ is a C^∞-bundle projection.

We can use these additional properties, together with smoothing theory as described in the next subsection, to complete the proof of Theorem 2.4 as follows. Let $\partial B \times [0, 1] \subset B$ be the G-equivariant collaring for ∂B in B given in §2.17. Set $B' = B - \partial B \times [0, 1)$, and set $A' = q^{-1}(B')$. Note that $q: A' \to B'$, $G \times A' \to A'$, $G \times B' \to B'$ are all C^∞-objects, and note also that $q/G: A'/G \to B'/G$ is topologically conjugate to $q/G: A/G \to B/G$. Thus by applying 2.15 we complete the proof of Theorem 2.4.

2.17. Equivariant smoothing of locally convex sets. In this subsection we let $q: A \to B$, $G \times A \to A$, and $G \times B \to B$ be any maps and actions by a finite group G which satisfy 2.15 and 2.16(a)–(c).

2.17.1. Lemma. *There is a G-equivariant collaring $\partial B \times [0, 1] \subset B$ for ∂B in B such that $\partial B \times 1$ is a smooth submanifold of $B - \partial B$.*

Proof of Lemma 2.17.1. We consider first the special case when G is the one element group. By using the local convexity property for B (cf. §2.3) we can choose a continuous map $f: \partial B \to B - \partial B$ which has the following property.

2.17.1.1. $d_B(f(x), x) \le \varepsilon$ for all $x \in B$, where $\varepsilon > 0$ may be chosen arbitrarily small (prior to the choice of f of course).

Define a vector field $V: \partial B \to S(B)$ (cf. §2.3 for $S(B)$) by letting $V(x)$ be the unit vector field tangent to the geodesic which starts at x and ends at $f(x)$. Note that if ε is chosen sufficiently small in 2.17.1.1 we may use the local convexity of B to show that $V: \partial B \to SB$ is well defined. Now let U be a small neighborhood for ∂B in B, and extend $V: \partial B \to SB$ to a vector field $V: U \to SB$. Approximate $V: U \to SB$ by a vector field $V': U \to SB$ which has the following properties.

2.17.1.2. (a) V' is smooth on all of U (cf. §2.3 for the definition of a smooth vector field on closed locally convex subsets).

(b) $V' \mid \partial B$ points into B, i.e., for each $p \in \partial B$ we have that $V'(p)$ is tangent to the geodesic which connects p to a point of $B - \partial B$.

Now integrate V' to get a partial flow $\psi: \partial B \times [0, \delta] \to B$ for sufficiently small δ. If δ is sufficiently small then ψ will be an embedding. Let s: Image$(\psi \mid \partial B \times (0, \delta)) \to S$ denote the quotient map which identifies each segment $\psi(b \times (0, \delta))$ to a point. Note that Image$(\psi \mid \partial B \times (0, \delta))$ inherits a C^∞-structure from $B - \partial B$, and that there is a unique C^∞-structure on S which makes s a smooth bundle projection. Choose a smooth cross section $c: S \to$ Image$(\psi \mid \partial B \times (0, \delta))$ for s. We can now easily obtain a collaring $\partial B \times [0, 1] \subset B$ such that $B \times 1 =$ Image(c).

This completes the proof of Lemma 2.17.1 for the special case when G is the one element group. To prove Lemma 2.17.1 in general we must use an equivariant version of the preceding argument. Details are left to the reader.

This completes the proof of Lemma 2.17.1.

3. $\mathscr{P}^c_*(f) \circ J_*: \mathscr{P}^c_*(p) \to \mathscr{P}_*(M)$ is surjective

Let $p: E \to G$ and $f: E \to M$ be as in Theorem 0.4. Let $\mathscr{P}^c_*(f): \mathscr{P}^c_*(E) \to \mathscr{P}_*(M)$ denote the map induced by $f: E \to M$ and let $J_*: \mathscr{P}^c_*(p) \to \mathscr{P}^c_*(E)$ denote the map of §1.4. The main result of this section is the following proposition.

3.1. Proposition. *The mapping $\mathscr{P}^c_*(f) \circ J_*: \mathscr{P}^c_*(p) \to \mathscr{P}_*(M)$ of Ω-spectra induces a surjection on the homotopy groups of the Ω-spectra.*

Before beginning the proof of 3.1 we need to introduce more notation and state two lemmas. This is done in the following subsections.

3.2. The northern hemisphere subbundle and geodesic flows. Recall that $\widehat{M}$ denotes the universal cover of M equipped with the Riemannian structure pulled back under the covering projection $\widehat{M} \to M$ from the given Riemannian structure on M. Set $N = M \times \mathbf{R}$, $\widehat{N} = \widehat{M} \times \mathbf{R}$, where each of these products is equipped with the product Riemannian structure. Let TX_y denote the tangent space of a smooth manifold X at a point $y \in X$. We let $u\colon \widehat{N} \to T\widehat{N}$ denote the unit vector field which points in the direction of the increasing $\mathbf{R}$-factor of $\widehat{N}$. For each point $y \in \widehat{N}$, or vector $v \in T\widehat{N}_y$, or path $r\colon [0, 1] \to \widehat{N}$, the splitting $\widehat{N} = \widehat{M} \times \mathbf{R}$ yields splittings $y = y_1 \times y_2$, $v = v_1 \times v_2 u$, $r = r_1 \times r_2$, where $r_1\colon [0, 1] \to \widehat{M}$ and $r_2\colon [0, 1] \to \mathbf{R}$. Let $S\widehat{N}$, $S\widehat{M}$ denote the unit sphere bundles for $\widehat{N}$, $\widehat{M}$, and set $S^+\widehat{N} = \{v \in S\widehat{N} : \langle v, u\rangle_{\widehat{N}} \geq 0\}$. We call $S^+\widehat{N}$ the *northern hemisphere subbundle* of $S\widehat{N}$. Let $\hat{g}^t\colon S\widehat{N} \to S\widehat{N}$, $t \in \mathbf{R}$, denote the geodesic flow on $S\widehat{N}$. Note that $\hat{g}^t$ leaves the subbundle $S^+\widehat{N}$ invariant. We let $\widehat{\mathscr{G}}^+$ denote the foliation of $S^+\widehat{N}$ by the orbits of the geodesic flow $\hat{g}^t\colon S^+\widehat{N} \to S^+\widehat{N}$, $t \in \mathbf{R}$. Note that the construction which gave the subbundle $S^+\widehat{N} \subset S\widehat{N}$ also applies to give a northern hemisphere subbundle $S^+N \subset SN$. Let $g^t\colon S^+N \to S^+N$, $t \in \mathbf{R}$, denote the restriction to S^+N of the geodesic flow on SN, and let $\mathscr{G}^+$ denote the foliation of S^+N by the orbits of $g^t\colon S^+N \to S^+N$, $t \in \mathbf{R}$. We leave as an exercise for the reader to show that there are Riemannian structures $\langle\ ,\ \rangle_{S^+\widehat{N}}$, $\langle\ ,\ \rangle_{S^+N}$ on $S^+\widehat{N}$, S^+N which have the following properties.

3.2.1. (a) The pairs $(S^+\widehat{N}, \widehat{\mathscr{G}}^+)$ and $(S^+N, \mathscr{G}^+)$ are of compact type (cf. Definition 1.5.2) with respect to the metrics $d_{S^+\widehat{N}}(\ ,\ )$ and $d_{S^+N}(\ ,\ )$ which are associated to $\langle\ ,\ \rangle_{S^+\widehat{N}}$ and $\langle\ ,\ \rangle_{S^+N}$.

(b) The Riemannian structure $\langle\ ,\ \rangle_{S^+\widehat{N}}$ is left invariant by the action of the deck transformations associated to the covering projection $S^+\widehat{N} \to S^+N$. Moreover the quotient of $\langle\ ,\ \rangle_{S^+\widehat{N}}$ by this action is equal to $\langle\ ,\ \rangle_{S^+N}$.

(c) For any $v \in S^+N$ define maps $f_v\colon \mathbf{R} \to N$ and $\bar{f}_v\colon \mathbf{R} \to S^+N$ by $f_v = \rho \circ \bar{f}_v$ (where $\rho\colon S^+N \to N$ is the standard projection) and by $\bar{f}_v(t) = g^t(v)$. Then we must have

$$\frac{1}{2}\left\langle \frac{df_v}{dt(0)}, \frac{df_v}{dt(0)}\right\rangle \leq \left\langle \frac{d\bar{f}_v}{dt(0)}, \frac{d\bar{f}_v}{dt(0)}\right\rangle_{S^+N} \leq 2\left\langle \frac{df_v}{dt(0)}, \frac{df_v}{dt(0)}\right\rangle_N .$$

Note that $\langle df_v/dt(0), df_v/dt(0)\rangle_N = 1$.

Remark. The Riemannian structure $\langle\ ,\ \rangle_N$ induces canonically a Riemannian structure $\langle\ ,\ \rangle_{SN}$ (cf. [7, p. 547–548]). By pulling $\langle\ ,\ \rangle_N$ and $\langle\ ,\ \rangle_{SN}$ back to $\langle\ ,\ \rangle_{\widehat{N}}$ and $\langle\ ,\ \rangle_{S\widehat{N}}$ we get an explicit construction for Riemannian structures which satisfy 3.2.1(a)–(c).

3.3. The path liftings $C^{r,v,\beta}$. Let $r\colon [0,1] \to \widehat{N}$ be a given path. A *lifting of* r *to* $S\widehat{N}$ is just a continuous vector field $V\colon [0,1] \to S\widehat{N}$ such that $V(t) \in S\widehat{N}_{r(t)}$ holds for all $t \in [0,1]$. In this subsection we describe a special way of lifting paths from $\widehat{N}$ to $S\widehat{N}$. Such liftings are uniquely determined by specifying the initial condition $V(0) = v$.

For a fixed but arbitrary number $\sigma > 0$ choose a C^∞-function $g\colon \mathbf{R} \to [0,1]$ which has the following properties.

3.3.1. (a) $g(t) = 0$ for all $t \le \sigma$, and $g(t) = 1$ for all $t \ge 2\sigma$.

(b) $dg/dt \ge 0$ everywhere. Choose a number $\kappa > 0$ such that $\kappa \ge dg/dt$ everywhere.

For any path $r\colon [0,1] \to \widehat{N}$, any vector $v \in S\widehat{N}_{r(0)}$, and any number $\beta > 0$, we define a lifting $A^{r,v,\beta}\colon [0,1] \to S\widehat{N}$ of r to $S\widehat{N}$ by Figure 3.3.2.

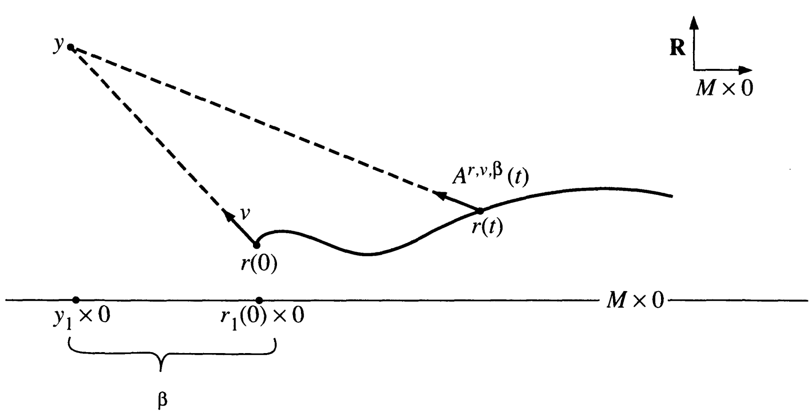

FIGURE 3.3.2.

In Figure 3.3.2 we see that the unit speed geodesics in $\widehat{N}$ which start at $r(0)$ with direction v, and which start at $r(t)$ with direction $A^{r,v,\beta}(t)$, intersect at the point $y \in \widehat{N}$ uniquely determined by r, v, and the equality $d_{\widehat{M}}(y_1, r_1(0)) = \beta$. Note that Lemma 2.0 assures us that the lifting $A^{r,v,\beta}$ is well defined by Figure 3.3.2 provided diameter(Image(r_1)) $< \beta$ in $\widehat{M}$, and $v \neq \pm u$. The product structure $\widehat{N} = \widehat{M} \times \mathbf{R}$ leads to the factoring $A^{r,v,\beta} = A_1^{r,v,\beta} \times A_2^{r,v,\beta} u$.

We also define a lifting $B^{r,v,\beta}\colon [0,1] \to S\widehat{N}$ of r to $S\widehat{N}$ by Figure 3.3.3.

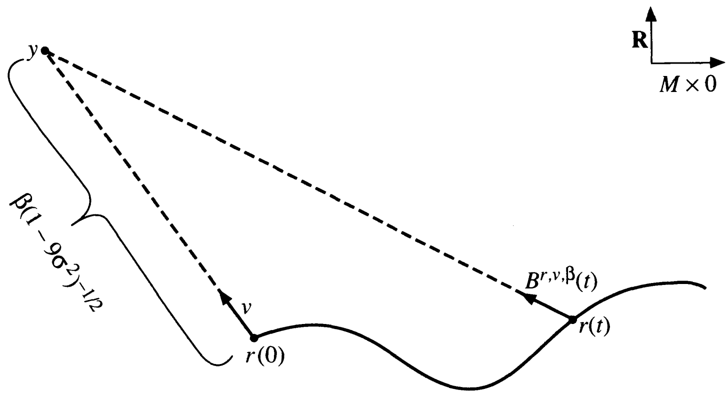

FIGURE 3.3.3.

In Figure 3.3.3 we see that the unit speed geodesics which start at $r(0)$ in the direction v, and at $r(t)$ in the direction $B^{r,v,\beta}(t)$, intersects at a point $y \in \widehat{N}$ which is uniquely determined by r, v, and the equality $d_{\widehat{N}}(y, r(0)) = \beta(1-9\sigma^2)^{-1/2}$. We note again that Lemma 2.0 assures us that the lifting $B^{r,v,\beta}$ is well defined by Figure 3.3.3 provided diameter(Image(r)) $< \beta(1-9\sigma^2)^{-1/2}$ in $\widehat{N}$, and $3\sigma < 1$.

The lifting in which we are really interested is a combination of the liftings $A^{r,v,\beta}$ and $B^{r,v,\beta}$, denoted by $C^{r,v,\beta}$, and defined as follows.

3.3.4. (a) If $v_2 \geq 3\sigma$ (where σ comes from 3.3.1) then we set $C^{r,v,\beta}(t) = B^{r,v,\beta}(t)$ for all $t \in [0, 1]$.

(b) If $0 \leq v_2 \leq 3\sigma$ then we set

$$C_2^{r,v,\beta}(t) = v_2 + g(v_2)(A_2^{r,v,\beta}(t) - v_2)$$

and

$$C_1^{r,v,\beta}(t) = A_1^{r,v,\beta}(t)(|A_1^{r,v,\beta}(t)|)^{-1}(1-(C_2^{r,v,\beta}(t))^2)^{1/2}$$

for all $t \in [0, 1]$. Here $C^{r,v,\beta} = C_1^{r,v,\beta} \times C_2^{r,v,\beta}u$ comes from the product structure $\widehat{N} = \widehat{M} \times \mathbf{R}$.

Remark. Note in 3.3.4 that $C^{r,v,\beta}(t)$ is well defined since $A^{r,v,\beta}(t) = B^{r,v,\beta}(t)$ holds for any v with $v_2 = 3\sigma$ and for all $t \in [0, 1]$. Note also that, as v_2 goes from σ to 2σ, $C^{r,v,\beta}(t)$ is a gradual tapering of the lifting

$$D^{r,v,\beta}(t) = (A_1^{r,v,\beta}(t)(|A_1^{r,v,\beta}(t)|^{-1}(1-v_2^2)^{1/2})) \times (v_2 u(r(t)))$$

to the lifting $A^{r,v,\beta}(t)$. Clearly if $v \in S^+N$ then $D^{r,v,\beta}(t) \in S^+N$ for all $t \in [0,1]$, and thus for β sufficiently large we will also have $C^{r,v,\beta}(t) \in S^+\widehat{N}$ for all $t \in [0,1]$ (cf. Lemma 3.4). This last property is (unfortunately) not satisfied by the liftings $B^{r,v,\beta}(t)$, which is our reason for introducing the more complicated liftings $C^{r,v,\beta}(t)$.

We will need the following two lemmas to prove Proposition 3.1. The proofs of these lemmas are given at the end of this section.

3.4. Lemma. (a) *$C^{r,v,\beta}(t)$ is continuous in r, v, β, and t.*

(b) *Given σ and g as in* 3.3.1, *and a number $\alpha > 0$, there exists a number $L > 0$ such that the following is true. Let $r\colon [0,1] \to \widehat{N}$ denote any path such that* diameter(Image(r)) $\le \alpha$, *and choose $\beta \ge L$. Then for any two vectors v, $w \in S^+\widehat{N}_{r(0)}$ we have*

$$C^{r,v,\beta}(t) = C^{r,w,\beta}(t) \quad \textit{for some } t \in [0,1]$$
$$\Leftrightarrow C^{r,v,\beta}(t) = C^{r,w,\beta}(t) \quad \textit{for all } t \in [0,1]$$

and

$$C^{r,v,\beta}(t),\ C^{r,w,\beta}(t) \in S^+\widehat{N} \quad \textit{for all } t \in [0,1].$$

3.5. Lemma. *Given any numbers $\varepsilon, \alpha > 0$, there is a number $\gamma > 0$. Given a function $g\colon R \to [0,1]$ which satisfies* 3.3.1 *for a fixed but arbitrary $\sigma \in [0,\gamma]$, there is a number L'. For any choice of $\beta > L'$ and for any path $r\colon [0,1] \to \widehat{N}$ which satisfies* Image(r) $\subset \widehat{M} \times [-\sigma,\sigma]$ *and* diameter(Image(r)) $\le \alpha$, *the following must hold. For all $v \in S^+\widehat{N}_{r(0)}$ the path $C^{r,v,\beta}$ is $(2\alpha,\varepsilon)$-controlled over $(S^+\widehat{N}, \widehat{\mathscr{G}}^+)$ with respect to the projection map $\hat{g}^{\beta'}\colon S^+\widehat{N} \to S^+\widehat{N}$, where $\hat{g}^t\colon S^+\widehat{N} \to S^+\widehat{N}$ is the geodesic flow on $S^+\widehat{N}$ and where $\beta' = (1-\sigma)\beta$ (cf.* §1.5 *for "control").*

Proof of Proposition 3.1. We will first complete the proof assuming that Hypothesis 2.14 is satisfied for all i. (Note that 2.14 holds for all i if M is a locally symmetric space.)

Let $h\colon S^k \to \mathscr{P}_j(M)$ represent an element in the $(k-j)$-dimensional homotopy group of the spectrum $\mathscr{P}_*(M)$, where k, $j > 0$. We must show that there is another map $h'\colon S^k \to \mathscr{P}_j^c(p)$ such that $\mathscr{P}_j^c(f) \circ J_j \circ h'$ and h are homotopic maps. We begin with the following definition.

3.6. Definition. A continuous map $h\colon X \to \mathscr{P}_j(M)$ is said to be *α-simply-controlled* if for each $x \in X$ and each $y \in M \times \mathbf{R}^j \times I^n$ each of the composite maps

$$[0,1] = y \times [0,1] \subset M \times \mathbf{R}^j \times I^n \times [0,1] \xrightarrow{h(x)} M \times \mathbf{R}^j \times I^n \times [0,1] \xrightarrow{\text{proj}} M$$

has a lifting $f\colon [0, 1] \to \widehat{M}$ to the universal covering space $\widehat{M}$ such that diameter(Image(f)) $\leq \alpha$.

We note that since S^k is compact, any given map $h\colon S^k \to \mathscr{P}_j(M)$ satisfies the following property.

3.7. There exists $\alpha > 0$ such that $h\colon S^k \to \mathscr{P}_j(M)$ is $\alpha/8$-simply-controlled (where α depends on h).

By composing $h\colon S^k \to \mathscr{P}_j(M)$ with the map $\mathscr{P}_j(M) \to \mathscr{P}_j(N)$, which is induced by the inclusion $M = M \times 0 \subset M \times \mathbf{R} = N$, we obtain a mapping $h^1\colon S^k \to \mathscr{P}_j(N)$ which we can arrange to have the following properties.

3.8. (a) For each $x \in S^k$ the support of the stable pseudoisotopy $h^1(x)$ lies over the subset $M \times [-\sigma, \sigma] \subset N$, where σ comes from 3.5.

(b) $h^1\colon S^k \to \mathscr{P}_j(N)$ is $\alpha/4$-simply-controlled.

The remainder of the proof of Proposition 3.1 is contained in the following subsections.

3.9. Transfers. let $\rho\colon \tau \to X$ denote a disc bundle over the manifold X, and let $h\colon Y \to \mathscr{P}_j(X)$ be a continuous map from the CW-complex Y. Recall that a *transfer of* h *in the bundle* $\tau \to X$ consists of a map $\bar{h}\colon Y \to \mathscr{P}_j(\tau)$ such that for each $y \in Y$ there is the following commutative diagram:

$$\begin{array}{ccc}
\tau \times (\mathbf{R}^j \times I^n \times [0, 1]) & \xrightarrow{\bar{h}(y)} & \tau \times (\mathbf{R}^j \times I^n \times [0, 1]) \\
{\scriptstyle \rho\times 1}\downarrow & & \downarrow{\scriptstyle \rho\times 1} \\
X \times (\mathbf{R}^j \times I^n \times [0, 1]) & \xrightarrow{h(y)} & X \times (\mathbf{R}^j \times I^n \times [0, 1])
\end{array}$$

Note that the transfer enjoys the following properties.

3.9.1. (a) For any $h\colon Y \to \mathscr{P}_j(X)$ there exists a transfer $\bar{h}\colon Y \to \mathscr{P}_j(\tau)$ of h in the bundle $\tau \to X$.

(b) If $\bar{h}_0, \bar{h}_1$ are two transfers of h in the bundle $\tau \to X$, then there is a homotopy $\bar{h}_t\colon Y \to \mathscr{P}_j(\tau)$, $t \in [0, 1]$, from $\bar{h}_0$ to $\bar{h}_1$ such that each $\bar{h}_t$ is a transfer of h in the bundle $\tau \to X$.

(c) Let $\zeta \to \tau$ be a disc bundle over τ such that the composite bundle projection $\zeta \to \tau \to X$ is equivalent to the trivial bundle $X \times I^k \to X$. Let $\bar{h}$ be a transfer of h in the bundle $\tau \to X$ and let $\tilde{h}$ denote a transfer of $\bar{h}$ in the bundle $\zeta \to \tau$. Then there is a homotopy $\tilde{h}_t\colon Y \to \mathscr{P}(X \times I^k)$, $t \in [0, 1]$, from $\tilde{h}$ to $h\colon Y \to \mathscr{P}_j(X) = \mathscr{P}_j(X \times I^k)$ such that each $\tilde{h}_t$ is a transfer of h in the trivial bundle $X \times I^k \to X$.

3.10. The special transfer $\bar{h}^1 : S^k \to \mathscr{P}_j(S^+N)$ of h^1 in $S^+N \to N$. We will now construct a special transfer $\bar{h}^1 : S^k \to \mathscr{P}_j(S^+N)$ in the bundle $S^+N \to N$ for the mapping h^1 of 3.8 by using the path liftings of §3.3. It will be more convenient to work with stable pseudoisotopies on $\widehat{N}$ and $S^+\widehat{N}$. Let $H^1 : S^k \to \mathscr{P}_j(\widehat{N})$ denote the map such that for each $y \in S^k$ the stable pseudoisotopy $H^1(y)$ of $\widehat{N}$ is obtained by pulling back the stable pseudoisotopy $h^1(y)$ of N along the covering projection $\widehat{N} \to N$. We will first construct a transfer $\overline{H}^1 : S^k \to \mathscr{P}_j(S^+\widehat{N})$ in the bundle $S^+\widehat{N} \to \widehat{N}$ for the map H^1. This transfer will have the property that for each $y \in S^k$ the stable pseudoisotopy $\overline{H}^1(y)$ of $S^+\widehat{N}$ is left invariant by the deck transformations of the covering $S^+\widehat{N} \to S^+N$. Thus the quotient of each $\overline{H}^1(y)$ under the deck transformations action yields the stable pseudoisotopy $\bar{h}^1(y)$ of S^+N, and hence the transfer $\bar{h}^1 : S^k \to \mathscr{P}_j(S^+N)$ in the bundle $S^+N \to N$ for the map h^1.

Choose the integer n sufficiently large so that for each $y \in S^k$ the stable pseudoisotopy $H^1(y)$ is a mapping $H^1(y) : \widehat{N} \times \mathbf{R}^j \times I^n \times [0, 1] \to \widehat{N} \times \mathbf{R}^j \times I^n \times [0, 1]$. For each $y \in S^k$ and each $z \in \widehat{N} \times \mathbf{R}^j \times I^n$, define two paths $p_{y,z,1} : [0, 1] \to \widehat{N}$ and $p_{y,z,2} : [0, 1] \to \mathbf{R}^j \times I^n \times [0, 1]$ to be the composite maps

$$[0, 1] = z \times [0, 1] \subset (\widehat{N} \times \mathbf{R}^j \times I^n) \times [0, 1] \xrightarrow{H^1(y)} \widehat{N} \times \mathbf{R}^j \times I^k \times [0, 1] \xrightarrow{\text{proj}} \widehat{N}$$

and

$$\begin{aligned}[0, 1] = z \times [0, 1] &\subset (\widehat{N} \times \mathbf{R}^j \times I^n) \times [0, 1] \\ &\xrightarrow{H^1(y)} \widehat{N} \times \mathbf{R}^j \times I^n \times [0, 1] \xrightarrow{\text{proj}} \mathbf{R}^j \times I^n \times [0, 1],\end{aligned}$$

respectively. We note that the map $H^1 : S^k \to \mathscr{P}_j(\widehat{N})$ can be reconstructed from the collections of paths $\{p_{y,z,i} : y \in S^k,\ z \in \widehat{N} \times \mathbf{R}^j \times I^n;\ i = 1, 2\}$ by using the definitions of these paths. Likewise we may reconstruct the desired transfer $\overline{H}^1 : S^k \to \mathscr{P}_j(S^+\widehat{N})$ for H^1 from a collection of paths $\{\bar{p}_{y,z,i} : y \in S^k,\ z \in S^+\widehat{N} \times \mathbf{R}^j \times I^n;\ i = 1, 2\}$ which are defined as follows.

3.10.1. For each $z \in S^+\widehat{N} \times \mathbf{R}^j \times I^n$ let z' and z'' denote the image of z under projection to $S^+\widehat{N}$ and $\widehat{N} \times \mathbf{R}^j \times I^n$, respectively. Define the mappings $\bar{p}_{y,z,1} : [0, 1] \to S^+\widehat{N}$ by $\bar{p}_{y,z,1} = C^{r,v,\beta}$, where $r = p_{y,z'',1}$, $v = z'$, and $\beta > \alpha$, and where $C^{r,v,\beta}$ comes from §3.3 and α is as in

3.8. Define the mapping $\bar{p}_{y,z,2}\colon [0, 1] \to \mathbf{R}^j \times I^n \times [0, 1]$ to be equal to the map $p_{y,z'',2}$.

Now the transfer $\overline{H}^1 : S^k \to \mathscr{P}_j(S^+\widehat{N})$ in the bundle $S^+\widehat{N} \to \widehat{N}$ of the map H^1 can be defined as follows.

3.10.2. For each $y \in S^k$, $z \in S^+\widehat{N} \times \mathbf{R}^j \times I^n$, and $t \in [0, 1]$, set

$$\overline{H}^1(y)(z, t) = \bar{p}_{y,z,1}(t) \times \bar{p}_{y,z,2}(t).$$

Note that it follows from Lemma 3.4 that each $\overline{H}^1(y)$ defined by 3.10.1, 3.10.2 is in fact a well-defined stable pseudoisotopy of $S^+\widehat{N}$ and that $\overline{H}^1(y)$ depends continuously on y (see, in particular, 3.4(b)). On the other hand, it can be deduced from Lemma 3.5 and 3.8 that the $\{\overline{H}^1(y) : y \in S^k\}$ have the following control properties.

3.10.3. For each $y \in S^k$ the stable pseudoisotopy $\overline{H}^1(y)\colon S^+\widehat{N} \times \mathbf{R}^j \times I^n \times [0, 1] \to S^+\widehat{N} \times \mathbf{R}^j \times I^n \times [0, 1]$ is $(\alpha/2, \delta)$-controlled over $(S^+\widehat{N}, \widehat{\mathscr{G}}^+)$ with respect to bundle projection $S^+\widehat{N} \xrightarrow{\hat{g}^{\beta'}} S^+\widehat{N}$, where $\hat{g}^t\colon S^+\widehat{N} \to S^+\widehat{N}$, $t \in \mathbf{R}$, is the geodesic flow on $S^+\widehat{N}$ (cf. Lemma 3.5 for β'). Here δ may be chosen arbitrarily small if β is chosen sufficiently large and if σ (of 3.5) is chosen sufficiently small.

We leave as an exercise for the reader to check that for each $y \in S^k$ the stable pseudoisotopy $\overline{H}^1(y)$ of $S^+\widehat{N}$ is left invariant by the deck transformations for the covering projection $S^+\widehat{N} \to S^+N$. So, as was noted at the outset of §3.10, we may define a stable pseudoisotopy $\bar{h}^1(y)$ on S^+N to be the quotient of $\overline{H}^1(y)$ by the deck transformation group action on $S^+\widehat{N}$. Thus we have the special transfer $\bar{h}^1\colon S^k \to \mathscr{P}_j(S^+N)$ in the bundle $S^+N \to N$ for the map h^1 of 3.8. The control properties of 3.10.3, and 3.8(a), 3.2.1 imply that the following hold.

3.10.4. (a) Each stable pseudoisotopy $\bar{h}^1(y)\colon S^+N \times \mathbf{R}^j \times I^n \times [0, 1] \to S^+N \times \mathbf{R}^j \times I^n \times [0, 1]$ is (α, δ)-controlled over $(S^+N, \mathscr{G}^+)$ with respect to the projection $S^+N \xrightarrow{g^{\beta'}} S^+N$. Here δ can be chosen arbitrarily small if β is chosen sufficiently large and if σ (of 3.5) is chosen sufficiently small.

(b) The support of each $\bar{h}^1(y)$ lies over $M \times [-\sigma, \sigma]$ with respect to the standard projection $S^+N \to N$.

3.11. Applying control theory to $\bar{h}^1$. First we note that every closed orbit for the geodesic flow $g^t\colon S^+N \to S^+N$, $t \in \mathbf{R}$, lies in a subset $S(M \times s) \subset S^+N$ consisting of all vectors of S^+N which are tangent to the subspace $M \times s \subset M \times \mathbf{R} = N$, for some number $s \in \mathbf{R}$. Thus the union of the closed orbits of the geodesic flow on S^+N are identified with the subset $\overline{E} \times \mathbf{R} \subset S^+N$, where $\overline{E} \subset SM$ equals the union of all closed orbits for the geodesic flow on SM. Let $\bar{p}: \overline{E} \to \overline{G}$ denote the quotient map obtained by collapsing every orbit in $\overline{E}$ to a point, and let $\bar{f}: \overline{E} \to M$ denote the composite map $\overline{E} \subset SM \xrightarrow{\text{proj}} M$.

There is a map $\overline{J}_*: \mathscr{P}^c_*(\bar{p}) \to \mathscr{P}^c_*(\overline{E})$ whose definition is analogous to the definition for J_* in §1.4. Let $\mathscr{P}^c_*(\bar{f}): \mathscr{P}^c_*(\overline{E}) \to \mathscr{P}_*(M)$ denote the map induced by $\bar{f}: \overline{E} \to M$. Note that the composite map $\mathscr{P}^c_*(\bar{f}) \circ \overline{J}_*: \mathscr{P}^c_*(\bar{p}) \to \mathscr{P}_*(M)$ factors through the composite map $\mathscr{P}^c_*(f) \circ J_*: \mathscr{P}^c_*(p) \to \mathscr{P}_*(M)$. Thus, to complete the proof of Proposition 3.1 it will suffice to show that the composite map $\mathscr{P}^c_*(\bar{f}) \circ \overline{J}_*: \mathscr{P}^c_*(\bar{p}) \to \mathscr{P}_*(M)$ induces a surjection on the homotopy groups of the Ω-spectra. In particular, it will suffice to find $h': S^k \to \mathscr{P}^c_j(\bar{p})$ such that h and $\mathscr{P}^c_j(\bar{f}) \circ \overline{J}_j \circ h'$ are homotopic.

The path components of $\overline{E}$ are exactly the subsets $\{\overline{E}_{i,j}\}$ discussed in §2.6 (cf. Theorem 2.4 and §2.6). Thus (by 2.4, 2.6) two closed orbits L_1, L_2 in $\overline{E}$ are in the same path component of $\overline{E}$ if there are freely homotopic orientation preserving immersions $g_i: S^1 \to L_i$, $i = 1, 2$ (where the orbits L_i, $i = 1, 2$, are oriented in the direction of the geodesic flow on SM). Using this criterion together with a compactness argument (SM is compact) it is an exercise to show that for any given number $\lambda > 0$ the following holds: there are only finitely many path components of $\overline{E}$ which contain an orbit of length less than or equal to λ.

In the special case that $\lambda = \eta\alpha$ (where η comes from 1.5.3 and where α comes from 3.10.4) we denote by $\overline{E}_1, \overline{E}_2, \cdots, \overline{E}_m$ the components of $\overline{E}$ which contain an orbit of length less than or equal to λ. Note that there are closed pairwise disjoint tubular neighborhoods $\tau_1, \tau_2, \cdots, \tau_m$ in S^+N for the corresponding components $\overline{E}_1 \times \mathbf{R}, \overline{E}_2 \times \mathbf{R}, \cdots, \overline{E}_m \times \mathbf{R}$ of $\overline{E} \times \mathbf{R}$, such that the subset $U \subset S^+N$ defined by $U = S^+N - \bigcup_i \tau_i$ satisfies 1.5.3(a), for sufficiently small ε in 1.5.3(a). This fact, together with 3.2.1 and 3.10.4, assure us that all the hypotheses are satisfied for applying Theorem 1.5.3 to Image$(\bar{h}^1)$ over the subset U, if β is chosen sufficiently large and if σ is chosen sufficiently small. So we apply Theorem 1.5.3 to get a homotopy $\bar{h}^1_t: S^k \to \mathscr{P}_j(S^+N)$, $t \in [0, 1]$, of $\bar{h}^1$ which has the following properties.

3.11.1. (a) For each $y \in S^k$, $t \in [0, 1]$, the stable pseudoisotopy $\bar{h}^1_t(y)$ is $(\eta\alpha, \varepsilon)$-controlled over $(S^+N, \mathscr{G}^+)$ with respect to the bundle projection $S^+N \xrightarrow{g^{\beta'}} S^+N$.

(b) For each number $s \geq 0$ and each integer i satisfying $1 \leq i \leq m$, let $\tau_{i,s}$ denote the restriction of τ_i to $\overline{E}_i \times [-s, s]$. Then for each $y \in S^k$ the support of the stable pseudoisotopy $\bar{h}_1^1(y)$ lies over the subset $\bigcup_i \tau_{i,s} \subset S^+N$ (with respect to the projection $S^+N \xrightarrow{g^{\beta'}} S^+N$), where $s = 4(\sigma + \eta\alpha + \beta')$.

For each $i = 1, 2, \cdots, m$ we choose a disc bundle $\zeta_i \to \tau_i$ such that the composite bundle $\zeta_i \to \tau_i \to \overline{E}_i \times \mathbf{R}$ is equivalent to the trivial bundle $\overline{E}_i \times \mathbf{R} \times I^a \to \overline{E}_i \times \mathbf{R}$, and for each number $s \geq 0$ we let $\zeta_{i,s}$ denote the restriction of ζ_i to $\tau_{i,s}$. By 3.11.1(b) we obtain a map $g_i\colon S^k \to \mathscr{P}_j(\tau_{i,s})$ by setting $g_i(y)$ equal to the restriction of $\bar{h}_1^1(y)$ to that part of its domain which lies over $g^{-\beta'}(\tau_{i,s})$, where we have identified $\mathscr{P}_j(\tau_{i,s})$ with $\mathscr{P}_j(g^{-\beta'}(\tau_{i,s}))$ under the homeomorphism $g^{\beta'}\colon g^{-\beta'}(\tau_{i,s}) \to \tau_{i,s}$. Let $\bar{g}_i\colon S^k \to \mathscr{P}_j(\overline{E}_i)$ denote a transfer for g_i in the bundle $\zeta_{i,s} \to \tau_{i,s}$, where we have identified $\mathscr{P}_j(\overline{E}_i)$ with $\mathscr{P}_j(\overline{E}_i \times [-s, s] \times I^a)$ under the stabilization map. Note that by 3.11.1(a), each $\bar{g}_i$ has the following property.

3.11.2. For each $y \in S^k$ the stable pseudoisotopy $\bar{g}_i(y)$ is ε'-controlled over $\overline{G}_i$ with respect to the projection $\bar{p}_i\colon \overline{E}_i \to \overline{G}_i$, where $\overline{G}_i = \bar{p}(\overline{E}_i)$, $\bar{p}_i = \bar{p} \mid \overline{E}_i$, and ε' is a positive number satisfying $\operatorname{limit}_{\varepsilon \to 0} \varepsilon' = 0$ with ε from 3.11.1.

By taking the disjoint union of the maps $\bar{g}_i\colon S^k \to \mathscr{P}_j(\overline{E}_i)$ we obtain a map $\bar{g}\colon S^k \to \mathscr{P}_j^c(\overline{E})$. It follows from 3.11.2 that $\bar{g}$ is ε'-controlled over $\overline{G}$ with respect to the bundle projection $\bar{p}\colon \overline{E} \to \overline{G}$. Thus we may apply Theorem 1.4.1 to get a mapping $h'\colon S^k \to \mathscr{P}_j^c(\bar{p})$ such that $\overline{J}_j \circ h' = \bar{g}$. (Actually, we apply 1.4.1 to each $\bar{g}_i$ forming h'_i and h' is the disjoint union of the h'_i.)

In order to complete the proof of Proposition 3.1 it remains to show that the maps $h\colon S^k \to \mathscr{P}_j(M)$ and $\mathscr{P}_j^c(\bar{f}) \circ \overline{J}_j \circ h'\colon S^k \to \mathscr{P}_j(M)$ are homotopic. To see this we first note that for sufficiently large $s > 0$ the support of each stable pseudoisotopy $\bar{h}_t^1(y)$ in 3.11.1 lies over the subset $M \times [-s, s]$ with respect to the standard projection $S^+N \to N$ (cf. 1.5.3). From this last remark and §3.9 it follows that by transferring $\bar{h}_1^1\colon S^k \to \mathscr{P}_j(S^+N)$ to a bundle $\tau \to S^+N$ (such that the composite bundle $\tau \to S^+N \to N$ is equivalent to the trivial bundle $N \times I^b \to N$), we get a transfer map $\tilde{h}_1^1\colon S^k \to \mathscr{P}_j(N \times I^b) = \mathscr{P}_j(N)$; then by restricting each image point $\tilde{h}_1^1(y)$, $y \in S^k$ to that part of its domain which lies over the subset $M \times [-s, s] \subset N$, we obtain a map $h'': S^k \to \mathscr{P}_j(M \times [-s, s]) = \mathscr{P}_j(M)$ which is homotopic to $h: S^k \to \mathscr{P}_j(M)$. On the other hand, another application of §3.9 shows that h'' and $\mathscr{P}_j^c(\bar{f}) \circ \overline{J}_j \circ h'$ are also homotopic. To verify this last fact the reader should consult [16; §1] to see how the transfer construction is related to the functorality of $\mathscr{P}_j^c(\ )$. Thus, h and $\mathscr{P}_j^c(\bar{f}) \circ \overline{J}_j \circ h'$ are homotopic as desired.

This completes the proof of Proposition 3.1 when Hypothesis 2.14 holds for all i.

There are only minor modifications (in 3.11) to be made on the preceding proof if Hypothesis 2.14 is not assumed to hold. Let $\overline{E}_1, \cdots, \overline{E}_m$ denote the components of $\overline{E}$ discussed in §3.11, and let $L_1, \cdots, L_m$ denote the leaves of the asymptotic foliation $\mathscr{A}$ that contain the $\overline{E}_1, \cdots, \overline{E}_m$ (cf. 2.6.1). Choose a number $r \geq 0$ sufficiently small so that the orthogonal projections $\overline{E}_i^r \to \overline{E}_i$, $1 \leq i \leq m$, are well defined, where $\overline{E}_i^r = \{x \in L_i : d_i^L(x, \overline{E}_i) < r\}$, and $d_i^L(\ ,\ )$ is the metric on L_i associated to the Riemannian structure pulled back from M by the covering projection $L_i \subset SM \xrightarrow{\text{proj}} M$. (Note that the sectional curvature restriction $K \leq 0$ on L_i and the local convexity of $\overline{E}_i$ (cf. 2.7.1) assure us that orthogonal projection to $\overline{E}_i$ is locally well defined in L_i (cf. [3, pp. 8–10])). Choose small tubular neighborhoods $\{\xi_i\}$ for the $\{\overline{E}_i^r\}$ in $S^+N \mid M \times 0$; thus each ξ_i is a smooth disk bundle (with corners) over $\overline{E}_i^r$. Now in §3.11 set $\tau_i = \xi_i \times \mathbf{R}$, and set $\tau_{i,s} = \xi_i \times [-s, s]$ for each $s \geq 0$. After applying the control Theorem 1.5.3, we get that 3.11.1 is true. To get the $\{\bar{g}_i : S^k \to \mathscr{P}_j(\overline{E}_i)\}$ of 3.11.2 we restrict each stable pseudoisotopy $\bar{h}_1^1(y)$, $y \in S^k$, to that part of its domain lying over $g^{-\beta'}(\tau_{i,s})$ (cf. 3.11.1 for s, β', $\bar{h}_1^1$), and then (having identified $\mathscr{P}_j^c(\tau_{i,s})$ with $\mathscr{P}_j^c(g^{-\beta'}(\tau_{i,s}))$ via the homeomorphism $g^{\beta'} : g^{-\beta'}(\tau_{i,s}) \to \tau_{i,s}$) project it into $\overline{E}_i$ by the composite map

$$\tau_{i,s} = \xi_i \times [-s, s] \xrightarrow{\text{proj}} \xi_i \xrightarrow{\text{proj}} \overline{E}_i^r \xrightarrow{\text{proj}} \overline{E}_i.$$

The rest of §3.11 is carried out as before.

This completes the proof of Proposition 3.1.

Proof of Lemma 3.4. The proof of 3.4(a) and of the second claim in 3.4(b) follow directly from the definition of $C^{r,v,\beta}$. The details are left as an exercise.

We divide the verification of the first claim in 3.4(b) into the following three cases.

3.12. Case I. $v_2 \geq 2\sigma$ and $w_2 \geq 2\sigma$.

Let $B(r_1(0), \beta)$ denote the closed ball of radius β centered at $r_1(0)$ in $\widehat{M}$, and let $B(r(0), \beta')$ denote the closed ball of radius $\beta' = \beta(1-9\sigma^2)^{-1/2}$ centered at $r(0)$ in $\widehat{N} = \widehat{M} \times \mathbf{R}$. Since both of the sets $B(r_1(0), \beta) \times \mathbf{R}$ and $B(r(0), \beta')$ are convex subsets of $\widehat{N}$ (cf. Lemma 2.2), it follows that $C = (B(r_1(0), \beta) \times \mathbf{R}) \cap B(r(0), \beta')$ is also a convex subset of $\widehat{N}$.

In Figure 3.12.1 we have indicated how the values of $C^{r,v,\beta}(t)$ and $C^{r,w,\beta}(t)$ are obtained by focusing to boundary points of C (cf. §3.3).

The convexity of C implies that $C^{r,v,\beta}(t) = C^{r,w,\beta}(t)$ holds for some $t \in [0, 1]$ if and only if $v = w$, provided Image$(r) \subset (C - \partial C)$, e.g., provided $L > \alpha$.

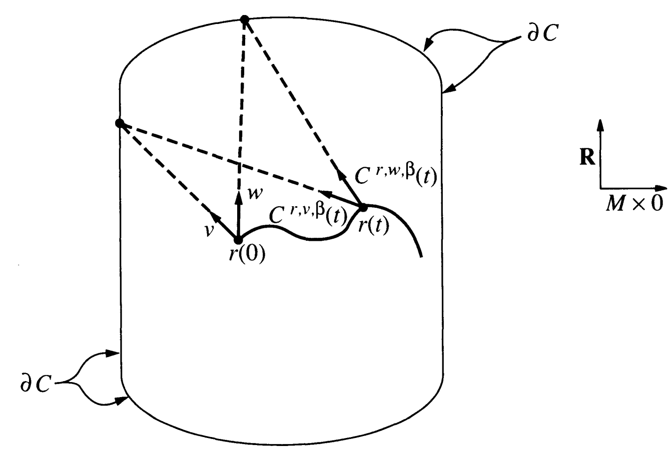

FIGURE 3.12.1.

3.13. Case II. $v_2 \leq 3\sigma$ and $w_2 \leq 3\sigma$.

If $\bar{v}_1 \neq \bar{w}_1$, then we show that $\overline{C}_1^{r,v,\beta}(t) \neq \overline{C}_1^{r,w,\beta}(t)$ for all $t \in [0, 1]$, where $\bar{z} = z|z|^{-1}$ for any $z \in T\widehat{M}$. In Figure 3.13.1 we have indicated how the values $C_1^{r,v,\beta}(t)$ and $C_1^{r,w,\beta}(t)$ are obtained by focusing to boundary points of the ball $B(r_1(0), \beta)$ (cf. §3.3).

The convexity of $B(r_1(0), \beta)$ implies that $\overline{C}_1^{r,v,\beta}(t) = \overline{C}_1^{r,w,\beta}(t)$ holds for some $t \in [0, 1]$ if and only if $\bar{v}_1 = \bar{w}_1$.

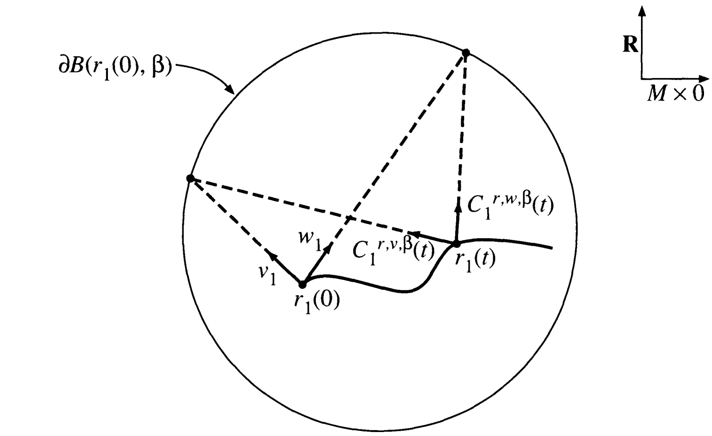

FIGURE 3.13.1.

If $\bar{v}_1 = \bar{w}_1$, then we must have that $v_2 \neq w_2$, assuming that $v \neq w$. Without any loss of generality we may suppose that $v_2 > w_2$. We shall complete the verification of 3.4(b) for Case II by proving the following claim.

3.13.2. Claim. $C_2^{r,v,\beta}(t) > C_2^{r,w,\beta}(t)$ holds for all $t \in [0, 1]$, provided β is chosen sufficiently large.

First note that our present assumptions, together with 3.3.1 and Figure 3.3.2, imply the truth of the following.

3.13.3. (a) $\bar{v}_1 = \bar{w}_1$ and $v_2 > w_2$.

(b) $g(v_2) \geq g(w_2)$.

(c) $A_2^{r,v,\beta}(t) > A_2^{r,w,\beta}(t)$ for all $t \in [0, 1]$.

Note that Claim 3.13.2 follows easily from 3.13.3 and 3.3.4 provided $A_2^{r,v,\beta}(t) \geq w_2$ holds for all $t \in [0, 1]$. It remains to verify 3.13.2 when the following inequality holds.

3.13.4. $w_2 > A_2^{r,v,\beta}(t)$ for the specific $t \in [0, 1]$ being considered in 3.13.2.

Note that for all v with $v_2 \leq 3\sigma$, all r which have diameter less than or equal to α in $\widehat{N}$, and all $t \in [0, 1]$, the following is true.

3.13.5. $\text{limit}_{\beta\to\infty}(v_2 - A_2^{r,v,\beta}(t)) = 0$ uniformly in r, v, and t.

We note that there is a smooth real-valued function $f(x, y, z)$ in three real variables, which has the following properties.

3.13.6. (a) $f(0, 0, z) = 0$.

(b) For the composite variables $\bar{x} = \beta^{-1}(d_{\widehat{M}}(r_1(t), y_1) - \beta)$ and $\bar{y} = \beta^{-1}(r_2(t) - r_2(0))$, where y_1, y are as in Figure 3.3.2, we have that

$$A_2^{r,v,\beta}(t) - v_2 = f(\bar{x}, \bar{y}, v_2).$$

By using 3.13.6(a) to help compute the second order Taylor polynomial for $f(x, y, w_2) - f(x, y, v_2)$ about the point $(0, 0, v_2)$—for fixed v_2 and variables x, y, w_2—we get the following equality.

3.13.7. $f(x, y, w_2) - f(x, y, v_2) = ax(w_2 - v_2) + by(w_2 - v_2)$, where a, b depend continuously on v_2, w_2, x, y.

The first order Taylor polynomial for $g(w_2)$ about v_2—here v_2 is fixed and w_2 is the variable—yields the following.

3.13.8. $g(w_2) = g(v_2) + c(w_2 - v_2)$, where $|c| \leq 2\kappa$. Here $\kappa > 0$ comes from 3.3.1.

Now we can complete the verification of Claim 3.13.2 when 3.13.3 and 3.13.4 hold. By combining 3.13.5, 3.13.6, 3.13.7, 3.13.8, and 3.3.4(b), we get the following equalities.

3.13.9. (a) $C_2^{r,w,\beta}(t) = w_2 + g(v_2)(A_2^{r,v,\beta}(t) - v_2) + R_1 + R_2$, where

$$R_1 = c(w_2 - v_2)(A_2^{r,w,\beta}(t) - w_2) \quad \text{and} \quad R_2 = g(v_2)(a\bar{x} + b\bar{y})(w_2 - v_2).$$

(b) $\text{limit}_{\beta \to \infty}(|R_1| + |R_2|)(w_2 - v_2)^{-1} = 0$ uniformly in r, v, w, t.

Now Claim 3.13.2 follows from 3.13.9 and 3.3.4.

3.14. Case III. $|v_2 - w_2| \geq \sigma$.

It is not difficult to deduce from 3.3 that all the limits $\text{limit}_{\beta \to \infty} C_2^{r,v,\beta}(t)$ and $\text{limit}_{\beta \to \infty} C_2^{r,w,\beta}(t)$ tend to v_2 and w_2 respectively, uniformly in r, v, w, and t, provided that diameter$(r) \leq \alpha$. Thus, for sufficiently large β, $C_2^{r,v,\beta}(t) \neq C_2^{r,w,\beta}(t)$ holds for all $t \in [0, 1]$.

This completes the proof of Lemma 3.4.

Proof of Lemma 3.5. We divide the proof into the following two cases.

3.15. Case I. $v_2 \geq 3\sigma$.

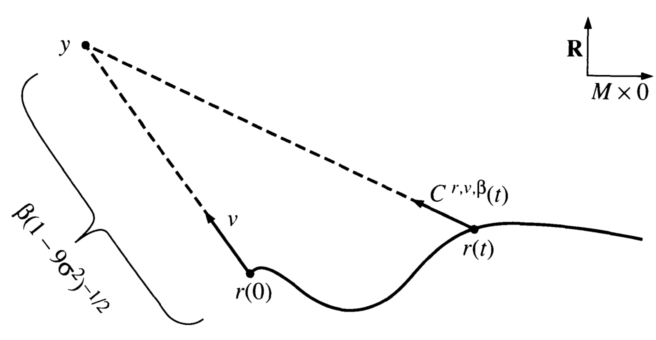

FIGURE 3.15.1.

In Figure 3.15.1 we have indicated how the values of $C^{r,v,\beta}$ are obtained by focusing at the point $y \in \widehat{N}$.

Note that the following equalities uniquely determine a one-parameter family of unit speed geodesics $f_t\colon \mathbf{R} \to N$, $t \in [0, 1]$ (cf. Lemma 2.0).

3.15.2. $f_t(0) = y$; $f_t(\beta_t) = r(t)$, where $\beta_t = d_{\widehat{N}}(y, r(t))$.

By definition of the geodesic flow $\hat{g}^t\colon S^+\widehat{N} \to S^+\widehat{N}$, $t \in \mathbf{R}$, and by 3.15.1, 3.15.2, we also have the following.

3.15.3. $\hat{g}^{\beta'} \circ C^{r,v,\beta}(t) = -df_t/ds(\beta_t - \beta')$, where $\beta' = (1-\sigma)\beta$, and for each value of t the map $f_t\colon \mathbf{R} \to N$ is a function of the variable s.

Now it follows from 3.15.2, 3.15.3, and 2.1, 3.2.1, that if σ is chosen small enough, and β is chosen large enough, then $\hat{g}^{\beta'} \circ C^{r,v,\beta}$ will be $(2\alpha, \varepsilon)$-controlled over $(S^+\widehat{N}, \widehat{\mathcal{G}}^+)$ for all r with diameter$(r) \le \alpha$. This completes the verification of Lemma 3.5 for Case I.

3.16. Case II. $v_2 < 3\sigma$.

In Figure 3.16.1 we have indicated how the values of $C^{r,v,\beta}$ are obtained by focusing at points $y^t \in \widehat{N}$. Note that the inequality of 3.16.2 follows from §3.3 and the hypothesis of Lemma 3.5 (used for the first time here) that Image$(r) \subset \widehat{M} \times [-\sigma, \sigma]$.

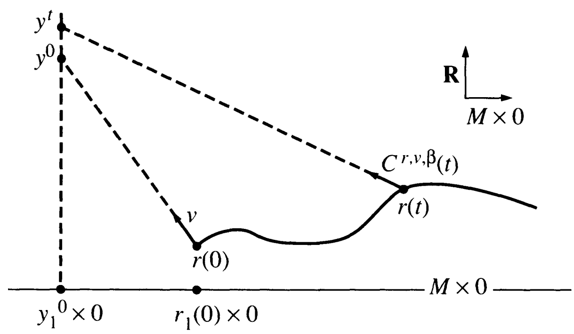

FIGURE 3.16.1

3.16.2. $d_{\widehat{N}}(y^0, y^t) \le 4(v_2\alpha + \sigma)$.

The following equalities uniquely determine a one-parameter family of unit speed geodesics $f_t\colon \mathbf{R} \to \widehat{N}$, $t \in [0, 1]$.

3.16.3. $f_t(0) = y^t$; $f_t(\beta_t) = r(t)$, where $\beta_t = d_{\widehat{N}}(y^t, r(t))$.

By the definition of the geodesic flow $\hat{g}^t\colon S^+\widehat{N} \to S^+\widehat{N}$, $t \in \mathbf{R}$, and by 3.16.1 and 3.16.3, we also have the following.

3.16.4. $\hat{g}^{\beta'} \circ C^{r,v,\beta}(t) = -df_t/ds(\beta_t - \beta')$, where $\beta' = (1-\sigma)\beta$.

Now it follows from 3.16.2–3.16.4 and 2.1, 3.2.1 that if σ is chosen small enough, and β is chosen sufficiently large, then $\hat{g}^{\beta'} \circ C^{r,v,\beta}$

will be $(2\alpha, \varepsilon)$-controlled over $(S^+\widehat{N}, \widehat{\mathscr{G}}^+)$ for all paths in r in $M \times [-\sigma, \sigma]$ with diameter$(r) \leq \alpha$.

This completes the proof of Lemma 3.5.

3.17. A control theorem for nonpositively curved manifolds. Before beginning the next section we state a lemma which will be needed in §5. Let M be a Riemannian manifold, and let $M_1 \subset M_2 \subset M_3 \subset \cdots$ be an increasing sequence of closed subspaces of M, all of which have the following properties.

3.17.1. (a) M is complete and of compact type (cf. 1.5.2).

(b) M has nonpositive sectional curvature everywhere.

(c) Each M_k is a codimension zero submanifold of M.

(d) The radius of injectivity for M at any point $q \in M - M_k$ is greater than or equal to k.

(e) For each k, $d_M(M - M_{k+m}, M_k) \to \infty$ as $m \to \infty$.

For any number $\alpha > 0$ and any integer $j \geq 0$ let $\mathscr{P}_j(M; \alpha)$ be the subspace of all stable pseudoisotopies in $\mathscr{P}(M \times \mathbf{R}^j)$ which are α-controlled over $M \times \mathbf{R}^j$ with respect to the identity projection $M \times \mathbf{R}^j \to M \times \mathbf{R}^j$ (cf. §1.2). Let $\mathscr{P}^b_j(M)$ denote the direct limit space $\text{limit}_{\alpha\to\infty} \mathscr{P}_j(M; \alpha)$ for any integer $j \geq 0$; and if $j < 0$ then set $\mathscr{P}^b_j(M) = \Omega^{-j}(\mathscr{P}^b_0(M))$. Hatcher's proof of the first half of 1.3.1 works, with only minor modifications, to show that the collection of spaces $\mathscr{P}^b_*(M) = \{\mathscr{P}^b_j(M) : j \in \mathbf{Z}\}$ is a Ω-spectrum called the spectrum of bounded stable pseudoisotopies on M. Of course the same construction can be used to obtain the Ω-spectrum $\mathscr{P}^b_*(M_k)$ for each integer $k \geq 0$.

3.17.2. Theorem. *Suppose that M and the M_k, $k = 1, 2, 3, \cdots$, satisfy* 3.17.1. *Then the direct limit Ω-spectrum* $\text{limit}_{k\to\infty} \mathscr{P}^b_*(M_k)$ *is weakly equivalent to the Ω-spectrum $\mathscr{P}^b_*(M)$ via the inclusion map*

$$\text{limit}_{k\to\infty} \mathscr{P}^b_*(M_k) \subset \mathscr{P}^b_*(M).$$

Proof of Theorem 3.17.2. We will show how to deform any $h \in \mathscr{P}^b_j(M)$ through a one-parameter family of stable pseudoisotopies $h_t \in \mathscr{P}^b_j(M)$, $t \in [0, 1]$, to $h_1 \in \mathscr{P}^b_j(M_k)$ for k sufficiently large. The remaining details are left to the reader.

Roughly speaking, to get h_t, $t \in [0, 1]$, we must reproduce the proof given above for Proposition 3.1. Set $N = M \times \mathbf{R}$ with the product metric. Let $h' \in \mathscr{P}^b_j(N)$ denote the image of h under the map induced by the inclusion $M = M \times 0 \subset M \times \mathbf{R} = N$ (cf. 3.8). Let $h'' \in \mathscr{P}^b_j(S^+N)$ denote a suitable transfer of h' in the bundle $S^+N \to N$ (cf. §3.10). Now by using the geodesic flow on S^+N we may deform h'' by h_t'', $t \in [0, 1/2]$, so that $h_{1/2}''$ is $(2\alpha, \delta)$-controlled over $(S^+N, \mathscr{G}^+)$ for a suitable small δ (cf. 3.10.4). Now 3.17.1 assures us that the foliated control Theorem 1.5.3 may be applied to $h_{1/2}''$ over $S^+N \mid ((M-M_k)\times\mathbf{R})$ (for suitably large k) to get a further deformation h_t'', $t \in [1/2, 1]$, of $h_{1/2}''$ such that the support of h_1'' lies over the subset $S^+N \mid (M_k \times \mathbf{R})$ of S^+N (cf. §3.11). Finally we transfer $h_t'' \in \mathscr{P}_j^b(S^+N)$, $t \in [0, 1]$, to $h_t''' \in \mathscr{P}_j^b(\tau)$, $t \in [0, 1]$, where $\tau \to S^+N$ is a disc bundle such that the composite bundle projection $\tau \to S^+N \to N$ is equivalent to the trivial bundle $N \times I^a \to N$. Note that the support of each h_t''', $t \in [0, 1]$, lies over the subset $M \times [-s, s] \subset N$ for sufficiently large $s > 0$, and the support of h_1''' lies over $M_k \times [-s, s]$. Thus, we may define the desired deformation h_t, $t \in [0, 1]$, of h to be the restriction of h_t''', $t \in [0, 1]$, to $M \times [-s, s] \times \mathbf{R}^j \times I^n \times [0, 1]$, where the domain of h_t''' is equal to $N \times \mathbf{R}^j \times I^n \times [0, 1]$.

This completes the proof of Theorem 3.71.2.

4. More geometry

The purpose of this section is to introduce more notation, and to state and prove several geometric lemmas (cf. Lemmas 4.6, 4.7, 4.11.1) which will be used in §6 in proving that the composite map $\mathscr{P}_*^c(f) \circ J_*: \mathscr{P}_*^c(p) \to \mathscr{P}_*(M)$ is injective on the homotopy groups of Ω-spectra.

In this section and the next the following hypothesis will be assumed true.

4.0. Hypothesis for §§4 and 5. *For each component E_i of E we have that $\partial E_i = \varnothing$. Equivalently, each D_i (of 2.7.1) is a complete Riemannian manifold with sectional curvature $K \le 0$ everywhere. These two equivalent conditions are satisfied if M is a compact locally symmetric space with sectional curvature $K \le 0$.*

4.1. The maps g_i, t_i, $t_{i,j}$, s_i, and the foliations $\widehat{F}_i$, $F_{i,j}$. Recall that $f_i: E_i \to M$ denotes the composite map $E_i \subset RPM \xrightarrow{\text{proj}} M$, where RPM is the real projective bundle for M, and E_i comes from Theorem 2.4. Note that f_i is not in general an embedding, however it follows from Lemma 2.7.1 that f_i is always a smooth immersion when Hypothesis 4.0 holds. Choose, for a sufficiently large integer $k > 0$, a smooth embedding $g_i: E_i \to M \times I^k$ such that the following diagram is commutative:

$$(4.1.1)\qquad \begin{tikzcd} E_i \arrow[rr, "g_i"] \arrow[dr, "f_i"'] & & M \times I^k \arrow[dl, "\text{proj}"] \\ & M & \end{tikzcd}$$

Since we shall be dealing with the embedding $g_i\colon E_i \to M \times I^k$ a lot, we often identify E_i with its image under g_i and denote g_i by $E_i \subset M \times I^k$. Choose a tubular neighborhood $T_i \subset M \times I^k$ for $E_i \subset M \times I^k$ in $M \times I^k$ with projections $t_i\colon T_i \to E_i$. Let $E_{i,j}, T_{i,j} \subset \widehat{M} \times I^k$, $j = 1, 2, \cdots$, denote all the path components of the preimages of E_i, T_i under the universal covering projection $\widehat{M} \times I^k \to M \times I^k$, and let $t_{i,j}\colon T_{i,j} \to E_{i,j}$ denote the bundle projections which cover the projection $t_i\colon T_i \to E_i$. There will be no loss of generality in assuming that the $\{t_{i,j}\}$ have the following properties.

4.1.2. (a) The projection map $\widehat{M} \times I^k \to \widehat{M}$ maps $E_{i,1}$ diffeomorphically onto the subset $\widehat{E}_i \subset \widehat{M}$ and maps $T_{i,1}$ onto a tubular neighborhood S_i for $\widehat{E}_i$ in $\widehat{M}$ (cf. 2.7.1 for the embedding $\widehat{E}_i \subset \widehat{M}$).

(b) The orthogonal projection $s_i\colon S_i \to \widehat{E}_i$ is a bundle projection (cf. [3]). Moreover we have the following commutative diagram:

$$\begin{tikzcd} T_{i,1} \arrow[r, "t_{i,1}"] \arrow[d] & E_{i,1} \arrow[d] \\ S_i \arrow[r, "s_i"] & \widehat{E}_i \end{tikzcd}$$

Note that there are canonical covering projections $\widehat{E}_i \to E_i$ and $E_{i,j} \to E_i$. We denote by $\widehat{F}_i$ the foliation for $\widehat{E}_i$ which covers the foliation F_i of E_i; recall that the leaves of F_i are the fibers of $p_i\colon E_i \to G_i$. We denote by $F_{i,j}$ the foliation of $E_{i,j}$ which covers the foliation F_i of E_i.

4.2. Stratified flat bundles. Recall that Γ_i denotes the fundamental group $\pi_1(E_i)$. Note that Γ_i acts on $\widehat{E}_i$ via deck transformations for the universal covering projection $\widehat{E}_i \to E_i$. Since $\widehat{E}_i = D_i \times \mathbf{R}$, Γ_i also acts on $D_i \times \mathbf{R}$: in fact, it is a consequence of 2.7.1(b), (c) that the action $\Gamma_i \times (D_i \times \mathbf{R}) \to D_i \times \mathbf{R}$ is the diagonal action for two separate actions $\Gamma_i \times D_i \to D_i$ and $\Gamma_i \times \mathbf{R} \to \mathbf{R}$. It is a consequence of 2.6.1, 2.7.1, 2.9 that the bundle projection $p_i\colon E_i \to G_i$ is equal to the quotient of the projection $D_i \times \mathbf{R} \to D_i$ under the group actions $\Gamma_i \times (D_i \times \mathbf{R}) \to D_i \times \mathbf{R}$ and $\Gamma_i \times D_i \to D_i$.

A map $p^W : W \to G_i$ is called a *stratified flat bundle* over G_i if there is a group action $\Gamma_i \times X \to X$ on a space X such that the diagonal action $\Gamma_i \times (D_i \times X) \to D_i \times X$ is a free and properly discontinuous action, and such that p^W is equal to the quotient of the standard projection $D_i \times X \to D_i$ under the group actions $\Gamma_i \times D_i \to D_i$ and $\Gamma_i \times (D_i \times X) \to D_i \times X$.

We have just seen that $p_i : E_i \to G_i$ is an example of a stratified flat bundle with $X = \mathbf{R}$. Other examples can be constructed by choosing subsets $X \subset \widehat{M}$ (or $X \subset \widehat{M} \times I^k$) which are left invariant by the deck transformation action $\Gamma_i \times \widehat{M} \to \widehat{M}$ (or by $\Gamma_i \times (\widehat{M} \times I^k) \to \widehat{M} \times I^k$). Note that such Γ_i-invariant subsets can be obtained by starting with any subset $Y \subset \widehat{M}$ (or $Y \subset \widehat{M} \times I^k$) and setting $X = \Gamma_i Y$, where $\Gamma_i Y$ denotes the orbit of Y under the Γ_i action. We get the following useful examples of stratified flat bundles

$$
\begin{aligned}
p_i^A &: A_i \to G_i, \\
p_i^B &: B_i \to G_i, \\
p_{i,j,q}^T &: T_{i,j,q} \to G_i, \\
p_{i,j,q}^E &: E_{i,j,q} \to G_i, \\
p_{i,i}^S &: S_{i,i} \to G_i, \\
p_{i,i}^{\widehat{E}} &: \widehat{E}_{i,i} \to G_i,
\end{aligned}
\tag{4.2.1}
$$

by choosing $X = \widehat{M} \times I^k$, $\widehat{M}$, $\Gamma_i T_{j,q}$, $\Gamma_i E_{j,q}$, S_i, or $\widehat{E}_i$, respectively. We can add to this list the projection

$$p_i^C : C_i \to G_i, \tag{4.2.2}$$

by choosing $X = D_i$. The projection p_i^C is not a stratified flat bundle projection because the diagonal action $\Gamma_i \times (D_i \times D_i) \to D_i \times D_i$ is not a free action. However, p_i^C will have its uses in what is to come.

4.3. Flat foliations. Let $p^W : W \to G_i$ be a stratified flat bundle constructed from the group action $\Gamma_i \times X \to X$. Suppose that X is a manifold. Then a foliation $\mathscr{F}$ for W is a *flat foliation* if there is a foliation $\mathscr{G}$ for X whose leaves are permuted by the action $\Gamma_i \times X \to X$, and $\mathscr{F}$ is the quotient under the diagonal action $\Gamma_i \times (D_i \times X) \to D_i \times X$ of the foliation of $D_i \times X$ with leaves equal to $\{b \times L : b \in D_i,\ L \in \mathscr{F}\}$. For example we get the flat foliations

$$F_{i,j,q} \text{ of } E_{i,j,q} \quad \text{and} \quad \widehat{F}_{i,i} \text{ of } \widehat{E}_{i,i}, \tag{4.3.1}$$

by choosing $(X, \mathscr{F}) = (\Gamma_i E_{j,q}, \Gamma_i F_{j,q})$ or $(\widehat{E}_i, \widehat{F}_i)$, respectively. Note that each fiber of $E_{i,j,q}$ and $\widehat{E}_{i,i}$ is the union of leaves in $F_{i,j,q}$ and $\widehat{F}_{i,i}$, and is thus foliated by the restrictions of these foliations.

4.4. Flat mappings between flat bundles. Let $p^W : W \to G_i$ and $p^{W'} : W' \to G_i$ be two stratified flat bundles formed from the group actions $\Gamma_i \times X \to X$ and $\Gamma_i \times X' \to X'$. A mapping $g: W \to W'$ is called a *flat mapping* if there is a Γ_i-equivariant map $r: X \to X'$ such that g is the quotient of $1 \times r: D_i \times X \to D_i \times X'$ under the diagonal actions $\Gamma_i \times (D_i \times X) \to D_i \times X$ and $\Gamma_i \times (D_i \times X') \to D_i \times X'$. Using this construction we get the following examples of flat maps:

$$\begin{gathered} \hat{p}_{i,i}: \widehat{E}_{i,i} \to C_i, \\ s_{i,i}: S_{i,i} \to \widehat{E}_{i,i}, \\ t_{i,j,q}: T_{i,j,q} \to E_{i,j,q}, \\ \widehat{E}_{i,i} \subset S_{i,i} \subset B_i \quad \text{and} \quad E_{i,j,q} \subset T_{i,j,q} \subset A_i \end{gathered} \tag{4.4.1}$$

by choosing $r: X \to X'$ to be equal to $\hat{p}_i: \widehat{E}_i \to \widehat{G}_i$, $s_i: S_i \to \widehat{E}_i$, $\Gamma_i t_{j,q}: \Gamma_i T_{j,q} \to \Gamma_i E_{j,q}$, $\widehat{E}_i \subset S_i \subset \widehat{M}$, or $\Gamma_i E_{j,q} \subset \Gamma_i T_{j,q} \subset \widehat{M} \times I^k$, respectively. Here $\hat{p}_i: \widehat{E}_i \to \widehat{G}_i$ denotes the standard projection $D_i \times \mathbf{R} \to D_i$ (cf. 2.7.1). Note that strictly speaking, $\widehat{p}_{i,i}$ is not a flat mapping since $p_i^C : C_i \to G_i$ is not a stratified flat bundle.

There are also the useful (nonflat) maps

$$I_i^A : E_i \to A_i, \qquad I_i^B : E_i \to B_i \tag{4.4.2}$$

defined as follows. Let $r: \widehat{E}_i \to D_i \times (\widehat{M} \times I^k)$ be given by $r(y) = (\hat{p}_i(y), \hat{g}_i(y))$ for each $y \in \widehat{E}_i$, where $\hat{g}_i: \widehat{E}_i \to \widehat{M} \times I^k$ is a fixed covering for the map $g_i: E_i \to M \times I^k$ such that $\text{Image}(\hat{g}_i) = E_{i,1}$. Then define I_i^A to be the quotient of the map r under the action $\Gamma_i \times \widehat{E}_i \to \widehat{E}_i$ and the diagonal action $\Gamma_i \times (D_i \times (\widehat{M} \times I^k)) \to D_i \times (\widehat{M} \times I^k)$; and define I_i^B to be the composite map

$$E_i \xrightarrow{I_i^A} A_i = B_i \times I^k \xrightarrow{\text{proj}} B_i.$$

Note that any flat map $g: W \to W'$ between stratified flat bundles, or either of the maps in (4.4.2), is fiber preserving.

4.5. Metrics on the fibers of flat bundles. For any stratified flat bundle $p^W: W \to G_i$ and any point $b \in G_i$ we let W_b denote the fiber lying over b. For any flat map $g: W \to W'$ between stratified flat bundles over G_i, and for any point $b \in G_i$, we denote by $g_b: W_b \to W'_b$ the restriction of g to W_b. For example we have, for each $b \in G_i$, the maps $\hat{p}_{i,i,b}: \widehat{E}_{i,i,b} \to C_{i,b}$, $s_{i,i,b}: S_{i,i,b} \to \widehat{E}_{i,i,b}$, and $t_{i,j,q,b}: T_{i,j,q,b} \to E_{i,j,q,b}$ from (4.4.1). Likewise we let $\widehat{F}_{i,i,b}$ and $F_{i,j,q,b}$ denote the restrictions to $\widehat{E}_{i,i,b}$ and $E_{i,j,q,b}$ of the foliations $\widehat{F}_{i,i}$ and $F_{i,j,q}$.

There are metrics

$$
\begin{aligned}
&d^A_{i,b}(\ ,\ ) \text{ on the fiber } A_{i,b},\\
(4.5.1)\qquad &d^B_{i,b}(\ ,\ ) \text{ on the fiber } B_{i,b},\\
&d^C_{i,b}(\ ,\ ) \text{ on the fiber } C_{i,b}
\end{aligned}
$$

defined as follows. Note that there are canonical covering projections $A_{i,b} \to M \times I^k$ and $B_{i,b} \to M$ via which $A_{i,b}$ and $B_{i,b}$ inherit Riemannian structures from M and $M \times I^k$ which has the product metric. We let $d^A_{i,b}(\ ,\ )$ and $d^B_{i,b}(\ ,\ )$ denote the metrics associated to these inherited Riemannian structures. To get the metric $d^C_{i,b}(\ ,\ )$ on the fiber $C_{i,b}$ we set

$$d^C_{i,b}(x, y) = \text{minimum}\{d^D_i(\bar{x}, \bar{y})\}$$

for all $x, y \in C_{i,b}$, where the minimum is taken over all preimages $\bar{x}, \bar{y}$ of x, y under the canonical "covering projection" $D_i \to C_{i,b}$.

We will denote by

$$(4.5.2)\qquad d^{\widehat{E}}_{i,i,b}(\ ,\ ),\ d^S_{i,i,b}(\ ,\ ),\ d^E_{i,j,q,b}(\ ,\ ),\ d^T_{i,j,q,b}(\ ,\ ),\ \text{etc.}$$

the restriction of $d^B_{i,b}(\ ,\ )$, $d^A_{i,b}(\ ,\ )$ to the subsets $\widehat{E}_{i,i,b}, S_{i,i,b}, E_{i,j,q,b}, T_{i,j,q,b}$, etc. of B, A.

We can now state and prove the main two lemmas of this section.

4.6. Lemma. *Suppose that M is compact. Then for any i, any $b \in G_i$, any j and any q, the foliation $F_{i,j,q,b}$ of §§4.3 and 4.5 is of compact type (cf. 1.5.2 for "compact type") with respect to the metric $d^E_{i,j,q,b}$. Moreover, if $i \neq j$, or if $i = j$ but $q \neq 1$, then $F_{i,j,q,b}$ has no compact leaves.*

We will need the following notation in the next lemma. For each $\alpha > 0$ and each $b \in G_i$ we let $\widehat{E}_{i,i,b}$ denote the set of all points in $B_{i,b}$ having a distance, with respect to the metric $d^B_{i,b}(\ ,\ )$, less than or equal to α from $\widehat{E}^{\alpha}_{i,i,b}$.

4.7. Lemma. *Suppose that M is compact. Then given any number $r > 0$ there is a number $\alpha > 0$, such that for any i, any $b \in G_i$, and all $x \in B_{i,b} - \widehat{E}^{\alpha}_{i,i,b}$, the radius of injectivity for $B_{i,b}$ at x is greater than r.*

Proof of Lemma 4.6. Note that $F_{i,j,q,b}$ is a covering for the foliation F_j of the space E_j. Since E_j is compact (cf. 2.4), it follows that $F_{i,j,q,b}$ is of compact type.

In showing that $F_{i,j,q,b}$ has no compact leaves, we first consider the case where $i \neq j$. Let $\Gamma_{i,b'}$ denote the isotropy subgroup for the action $\Gamma_i \times D_i \to D_i$ at an arbitrary point b' in the preimage under the projection $D_i \to D_i/\Gamma_i = G_i$ of the point $b \in G_i$. Note that $\Gamma_{i,b'}$ is an infinite cyclic group with generator which we denote by $g_{b'}$ (cf. 2.4, 2.6.1, 2.7.1, 2.9); thus any compact leaf $L_1 \in F_{i,j,q,b}$ is in the same free homotopy class (in $A_{i,b}$) as $(g_{b'})^n$ for some integer n. Under the canonical covering projection $E_{i,j,q,b} \to E_j$, the leaf L_1 is mapped onto a leaf L_2 of F_j which is in the same free homotopy class (in M) as $(g_{b'})^m$ for some integer m. On the other hand, the leaf $E_{i,b}$ of F_i has the same free homotopy class (in M) as $g_{b'}$. Now it follows from the definition for F_i, F_j, E_i, E_j given in §2 (just prior to 2.4) that $E_i = E_j$, i.e., $i = j$, which contradicts our assumption that $i \neq j$.

Now we consider the case where $i = j$ but $q \neq 1$. For $g_{b'}$ as in the previous paragraph we know that any compact leaf $L_1 \in F_{i,i,q,b}$ is in the same free homotopy class (in $A_{i,b}$) as $(b_{b'})^n$. Note that $I_i^A(E_{i,b})$ is a compact leaf of $F_{i,i,1,b}$ in the same free homotopy class (in $A_{i,b}$) as $g_{b'}$ (cf (4.4.2) for I_i^A). Let $h: S^1 \times [0, 1] \to A_{i,b}$ be a homotopy which connects L_1 to a power of $I_i^A(E_{i,b})$, and let $\pi_{i,b}: A_{i,b} \to B_{i,b}$ denote the map $A_{i,b} = B_{i,b} \times I^k \xrightarrow{\text{proj}} B_{i,b}$. Apply Lemma 2.5.1 to $\pi_{i,b} \circ h: S^1 \times [0, 1] \to B_{i,b}$ to get a homotopy $(\pi_{i,b} \circ h)_t: S^1 \times [0, 1] \to B_{i,b}$, $t \in [0, 1]$, such that $(\pi_{i,b} \circ h)_1: S^1 \times [0, 1] \to B_{i,b}$ is a flat band. There is a unique lifting of $(\pi_{i,b} \circ h)_1: S^1 \times [0, 1] \to B_{i,b}$ to $H: S^1 \times [0, 1] \to A_{i,b}$ which satisfies the following properties: each $H \mid S^1 \times t$ parametrizes a leaf of $\bigcup_q F_{i,i,q,b}$; $H \mid S^1 \times 0$ parametrizes L_1; $H \mid S^1 \times 1$ parametrizes a power of $I_i^A(E_{i,b})$. It follows that $q = 1$, which contradicts our hypothesis that $q \neq 1$.

This completes the proof for Lemma 4.6.

Proof of Lemma 4.7. For a given $b \in G_i$ we suppose that there is no such $\alpha > 0$. Then for any given $\alpha > 0$ there is an essential smooth map

$h: S^1 \to B_{i,b}$ which has the following properties.

4.8. (a) $\text{length}(h) \le 2r$ in $B_{i,b}$.

(b) The distance from $h(1)$ to $\widehat{E}_{i,i,b}$ is greater than α in $B_{i,b}$.

Since any essential closed loop in a compact nonpositively curved manifold is homotopic to a closed geodesic (cf. [3] or [16]), from 4.8(a) and the compactness of M it follows that there is a homotopy $H: S^1 \times [0, 1] \to M$ of the composite map $S^1 \xrightarrow{h} B_{i,b} \xrightarrow{\text{proj}} M$ which has the following properties.

4.9. (a) H is a smooth map, and the lengths of $H \mid S^1 \times 1$ and $H \mid 1 \times [0, 1]$ are both less than r', where r' is independent of α.

(b) $H \mid S^1 \times 1$ is a geodesic in M.

Choose a lifting $\widehat{H}: S^1 \times [0, 1] \to B_{i,b}$ of H such that $H \mid S^1 \times 0 = h$. Then we deduce from 4.8, 4.9 that the following is true.

4.10. There is a closed geodesic in $B_{i,b} - E^{\alpha-2r'}_{i,i,b}$, where r' is independent of α.

On the other hand, there is the closed geodesic $I_i^B(E_{i,b})$ in $\widehat{E}_{i,i,b}$ (cf. (4.4.2) for I_i^B); note that $I_i^B(E_{i,b})$ is also a compact leaf in $\widehat{F}_{i,i,b}$. The fundamental group $\pi_1(B_{i,b})$ is an infinite cyclic group with generator $g_{b'}$ (cf. the proof of 4.6 and note that $A_{i,b} = B_{i,b} \times I^k$); moreover any closed path representing $g_{b'}$ is freely homotopic to $I_i^B(E_{i,b})$ in $B_{i,b}$. The closed geodesic of 4.10 must be freely homotopic to $(g_{b'})^n$ for some integer n. Thus by applying Lemma 2.5.1 in $B_{i,b}$ to the homotopy which connects the geodesic of 4.10 to a power of $I_i^B(E_{i,b})$ we see that the geodesic of 4.10 must be a leaf of $\widehat{F}_{i,i,b}$, which contradicts 4.10.

We have shown that for each $b \in G_i$ and each $r > 0$ there is an α satisfying the conclusions of Lemma 4.7. To see that α may be chosen independent of $b \in G_i$ note that, for a fixed i, there is only a finite number of different isometry types for the pairs $(B_{i,b}, \widehat{E}_{i,i,b})$, $b \in G_i$, with respect to the metric $d^B_{i,b}$ given in (4.5.1).

This completes the proof of Lemma 4.7.

4.11. One-parameter families of homeomorphisms $\phi_t: B_i \to B_i$ and $\psi_t: C_i \to C_i$, $t \in (0, 1]$. For each $b \in G_i$ let $\phi_{t,b}: B_{i,b} \to B_{i,b}$, $t \in [0, 1]$, be the unique map satisfying the following properties for each point $x \in B_{i,b}$: $d^B_{i,b}(\phi_{t,b}(x), \widehat{E}_{i,i,b}) = t d^B_{i,b}(x, \widehat{E}_{i,i,b})$; let $g: [0, \infty) \to B_{i,b}$ be the geodesic ray which starts at the point $g(0) \in B_{i,b}$ and contains both x and $\phi_{t,b}(x)$. Then g meets $\widehat{E}_{i,i,b}$ perpendicularly at $g(0)$.

Note that each $\phi_{t,b}\colon B_{i,b} \to B_{i,b}$, $t \in (0, 1]$, is a homeomorphism, and that $\phi_{0,b}\colon B_{i,b} \to \widehat{E}_{i,i,b}$ is the orthogonal projection onto $\widehat{E}_{i,i,b}$. In order to verify that $\phi_{t,b}\colon B_{i,b} \to B_{i,b}$ is a well-defined map for all $t \in [0, 1]$ we must use an equivariant version of [3, pp. 8–10] as well as all of the following properties: $B_{i,b}$ is a complete Riemannian manifold with nonpositive curvature everywhere; $\widehat{E}_{i,i,b}$ is a closed and connected locally convex subset of $B_{i,b}$; the inclusion map $E_{i,i,b} \subset B_{i,b}$ induces an isomorphism of fundamental groups.

Although $C_{i,b}$ need not be a manifold we can still define a geodesic in $C_{i,b}$ to be the images of geodesics in D_i under the "canonical covering" projection $D_i \to C_{i,b}$. For each $b \in G_i$ let $\psi_{t,b}\colon C_{i,b} \to C_{i,b}$, $t \in [0, 1]$, be the unique map which satisfies the following properties for each point $x \in C_{i,b}$: $d^C_{i,b}(\psi_{t,b}(x), \hat{p}_{i,i}(I^B_i(E_{i,b}))) = td^C_{i,b}(x, \hat{p}_{i,i}(I^B_i(E_{i,b})))$; the geodesic which connects x to $\hat{p}_{i,i}(I^B_{i,b}(E_{i,b}))$ contains the point $\psi_{t,b}(x)$. Note that $C_{i,b}$ is the orbit space of a finite cyclic group action $\mathbf{Z}_m \times D_i \to D_i$ by isometries (cf. 2.7.1, 2.7.2 and use the fact that $\pi_1(\widehat{E}_{i,i,b})$ is an infinite cyclic group); moreover there is a fixed point of the action $\mathbf{Z}_m \times D_i \to D_i$ which is sent to $\hat{p}_{i,i}(I^B_i(E_{i,b}))$ under the quotient map $D_i \to D_i/\mathbf{Z}_m = C_{i,b}$; thus by Hypothesis 4.0 there is for any point $x \in C_{i,b}$ a unique geodesic segment in $C_{i,b}$ connecting x to $\hat{p}_{i,i}(I^B_i(E_{i,b}))$. Note that each $\psi_{t,b}\colon C_{i,b} \to C_{i,b}$, $t \in (0, 1]$, is a homeomorphism, and that $\mathrm{Image}(\psi_{0,b}) = \hat{p}_{i,i}(I^B_i(E_{i,b}))$.

Now set

$$\phi_t = \bigcup_{b \in G_i} \phi_{t,b}, \qquad \psi_t = \bigcup_{b \in G_i} \psi_{t,b}.$$

4.11.1. Lemma. *For all i, $b \in G_i$, $x, x' \in B_{i,b}$, $y, y' \in C_{i,b}$, and for all $t \in [0, 1]$, the following are true.*

(a) $d^B_{i,b}(x, x') \geq d^B_{i,b}(\phi_t(x), \phi_t(x'))$.

(b) $td^C_{i,b}(y, y') \geq d^C_{i,b}(\psi_t(y), \psi_t(y'))$.

(c) *Let $q_{i,i,b}\colon B_{i,b} \to C_{i,b}$ denote the composite map $B_{i,b} \xrightarrow{\phi_{0,b}} \widehat{E}_{i,i,b} \xrightarrow{\hat{p}_{i,i,b}} C_{i,b}$. Then $d^C_{i,b}(q_{i,i,b}(x), q_{i,i,b}(x')) \leq d^B_{i,b}(x, x')$.*

Proof of Lemma 4.11.1. To prove part (a) we use the convexity of the distance function $d^B_{i,b}\colon B_{i,b} \times B_{i,b} \to [0, \infty)$, together with the following fact: if $g, f\colon \mathbf{R} \to B_{i,b}$ are two geodesics in $B_{i,b}$ which meet $\widehat{E}_{i,i,b}$

perpendicularly at $g(0)$, $f(0)$, then the minimal distance from Image(g) to Image(f) is equal to $d^B_{i,b}(g(0), f(0))$. This last condition is equivalent to the fact that the orthogonal projection $\phi_{0,b}\colon B_{i,b} \to \widehat{E}_{i,i,b}$ is not distance increasing (cf. [3, pp. 8–10]).

To prove part (b) we recall that D_i is a complete Riemannian manifold having nonpositive sectional curvature everywhere (cf. Hypothesis 4.0), and $C_{i,b}$ is the orbit space of a finite cyclic group action $\mathbf{Z}_m \times D_i \to D_i$ by isometries. If the cyclic action $\mathbf{Z}_m \times D_i \to D_i$ were trivial, then 4.11.1(b) would follow from Lemma 2.1(a). In general there is a one-parameter family of maps $\hat{\psi}_{t,b}\colon D_i \to D_i$, $t \in [0, 1]$, which commute with the action $\mathbf{Z}_m \times D_i \to D_i$, such that the $\psi_{t,b}$ are just the quotients of the $\hat{\psi}_{t,b}$ under the $\mathbf{Z}_m$-action. Now 4.11.1(b) holds for the $\hat{\psi}_{t,b}$ (by 2.1(a)), so 4.11.1(b) must also hold for the $\psi_{t,b}$.

Part (c) of this lemma follows from the fact that the orthogonal projection $\phi_{0,b}\colon B_{i,b} \to \widehat{E}_{i,i,b}$ satisfies $d^{\widehat{E}}_{i,i,b}(\phi_{0,b}(x), \phi_{0,b}(x')) \le d^B_{i,b}(x, x')$ (cf. [3]), and the map $\hat{p}_{i,i,b}\colon \widehat{E}_{i,i,b} \to C_{i,b}$ satisfies $d^C_{i,b}(\hat{p}_{i,i,b}(z), \hat{p}_{i,i,b}(z')) \le d^{\widehat{E}}_{i,i,b}(z, z')$ for all $z, z' \in \widehat{E}_{i,i,b}$ (cf. 2.7.1).

This completes the proof of Lemma 4.11.1.

5. Some equivalences of Ω-spectra

In this section we introduce more stable pseudoisotopy Ω-spectra. The main results of this section state that various of these Ω-spectra are equivalent to one another (cf. 5.3, 5.5, 5.9).

In the rest of this paper we adhere strictly to the following convention.

5.0. Convention for §§5 and 6. *Any spectrum $\mathcal{S}_* = \{\mathcal{S}_j : j \in \mathbf{Z}\}$ with structure maps $\{h_j\colon \mathcal{S}_j \to \Omega\mathcal{S}_{j+1}\}$ will be identified with the singular complexes of the spaces $\{\mathcal{S}_j : j \in \mathbf{Z}\}$ and with the maps induced on the singular complexes by the structure maps. Thus if each $h_j\colon \mathcal{S}_j \to \Omega\mathcal{S}_{j+1}$ induces an isomorphism on homotopy groups, then $\mathcal{S}_*$ is an Ω-spectrum, and also any weak equivalence of Ω-spectra is an equivalence of Ω-spectra.*

5.1. The Ω-spectra $\widetilde{\mathscr{P}}_*(p^W)$ and $\widetilde{\mathscr{P}}^b_*(p^W)$. Let $p^W\colon W \to G_i$ denote any stratified flat bundle of §4.2. For each integer $j \ge 0$ we shall denote by $\widetilde{\mathscr{P}}_j(p^W)$ the subspace of all stable pseudoisotopies $h \in \mathscr{P}_j(W)$ which has the following property: $h(W_b \times \mathbf{R}^j \times I^n \times [0, 1]) \subset W_b \times \mathbf{R}^j \times I^n \times [0, 1]$ holds for all $b \in G_i$, where $W_b = (p^W)^{-1}(b)$, and $W \times \mathbf{R}^j \times I^n \times [0, 1]$ is the domain of h.

Each of the fibers W_b of the projection p^W is provided with a metric $d_b^W(\ ,\ )$ in §4.5. For each number $\alpha > 0$ and each integer $j \geq 0$ we let $\widetilde{\mathscr{P}}_j(p^W;\alpha)$ denote the subspace of all stable pseudoisotopies $h \in \widetilde{\mathscr{P}}_j(p^W)$ such that for all $b \in G_i$ the restricted stable pseudoisotopy $h \mid W_b \times \mathbf{R}^j \times I^n \times [0, 1]$ is α-controlled over $W_b \times \mathbf{R}^j$ (cf. §1.2 for "control"). We define $\widetilde{\mathscr{P}}_j^b(p^W)$ to be the direct limit space $\operatorname{limit}_{\alpha\to\infty} \widetilde{\mathscr{P}}_j^b(p^W;\alpha)$. For each integer $j < 0$ we set $\widetilde{\mathscr{P}}_j(p^W) = \Omega^{-j}(\widetilde{\mathscr{P}}_0(p^W))$ and $\widetilde{\mathscr{P}}_j^b(p^W) = \Omega^{-j}(\widetilde{\mathscr{P}}_0^b(p^W))$. We let $\widetilde{\mathscr{P}}_*(p^W)$ and $\widetilde{\mathscr{P}}_*^b(p^W)$ denote the collection of spaces $\{\widetilde{\mathscr{P}}_j(p^W) : j \in \mathbf{Z}\}$ and $\{\widetilde{\mathscr{P}}_j^b(p^W) : j \in \mathbf{Z}\}$.

5.2. Lemma. *$\widetilde{\mathscr{P}}_*(p^W)$ and $\widetilde{\mathscr{P}}_*^b(p^W)$ are Ω-spectra.*

Proof of Lemma 5.2. Note that it follows from Theorem 2.4 and Hypothesis 4.0 that there is a finite triangulation L for G_i which has the following properties.

5.2.1. (a) Let $G_{i,k}$ denote the union of all strata in G_i having dimension less than or equal to k. Then for each simplex $\Delta \in L$, $\Delta \cap G_{i,k}$ is also a simplex of L.

(b) There is a piecewise smooth triangulation $\widehat{L}$ of D_i such that the projection $D_i \to G_i$ maps each simplex of $\widehat{L}$ homeomorphically onto a simplex of L.

We will first show that $\widetilde{\mathscr{P}}_*(p^W)$ is a Ω-spectrum. To do this it will suffice to show that for each integer $j \geq 0$ the spaces $\widetilde{\mathscr{P}}_j(p^W)$ and $\Omega\widetilde{\mathscr{P}}_{j+1}(p^W)$ are homotopy equivalent. Note that for each subset $K \subset G_i$ we can define $\widetilde{\mathscr{P}}_*(p_K^W)$ as in §5.1, where $p_K^W : W_K \to K$ denotes the restriction of p^W to the subset $W_K \subset W$, and where $W_K = (p^W)^{-1}(K)$. For each $j \geq 0$ there is a map $f_{K,j} : \widetilde{\mathscr{P}}_j(p_K^W) \to \Omega\widetilde{\mathscr{P}}_{j+1}(p_K^W)$ defined as follows. Let $i : \widetilde{\mathscr{P}}_j(p_K^W) \to \widetilde{\mathscr{P}}_{j+1}(p_K^W)$ be induced by the inclusion $[0, 1] \subset \mathbf{R}$. Two null homotopies of i are obtained by translating $[0, 1]$ to $+\infty$ and to $-\infty$. This defines for each $h \in \widetilde{\mathscr{P}}_j(p_K^W)$ the loop $f_{K,j}(h)$ of stable pseudoisotopies in $\widetilde{\mathscr{P}}_{j+1}(p_K^W)$.

A. Hatcher has proven in [17, Appendix II] that $f_{K,j}$ is a homotopy equivalence if K is any point of G_i. Thus we may proceed by induction over the skeleta of L to show that $f_{K,j}$ is a homotopy equivalence for each subcomplex $K \subset L$. In more detail let $K_1 \subset K_2 \subset K_3 \subset \cdots \subset K_m = L$ be subcomplexes of L such that $K_{r+1} = K_r \cup \Delta_r$, where Δ_r is a simplex of L such that $\dim(\Delta_r) \geq \dim(K_r)$ and $\partial\Delta_r \subset K_r$. For the induction step we consider the following commutative diagram:

$$\begin{tikzcd}
\Omega^{a_r+1}\widetilde{\mathscr{P}}_{j+1}(p^W_{b_r}) \arrow[r] & \Omega\widetilde{\mathscr{P}}_{j+1}(p^W_{K_{r+1}}) \arrow[r, "\Omega\phi_{r+1,j}"] & \Omega\widetilde{\mathscr{P}}_{j+1}(p^W_{K_r}) \\
\Omega^{a_r}\widetilde{\mathscr{P}}_j(p^W_{b_r}) \arrow[r] \arrow[u, "\Omega^{a_r}(f_{b_r,j})"] & \widetilde{\mathscr{P}}_j(p^W_{K_{r+1}}) \arrow[r, "\phi_{r+1,j}"'] \arrow[u, "f_{K_{r+1},j}"] & \widetilde{\mathscr{P}}_j(p^W_{K_r}) \arrow[u, "f_{K_r,j}"]
\end{tikzcd} \tag{5.2.2}$$

The map $\phi_{r+1,j}$ in (5.2.2) is obtained by restriction; in (5.2.2) the rows are fibrations, the map $f_{K_r,j}$ is an equivalence by induction hypothesis, and the map $\Omega^{a_r}(f_{b_r,j})$ is an equivalence by Hatcher's result, where $a_r = \dim(\Delta_r)$, and b_r is the barycenter of Δ_r. Thus, $f_{K_{r+1},j}$ is also an equivalence (cf. Convention 5.0).

This completes the proof of Lemma 5.2 for $\widetilde{\mathscr{P}}_*(p^W)$. The proof for $\widetilde{\mathscr{P}}^b_*(p^W)$ is carried out in exactly the same way.

This completes the proof of Lemma 5.2.

5.3. Lemma. *There is an equivalence of Ω-spectra $\widetilde{\mathscr{P}}^b_*(p^A_i) \cong \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$, which is induced by the inclusion map $T_{i,i,1} \subset A_i$. Here $p^A_i\colon A_i \to G_i$ and $p^T_{i,i,1}\colon T_{i,i,1} \to G_i$ are the stratified flat bundles of* §4.2.

Proof of Lemma 5.3. First note that $\widetilde{\mathscr{P}}^b_*(p^A_i) = \widetilde{\mathscr{P}}^b_*(p^B_i)$ because p^A_i is just the composite map $A_i = B_i \times I^k \xrightarrow{\text{proj}} B_i \xrightarrow{p^B_i} G_i$. We also have an equivalence of Ω-spectra $\widetilde{\mathscr{P}}^b_*(p^T_{i,i,1}) \cong \widetilde{\mathscr{P}}^b_*(p^S_{i,i})$. To verify this last equivalence we note that the inclusions $\widehat{E}_{i,i} \subset S_{i,i}$ and $E_{i,i,1} \subset T_{i,i,1}$ induce equivalences of Ω-spectra $\widetilde{\mathscr{P}}^b_*(p^{\widehat{E}}_{i,i}) \cong \widetilde{\mathscr{P}}^b_*(p^S_{i,i})$ and $\widetilde{\mathscr{P}}^b_*(p^E_{i,i,1}) \cong \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$, because $S_{i,i}$ is the fiberwise tubular neighborhood for $\widehat{E}_{i,i}$ in B_i, and $T_{i,i,1}$ is the fiberwise tubular neighborhood for $E_{i,i,1}$ in A_i. Moreover, the restriction of the projection $A_i = B_i \times I^k \xrightarrow{\text{proj}} B_i$ to $E_{i,i,1}$ yields a diffeomorphism $E_{i,i,1} \to \widehat{E}_{i,i}$ (cf. 4.1.2), showing that $\widetilde{\mathscr{P}}^b_*(p^{\widehat{E}}_{i,i}) \cong \widetilde{\mathscr{P}}^b_*(p^E_{i,i,1})$. Thus to complete the verification of 5.3 it will suffice to show that the inclusion map $S_{i,i} \subset B_i$ induces an equivalence of Ω-spectra $\widetilde{\mathscr{P}}^b_*(p^S_{i,i}) \cong \widetilde{\mathscr{P}}^b_*(p^B_i)$.

Let L be a triangulation for G_i as in 5.2.1. For each subcomplex $K \subset L$ the inclusion $S_{i,i,K} \subset B_{i,K}$ induces maps of Ω-spectra $g_{K,*}\colon \widetilde{\mathscr{P}}^b_*(p^S_{i,i,K}) \to \widetilde{\mathscr{P}}^b_*(p^B_{i,K})$, where $S_{i,i,K} = (p^S_{i,i})^{-1}(K)$, $p^S_{i,i,K} = p^S_{i,i} \mid S_{i,i,K}$, $B_{i,K} = (p^B_i)^{-1}(K)$, and $p^B_{i,K} = p^B_i \mid B_{i,K}$. To complete the proof of 5.3 it is enough to show that for each subcomplex $K \subset L$ the map $g_{K,*}$ is an equivalence of Ω-spectra. On the other hand the argument given in the proof of 5.2 shows that it suffices to consider only those K which are single vertices of L. Let $b \in G_i$ denote a vertex of L. For any number $\alpha > 0$ we will denote by $p_{i,i}^{\widehat{E}^\alpha}: \widehat{E}_{i,i}^\alpha \to G_i$ the stratified flat subbundle of $p_i^B: B_i \to G_i$ whose fiber $\widehat{E}_{i,i,x}^\alpha$ over any point $x \in G_i$ consists of all points in $B_{i,x}$ which are at a distance less than or equal to α from $\widehat{E}_{i,i,x}$ in $(B_{i,x}, d_{i,x}^B(\ ,\ ))$. Now we apply 4.7 and 3.17.2 (see also Convention 5.0) to get a deformation of $\widetilde{\mathscr{P}}_*^b(p_{i,b}^B)$ into its direct limit subspace $\operatorname{limit}_{\alpha\to\infty} \widetilde{\mathscr{P}}_*^b(p_{i,i,b}^{\widehat{E}^\alpha})$.

Thus, to complete the proof of 5.3, it will now suffice to show that the inclusion map $\widetilde{\mathscr{P}}_*^b(p_{i,i,b}^S) \to \operatorname{limit}_{\alpha\to\infty} \widetilde{\mathscr{P}}_*^b(p_{i,i,b}^{\widehat{E}^\alpha})$ is an equivalence. To verify this we shall make the assumption that there is a sufficiently small number $\varepsilon > 0$ such that $S_{i,i,x} = \widehat{E}_{i,i,x}^\varepsilon$ holds for all $x \in G_i$; there is no loss of generality in making this assumption (cf. 4.1.2). For each $\alpha > \varepsilon$ a homotopy inverse to the inclusion $\widetilde{\mathscr{P}}_*^b(p_{i,i,b}^S) \to \widetilde{\mathscr{P}}_*^b(p_{i,i,b}^{\widehat{E}^\alpha})$ can be defined by sending any stable pseudoisotopy $h: \widehat{E}_{i,i,b}^\alpha \times \mathbf{R}^j \times I^n \times [0, 1] \to \widehat{E}_{i,i,b}^\alpha \times \mathbf{R}^j \times I^n \times [0, 1]$ in $\widetilde{\mathscr{P}}_*^b(p_{i,i,b}^{\widehat{E}^\alpha})$ to the stable pseudoisotopy $(\phi_t \times \mathrm{id}) \circ h \circ (\phi_t^{-1} \times \mathrm{id})$ in $\widetilde{\mathscr{P}}_*^b(p_{i,i,b}^S)$, where $t = \varepsilon\alpha^{-1}$, $\phi_t: B_{i,b} \to B_{i,b}$ comes from §4.11, and $\mathrm{id}: \mathbf{R}^j \times I^n \times [0, 1] \to \mathbf{R}^j \times I^n \times [0, 1]$ is the identity map. Thus 4.11.1(a) guarantees that $(\phi_t \times \mathrm{id}) \circ h \circ (\phi_t^{-1} \times \mathrm{id})$ is in $\widetilde{\mathscr{P}}_*^b(p_{i,i,b}^S)$.

This completes the proof of Lemma 5.3.

5.4. The Ω-spectrum $\widetilde{\mathscr{P}}_*(p_{i,i,1}^T; q_i)$. In the remainder of this section we let $q_i: T_{i,i,1} \to C_i$ denote the composite map

$$T_{i,i,1} \subset A_i = B_i \times I^k \xrightarrow{\text{proj}} B_i \supset S_{i,i} \xrightarrow{s_{i,i}} \widehat{E}_{i,i} \xrightarrow{\hat{p}_{i,i}} C_i,$$

where $s_{i,i}$ and $\hat{p}_{i,i}$ come from §4.4. For each integer $j \geq 0$ we define $\widetilde{\mathscr{P}}_j(p_{i,i,1}^T; q_i)$ to be the space of all maps $g: [0, \infty) \to \widetilde{\mathscr{P}}_j^b(p_{i,i,1}^T)$ which have the following property.

5.4.1. There is a number $\alpha > 0$ depending on g such that for all $t \geq 0$ and all $b \in G_i$ the restricted stable pseudoisotopy $g(t): T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1] \to T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1]$ is $\alpha(1+t)^{-1}$-controlled over $C_{i,b}$ with respect to the projection $q_{i,b}: T_{i,i,1,b} \to C_{i,b}$ and the metric $d_{i,b}^C(\ ,\ )$ of §4.5 (cf. §1.2 for "control").

For each integer $j < 0$ we set $\widetilde{\mathscr{P}}_j(p^T_{i,i,1}; q_i) = \Omega^{-j}\widetilde{\mathscr{P}}_0(p^T_{i,i,1}; q_i)$. Let $\widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q_i)$ denote the collection $\{\widetilde{\mathscr{P}}_j(p^T_{i,i,1}; q_i) : j \in \mathbf{Z}\}$.

5.4.2. Lemma. *$\widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q_i)$ is an Ω-spectrum.*

Proof of Lemma 5.4.2. The proof consists of the argument given for the proof of Lemma 5.2 but with the following change. Where in the proof of 5.2 we appeal to a result of Hatcher [17], we now substitute an appeal to a result of Quinn [25, Theorem 5.9]. Recall that although Quinn's result is stated for spaces of stable embeddings, it also holds for spaces of stable pseudoisotopies (cf. Lemma 1.3.1). The remaining details are left to the reader.

This completes the proof of Lemma 5.4.2.

5.5. Lemma. *The map $r_*: \widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q_i) \to \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$, which sends each $g: [0, \infty) \to \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$ of* 5.4.1 *to the stable pseudoisotopy $g(0) \in \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$, is an equivalence of Ω-spectra.*

Proof of Lemma 5.5. We shall complete the proof by constructing a homotopy inverse r'_* to r_* in the category of Ω-spectra. Let $\psi_t: C_i \to C_i$, $t \in (0, 1]$, be the one-parameter family of homeomorphisms from §4.11. Choose "liftings" $\hat{\psi}_t: T_{i,i,1} \to T_{i,i,1}$, $t \in (0, 1]$, of the ψ_t, $t \in (0, 1]$, so that the following properties are satisfied.

5.5.1. (a) $\hat{\psi}_t: T_{i,i,1} \to T_{i,i,1}$ is a homeomorphism for each value of $t \in (0, 1]$, and is continuous in t.

(b) $q_i \circ \hat{\psi}_t = \psi_t \circ q_i$ holds for all $t \in (0, 1]$.

Now we construct a map $r'_*: \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1}) \to \widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q)$ as follows. For each stable pseudoisotopy $h: T_{i,i,1} \times \mathbf{R}^j \times I^n \times [0, 1] \to T_{i,i,1} \times \mathbf{R}^j \times I^n \times [0, 1]$ in $\widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$ we define $r'_*(h): [0, \infty) \to \widetilde{\mathscr{P}}^b_*(p^T_{i,i,1})$ to be the map whose value at $t \in [0, \infty)$ is equal to the stable pseudoisotopy $(\hat{\psi}_{t'} \times \mathrm{id}) \circ h \circ (\hat{\psi}_{t'}^{-1} \times \mathrm{id})$, where $\mathrm{id}: \mathbf{R}^j \times I^n \times [0, 1] \to \mathbf{R}^j \times I^n \times [0, 1]$ is the identity map and $t' = (1+t)^{-1}$. Thus from 4.11.1(b), 5.4.1, and 5.5.1 it follows that r'_* is a well-defined map with range equal to $\widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q_i)$. The map r'_* is in fact a homotopy inverse to r_*, as the reader can easily check.

This completes the proof of Lemma 5.5.

5.6. Definition. A *compact submanifold pair in* G_i consists of a pair of subsets $(X, \partial X)$ of G_i, which have the following properties.

(a) X and ∂X are compact subsets of G_i, and $\partial X \subset X$.

(b) Let $\widehat{X}$ and $\partial \widehat{X}$ denote the preimage of X and ∂X under the projection map $D_i \to D_i/\Gamma_i = G_i$. Then $(\widehat{X}, \partial \widehat{X})$ is a piecewise smooth submanifold pair of D_i (recall that D_i is a smooth manifold by 2.7.1 and 4.0) which intersects all the fixed point strata of the group action $\Gamma_i \times D_i \to D_i$ transversely.

The *codimension of X in G_i* is defined to be the codimension of $\widehat{X}$ in D_i, and ∂X is called the *boundary* of X.

5.7. The Ω-spectrum $\widetilde{\mathscr{P}}_*(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$. Let $(Z, \partial Z) \subset G_i$ be a compact codimension-zero submanifold pair of G_i such that $\partial Z = \partial_1 Z \cup \partial_2 Z$ and $\Lambda Z = \partial_1 Z \cap \partial_2 Z$, where $(\partial_1 Z, \Lambda Z)$ and $(\partial_2 Z, \Lambda Z)$ are compact codimension-one submanifolds of G_i, and ΛZ is a compact codimension-two submanifold of G_i without boundary. Set

$$p^{T,1}_{i,i,1} = p^T_{i,i,1} \mid (p^T_{i,i,1})^{-1}(Z - \partial_1 Z),$$
$$p^{T,2}_{i,i,1} = p^T_{i,i,1} \mid (p^T_{i,i,1})^{-1}(\partial_2 Z - \Lambda Z).$$

For each integer $j \geq 0$ we define $\widetilde{\mathscr{P}}_j(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ to be the space of all maps $g\colon [0, \infty) \to \widetilde{\mathscr{P}}^b_j(p^{T,1}_{i,i,1})$ which satisfy the following properties.

5.7.1. (a) There is a number $\alpha > 0$ depending on g such that 5.4.1 is satisfied for all $t \in [0, \infty)$ and all $b \in (Z - \partial_1 Z)$.

(b) For each $b \in (\partial_2 Z - \Lambda Z)$ and all $t \in [0, \infty)$, the restricted stable pseudoisotopy $g(t)\colon T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1] \to T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1]$ is the identity map.

For each integer $j < 0$ set

$$\widetilde{\mathscr{P}}_j(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i) = \Omega^{-j}\widetilde{\mathscr{P}}_0(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i),$$

$\widetilde{\mathscr{P}}_*(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i) = \{\widetilde{\mathscr{P}}_j(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i) : j \in \mathbf{Z}\}$.

The following lemma is proven by using the arguments contained in the proofs of Lemmas 5.2 and 5.4.2.

5.7.2. Lemma. *$\widetilde{\mathscr{P}}_*(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ is an Ω-spectrum.*

5.8. The Ω-spectrum $\widetilde{\mathscr{E}}_*(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$. Let $s\colon (Z - \partial_1 Z) \to (0, \infty)$ be a given continuous map. For each $b \in (Z - \partial_1 Z)$ let $V_b \subset C_{i,b}$ denote the closed ball of radius $s(b)$ centered at the point $\hat{p}_{i,i}(I^B_i(E_{i,b}))$ in $C_{i,b}$. Set $U_b = q_1^{-1}(V_b)$, and $U = \bigcup_{b \in (Z - \partial_1 Z)} U_b$.

Roughly speaking the space $\widetilde{\mathscr{E}}_*(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ is defined just as was the space $\widetilde{\mathscr{P}}_*(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$, except all relevant pseudoisotopies have the restricted domain $U \times \mathbf{R}^j \times I^n \times [0, 1]$ instead of all of $(p^T_{i,i,1})^{-1} \times (Z - \partial_1 Z) \times \mathbf{R}^j \times I^n \times [0, 1]$ for domain, but they still have $(p^T_{i,i,1})^{-1} \times (Z - \partial_1 Z) \times \mathbf{R}^j \times I^n \times [0, 1]$ for range. In more detail, for each number

$\alpha > 0$ and each integer $j \geq 0$ let $\widetilde{\mathscr{E}}_j(U, p^{T,1}_{i,i,1}; \alpha)$ denote the space of all "stable" embeddings $h: U \times \mathbf{R}^j \times I^n \times [0, 1] \to (p^T_{i,i,1})^{-1}(Z - \partial_1 Z) \times \mathbf{R}^j \times I^n \times [0, 1]$ which have the following properties.

5.8.1. (a) $h(U_b \times \mathbf{R}^j \times I^n \times [0, 1]) \subset T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1]$ and $h((U_b \cap \partial T_{i,i,1,b}) \times \mathbf{R}^j \times I^n \times [0, 1]) \subset \partial T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1]$ hold for all $b \in (Z - \partial_1 Z)$.

(b) For each $b \in (Z - \partial_1 Z)$ and each $y \in U_b \times \mathbf{R}^j \times I^n$ the composite path

$$[0, 1] = y \times [0, 1] \subset U_b \times \mathbf{R}^j \times I^n \times [0, 1]$$
$$\xrightarrow{h} T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times [0, 1] \xrightarrow{\text{proj}} T_{i,i,1,b} \times \mathbf{R}^j$$

must have diameter less than or equal to α in $A_{i,b} \times \mathbf{R}^j$, where $A_{i,b} \times \mathbf{R}^j$ is equipped with the product of the metric $d^A_{i,b}(\ ,\ )$ from §4.5 and with the Euclidean metric on $\mathbf{R}^j$.

(c) $h \mid U_b \times \mathbf{R}^j \times I^n \times 0 =$ inclusion; $h(U_b \times \mathbf{R}^j \times I^n \times 1) \subset T_{i,i,1,b} \times \mathbf{R}^j \times I^n \times 1$ and $h(U_b \times \mathbf{R}^j \times \partial I^n \times [0, 1]) \subset T_{i,i,1,b} \times \mathbf{R}^j \times \partial I^n \times [0, 1]$ hold for all $b \in (Z - \partial_1 Z)$.

Let $\widetilde{\mathscr{E}}^b_j(U, p^{T,1}_{i,i,1})$ denote the direct limit space $\text{limit}_{\alpha \to \infty} \widetilde{\mathscr{E}}_j(U, p^{T,1}_{i,i,1}; \alpha)$, and $\widetilde{\mathscr{E}}_j(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ the space of all maps

$$g: [0, \infty) \to \widetilde{\mathscr{E}}^b_j(U, p^{T,1}_{i,i,1})$$

which satisfy properties analogous to 5.7.1. Let $\widetilde{\mathscr{E}}_*(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ denote the collection of spaces $\{\widetilde{\mathscr{E}}_j(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i) : j \in \mathbf{Z}\}$, where $\widetilde{\mathscr{E}}_j(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ for $j < 0$ is equal to $\Omega^{-j}\widetilde{\mathscr{E}}_0(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$.

The next lemma is proven by using arguments similar to those contained in the Proofs of Lemmas 5.2 and 5.4.2.

5.8.2. Lemma. *$\widetilde{\mathscr{E}}_*(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$ is an Ω-spectrum.*

Note that by restricting stable pseudoisotopies defined on $(p^T_{i,i,1})^{-1} \times (Z - \partial_1 Z) \times \mathbf{R}^j \times I^n \times [0, 1]$ to the subspace $U \times \mathbf{R}^j \times I^n \times [0, 1]$ we obtain a map $w_*: \widetilde{\mathscr{P}}_*(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i) \to \widetilde{\mathscr{E}}_*(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$.

5.9. Lemma. *The map*

$$w_*: \widetilde{\mathscr{P}}_*(p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i) \to \widetilde{\mathscr{E}}_*(U, p^{T,1}_{i,i,1}, p^{T,2}_{i,i,1}; q_i)$$

is an equivalence of Ω-spectra.

Proof of Lemma 5.9. Each inclusion $V_b \subset C_{i,b}$ for $b \in G_i$ is a homotopy equivalence; in fact V_b is a deformation retract of $C_{i,b}$. Thus it follows from [25, §5] that $w_{*,b}$ is a weak equivalence of Ω-spectra, where $w_{*,b}$ denotes w_* in the special case where Z is replaced by b, and $\partial_1 Z$ is empty. Thus we may argue by induction over various subcomplexes $K \subset L$ in a triangulation L for $Z - \partial_1 Z$ and imitating the arguments in the proof of Lemma 5.2, to show that $w_{*,K}$ is an equivalence for every subcomplex $K \subset L$, where $w_{*,K}$ denotes w_* in the special case where Z is replaced by K, and $\partial_i Z$, ΛZ are replaced by $\partial_i Z \cap K$, $\Lambda Z \cap K$.

This completes the proof of Lemma 5.9.

6. $\mathscr{P}_*(f) \circ J_* : \mathscr{P}_*(p) \to \mathscr{P}_*(M)$ is injective

Let M, $p: E \to G$, $f: E \to M$, be as in Theorem 0.4, and let $\mathscr{P}^c_*(f): \mathscr{P}^c_*(E) \to \mathscr{P}_*(M)$ be the map of stable pseudoisotopy spectra which is induced by f. Let $J_*: \mathscr{P}^c_*(p) \to \mathscr{P}^c_*(E)$ be as in §1.4. The main result of this section is the following proposition.

6.1. Proposition. *The composite map $\mathscr{P}^c_*(f) \circ J_*: \mathscr{P}^c_*(p) \to \mathscr{P}_*(M)$ of Ω-spectra induces an injection on the homotopy groups of the Ω-spectra.*

Proof of Proposition 6.1. We first prove 6.1 when Hypothesis 4.0 is assumed to hold. Since 4.0 is in effect, we may use all of the results from §§4 and 5 in our proof of 6.1.

For integers $i \geq 1$, $j \geq 0$ we define a map $r^i_j: \mathscr{P}_j(M) \to \widetilde{\mathscr{P}}^b_j(p^B_i)$ as follows. Let $h: M \times \mathbf{R}^j \times I^n \times [0, 1] \to M \times \mathbf{R}^j \times I^n \times [0, 1]$ represent a point in $\mathscr{P}_j(M)$, and let $\hat{h}: \widehat{M} \times \mathbf{R}^j \times I^n \times [0, 1] \to \widehat{M} \times \mathbf{R}^j \times I^n \times [0, 1]$ be the unique lifting of h to a stable pseudoisotopy of the universal covering space $\widehat{M} \times \mathbf{R}^j$. By taking the product of $\hat{h}$ with the identity map $D_i \to D_i$ we obtain a stable pseudoisotopy $\bar{h} \in \mathscr{P}_j(D_i \times \widehat{M})$. Note that $\bar{h}$ is left invariant by the diagonal action $\Gamma_i \times (D_i \times \widehat{M}) \to D_i \times \widehat{M}$, and recall that $p^B_i: B_i \to G_i$ is the quotient of the projection map $D_i \times \widehat{M} \to D_i$ by the group action of Γ_i on $D_i \times \widehat{M}$ and D_i (cf. §4.2). Moreover, for each $b \in D_i$, $\bar{h}$ maps the subset $b \times \widehat{M} \times \mathbf{R}^i \times I^n \times [0, 1]$ into itself. Thus we may define $r^i_j(h) \in \widetilde{\mathscr{P}}^b_j(p^B_i)$ to be the quotient of $\bar{h}$ under the action $\Gamma_i \times (D_i \times \widehat{M}) \to D_i \times \widehat{M}$. For each integer $j < 0$ we define $r^i_j: \mathscr{P}_j(M) \to \widetilde{\mathscr{P}}^b_j(p^B_i)$ to be the j-fold looping of $r^i_0: \mathscr{P}_0(M) \to \widetilde{\mathscr{P}}^b_0(p^B_i)$. Then the collection of maps $\{r^i_j : j \in \mathbf{Z}\}$, which is denoted by $r^i_*: \mathscr{P}_*(M) \to \widetilde{\mathscr{P}}^b_*(p^B_i)$, is a map in the category of Ω-spectra.

Clearly the verification of the following claim would complete the proof of 6.1. Recall that $p_j: E_j \to G_j$, $j = 1, 2, 3, \cdots$, denote the components of $p: E \to G$. Let $f_j: E_j \to M$ denote the restriction of f to E_j, $\mathscr{P}_*(f_j): \mathscr{P}_*(E_j) \to \mathscr{P}_*(M)$ the map induced by f_j, and $J^j_*: \mathscr{P}_*(p_j) \to \mathscr{P}_*(E_j)$ the restriction of J_* to $\mathscr{P}_*(p_j)$.

6.2. Claim. (a) For any map $g: X \to \mathscr{P}_*(p_j)$ of a finite CW-complex X, and for any positive integers i, j with $i \neq j$, the following composite map is null homotopic:

$$X \xrightarrow{g} \mathscr{P}_*(p_j) \xrightarrow{\mathscr{P}_*(f_j)\circ J^j_*} \mathscr{P}_*(M) \xrightarrow{r^i_*} \widetilde{\mathscr{P}}^b_*(p^B_i).$$

(b) For any positive integer i the composite map

$$\mathscr{P}_*(p_i) \xrightarrow{\mathscr{P}_*(f_i)\circ J^i_*} \mathscr{P}_*(M) \xrightarrow{r^i_*} \widetilde{\mathscr{P}}^b_*(p^B_i)$$

is a weak equivalence of Ω-spectra.

First we will verify 6.2(a). We will need the following definition. Recall that F_j is a foliation of E_j by the fibers of the projection $p_j: E_j \to G_j$, and $T_j \subset M \times I^k$ is defined in §4.1.

6.3. Definition. A stable pseudoisotopy $h: T_j \times \mathbf{R}^i \times I^n \times [0, 1] \to T_j \times \mathbf{R}^i \times I^n \times [0, 1]$ is (α, δ)*-simply-controlled over* (E_j, F_j), with respect to the projection $t_j: T_j \to E_j$ of §4.1, if the unique lifting $\hat{h}: \widehat{T}_j \times \mathbf{R}^i \times I^n \times [0, 1] \to \widehat{T}_j \times \mathbf{R}^i \times I^n \times [0, 1]$ of h to the universal covering space $\widehat{T}_j$ is (α, δ)-controlled over $(\widehat{E}_j, \widehat{F}_j)$ with respect to the projection $\hat{t}_j: \widehat{T}_j \to \widehat{E}_j$ (cf. §1.5 for "(α, δ)-control"). Here $\widehat{E}_j, \widehat{F}_j$ are the preimages of E_j, F_j under the covering projection $\widehat{T}_j \to T_j$, and $\hat{t}_j$ is the unique lifting of t_j to $\widehat{T}_j$ which satisfies $\hat{t}_j \mid \widehat{E}_j = \text{identity}$.

It will be convenient to identify $\mathscr{P}_*(p_j)$ with the space of all maps $u: [0, \infty) \to \mathscr{P}_*(M \times I^k)$ which have the following properties.

6.4. (a) The support of each $u(t)$ lies over the subset $T_j \subset M \times I^k$.

(b) Each $u(t)$ is $(\alpha, (1+t)^{-1})$-simply-controlled over (E_j, F_j) with respect to the projection $t_j: T_j \to E_j$ of §4.1. Here α is a positive number depending only on u.

6.5. Remark. The identification of $\mathscr{P}_*(p_j)$ with the space of maps given in 6.4 would be routine if we allowed for a dependence of α on t in 6.4(b) (as well as a dependence of u), and if "(α, δ)-simply-controlled" were replaced by "(α, δ)-controlled". Let U^1_* denote the space of all maps given in 6.4, and U^2_* the larger space of all maps given by 6.4, where in 6.4(b) we allow α to depend on both u and t, and "(α, δ)-simply-controlled" is replaced by "(α, δ)-controlled." Then U^1_* is a deformation retract of U_*^2, as can be seen by reviewing the arguments in [25, §4] with the stable pseudoisotopy space example kept in mind.

Note that $\mathscr{P}_*(M) = \mathscr{P}_*(M \times I^k)$, and since $p_i^A\colon A_i \to G_i$ of §4.2 is just the composite map $A_i = B_i \times I^k \xrightarrow{\text{proj}} B_i \xrightarrow{p_i^B} G_i$, we also have that $\widetilde{\mathscr{P}}_*^b(p_i^B) = \widetilde{\mathscr{P}}_*^b(p_i^A)$. Thus $r_*^i\colon \mathscr{P}_*(M) \to \widetilde{\mathscr{P}}_*^b(p_i^B)$ becomes $r_*^i\colon \mathscr{P}_*(M \times I^k) \to \widetilde{\mathscr{P}}_*^b(p_i^A)$, and $\mathscr{P}_*(f_j) \circ J_*^j\colon \mathscr{P}_*(p_j) \to \mathscr{P}_*(M)$ becomes the inclusion $U_*^1(0) \subset \mathscr{P}_*(M \times I^k)$, where $U_*^1(0) = \{u(0) : u \in U_*^1\}$. Applying these substitutions and also 6.4, we see that the composite map of 6.2(a) satisfies the following properties.

6.6. (a) Define a homotopy $(\mathscr{P}_*(f_j) \circ J_*^j)_t\colon \mathscr{P}_*(p_j) \to \mathscr{P}_*(M)$, $t \in [0, \infty)$, of $\mathscr{P}_*(f_j) \circ J_*^j$ by $(\mathscr{P}_*(f_j) \circ J_*^j)_t(u) = u(t)$ for each $u \in U_*^1$. Then for each $x \in X$ and each $t \geq 0$ the support of the stable pseudoisotopy $r_*^i \circ (\mathscr{P}_*(f_j) \circ J_*^j)_t \circ g(x)$ lies over the subset $\bigcup_q T_{i,j,q} \subset A_i$.

(b) Since X is a finite complex there is a number $\alpha > 0$ such that for all $x \in X$, all $b \in G_i$, and all $t \geq 0$, the restricted stable pseudoisotopy $r_*^i \circ (\mathscr{P}_*(f_j) \circ J_*^j)_t \circ g(x) \mid ((\bigcup_q T_{i,j,q,b}) \times \mathbf{R}^a \times I^n \times [0, 1])$ is $(\alpha, (1+t)^{-1})$-simply-controlled over $(\bigcup_q E_{i,j,q,b}, \bigcup_q F_{i,j,q,b})$ with respect to the projection $(\bigcup_q t_{i,j,q,b})\colon \bigcup_q T_{i,j,q,b} \to \bigcup_q E_{i,j,q,b}$. Here we assume, with no loss of generality, that $\operatorname{Image}(g) \subset \mathscr{P}_a(p_j)$ with $a \geq 0$.

If t in 6.6(b) is chosen sufficiently large, then we may use Theorem 1.5.3, in conjunction with 6.6 and 4.6 (for $i \neq j$), to get for each $y \in G_i$ and each $x \in X$ a one-parameter family of stable pseudoisotopies $h_{x,y,s} \in \mathscr{P}_*^b(A_{i,y})$, $s \in [0, 1]$, such that

$$h_{x,y,0} = [r_*^i \circ (\mathscr{P}_*(f_j) \circ J_*^j)_t \circ g(x)] \mid A_{i,y} \times \mathbf{R}^a \times I^n \times [0, 1]$$

and such that $h_{x,y,1} = \text{identity}$. Now a homotopy from the composite map in 6.2(a) to a constant map is obtained by concatenating the homotopy $r_*^i \circ (\mathscr{P}_*(f_j) \circ J_*^j)_s \circ g$, $s \in [0, t]$, with the homotopy $\bigcup_{x,y} h_{x,y,s}$, $s \in [0, 1]$. Note that there is some work involved in showing that the $h_{x,y,s}$, $s \in [0, 1]$, can be chosen so that the union $\bigcup_{y \in G_i} h_{x,y,s}$, $s \in [0, 1]$, is a one-parameter family of stable pseudoisotopies in $\widetilde{\mathscr{P}}_*^b(p_i^A)$ continuous in x. To see this we first choose a triangulation L for G_i as in 5.2.1, and then proceed by induction over the skeleta of L; our (mth) induction hypothesis assumes that the $h_{x,y,s}$, $s \in [0, 1]$, have been chosen so that the union $\bigcup_{y \in G_i} h_{x,y,s}$, $s \in [0, 1]$, is a one-parameter family of stable pseudoisotopies in $\widetilde{\mathscr{P}}_*^b(p_i^A)$ which is continuous in x, that for each $x \in X$

and each $y \in G_i$ we have $h_{x,y,0} = [r^i \circ (\mathscr{P}_*(f_j) \circ J_*^j)_t \circ g(x)] \mid A_{i,y} \times \mathbf{R}^a \times I^n \times [0, 1]$, and that $h_{x,y,1} = 1$ for each $x \in X$ and each $y \in L^m$ where $L^m = m$-skeleton of L. Furthermore, we assume that $h_{x,y,s}$ is supported on $U_q T_{i,j,q,y}$. Now for each $(m+1)$-simplex $\Delta \in L$ we choose a map $h_\Delta \colon \Delta \times A_{i,b_\Delta} \to A_i$, where b_Δ is the barycenter of Δ, having the following properties: $p_\Delta = p_i^A \circ h_\Delta$, where $p_\Delta \colon \Delta \times A_{i,b_\Delta} \to \Delta$ is the standard projection; for each $y \in \Delta - \partial\Delta$ the restricted map $h_\Delta \colon y \times A_{i,b_\Delta} \to A_{i,y}$ is an isometry with respect to the metrics of §4.5; for each $y \in \partial\Delta$ the restricted map $h_\Delta \colon y \times A_{i,b_\Delta} \to A_{i,y}$ is a local isometry and a covering space projection; and "local trivialization" $h_\Delta \colon \Delta \times A_{i,b_\Delta} \to A_i$ for $p_i^A \colon A_i \to G_i$ over Δ is consistent with the local flat structure for $p_i^A \colon A_i \to G_i$ given in §4.2. For each $x \in X$ and each $y \in \Delta$ we let $\hat{h}_{x,y,1}$ denote the pullback along the covering space projection $h_\Delta \colon y \times A_{i,b_\Delta} \to A_{i,y}$ of $h_{x,y,1}$. Note that $\{\hat{h}_{x,y,1} : x \in X,\ y \in \Delta\}$ is a continuous (in x, y) family of stable pseudoisotopies in $\mathscr{P}_*^b(A_{i,b_\Delta})$ to which we may apply Theorem 1.5.3 (as we applied Theorem 1.5.3 to each individual $h_{x,y,0}$ above) to get a one-parameter family $\{\hat{h}_{x,y,s} : x \in X,\ y \in \Delta,\ s \in [1, 2]\}$ of stable pseudoisotopies in $\mathscr{P}_*^b(A_{i,b_\Delta})$, which are continuous in x, y, s, and satisfy $\hat{h}_{x,y,s} = \hat{h}_{x,y,1}$ for all $x \in X$, all $y \in \partial\Delta$, and all $s \in [1, 2]$. We do this for each $(m+1)$-simplex Δ in K_i, and then push the results back into A_i under the $\{h_\Delta\}$ to get a one-parameter family $\{h_{x,y,s} : x \in X,\ y \in L^{m+1},\ s \in [0, 2]$; or $x \in X,\ y \in G_i,\ s \in [0, 1]\}$ which is continuous in x, y, s and which satisfies $h_{x,y,2} = 1$ for all $x \in X$ and $y \in L^{m+1}$. Note that this one-parameter family extends to a one-parameter family $\{h_{x,y,s} : x \in X,\ y \in G_i,\ s \in [0, 2]\}$ of stable pseudoisotopies in the $\{\mathscr{P}_*^b(A_{i,y}) : y \in G_i\}$ which is continuous in x, y, s; this extension can easily be chosen so that each of the $h_{x,y,s}$ has good control properties (analogous to 6.6(b)); however $h_{x,y,2} = 1$ does not necessarily hold if $y \notin L^{m+1}$. Our induction step is now completed by simply reparametrizing the one-parameter family $\{h_{x,y,s} : x \in X,\ y \in G_i,\ s \in [0, 2]\}$ by $s \in [0, 1]$ instead of by $s \in [0, 2]$.

This completes the verification of 6.2(a). We turn now to the verification of 6.2(b).

6.7. The homology functors $\mathbb{H}_*^1(\ ,\ )$ and $\mathbb{H}_*^2(\ ,\ )$.

6.7.1. Definition. A pair of subsets (X, Y) of G_i is called an *admissible pair* if they have the following properties.

(a) $Y \subset X$; moreover, X, Y, and $Z = \text{closure}(X - Y)$ are all compact codimension-zero submanifolds of G_i with boundaries (cf. 5.6).

(b) $\partial_1 Z = Y \cap Z$ is compact codimension-one submanifold of G_i with boundary. Set $\partial_2 Z = \text{closure}(\partial Z - \partial_1 Z)$ and $\Lambda Z = \partial_1 Z \cap \partial_2 Z$; then $\partial_2 Z$ and ΛZ are also compact submanifolds of G_i having codimension one and two respectively, with $\partial(\Lambda Z) = \varnothing$.

We define two Ω-spectra valued functors $\mathbb{H}^1_*(\ , \ )$, $\mathbb{H}^2_*(\ , \ )$ from admissable pairs in G_i as follows. For any admissible pair (X, Y) in G_i, and Z, $\partial_1 Z$, $\partial_2 Z$, ΛZ as in 6.7.1, we set

$$\mathbb{H}^1_*(X, Y) = \mathscr{P}_*(p_i^{T,1}) \quad (\text{cf. §1.2}),$$
$$\mathbb{H}^2_*(X, Y) = \widetilde{\mathscr{P}}_*(p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i) \quad (\text{cf. §5.7}),$$

where $p_{i,i,1}^{T,1} = p_{i,i,1}^{T} \mid (p_{i,i,1}^{T})^{-1}(Z - \partial_1 Z)$ (cf. §4.2 and 4.1.2 for $p_{i,i,1}^{T}$), $p_{i,i,1}^{T,2} = p_{i,i,1}^{T} | (p_{i,i,1}^{T})^{-1}(\partial_2 Z - \Lambda)$, and $p_i^{T,1} = p_i \circ t_i \mid (p_i \circ t_i)^{-1}(Z - \partial_1 Z)$ (cf. §4.1 for $t_i \colon T_i \to E_i$). If $Y = \varnothing$ then we write $\mathbb{H}^i_*(X)$ for $\mathbb{H}^i_*(X, Y)$.

We note that the functors $\mathbb{H}^1_*(\ , \ )$ and $\mathbb{H}^2_*(\ , \ )$ satisfy the usual axioms of homology theory. This is a deep result proven by F. Quinn in [24] for the functor $\mathbb{H}^1_*(\ , \ )$, and is an easy exercise for the functor $\mathbb{H}^2_*(\ , \ )$.

6.8. A natural map $w_*(\ , \ ) \colon \mathbb{H}^1_*(\ , \ ) \to \mathbb{H}^2_*(\ , \ )$. We define a natural map $w_*(\ , \ ) \colon \mathbb{H}^1_*(\ , \ ) \to \mathbb{H}^2_*(\ , \ )$ between functors as follows. Let

$$\begin{aligned} h_t \colon & ((p_i \circ t_i)^{-1}(Z - \partial_1 Z)) \times \mathbf{R}^a \times I^n \times [0, 1] \\ & \to ((p_i \circ t_i)^{-1}(Z - \partial_1 Z)) \times \mathbf{R}^a \times I^n \times [0, 1], \qquad t \in [0, \infty), \end{aligned}$$

be a one-parameter family of stable pseudoisotopies representing a point $\{h_t\} \in \mathbb{H}^1_*(X, Y)$. Without loss of generality we may assume that the h_t, $t \in [0, \infty)$, have the following property.

6.8.1. There is a neighborhood N for $\partial_2 Z - \Lambda Z$ in $Z - \partial_1 Z$ such that for each $t \geq 0$ the restriction $h_t \mid ((p_i \circ t_i)^{-1}(N)) \times \mathbf{R}^a \times I^n \times [0, 1]$ is the inclusion map.

Now we choose the map $s \colon Z - \partial_1 Z \to (0, \infty)$ of §5.8 so that the following holds.

6.8.2. (a) There is another neighborhood N' for $\partial_2 Z - \Lambda Z$ in $Z - \partial_1 Z$, such that N is also a neighborhood for N' in $Z - \partial_1 Z$.

(b) For each $b \in (Z - \partial_1 Z - N')$ the distance in G_i from b to $G_i - Z$ is greater than $s(b)$. The distance function $d_i^G \colon G_i \times G_i \to [0, \infty)$ which we are using here is defined as follows. For any $x, y \in G_i$ set

$d_i^G(x, y) = \text{minimum}\{d_i^D(\bar{x}, \bar{y})\}$, where $\bar{x}, \bar{y}$ are any preimage points for x, y under the projection $D_i \to D_i/\Gamma_i = G_i$, and $d_i^D(\ ,\ )$ is the metric which D_i inherits from $\widehat{M}$ (cf. 2.7.1(b)).

(c) Let $\partial N'$ denote the topological boundary for N' in $Z - \partial_1 Z$. Then for each $b \in \partial N'$ the closed ball of radius $s(b)$ centered at b in G_i is contained in N.

Let $U \subset ((p_{i,i,1}^T)^{-1}(Z - \partial_1 Z))$ denote the subset constructed in §5.8, where in §5.8 the function $s\colon Z - \partial_1 Z \to (0, \infty)$ satisfies 6.8.2. Construct from $\{h_t\}$ of 6.8.1 a one-parameter family of stable embeddings $h'_t\colon U \times \mathbf{R}^a \times I^n \times [0, 1] \to (p_{i,i,1}^T)^{-1}(Z - \partial_1 Z) \times \mathbf{R}^a \times I^n \times [0, 1]$, $t \in [0, \infty)$, as follows. Let W_b denote the closed ball of radius $s(b)$ centered at b in G_i. Note that it follows from 6.8.2 that for any $b \in (Z - \partial_1 Z - N')$ the restricted family of maps $h_t \mid ((p_i \circ t_i)^{-1}(W_b)) \times \mathbf{R}^a \times I^n \times [0, 1]$, $t \in [0, \infty)$, lifts to a unique family of embeddings $h'_{t,b}\colon U_b \times \mathbf{R}^a \times I^n \times [0, 1] \to T_{i,i,1,b} \times \mathbf{R}^a \times I^n \times [0, 1]$, $t \in [0, \infty)$, satisfying

$$h'_{t,b}|(U_b \times \mathbf{R}^a \times I^n \times 0) = \text{inclusion}.$$

If $b \in N'$, then we define $h'_{t,b}\colon U_b \times \mathbf{R}^a \times I^n \times [0, 1] \to T_{i,i,1,b} \times \mathbf{R}^a \times I^n \times [0, 1]$, $t \in [0, \infty)$, to equal the inclusion for all $t \geq 0$. Set

$$h'_t = \bigcup_{b \in (Z - \partial_1 Z)} h'_{t,b}, \qquad t \in [0, \infty).$$

Note that the construction $\{h_t\} \to \{h'_t\}$ yields a well-defined map between Ω-spectra

$$u_*(X, Y)\colon \mathbb{H}_*^1(X, Y) \to \widetilde{\mathscr{E}}_*(U, p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i) \quad \text{(cf. 6.8.2 and §5.8)}.$$

We now define the map $w_*(X, Y)\colon \mathbb{H}^1(X, Y) \to \mathbb{H}_*^2(X, Y)$ to be the composition of $u_*(X, Y)$ with the equivalence of Ω-spectra

$$\widetilde{\mathscr{E}}_*(U, p_{i,i,1}^{T,1} p_{i,i,1}^{T,2}; q_i) \cong \widetilde{\mathscr{P}}_*(p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i)$$

given by Lemma 5.9. If $Y = \varnothing$, then we write $w_*(X)$ for any $w_*(X, Y)$.

6.8.3. Lemma. *In the special case where $X = G_i$ and $Y = \varnothing$ we will denote the map $w_*(X, Y)$ by $w_*\colon \mathscr{P}_*(p_i \circ t_i) \to \widetilde{\mathscr{P}}_*(p_{i,i,1}^T; q_i)$. Then the map w_* is an equivalence of Ω-spectra.*

Proof of Lemma 6.8.3. Let K denote a triangulation for G_i as in 5.2.1, and K^* the "dual cell complex" for K. That is, for each simplex $\Delta \in K$ the "dual cell" $\Delta^* \in K^*$ is defined to be the union of all simplices $e \in K^{(1)}$ in the first barycentric subdivision of K such that $e \cap \Delta = b(\Delta)$, where $b(\Delta)$ is the barycenter of Δ. The "dual cells" $\Delta^* \in K^*$ are always cone spaces, but they are not always PL cells because G_i is not in general a PL manifold. Let $K_1^* \subset K_2^* \subset \cdots \subset K_n^* = K^*$ denote an increasing sequence of subcomplexes of K^* such that for all $i < n$, $K_{i=1}^* = K_i^* \cup \Delta_i^*$ for some $\Delta_i^* \in K^*$ with $\dim(\Delta_i^*) \geq \dim(K_i^*)$ and $\partial \Delta_i^* \subset K_i^*$. Define an increasing sequence of subsets $S_1 \subset S_2 \subset \cdots \subset S_n = G_i$ by induction as follows: S_{i+1} is the union of S_i with all simplices $e \in K^{(i+2)}$ in the $(i+2)$-fold barycentric subdivision of K which intersect with K_{i+1}^*. Note that each S_i is a regular neighborhood for K_i^* in G_i, and that each pair (S_{i+1}, S_i) is an admissable pair.

We complete the proof of 6.8.3 by showing (by induction over i) that each map $w_*(S_i)\colon \mathbb{H}^1(S_i) \to \mathbb{H}_*^2(S_i)$ is an equivalence of Ω-spectra. Suppose that this is true for all $i \leq r$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{H}_*^2(S_r) & \xrightarrow{\phi^2} & \mathbb{H}_*^2(S_{r+1}) & \xrightarrow{\psi^2} & \mathbb{H}_*^2(S_{r+1}, S_r) \\
\uparrow{\scriptstyle w_*(S_r)} & & \uparrow{\scriptstyle w_*(S_{r+1})} & & \uparrow{\scriptstyle w_*(S_{r+1}, S-r)} \\
\mathbb{H}_*^1(S_r) & \xrightarrow{\phi^1} & \mathbb{H}_*^1(S_{r+1}) & \xrightarrow{\psi^1} & \mathbb{H}_*^1(S_{r+1}, S_r)
\end{array}$$

The horizontal maps in this diagram, which are induced by the inclusion maps $S_r \subset S_{r+1}$ and $(S_{r+1}, \varnothing) \subset (S_{r+1}, S_r)$ are fibrations in the category of Ω-spectra (this is one of the homology axioms for $\mathbb{H}_*^1(\ ,\ )$ and $\mathbb{H}_*^2(\ ,\ )$ (cf. [25, Appendix])). Thus to complete the induction step it will suffice to show that $w_*(S_{r+1}, S_r)$ is an equivalence of Ω-spectra (recall that Convention 5.0 is in effect).

Let $G_{i,r'}$ denote the stratum of the stratified space G_i which contains the barycenter $b(\Delta_r)$ of Δ_r. Let $D_{i,r'} \subset D_i$ be a connected component of the preimage of $G_{i,r'}$ under the "covering projection" $D_i \to D_i/\Gamma_i = G_i$, and choose a point $b_r \in D_{i,r'}$ in the preimage of $b(\Delta_r)$. Let $\Gamma_i^r \subset \Gamma_i$ be the isotropy subgroup at b_r of the action $\Gamma_i \times D_i \to D_i$. Let $V_r \subset TD_i$ denote the subset of all vectors in the tangent space of D_i at b_r which are perpendicular to $D_{i,r'}$ at the point b_r; let $\Gamma_i^r \times V_r \to V_r$ be the action induced by $\Gamma_i^r \times D_i \to D_i$; let $\Gamma_i^r\colon (V_r \times \mathbf{R}) \to V_r \times \mathbf{R}$ denote the diagonal action, where Γ_i^r acts on $\mathbf{R}$ through the homomorphism $h_{i,2}\colon \Gamma_i \to \Gamma_{i,2}$ of 2.10; and let $\rho_r\colon (V_r \times \mathbf{R})/\Gamma_i^r \to V_r/\Gamma_i^r$ denote the quotient of the standard projection map $V_r \times \mathbf{R} \to V_r$ under the actions by Γ_i^r. Note that the projection map $p_i\colon p_i^{-1}(S_{r+1} - S_r, \partial(S_{r+1} - S_r)) \to (S_{r+1} - S_r, \partial(S_{r+1} - S_r))$ is topologically equivalent to the projection map

$\rho_r \times \mathrm{id}\colon ((V_r \times R)/\Gamma_i^r) \times \mathbf{R}^u \times (I^v, \partial I^v) \to (V_r/\Gamma_i^r) \times \mathbf{R}^u \times (I^v, \partial I^v)$, where id: $\mathbf{R}^u \times (I^v, \partial I^v) \to \mathbf{R}^u \times (I^v, \partial I^v)$ is the identity map, $v = \dim(\Delta_r)$, and $u = (\dim(G_{i,r'}) - v)$. Thus (by 1.3.1) there is the following equivalence of Ω-spectra.

6.8.3.1. $\mathbb{H}^1_*(S_{r+1}, S_r) \cong \mathscr{P}_{*+u}(\rho_r)$.

Note that the projection $q_{i,b(\Delta_r)}\colon E_{i,i,1,b(\Delta_r)} \to C_{i,b(\Delta_r)}$ (cf. §5.4) is topologically equivalent to the projection map $\rho_r \times \mathrm{id}\colon ((V_r \times R)/\Gamma_i^r) \times \mathbf{R}^{u+v} \to (V_r/\Gamma_i^r) \times \mathbf{R}^{u+v}$, where id: $\mathbf{R}^{u+v} \to \mathbf{R}^{u+v}$ is the identity map; and the intersection $G_{i,r'} \cap (S_{r+1} - S_r, \partial(S_{r+1} - S_r))$ is homeomorphic to $\mathbf{R}^u \times (I^v, \partial I^v)$. Thus (by 1.3.1) there is the following equivalence of Ω-spectra.

6.8.3.2. $\mathbb{H}^2_*(S_{r+1}, S_r) \cong \Omega^v(\mathscr{P}_{*+u+v}(\rho_r))$.

It follows from 6.8.3.1 and 6.8.3.2 that $\mathbb{H}^1_*(S_{r+1}, S_r)$ and $\mathbb{H}^2_*(S_{r+1}, S_r)$ are equivalent Ω-spectra. The remaining details, in verifying that $w_*(S_{r+1}, S_r)$ is actually an equivalence of Ω-spectra, are left to the reader. This completes the proof of Lemma 6.8.3.

In light of Lemma 6.8.3 it is clear that the verification of 6.2(b) is completed by the next lemma.

6.9. Lemma. *The composite map of* 6.2(b), *when restricted to any compact subset $C \subset \mathscr{P}_*(p_i)$, is homotopic to the restriction of the following composition of Ω-spectra equivalences*:

$$\mathscr{P}_*(p_i) \xrightarrow{e^1_*} \mathscr{P}_*(p_i \circ t_i) \xrightarrow{w_*} \widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q_i)$$
$$\xrightarrow{e^2_*} \mathscr{P}^b_*(p^T_{i,i,1}) \xrightarrow{e^3_*} \mathscr{P}^b_*(p^A_i) = \mathscr{P}^b_*(p^B_i).$$

Here e^1_* exists because T_i is a tubular neighborhood for E_i in $M \times I^k$ with projection map $t_i\colon T_i \to E_i$ (cf. §4.1), e^2_* is the equivalence of Lemma 5.5, and e^3_* is the equivalence of Lemma 5.3.

Proof of Lemma 6.9. First, as with the verification of 6.2(a), it will be useful to identify $\mathscr{P}_*(p_i)$ and $\mathscr{P}_*(p_i \circ t_i)$ with U^1_* of 6.5. Now we shall construct a homotopy $w_{*,t}\colon U^1_* \to \widetilde{\mathscr{P}}_*(p^T_{i,i,1}; q_i)$, $t \in [0, 1]$, of w_*, and a homotopy $(r^i_* \circ \mathscr{P}_*(f_i) \circ J^i_*)_t\colon U^1_* \to \widetilde{\mathscr{P}}_*(p^A_i)$, $t \in [0, 1]$, of $r^i_* \circ \mathscr{P}_*(f_i) \circ J^i_*$, such that $e^3_* \circ e^2_* \circ w_{*,1} \mid C = (r^i_* \circ \mathscr{P}_*(f_i) \circ J^i_*)_1 \mid C$, where C is a fixed but arbitrary compact subset of U^1_*. Clearly this will complete the proof of Lemma 6.9.

To get $w_{*,t}$, $t \in [0, 1]$, we define functions $s_t\colon G_i \to (0, \infty]$ by $s_t(b) = (1 - t)^{-1} s(b)$ for all $b \in G_i$, where $s\colon G_i \to (0, \infty)$ is the function given 6.8.2 when $X = G_i$ and $Y = \varnothing$ in 6.8.2. Note that by substituting s_t for s in the construction of the map $u_*(X, Y): U_*^1 \to \widetilde{\mathscr{E}}_*(U, p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i)$ in §6.8, we obtain a homotopy $u_*(X, Y)_t: U_*^1 \to \widetilde{\mathscr{E}}_*(U_t, p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i)$, $t \in [0, 1]$, of $u_*(X, Y)$. We set $w_{*,t}$ equal to the composite of $u_*(X, Y)_t$ with the Ω-spectra equivalence

$$\widetilde{\mathscr{E}}_*(U_t, p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i) \cong \widetilde{\mathscr{P}}_*(p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i)$$

of 5.9.

Now we will construct the homotopy $(r_*^i \circ \mathscr{P}_*(f_i) \circ J_*^i)_t$, $t \in [0, 1]$. Let $(\mathscr{P}_*(f_i) \circ J_*^i)_t$, $t \in (0, \infty)$, be the homotopy of $\mathscr{P}_*(f_i) \circ J_*^i$ given in 6.6(a). Note that the composite homotopy $r_*^i \circ (\mathscr{P}_*(f_i) \circ J_*^i)_t$, $t \in [0, \infty)$, has the following properties.

6.9.1. For each $b \in G_i$, each $x \in U_*^1$, and each $t \geq 0$, the support of the restricted stable pseudoisotopy $r_*^i \circ (\mathscr{P}_*(f_i) \circ J_*^i)_t(x): A_{i,b} \times \mathbf{R}^a \times I^n \times [0, 1] \to A_{i,b} \times \mathbf{R}^a \times I^n \times [0, 1]$ lies over the subset $\bigcup_q T_{i,i,q,b} \subset A_{i,b}$.

(b) Given any compact subset $C \subset U_*^1$ and any $\delta > 0$, if the number $\lambda > 0$ is chosen sufficiently large, then for each $b \in G_i$ and each $x \in G$ the restricted stable pseudoisotopy

$$r_*^i \circ (\mathscr{P}_*(f_i) \circ J_*^i)_\lambda(x) \left| \left(\bigcup_q T_{i,i,q,b} \right) \times \mathbf{R}^a \times I^n \times [0, 1] \right.$$

will be (α, δ)-controlled over $(\bigcup_q E_{i,i,q,b}, \bigcup_q F_{i,i,q,b})$ with respect to the projection $\bigcup_q t_{i,i,q,b}: \bigcup_q T_{i,i,q,b} \to \bigcup_q E_{i,i,q,b}$, where α depends on C, but does not depend on δ.

If λ in 6.9.1(b) is chosen sufficiently large (making δ sufficiently small in 6.9.1(b)), then we may use Theorem 1.5.3, in conjunction with 6.9.1(a), (b) and 4.6 (for $i = j$), to find for each $y \in G_i$ and each $x \in C$ a one-parameter family of stable pseudoisotopies $g_{y,x,t}: A_{i,y} \times \mathbf{R}^a \times I^n \times [0, 1] \to A_{i,y} \times \mathbf{R}^a \times I^n \times [0, 1]$, $t \in [0, 1]$, which has the following properties.

6.9.2. (a) $g_{y,x,0} = r_*^i \circ (\mathscr{P}_*(f_i) \circ J_*^i)_\lambda(x) \mid A_{i,y} \times \mathbf{R}^a \times I^n \times [0, 1]$.

(b) Set $g_t(x) = \bigcup_{y \in G_i} g_{y,x,t}$ for all $t \in [0, 1]$. Then $g_t(x)$, $t \in [0, 1]$, is a one-parameter family of stable pseudoisotopies in $\widetilde{\mathscr{P}}_*^b(p_i^A)$ which depend continuously on x.

(c) $g_t(x) \mid T_{i,i,1} \times \mathbf{R}^a \times I^n \times [0, 1]$

$$= (r_*^i \circ (\mathscr{P}_*(f_i) \circ J_*^i)_{\lambda - \lambda t})(x) \mid T_{i,i,1} \times \mathbf{R}^a \times I^n \times [0, 1]$$

holds for all $t \in [0, 1]$ and all $x \in C$.

(d) The support of each $g_t(x)$ lies over the subset $\bigcup_q T_{i,i,q} \subset A_i$, and the support of each $g_1(x)$ lies over $T_{i,i,1}$.

Now the desired homotopy $(r^i_* \circ \mathscr{P}_*(f_i) \circ J^i_*)_t$, $t \in [0, 1]$, of $r^i_* \circ \mathscr{P}_*(f_i) \circ J^i_*$ is defined to be the composite of the homotopy $r^i_* \circ (\mathscr{P}_*(f_i) \circ J^i_*)_t$, $t \in [0, \lambda]$, and 6.6(a) with the homotopy g_t, $t \in [0, 1]$, given in 6.9.2.

This completes the proof of Lemma 6.9, and therefore also the proof of Proposition 6.1, when Hypothesis 4.0 is assumed to be in effect.

We now discuss the modifications that must be made on the preceding proof of 6.1 if Hypothesis 4.0 does not hold.

The guiding step in our modified proof is the replacement of the maps $r^i_j: \mathscr{P}_j(M) \to \widetilde{\mathscr{P}}^b_j(p^B_i)$ of 6.2 by maps $\hat{r}^i_j: \mathscr{P}_j(M) \to \widetilde{\mathscr{P}}_j(p^{\widehat{E}^0}_{i,i}, p^{\widehat{E}^0}_{i,i,\partial}; \hat{p}_{i,i})$ which are defined as follows.

Set $\widehat{E}^0_i = \widehat{E}_i - \partial\widehat{E}_i$, and let $p^{\widehat{E}^0}_{i,i}: \widehat{E}^0_{i,i} \to G_i$ denote the stratified flat bundle obtained as in §4.2 from the group action $\Gamma_i \times \widehat{E}^0_i \to \widehat{E}^0_i$. Let $\widetilde{\mathscr{P}}_j(p^{\widehat{E}^0}_{i,i}, p^{\widehat{E}^0}_{i,i,\partial}; \hat{p}_{i,i})$ denote the space of all maps $g: [0, \infty) \to \widetilde{\mathscr{P}}^b_j(p^{\widehat{E}^0}_{i,i})$ such that for some $\alpha > 0$ (α depends on g) and all $t \in [0, \infty)$, $y \in G_i$ the stable pseudoisotopy $g(t) \mid (\widehat{E}^0_{i,i,y} \times \mathbf{R}^j \times I^n \times [0, 1])$ is $\alpha(1+t)^{-1}$-controlled over $C_{i,y}$ with respect to the projection $\hat{p}_{i,i}$: $\widehat{E}_{i,i,y} \to C_{i,y}$ of §4.4, and is equal to the identity map if $y \in \partial G_i$. There is also an analogous space $\widetilde{\mathscr{E}}_j(U, p^{\widehat{E}^0}_{i,i}, p^{\widehat{E}^0}_{i,i,\partial}; \hat{p}_{i,i})$ for embeddings, where $U = \bigcup_{y \in G_i}(\hat{p}^{-1}_{i,i}(V_y) \cap \widehat{E}^0_{i,i,y})$, and V_y comes from §5.8. Finally there is the space $\widetilde{\mathscr{P}}_j(p^{\widehat{E}}_{i,i}; \hat{p}_{i,i})$ of all continuous maps $g: [0, \infty) \to \widetilde{\mathscr{P}}^b_j(p^{\widehat{E}}_{i,i})$ such that for some $\alpha > 0$ (α depends on g) and all $t \in [0, \infty)$, $y \in g_i$, $g(t) \mid (\widehat{E}_{i,i,y} \times \mathbf{R}^j \times I^n \times [0, 1])$ is $\alpha(1+t)^{-1}$-controlled over $C_{i,y}$ with respect to the projection $\hat{p}_{i,i}: \widehat{E}_{i,i,y} \to C_{i,y}$.

Now proceeding as in the proof of Lemma 5.3 we can argue that there is an equivalence of Ω-spectra $f^1_*: \widetilde{\mathscr{P}}^b_*(p^B_i) \to \widetilde{\mathscr{P}}^b_*(p^{\widehat{E}}_{i,i})$.

Proceeding as in the proof of Lemma 5.5 we can get an equivalence of Ω-spectra $f^2_*: \widetilde{\mathscr{P}}^b_*(p^{\widehat{E}}_{i,i}) \to \widetilde{\mathscr{P}}_*(p^{\widehat{E}}_{i,i}; \hat{p}_{i,i})$.

There is a "restriction" map

$$f^3_*: \widetilde{\mathscr{P}}_*(p^{\widehat{E}}_{i,i}; \hat{p}_{i,i}) \to \widetilde{\mathscr{E}}_*(U, p^{\widehat{E}^0}_{i,i}, p^{\widehat{E}^0}_{i,i,\partial}; \hat{p}_{i,i}),$$

the construction of which is similar to the construction of "$u(X, Y)$" given in §6.8.

Finally, by proceeding as in the proof of Lemma 5.9, we can get an equivalence of Ω-spectra

$$f_*^4 : \widetilde{\mathscr{E}}_*(U, p_{i,i,}^{\widehat{E}^0}, p_{i,i,\partial}^{\widehat{E}^0}; \hat{p}_{i,i}) \to \widetilde{\mathscr{P}}_*(p_{i,i}^{\widehat{E}^0}, p_{i,i,\partial}^{\widehat{E}^0}; \hat{p}_{i,i}).$$

Now set $\hat{r}_j^i = f_j^4 \circ f_j^3 \circ f_j^2 \circ f_j^1 \circ r_j^i$.

The proof of Proposition 6.1, when 4.0 is not satisfied, is completed by showing that Claim 6.2, modified by replacing the r_j^i by the $\hat{r}_j^i$ is true.

The verification of 6.2(a) proceeds much as before for the r_j^i; clearly the truth of 6.2(a) for the r_j^i implies the truth of 6.2 for the $\hat{r}_j^i$. In verifying that 6.2(b) is satisfied by the $\hat{r}_j^i$ we use a homology argument as in §§6.7, 6.8. In carrying out the details all arguments should be made relative to the boundary ∂G_i, over which all relevant stable pseudoisotopies may be assumed to equal the identity. This requires the following changes in §§6.7, 6.8 and also in 5.6–5.9. We need in 5.6–5.9 and §§6.7, 6.8 that every submanifold pair $(X, \partial X) \subset G_i$ meet the boundary ∂G_i "transversely," i.e., that the preimage of $(X, \partial X)$ under the projection map $D_i \to D_i/\Gamma_i = G_i$ must meet ∂D_i transversely in a piecewise smooth sense. In 5.7–5.9 we must replace $\widetilde{\mathscr{P}}_*(p_{i,i,1}^{T,1}, p_{i,i,1}^{T,2}; q_i)$ and $\widetilde{\mathscr{E}}_*(U, p_{i,i}^{T,1}, p_{i,i}^{T,2}, p_{i,i,1}^{T,2}; q_i)$ by $\widetilde{\mathscr{P}}_*(p_{i,i}^{\widehat{E}^0,1}, p_{i,i}^{\widehat{E}^0,2}; \hat{p}_{i,i})$ and $\widetilde{\mathscr{E}}_*(U, p_{i,i}^{\widehat{E}^0,1}, p_{i,i}^{\widehat{E}^0,2}; \hat{p}_{i,i})$ respectively, where

$$p_{i,i}^{\widehat{E}^0,1} = p_{i,i}^{\widehat{E}^0} \mid (p_{i,i}^{\widehat{E}^0})^{-1}(Z - \partial_1 Z),$$

$$p_{i,i}^{\widehat{E}^0,2} = p_{i,i}^{\widehat{E}^0} \mid (p_{i,i}^{\widehat{E}^0})^{-1}((\partial_2 Z \cup (Z \cap \partial G_i)) - \partial_1 Z).$$

The Ω-spectra $\widetilde{\mathscr{P}}_*(p_{i,i}^{\widehat{E}^0,1}, p_{i,i}^{\widehat{E}^0,2}; \hat{p}_{i,i})$ and $\widetilde{\mathscr{E}}_*(U, p_{i,i}^{\widehat{E}^0,1}, p_{i,i}^{\widehat{E}^0,2}, \hat{p}_{i,i})$ are defined in a fashion analogous to 5.7.1 and 5.8.1, where

$$U = \bigcup_{y \in (Z - \partial_1 Z)} ((\hat{p}_{i,i})^{-1}(V_y) \cap \widehat{E}^0_{i,i,y})$$

and the V_y come from §5.8. In §§6.7 and 6.8 we let $\mathbb{H}_*^1(X, Y)$ be as before but set $\mathbb{H}_*^2(X, Y) = \widetilde{\mathscr{P}}_*(p_{i,i}^{\widehat{E}^0,1}, p_{i,i}^{\widehat{E}^0,2}; \hat{p}_{i,i})$. The remaining details in the verification of 6.2(b) for the $\hat{r}_j^i$ are left to the reader to sort out.

This completes the proof of Proposition 6.1.

References

[1] D. R. Anderson & W.-C. Hsiang, *The functors K_{-i} and pseudoisotopies of polyhedra*, Ann. of Math. (2) **105** (1977) 201–223.

[2] W. Ballmann, M. Brin & P. Eberlein, *Structure of manifolds of non-positive curvature.* I, Ann. of Math. (2) **122** (1985) 171–203.

[3] W. Ballmann, M. Gromov & V. Schroeder, *Manifolds of non-positive curvature*, Birkhauser, Boston, 1985.

[4] J. Cheeger & D. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975.

[5] P. Eberlein & B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973) 45–110.

[6] F.T. Farrell & W.-C. Hsiang, *Whitehead groups of poly-(finite or cyclic) groups*, J. London Math. Soc. **24** (1981) 308–324.

[7] F. T. Farrell & L. E. Jones, *K-theory and dynamics*. I, Ann. of Math. **124** (1986) 531–569.

[8] ——, *Algebraic K-theory of spaces and stratified fibered over hyperbolic orbifolds*, Proc. nat. Acad. Sci. U.S.A. **83** (1986) 5364–5366.

[9] ——, *K-theory and dynamics*. II, Ann. of Math. (2) **126** (1987) 451–493.

[10] ——, *Algebraic K-theory of discrete subgroups of Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **84** (1987) 3095–3096.

[11] ——, *Foliated control theory*. I, K-Theory **2** (1988) 357–399.

[12] ——, *A toplogical analogue of Mostow's rigidity theorem*, J. Amer. Math. Soc. **2** (1989) 257–370.

[13] ——, *Rigidity and other topological aspects of compact non-positively curved manifolds*, Bull. Amer. Math. Soc. (N.S.) **22** (1990) 59–64.

[14] ——, *Computations of stable pseudoisotopy spaces for compact aspherical manifolds*, Proc. Internat. Conf. on Algebraic Topology (Poznan, Poland, June, 1989), to appear.

[15] ——, *Isomorphism conjectures in algebraic K-theory*, in preparation.

[16] P. Hartman, *On homotopic harmonic maps*, Canad. J. Math. **19** (1967) 673–687.

[17] A. Hatcher, *Concordance spaces, higher simple homotopy theory, and applications*, Proc. Sympos. Pure Math., Vol. 32, Amer. Math. Soc., Providence, RI, 1978, 3–21.

[18] A.E. Hatcher & J. B. Wagoner, *Pseudoisotopies of compact manifolds*, Asterisque **6** (1973).

[19] E. Heintz & H.C. ImHof, *Geometry of horospheres*, J. Differential Geometry **12** (1977) 481–491.

[20] B. Hu, *H-cobordisms over certain non-positively curved manifolds*, Thesis, State University of New York at Stony Brook, 1989.

[21] K. Igusa, *On the algebraic K-theory of A_∞-ring spaces*, Lecture Notes in Math., Vol. 967, Springer, Berlin, 1982, 146–194.

[22] I.M. James, *Ex-homotopy theory*, Illinois J. Math. **15** (1971) 324–338.

[23] A. J. Nicas, *On the higher Whitehead groups of a Bieberbach group*, Trans. Amer. Math. Soc. **287** (1985) 853–859.

[24] D. Quillen, *Higher algebraic K-theory*. I, Algebraic K-theory. I, Lecture Notes in Math., Vol. 341, Springer, Berlin, 1973, 85–147.

[25] F. Quinn, *Ends of maps*. II, Invent. Math. **68** (1982) 353–424.

[26] ——, *Algebraic K-theory of poly-(finite or cyclic) groups*, Bull. Amer. Math. Soc. (N.S.) **12** (1985) 221–226.

[27] I. A. Volodin, *Generalized Whitehead groups and pseudoisotopies*, Uspekhi Mat. Nauk **27** (1972) 229–230.

[28] F. Waldhausen, *Algebraic K-theory of generalized free products*, Ann. of Math. (2) **108** (1978) 135–256.

[29] ——, *Algebraic K-theory of topological spaces*. I, Proc. Sympos. Pure Math., Vol. 32, Amer. Math. Soc., Providence, RI, 1978, 35–60.

Columbia University
State University of New York, Binghamton
State University of New York, Stony Brook