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SPECIAL METRICS AND STABILITY FOR HOLOMORPHIC BUNDLES WITH GLOBAL SECTIONS

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Abstract

In this paper we describe canonical metrics on holomorphic bundles in which there are global holomorphic sections. Such metrics are defined by a constraint on the curvature of the corresponding metric connection. The constraint is in the form of a P.D.E which looks like the Hermitian-Yang-Mills equation with an extra zeroth order term. We identify the necessary and sufficient condition for the existence of solutions to this equation. This condition is given in terms of the slopes of subsheaves of the bundle and defines a property similar to stability. We show that if a holomorphic bundle meets this stability-like criterion, then its Chern classes are constrained by an inequality similar to the Bogomolov-Gieseker inequality for stable bundles.

0. Introduction

A fundamental feature of the geometry of holomorphic vector bundles is the existence of so-called *hermitian* or *metric* connections. These are the connections which are compatible both with the holomorphic structure of the bundle and with a hermitian bundle metric. In fact, given a hermitian bundle metric on a holomorphic bundle, the metric connection is uniquely determined. The correspondence thus obtained between metrics and connections is somewhat analogous to the relationship between riemannian metrics on smooth manifolds and the Levi-Civita connections on their tangent bundles. This relationship allows one (for example in the Yamabe problem) to define special riemannian metrics in terms of constraints on the curvatures of the corresponding Levi-Civita connections. The same can be done for holomorphic vector bundles; the Hermitian-Einstein (or Hermitian-Yang-Mills) equation [11], [12] can be thought of as a constraint on metrics in this way. It provides a criterion formulated in terms of curvatures whereby the holomorphic structure of a complex vector bundle determines a preferred hermitian bundle metric.

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In this paper we consider holomorphic vector bundles on which additional data in the form of a prescribed holomorphic global section is given. We consider the question as to whether there is any sense in which these two data (viz. the holomorphic structure and the holomorphic section) determine a preferred hermitian metric and if so when such metrics exist.

The objects we will consider thus fall into the general category of bundles whose description includes more than the specification of a holomorphic structure. Another example from this category is the Higgs bundle. A Higgs bundle is a holomorphic bundle, E, together with a given holomorphic linear map

$$\theta: \Omega^0(X, E) \to \Omega^{1,0}(X, E)$$

which satisfies $\theta^2 = 0$. These have been studied in a general setting by Simpson [18], [19] and a special case has been investigated by Hitchin [10]. Their results show that one can indeed identify preferred hermitian metrics on a Higgs bundle. The criterion one uses is again in the form of a constraint on the curvature of certain connections. The constraint equation is in fact formally identical to the Hermitian-Einstein equation. The difference is that whereas the Hermitian-Einstein equation constrains the curvature of the metric connection, in the case of Higgs bundles, the curvature is taken to be that of a connection constructed from the metric and the map θ.

There is an important difference between the case which we will consider and that of the Higgs bundles. It is the fact that while the Higgs field θ can be used to construct a connection, there does not seem to be any way to do the same using a section of E. Hence we cannot expect to formulate a constraint on metrics which formally resembles the Hermitian-Yang-Mills equation. Instead, we propose a criterion for selecting canonical metrics in the form of an equation which we call the *vortex equation.* Like the Hermitian-Einstein equation, this equation gives a constraint on the curvature of the metric connection associated to a hermitian metric. It novelty lies in the fact that the equation has a term coming from the prescribed holomorphic section. To understand when a bundle can support such special metrics we have studied the necessary and sufficient conditions for the existence of solutions to the vortex equation.

It is now well known that existence of solutions to the Hermitian-Einstein equation is governed by the *stability* properties of the holomorphic bundle—or rather of the sheaf of germs of holomorphic sections (cf. [13], [15], [3], [21]). Originally introduced by Mumford in connection with the moduli space of holomorphic bundles over algebraic curves, stability is a property defined in terms of the slopes (or normalized degrees) of subsheaves. In the case of the Higgs bundles, Simpson [18], [19] has shown that here too existence of "canonical" metrics signifies "stability", except now the appropriate notion of stability is that of stability of the bundle as a Higgs bundle. This is still a condition on the slopes of reflexive subsheaves of E but constrains only those subsheaves that have the structure of Higgs subsheaves (i.e., on which θ can be made to act by restriction).

We have found that existence of solutions to the vortex equation is governed by a similar property which we call ϕ-stability. The definition of ϕ-stability is, as in the other two cases, a condition on the slopes of subsheaves. It is however a somewhat more delicate notion than stability, and one in which subsheaves containing the prescribed section play a special role. Nevertheless, our results accord well with expectations based on our understanding of Hermitian-Einstein metrics on holomorphic bundles and of the analogous metrics on Higgs bundles. They provide a further instance where an algebraic property of a holomorphic bundle is expressable in terms of solutions to a partial differential equation and is indicated by the existence of canonical bundle metrics.

We define ϕ-stability and the vortex equation in §2. The bulk of the paper (§§2 and 3) is devoted to proving that ϕ-stability is a necessary and sufficient condition for the existence of solutions to the vortex equation. We conclude in §4 by showing that if a holomorphic bundle with prescribed section satisfies the requirements of ϕ-stability, then the Chern classes of the bundle satisfy an inequality similar to the Bogomolov-Gieseker inequality for stable bundles [1], [6].

1. Background and notation

The setting for the work in this paper is that of holomorphic vector bundles over compact Kähler manifolds. There is a considerable amount of machinery associated with such structures. It will be convenient to collect together the various definitions and basic facts that we will need. These are given in greater detail in [2], and good references for this material include [22], [9], [12].

1.1. Let E be a rank R holomorphic vector bundle over X, a compact Kähler manifold of complex dimension n. Let $\operatorname{End} E$ denote the endomorphism bundle associated with E, i.e., $\operatorname{End} E$ is the bundle whose global sections are bundle endomorphisms of E. Let $\omega \in \Omega^{1,1}(X, \mathbf{R})$ be a (fixed) Kähler form on X. The bundles E and $\operatorname{End} E$ carry various geometric structures. We will be primarily concerned with hermitian metrics and connections. We denote the space of hermitian metrics by $\mathscr{Met}$, and the space of connections of $\mathscr{A}$. If K is a fixed hermitian metric on E, then any other such metric, H, is related to K by $H = Kh$ where h is a smooth section of $\operatorname{End} E$ such that $h^{*_K} = h$. Here h^{*_K} denotes the adjoint with respect to the metric K, and by $H = Kh$ we mean that for any sections $s, t \in \Omega^0(X, E)$

$$(s, t)_H = (hs, t)_K. \tag{1.1}$$

We point out that the general linear gauge group $\mathrm{GL}(E)$ acts transitively on $\mathscr{M}et$ by the action $g(K) = Kg^*g$. In fact

$$\mathscr{M}et \approx \mathrm{GL}(E)/\mathrm{U}(K),$$

where $U(K)$ is the group of gauge transformations (i.e., bundle automorphisms) that are unitary with respect to K.

1.2. Given a Kähler form ω on X and an hermitian metric H on E, there are a variety of structures that can be erected. In particular, the following sesquilinear pairings are defined in the obvious way:

$$(\ , \ )_\omega : \Omega^{p,q}(X, \mathbf{C}) \times \Omega^{p,q}(X, \mathbf{C}) \to C^\infty(X, \mathbf{C}), \tag{1.2}$$

$$(\ , \ )_H : \Omega^k(X, E) \times \Omega^1(X, E) \to \Omega^{k+1}(X, \mathbf{C}), \tag{1.3}$$

$$((\ , \ )) : \Omega^{p,q}(X, E) \times \Omega^{p,q}(X, E) \to C^\infty(X, \mathbf{C}). \tag{1.4}$$

Here $\Omega^{p,q}(X, \mathbf{C})$ is the space of global sections of $\Lambda^{p,q}(X, \mathbf{C})$, the sheaf of germs of smooth complex-valued (p, q) forms on X. Similarly, $\Lambda^{p,q}(X, E)$ is the sheaf of germs of smooth (p, q) forms on X with values in E and $\Omega^{p,q}(X, E)$ is the space of its global sections. We can use $(\ , \ )_\omega$ and $((\ , \ ))$ to define L^2 inner products on $\Omega^{p,q}(X, \mathbf{C})$ and $\Omega^{p,q}(X, E)$. These are given by

$$\langle \alpha, \beta \rangle = \int_X (\alpha, \beta)_\omega \omega^{[n]} \quad \text{for } \alpha, \beta \in \Omega^{p,q}(X, \mathbf{C}), \tag{1.5}$$

and

$$\langle A, B \rangle = \int_X ((A, B))\omega^{[n]} \quad \text{for } A, B \in \Omega^{p,q}(X, E). \tag{1.6}$$

In both cases $\omega^{[n]} = \omega^n/n!$ is the volume form on X.

1.3. The Kähler form on X gives rise to a linear map

$$L : \Omega^{p,q}(X, E) \to \Omega^{p+1,q+1}(X, E).$$

where L is defined by

$$L(\alpha) = \alpha \wedge \omega. \tag{1.7}$$

We use the notation

(1.8) $$\Lambda = L^*.$$

for the adjoint with respect to the L^2 inner product defined above. If $\alpha \in \Omega^{1,1}(X, E)$, then

(1.9) $$\Lambda\alpha = ((\alpha, \omega)).$$

1.4. The space of connections is an affine space: if D_0 is a fixed connection on E, then

$$\mathscr{A} = D_0 + \Omega^1(X, \operatorname{End} E).$$

Using the splitting $\Omega^1(X, E) = \Omega^{0,1}(X, E) \oplus \Omega^{1,0}(X, E)$ coming from the complex structure on X, we can split a connection D as

(1.11) $$D = D^{0,1} + D^{1,0}.$$

As the notation suggests,

$$D^{0,1}: \Omega^0(X, E) \to \Omega^{0,1}(X, E)$$

and

$$D^{1,0}: \Omega^0(X, E) \to \Omega^{1,0}(X, E).$$

Both $D^{0,1}$ and $D^{1,0}$ extend in the usual way to act on $\Omega^{p,q}(X, E)$, and

(1.12) $$F_D = D^2$$

is the curvature of D.

Definition 1.4.1. A connection D is called *integrable* if $(D^{0,1})^2 = 0$. We denote by $\mathscr{A}^{1,1}$ the space of all integrable connections.

1.5. It is well known that the $(0, 1)$ part of an integrable connection defines a holomorphic structure on E. The converse to this theorem is also true, i.e., every holomorphic structure on E comes from a $\mathbf{C}$-linear $\bar\partial$ *operator*

$$\bar\partial_E: \Omega^0(X, E) \to \Omega^{0,1}(X, E)$$

which satisfies the following two properties:

(i) If $f \in C^\infty(X, \mathbf{C})$ and $\xi \in \Omega^0(X, E)$, then $\bar\partial_E(f\xi) = \bar\partial f \otimes \xi + f\bar\partial_E(\xi)$,

(ii) $(\bar\partial_E)^2 = 0$.

By this we mean that given a holomorphic structure on E, there is a $\bar\partial$ operator, $\bar\partial_E$, such that a section $\xi \in \Omega^0(X, E)$ is holomorphic if and only if $\bar\partial_E \xi = 0$.

Definition 1.5.1. Suppose that the holomorphic structure on E comes from a particular operator $\overline{\partial}_E$. A connection is called *compatible* with this complex structure if

$$D^{0,I} = \overline{\partial}_E. \tag{1.13}$$

Suppose that H is a hermitian metric on the bundle E. A connection D is unitary with respect to H, or is compatible with H, if for all $\xi, \eta \in \Omega^0(X, E)$,

$$d(\xi, \eta)_H = (D\xi, \eta)_H + (\xi, D\eta)_H. \tag{1.14}$$

The space of connections compatible with H is denoted by $\mathscr{A}(H)$.

1.6. It is an important fact that given $\overline{\partial}_E$ and H, there is a unique connection on E that is compatible with *both.* We will denote this so-called *metric connection* by $D_{\overline{\partial}_E, H}$. Its $(0, 1)$ part is clearly $\overline{\partial}_E$, and we will denote its $(1, 0)$ part by D'_H. Hence

$$D_{\overline{\partial}_E, H} = \overline{\partial}_E + D'_H. \tag{1.15}$$

Proposition 1.6.1. *Suppose that two hermitian metrics H and K are related by $H = Kh$ where h is a positive self-adjoint section of* $\operatorname{End} E$. *Then*

$$D_{\overline{\partial}_E, H} = D_{\overline{\partial}_E, K} + h^{-1} D'_K(h). \tag{1.16}$$

The curvatures $F_{\overline{\partial}_E, K}$ and $F_{\overline{\partial}_E, H}$ are related by

$$F_{\overline{\partial}_E, H} = F_{\overline{\partial}_E, K} + \overline{\partial}_E(h^{-1} D'_K(h)). \tag{1.17}$$

Proof. This can be verified via a local calculation using a holomorphic frame for E. See [2] for details.

Proposition 1.6.2. *The components of $D_{\overline{\partial}_E, H}$ satisfy the Kähler identities*

$$\sqrt{-1}[\Lambda, \overline{\partial}_E] = D_H'^{*_H}, \tag{1.18a}$$

$$-\sqrt{-1}[\Lambda, D'_H] = \overline{\partial}_E^{*_H}. \tag{1.18b}$$

Proof. Cf. [9].

1.7. These relations between the $(0, 1)$ and $(1, 0)$ pieces of a connection hold more generally than just for metric connections. In fact, as the next result shows, they constitute an alternative characterization of unitary connections.

Proposition 1.7.1. *Let $D = D^{0,1} + D^{1,0}$ be a connection on a holomorphic bundle E. D is compatible with an hermitian metric H if and only if*

$$\sqrt{-1}[\Lambda, D^{0,1}] = (D^{1,0})^{*_H} \tag{1.19a}$$

and

$$-\sqrt{-1}[\Lambda, D^{1,0}] = (D^{0,1})^{*_H}. \tag{1.19b}$$

Proof. Cf. Proposition 2.4(3) in [2].

1.8. If $D = D^{0,1} + D^{1,0}$ is an integrable unitary connection, we can define the following Laplace operators.

Definition 1.8.1.

$$\Delta = D^* D + D D^*, \tag{1.20a}$$

$$\Delta' = (D^{1,0})^* D^{1,0} + D^{1,0}(D^{1,0})^*, \tag{1.20b}$$

$$\Delta'' = (D^{0,1})^* D^{0,1} + D^{0,1}(D^{0,1})^*. \tag{1.20c}$$

The adjoints are all with respect to the given hermitian bundle metric.

These are all second order elliptic operators. The Kähler identities give the following formulas.

Lemma 1.8.2.

$$\Delta' = \Delta'' + \sqrt{-1}[\Lambda, F_D], \tag{1.21a}$$

$$\Delta = \Delta' + \Delta''$$

$$= 2\Delta' + \sqrt{-1}[\Lambda, F_D] \tag{1.21b}$$

$$= 2\Delta'' - \sqrt{-1}[\Lambda, F_D]. \tag{1.21c}$$

When acting on sections of E these formulas simplify to

$$\Delta' = \sqrt{-1}\Lambda D^{0,1} D^{1,0}, \tag{1.22a}$$

$$\Delta'' = -\sqrt{-1}\Lambda D^{1,0} D^{0,1}, \tag{1.22b}$$

$$\Delta = \sqrt{-1}\Lambda(D^{0,1} D^{1,0} - D^{1,0} D^{0,1}). \tag{1.22c}$$

Proof. Use the Kähler identities (1.17) and the fact that if D is integrable, then $(D^{0,1})^2 = (D^{1,0})^2 = 0$.

2. Necessary conditions

We will assume that E has a fixed holomorphic structure, i.e., we will take as given an operator

$$\bar\partial_E \colon \Omega^0(X, E) \to \Omega^{0,1}(X, E)$$

as in §1.5. We assume further that E has nontrivial holomorphic global sections. Let $\phi \in \Omega^0(X, E)$ be such a prescribed section of E. So ϕ satisfies

$$\overline{\partial}_E \phi = 0. \tag{2.1}$$

Our goal is to look for a reasonable criterion whereby $\overline{\partial}_E$ and ϕ together might determine a preferred hermetian metric on E. We will then seek to understand this criterion by studying the necessary and sufficient conditions for it to be satisfied.

We approach the first problem, i.e., of finding hermitian metrics determined by ϕ and $\overline{\partial}_E$, as follows. We use the fact that hermitian metrics on holomorphic bundles determine connections. Hence a criterion for picking out a metric can be formulated in terms of an equation to be satisfied by the curvature of the corresponding metric. This ties the data (viz. $\overline{\partial}_E$ and ϕ) directly to the geometry of E and, via the Chern-Weil homomorphism, to its topology. The equation we propose is

$$\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \mathrm{const.}\,\mathbf{I} = 0. \tag{2.2}$$

Here F_H is the curvature of the metric connection determined by $\overline{\partial}_E$ and a hermitian metric H, ϕ^{*H} is the adjoint of ϕ with respect to H, and $\mathbf{I} \in \Omega^0(X, \mathrm{End}\, E) \approx \Omega^0(X, E \otimes E^*)$ is the identity section.

Without the term involving ϕ, (2.2) is the Hermitian-Einstein equation. In other words, if $\phi = 0$ this equation is the criterion according to which the holomorphic structure alone determines preferred bundle metrics. Equation (2.2) is thus, at the very least, a reasonable candidate for a criterion by which to select preferred metrics on E. In fact, as the results of the next few subsections show, (2.2) does indeed pick out an interesting class of bundle metrics. We have called this equation *the vortex equation.* The reason for the name has to do with the role of the equation as one of the equations governing the minima of a functional in Gauge Theory (cf. [2]). We will say more about this in §4. We will write the constant in (2.2) as $\frac{\sqrt{-1}}{2}\tau$.

The problem is now that of understanding the necessary and sufficient conditions for the existence of solutions to the vortex equation. We first discuss the necessary conditions. We show that if the system $(E, \overline{\partial}_E, \phi)$, i.e., the bundle E with holomorphic structure given by the operator $\overline{\partial}_E$ and with prescribed section ϕ, supports a solution to the vortex equation, then a constraint is imposed on subsheaves of the sheaf of germs of holomorphic sections of E. The constraint is expressed in terms of the slopes, or normalized degrees, of the subsheaves.

2.1. Statement of result.

Definition 2.1.1. Let $\mathscr{F}$ be a rank R torsion free coherent analytic sheaf over a compact Kähler manifold X. Let ω be the Kähler form on X.

The first Chern class of $\mathscr{F}$ is defined to be the first Chern class of the determinant line bundle $\det(\mathscr{F})$, i.e.,

$$c_1(\mathscr{F}) = c_1(\det(\mathscr{F})). \tag{2.3}$$

We will use $c_1(\mathscr{F})$ to denote both the element in cohomology and a real closed $(1, 1)$ form on X which represents it.

The degree of $\mathscr{F}$ depends on the Kähler form ω and is defined by

$$\deg(\mathscr{F}, \omega) = \int_X c_1(E) \wedge \omega^{[n-1]}, \tag{2.4}$$

where $\omega^{[m]} = \omega^m/m!$.

The slope, $\mu(\mathscr{F})$, is defined by

$$\mu(\mathscr{F}) = \frac{\deg(\mathscr{F}, \omega)}{R}. \tag{2.5}$$

Definition 2.1.2. A reflexive sheaf $\mathscr{E}$ of rank R is called semistable if for every subsheaf $\mathscr{E}'$ such that the quotient $\mathscr{E}/\mathscr{E}'$ is torsion free, we have

$$\mu(\mathscr{E}') \leq \mu(\mathscr{E}). \tag{2.6}$$

$\mathscr{E}$ is called stable if the inequality is strict whenever $0 < \operatorname{rank}(\mathscr{E}') < R$.

Note. It is enough (cf. [16]) to consider reflexive subsheaves in the definition of stability. This is because if $\mathscr{E}/\mathscr{E}'$ is torsion free, then $\mathscr{E}'$ is reflexive.

As is the case for the Hermitian-Einstein equation, we can likewise expect solutions of the vortex equation to be related to stability properties of the system $(E, \overline{\partial}_E, \phi)$. This is in fact so, but the connection between existence of solutions and stability of the bundle is somewhat more subtle than in the former case. We need to define the following parameters on $(E, \overline{\partial}_E, \phi)$:

Definitions 2.1.3. Let $\mathscr{E}$ be the sheaf of germs of holomorphic sections of E, R its rank, and $\mu = \mu(\mathscr{E})$ its slope. Let

$$\hat{\mu}_M = \operatorname{Sup}\{\mu(\mathscr{E}')/\mathscr{E}' \subset \mathscr{E} \text{ is a reflexive subsheaf of rank } R' \text{ with } 0 < R' < R\},$$

$$\mu_M = \max\{\hat{\mu}_M, \mu\},$$

$$\mu_m(\phi) = \operatorname{Inf}\left\{ \frac{R\mu - R'\mu(\mathscr{E}')}{R - R'} / \mathscr{E}' \subset \mathscr{E} \text{ is a reflexive subsheaf} \right.$$

$$\left. \text{of rank } R' \text{ with } 0 < R' < R \text{ and } \phi \in \mathscr{E}' \right\}.$$

The next proposition shows that the parameters μ_M and $\mu_M(\phi)$ can indeed be interpreted as measures of stability properties. The parameter μ_M carries information about the stability of the bundle E itself, while the parameter $\mu_m(\phi)$ measures the stability of the quotient of $\mathscr{E}$ by the sheaf generated by the section ϕ. The section ϕ generates a renk one subsheaf of $\mathscr{E}$ via the injection

$$\phi: \mathbf{O}_X \to \mathscr{E},$$

where $\mathbf{O}_X$ is the structure sheaf on X. The quotient $\mathscr{E}/\phi(\mathscr{O}_X)$ is not necessarily torsion free, however one can always extend $\phi(\mathbf{O}_X)$ to a rank one torsion free subsheaf, $[\phi]$, such that the quotient $\mathscr{E}/[\phi]$ *is* torsion free. The subsheaf $[\phi]$ is called the saturation of $\phi(\mathbf{O}_X)$. The parameter $\mu_m(\phi)$ is a measure of the stability of $\mathscr{E}/[\phi]$.

Proposition 2.1.4. (i) $\mu_M \geq \mu$, *with equality occurring if and only if the bundle* E *is semistable.*

(ii) $\mu_m(\phi) \leq \mu(\mathscr{E}/[\phi])$, *with equality occurring if and only if* $\mathscr{E}/[\phi]$ *is semistable.*

Proof. (i) This follows straight from the definitions of stability and μ_M.

(ii) The inequality $\mu_m(\phi) \leq \mu(\mathscr{E}/[\phi])$ holds by definition. Furthermore, having $\mu_m(\phi) = \mu(\mathscr{E}/[\phi])$ is equivalent to having

$$\frac{R\mu - R'\mu(\mathscr{E}')}{R - R'} \geq \mu(\mathscr{E}/[\phi]), \tag{2.7}$$

whenever $\mathscr{E}'$ is a reflexive subsheaf of $\mathscr{E}$ with $0 < R' < R$ and $\phi \in \mathscr{E}'$.

By using the fact that

$$\mu(\mathscr{E}/[\phi]) = \frac{R\mu - \mu([\phi])}{R - 1}, \tag{2.8}$$

inequality (2.7) can be rewritten as

$$\frac{R\mu - \mu([\phi])}{R - 1} \geq \frac{R'\mu(\mathscr{E}) - \mu([\phi])}{R' - 1}. \tag{2.9}$$

That is,

$$\mu(\mathscr{E}/[\phi]) \geq \mu(\mathscr{E}'/[\phi]). \tag{2.10}$$

Since this is precisely the condition for the semistability of $\mathscr{E}/[\phi]$, the proposition is proved.

Definition 2.1.5. A holomorphic bundle E with prescribed holomorphic section ϕ is ϕ-stable if and only if

$$\mu_M < \mu_m(\phi). \tag{2.11}$$

It is apparent that ϕ-stability is a property of a bundle endowed with a holomorphic structure and a prescribed holomorphic section. We will thus talk of ϕ-stability of the triple $(E, \overline{\partial}_E, \phi)$. If the holomorphic structure is implied, we will sometimes omit the $\overline{\partial}$ operator and talk of the ϕ-stability of the pair (E, ϕ).

The main result of this section can now be stated:

Theorem 2.1.6. *Let E be a holomorphic bundle with a prescribed holomorphic section ϕ. Suppose that for a given value of the real parameter $\tau > 0$ there exists a hermitian metric H satisfying*

$$\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\tau \mathbf{I} = 0. \tag{2.12}$$

Then either

(a) *(E, ϕ) is ϕ-stable and $\mu_M < \tau \operatorname{Vol}(X)/4\pi < \mu_m(\phi)$, or*

(b) *the bundle splits as $E = E_\phi \oplus E'$ into a direct sum of holomorphic bundles, one of which contains the section ϕ.*

In case (b), *the part containing ϕ (denoted E_ϕ) is ϕ-stable and satisfies the inequality in* (a). *The remaining summands (which together comprise E') are all stable and each of slope $\tau \operatorname{Vol}(X)/(4\pi)$.*

To prove this theorem we need to examine the slopes of subsheaves of the sheaf of germs of holomorphic sections of E. We begin by considering holomorphic *subbundles* of E. Later we will show that the conclusions we draw for subbundles apply unchanged to reflexive subsheaves. First we recall some general facts about the geometry of holomorphic subbundles (cf. [9]).

2.2. Background. Suppose that the holomorphic structure on a bundle E is given by the operator $\overline{\partial}_E$, that H is a metric on E, and that $D_{\overline{\partial}_E, H}$ is the corresponding metric connection. Denote the rank of E by R. Let E' be a holomorphic subbundle of rank $R' < R$. Let $\{e_1, \cdots, e_R\}$ be a local unitary frame chosen such that $\{e_1, \cdots, e_{R'}\}$ is a local basis for sections of E'. Let

$$D_{\overline{\partial}_E, H} e_a = \theta_{ab} e_b. \tag{2.13}$$

With respect to this local frame, the matrix θ splits into four blocks as

$$\theta = \begin{pmatrix} \theta' & A \\ -\overline{A}^{\mathrm{T}} & \theta^{\perp} \end{pmatrix}. \tag{2.14}$$

Here θ' is the connection matrix for the metric connection coming from the restriction of H and $\overline{\partial}_E$ to E', and $\theta^{\perp}$ gives a connection on the orthogonal complement of E'. The matrix valued 1-form A is the second fundamental form and $\overline{A}^{\mathrm{T}}$ is its conjugate transpose (cf. [9]). If

$$\pi: \Omega^0(X, E) \to \Omega^0(X, E')$$

denotes projection onto the subbundle, then

$$A \equiv (1-\pi) D_{\overline{\partial}_E, H}(\pi). \tag{2.15}$$

A is of type $(1, 0)$ since E' is a *holomorphic* subbundle.

Let F_H, F', and $F^{\perp}$ be the curvatures of $D_{\overline{\partial}_E, H}$, the metric connection on E', and the connection on $E^{\perp}$ respectively. Then

$$F_H = d\theta - \theta \wedge \theta. \tag{2.16}$$

From (2.14) we see that

$$F_H = \begin{pmatrix} F' + A \wedge \overline{A}^{\mathrm{T}} & * \\ * & F^{\perp} + \overline{A}^{\mathrm{T}} \wedge A \end{pmatrix}. \tag{2.17}$$

For future reference, we define

$$c_A \equiv \operatorname{Tr} \sqrt{-1} \Lambda (A \wedge \overline{A}^{\mathrm{T}}), \tag{2.18a}$$

$$c_A^* \equiv \operatorname{Tr} \sqrt{-1} \Lambda (\overline{A}^{\mathrm{T}} \wedge A), \tag{2.18b}$$

$$C_A \equiv \int_X c_A \, d\,\mathrm{vol} = \int_X \operatorname{Tr} \sqrt{-1} \Lambda (A \wedge \overline{A}^{\mathrm{T}}) \, d\,\mathrm{vol}. \tag{2.18c}$$

We will need the following result.

Lemma 2.2.1. (i) $C_A \geq 0$.

(ii) $\int_X c_A^* d\,\mathrm{vol} = \int_X \operatorname{Tr} \sqrt{-1} \Lambda (\overline{A}^{\mathrm{T}} \wedge A) \, d\,\mathrm{vol} = -C_A$.

Proof. Both results follow from the pointwise properties of the integrands. Fix a point x_0 in X and choose local coordinates $\{z_i\}_{i=1}^n$ in a neighborhood U_x of x_0 such that at x_0, $\{dz_i\}_{i=1}^n$ and $\{d\overline{z}_i\}_{i=1}^n$ are orthogonal with respect to the Kähler metric. Normalize so that

$$|dz_i|^2 = |d\overline{z}_i|^2 = 2. \tag{2.19}$$

Then at x_0

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i. \tag{2.20}$$

(i) Since A is of type $(1, 0)$, we can write

$$A(x_0) = A^i(x_0)\, dz_i \tag{2.21}$$

where $A^i \in \Omega^0(U_x, \operatorname{End} E)$. A^i is thus a matrix-valued function on U_x. We get

$$\overline{A}^{\mathrm{T}}(x_0) = A^{i*}(x_0)\, d\overline{z}_i \tag{2.22}$$

and

$$\operatorname{Tr}\sqrt{-1}\Lambda(A \wedge \overline{A}^{\mathrm{T}})(x_0) = \frac{1}{2}\sum_{i=1}^{n} \operatorname{Tr} A^i(x_0)A^{i*}(x_0) = \frac{1}{2}\sum_{i=1}^{n} |A^i(x_0)|^2.$$

(ii) This follows immediately from the fact that $dz_i \wedge d\overline{z}_j = -d\overline{z}_j \wedge dz_i$.

2.3 Calculations in local coordinates. Notice that with respect to the basis $\{e_a\}$, ϕ can be written

$$\phi = \sum_{a=1}^{R} \phi_a e_a. \tag{2.23}$$

Let

$$\phi' = \sum_{a=1}^{R'} \phi_a e_a \quad \text{and} \quad \phi^{\perp} = \sum_{a=R'+1}^{R} \phi_a e_a. \tag{2.24}$$

Then in local coordinates

$$\sqrt{-1}\Lambda F_H + \tfrac{1}{2}\phi \otimes \phi^* - \tfrac{\tau}{2}\mathbf{I} = 0$$

has the form

$$\begin{pmatrix} \sqrt{-1}\Lambda F' + \sqrt{-1}\Lambda(A \wedge \overline{A}^{\mathrm{T}}) & * \\ * & \sqrt{-1}\Lambda F^{\perp} + \sqrt{-1}\Lambda(\overline{A}^{\mathrm{T}} \wedge A) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \phi' \otimes \phi'^* & * \\ * & \phi^{\perp} \otimes \phi^{\perp *} \end{pmatrix} - \frac{\tau}{2}\mathbf{I} = 0. \tag{2.25}$$

We thus get the following two equations:

$$\sqrt{-1}\Lambda F' + \sqrt{-1}\Lambda(A \wedge \overline{A}^{\mathrm{T}}) + \tfrac{1}{2}(\phi' \otimes \phi'^*) - \tfrac{\tau}{2}\mathbf{I}_{R'} = 0, \tag{2.26a}$$

$$\sqrt{-1}\Lambda F^{\perp} + \sqrt{-1}\Lambda(\overline{A}^{\mathrm{T}} \wedge A) + \tfrac{1}{2}(\phi^{\perp} \otimes \phi^{\perp *}) - \tfrac{\tau}{2}\mathbf{I}_{R-R'} = 0. \tag{2.26b}$$

2.4. Proof of Theorem 2.1.6 for subbundles. If we take the trace of equations (2.26) and let

$$|\phi'|^2 = \sum_{a=1}^{R'} |\phi_a|^2, \qquad |\phi^{\perp}|^2 = \sum_{a=R'+1}^{R} |\phi_a|^2, \tag{2.27}$$

then we get the equations

$$\text{(2.28a)} \qquad \operatorname{Tr}(\sqrt{-1}\Lambda F') + c_A + \frac{1}{2}|\phi'|^2 - \frac{R'\tau}{2} = 0,$$

$$\text{(2.28b)} \qquad \operatorname{Tr}(\sqrt{-1}\Lambda F^{\perp}) + c_A^* + \frac{1}{2}|\phi^{\perp}|^2 - (R-R')\frac{\tau}{2} = 0.$$

Furthermore, since $E' \oplus E^{\perp} = E$ (topologically),

$$\text{(2.28c)} \qquad \operatorname{Tr}(\sqrt{-1}\Lambda F') + \operatorname{Tr}(\sqrt{-1}\Lambda F^{\perp}) = \operatorname{Tr}\sqrt{-1}\Lambda F_H.$$

Recall that if ω is the Kähler form on X, then

$$\deg(E', \omega) = \int_X \Lambda c_1(E')\, d\,\mathrm{vol} = \frac{1}{2\pi}\int_X \operatorname{Tr}(\sqrt{-1}\Lambda F')\, d\,\mathrm{vol},$$

with similar formulas for $\deg(E^{\perp}, \omega)$ and $\deg(E, \omega)$. (2.28a) and part (i) of Lemma 2.2.1 thus establish that either $\mu(E') < \tau\operatorname{Vol}(X)/(4\pi)$, or C_A and $\int_X |\phi'|^2\, d\,\mathrm{vol}$ are both zero. In the latter case E splits holomorphically as $E = E' \oplus E^{\perp}$ and ϕ is a section on $E^{\perp}$.

It follows from (2.28b) and (2.28c) that if $|\phi^{\perp}|^2 = 0$, i.e., ϕ is a section of E', then either

$$\frac{R\mu - R'\mu(E')}{R - R'} > \frac{\tau\operatorname{Vol}(X)}{4\pi},$$

or $C_A = 0$, in which case E splits holomorphically as $E = E' \oplus E^{\perp}$.

Finally, by taking the trace of the original vortex equation and integrating over X, we see that

$$\mu < \frac{\tau\operatorname{Vol}(X)}{4\pi}.$$

This concludes the proof of part (a) of the theorem where μ_M and $\mu_m(\phi)$ are calculated using subbundles of E. Part (b) is now clear since if E splits holomorphically as $E = E_\phi \oplus E''$, then the vortex equation splits into a vortex equation with the same ϕ and τ on E_ϕ and a Hermitian-Yang-Mills equation

$$\text{(2.29)} \qquad \sqrt{-1}\Lambda F_H = \tfrac{\tau}{2}\mathbf{I}$$

on E''. The result follows from the fact that a bundle which supports a solution to (2.29) must be a direct sum of stable bundles, each of slope $\tau\operatorname{Vol}(X)/(4\pi)$ (cf. [13]).

2.5. Extension to subsheaves. We now show that the above calculations (and therefore the conclusions) can be extended to reflexive subsheaves of $\mathscr{E}$. The basic reason is because off a set of complex codimension 2 in X, a reflexive subsheaf is locally free, i.e., is a subbundle:

Lemma 2.5.1 [16]. *Let $\mathscr{E}' \subset \mathscr{E}$ be a reflexive subsheaf of $\mathscr{E}$ and let Σ, X be the singularity set of $\mathscr{E}'$. Then the (complex) codimension of Σ is at least 2. If $E' = \mathscr{E}|_{X-\Sigma}$, then E' is locally free, i.e., is a subbundle of E restricted to $X - \Sigma$.*

Definitions 2.5.2. Let $\pi\colon \mathscr{E}' \to \mathscr{E}$ be projection of E onto E' where E' is defined (i.e., on $X - \Sigma$) and let

$$C_1(\mathscr{E}') = \frac{\sqrt{-1}}{2\pi}(\operatorname{Tr}(\pi F_E) + \operatorname{Tr}(\overline{\partial}_E(\pi) \wedge D'(\pi)). \tag{2.30}$$

Lemma 2.5.3. (i) $\pi^2 = \pi$, $\pi^* = \pi$, *and* $(1-\pi)\overline{\partial}_E(\pi) = 0$.

(ii) *If K is a metric on E and F'_K is the curvature of the metric connection on E' induced by K and $\overline{\partial}_E$, then* $\sqrt{-1}\Lambda C_1(E') = \sqrt{-1}\operatorname{Tr}(\Lambda F'_K)$.

(iii) $\sqrt{-1}\int_X C_l(\mathscr{E}') = \deg(\mathscr{E}', \omega)$.

(iv) $\pi \in L_1^2(\operatorname{End} E)$.

Proof. (i) This is clear since on $X - \Sigma$ π is a projection onto a holomorphic subbundle.

(ii) On $X - \Sigma$, where E' is defined, π is smooth and thus the result follows directly from the formula for $\operatorname{Tr}(F'_K)$, viz;

$$\operatorname{Tr}(F'_K) = \operatorname{Tr}(\pi \cdot F_K \cdot \pi) + \operatorname{Tr}(\overline{\partial}_E(\pi) \wedge D'_K(\pi)). \tag{2.31}$$

(iii)–(iv) are proved by Simpson in [19]. One shows that $C_1(\mathscr{E}')$ is a closed current on X and that it represents the first Chern class of $\mathscr{E}$ in $H^2(X, \mathbf{C})$. The fact that $\operatorname{codim}(\Sigma) \geq 2$ is crucial here.

Remark. There is a converse to Lemma 2.5.3 which will be important when we look at the sufficient conditions for solutions to the vortex equation. The result, due to Uhlenbeck and Yau [21], gives an analytic characterization of reflexive subsheaves in terms of "L_1^2 projections, π" which satisfy properties (i) and (iv) of Lemma 2.5.3 (cf. §3.11).

2.6. Proof of Theorem 2.1.6 for subsheaves. Let $\mathscr{E}'$ be a reflexive subsheaf of $\mathscr{E}$. By Lemma 2.5.1 $E' = \mathscr{E}'|_{X-\Sigma}$ is a subbundle of $E|_{X-\Sigma}$. The results of §2.4 apply to E', and by Lemma 2.5.3 the following conclusions carry over to $\mathscr{E}'$:

(a) $\mu(\mathscr{E}') \leq \tau \operatorname{Vol}(X)/4\pi$.

(b) Suppose $\phi \in \mathscr{E}'$ (and therefore $\phi \in \Omega^0(E')$). Then

$$\frac{R\mu - R'\mu(E')}{R - R'} \geq \frac{\tau \operatorname{Vol}(X)}{4\pi}.$$

(c) The conditions for equality in (a) or (b) are the same as in §2.4, viz. $c_A = 0$ and $\phi \in \Omega^0(X, E^\perp)$.

Now $c_A = 0$ implies that $A = 0$, i.e., $(1 - \pi)D'_H(\pi) = 0$. It follows by the Kähler identities and the fact that $\pi^* = \pi$ that $\pi\overline{\partial}_E(1 - \pi) = 0$. In fact Lemma 2.5.3 holds with $\pi^\perp = (1 - \pi)$ in place of π, and thus (cf. Theorem 3.11.2) $\pi^\perp$ defines a reflexive subsheaf $\mathscr{E}^\perp$ which is equal to the orthogonal complement of E' on $X - \Sigma$. Furthermore, $\overline{\partial}_E(\pi) = 0$ and hence $\Delta''(\pi) = 0$, where the $\overline{\partial}$-Laplacian Δ'' is $\overline{\partial}_E * \overline{\partial}_E$ on 0-forms. Elliptic regularity for linear operators (cf. [7]) can now be used to show that π is in fact smooth. This means that $\mathscr{E}'$ and $\mathscr{E}^\perp$ correspond to subbundles of E and that equality in (1) or (2) occurs only when E splits as $E = E' \oplus E^\perp$ and $\phi \in E^\perp$.

3. Sufficient conditions

In this section we will prove the following theorem.

Theorem 3.1.1. *Let E be a holomorphic vector bundle over a closed Kähler manifold X. Let ϕ be a prescribed holomorphic section of E and suppose that (E, ϕ) is ϕ-stable, i.e., $\mu_M < \mu_m(\phi)$. Let τ be any real number satisfying $\mu_M < \tau \operatorname{Vol}(X)/4\pi < \mu_m(\phi)$. Then there exists a smooth hermitian metric H such that the vortex equation* (2.12) *is satisfied, i.e.,*

$$\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*_H} + \frac{\sqrt{-1}}{2}\tau \mathbf{I} = 0.$$

3.1 Outline of proof. Before presenting the details, we briefly describe the structure of the proof. The proof of this theorem is very similar to Simpson's proof in [18] (cf. also [19]) that a stable Higgs bundle supports a metric compatible with the holomorphic structure and the map $\theta\colon \Omega^0(X, E) \to \Omega^{1,0}(X, E)$ which define the bundle. When appropriate we will refer the reader to [18] for the details that carry over directly to this proof.

Our goal is to find a Hermitian bundle metric which satisfies the vortex equation (2.12). For this purpose we introduce a real valued functional $\mathbf{M}_{\phi,\tau}(K, H)$ defined on pairs of bundle metrics and with the following two properties. With the first argument fixed, the functional $\mathbf{M}_{\phi,\tau}(K, \text{-})$ becomes a convex functional, and is minimum corresponds to a zero of the vortex equation. We can thus study solutions of the vortex equation by studying this functional. In particular we can understand the necessary conditions for the existence of vortex solutions by investigating the necessary conditions for $\mathbf{M}_{\phi,\tau}(K, \text{-})$ to attain its minimum.

To facilitate the analysis, we choose $p > 2n$ and fix a smooth background metric K. Recall; that any metric H can be related to K by a positive, self-adjoint bundle endomorphism. We define

$$\mathcal{M}et_2^p = \{H = Ke^s | s \in L_2^p(S(K))\}, \tag{3.1}$$

where

$$S(K) = \{s \in \Omega^0(X, \operatorname{End} E) | s^{*_K} = s\}, \tag{3.2}$$

and $L_2^p(S(K))$ denotes those elements of $S(K)$ with finite Sobolev L_2^p norm.

Initially we look for a minimum of $\mathbf{M}_{\phi,\tau}(K, \text{-})$ (and therefore a zero of the vortex equation) on $\mathcal{M}et_2^p$. We then use elliptic regularity to show that the metric obtained is in fact smooth. An important tool is an estimate of the C^0 norm for $s \in L_2^p(S(K))$. The estimate is given by the functional $\mathbf{M}_{\phi,\tau}(K, \text{-})$ and is of the form

$$\sup|s| \le C_1\mathbf{M}_{\phi,\tau}(K, Ke^s) + C_2. \tag{3.3}$$

Following Simpson [18] we show that either constants C_1, C_2 can be found such that an estimate of this sort holds or the bundle is not ϕ-stable. If such an estimate does hold then $\mathbf{M}_{\phi,\tau}(K, \text{-})$ is bounded below and we can show that taking a minimizing sequence $\{s_i\}_{i=1}^{\infty}$ leads to a minimizing metric $H = Ke^{s_\infty}$.

3.2. Definition of $\mathbf{M}_{\phi,\tau}$. Let $\mathbf{M}_D(\ ,\ )$ denote the functional defined by Donaldson [4, 5] on pairs of metrics. Fix a smooth background metric K on E and let $\mathbf{M}_D\colon \mathcal{M}et_2^p \to \mathbf{R}$ be given by

$$\mathbf{M}_D(H) = \mathbf{M}_D(K, H). \tag{3.4}$$

The functional $\mathbf{M}_D$ is defined in terms of Bott-Chern classes. Its most important property is that it acts as a potential function for the vector field $-2\sqrt{-1}\Lambda F$. By this we mean the following: If δH and $\delta\mathbf{M}_D$ denote the variation of H and $\mathbf{M}_D$ respectively, then

$$\delta\mathbf{M}_D = 2\sqrt{-1}\int_X H^{-1}\delta H \Lambda F_H. \tag{3.5}$$

Definition 3.2.1. Let ϕ be a prescribed section of E and let τ be a real parameter. Define $\mathbf{M}_{\phi,\tau}(\text{-}, \text{-})$ on pairs of metrics by

$$\mathbf{M}_{\phi,\tau}(K, H) = \mathbf{M}_D(K, H) + \|\phi\|_H^2 - \|\phi\|_K^2 - \tau\int_X \operatorname{Tr}(\log K^{-1}H), \tag{3.6}$$

where $\|\phi\|_H^2 = \int_X(\phi, \phi)_H$. If K is a fixed smooth background metric on E, let $\mathbf{M}_{\phi,\tau}\colon \mathcal{M}et_2^p \to \mathbf{R}$ be given by

$$\mathbf{M}_{\phi,\tau}(H) = \mathbf{M}_{\phi,\tau}(K, H). \tag{3.7}$$

There are explicit forms for $\mathbf{M}_D(H)$ and $\mathbf{M}_{\phi,\tau}(H)$ when $H = Ke^s$ and $s \in L_2^p(S(K))$ (i.e., when $H \in \mathcal{M}et_2^p$). To describe these it is convenient to make use of the following constructions on $S(K)$ (cf. [18]).

Let $f\colon \mathbf{R} \to \mathbf{R}$ be a smooth function. Define $f\colon S(K) \to S(K)$ as follows. Suppose that $s \in S(K)$. At each point x on X chose an orthonormal basis $\{e_a\}_{a=1}^R$ for local sections of E such that $s(e_a) = \sum_{a=1}^R \lambda_a e_a$. So the λ_a are the eigenvalues of s. Set

$$f(s)(e_a) = \sum_{a=1}^{R} f(\lambda_a)e_a. \tag{3.8}$$

Let $F\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a smooth function of two variables. In a similar way we can define the smooth map $F\colon S(K) \to S_K(\operatorname{End} E)$, where $S_K(\operatorname{End} E) = \{A \in \operatorname{End}(\operatorname{End} E) | A^* = A$ with respect to the metric induced by $K\}$. If $\{e_a^*\}_{a=1}^R$ is the dual bases for local sections of E^*, then $A \in \Omega^0(X, \operatorname{End} E)$ can be written

$$A = \sum_{a,b=1}^{R} A_{ab} e_a^* \otimes e_b. \tag{3.9}$$

Define

$$F(s)A = \sum_{a,b=1}^{R} F(\lambda_a, \lambda_b) A_{ab} e_a^* \otimes e_b. \tag{3.10}$$

We will need to know how these constructions behave on L_k^p spaces. The relevant properties are given in the next proposition (cf. [18], also [17]).

Proposition 3.2.2. *Let $L_{k,b}^p \subset L_k^p$ be the subset of sections $s \in L_k^p$ such that $|s| \le b$ almost everywhere. Let $f\colon S(K) \to S(K)$ and $F\colon S(K) \to S_K(\operatorname{End} E)$ be as above. These maps are smooth on $S(K)$. Furthermore, we have the following.*

(i) *The map f extends to a continuous map*

$$f\colon L_{0,b}^p(S(K)) \to L_{0,b'}^p(S(K))$$

for some b'.

(ii) *The map F extends to a map*

$$F\colon L_{0,b}^p(S(K)) \to \operatorname{Hom}(L^p(\operatorname{End} E), L^q(\operatorname{End} E))$$

for $q \le p$. For $q < p$ the map is continuous in the operator norm topology.

(iii) *If* L *denotes a Sobolev space such that* $L \subset C^0$, *then the maps*

$$f\colon L(S(K)) \to L(S(K)),$$
$$F\colon L(S(K)) \to \mathrm{Hom}(L(\Gamma(\mathrm{End}\,E)), L(\Gamma(\mathrm{End}\,E)))$$

are smooth.

(iv) *Define* $df\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *by*

$$df(x, y) = \begin{cases} (f(x) - f(y))/(x-y) & \text{if } x \neq y, \\ df/dx & \text{if } x = y. \end{cases}$$

Let $D^{0,1}$ *be the* $(0, 1)$ *part of a connection on* E. *Then for* $s \in L^p_{1,b}(S(K))$,

$$D^{0,1} f(s) = df(s)(D^{0,1}s).$$

In the next lemma and in the rest of the proof, p is chosen such that $p > 2n$ and p is even.

Lemma 3.2.3. *Define a smooth function* $\psi\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *by*

$$\Psi(x, y) = \frac{e^{y-x} - (y-x) - 1}{(y-x)^2}. \tag{3.11}$$

Let Ψ *define a map* $\Psi\colon S(K) \to S_K(\mathrm{End}\,E)$ *as in* (3.10). *Then for any* $s \in L^p_2(S(K))$,

$$\mathbf{M}_D(Ke^s) = 2\sqrt{-1}\int_X \mathrm{Tr}\, s\Lambda F_K + 2\int_X (\Psi(s)\bar\partial_E s, \bar\partial_E s)_K. \tag{3.12}$$

Proof. Essentially this result can be found in [5] and also as Lemma 5.2.1 in [18]. The only difference is that there s is constrained to be *trace free.* However the trace of s does not enter into the proof in any way and the conclusion is valid for any $s \in L^p_2(S(K))$.

The proof is a calculation based on the fact that along a path $H_t = Ke^{ts}$ in $\mathscr{M}et^p_2$ we have

$$\frac{d}{dt}\mathbf{M}_D(H_t) = 2\sqrt{-1}\int_X \mathrm{Tr}(s\Lambda F_{H_t}).$$

Furthermore, using (1.17) we calculate that

$$\frac{d^2}{dt^2}\mathbf{M}_D(H_t) = 2\int_X |\bar\partial_E s|^2_{H_t}. \tag{3.13}$$

If we define $\mathscr{F}_t\colon \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ by $\mathscr{F}_t(x, y) = e^{t(x-y)}$, then we can write

$$\frac{d^2}{dt^2}\mathbf{M}_D(H_t) = 2\int_X (\mathscr{F}_t(s)\bar\partial_E(s), \bar\partial_E(s))_K.$$

Integrating this expression twice completes the proof.

Remark. We point out that (3.13) shows that $\mathbf{M}_D$ is a *convex* functional, i.e., along any path $H_t = Ke^{ts}$, it satisfies $(d^2/dt^2)\mathbf{M}_D(H_t) \geq 0$.

Lemma 3.2.4. *If* $H = Ke^s$ *with* $s \in L_2^p(S(K))$, *then*

$$\mathbf{M}_{\phi,\tau}(H) = 2\sqrt{-1}\int_X \operatorname{Tr} s\Lambda F_K + 2\int_X (\Psi(s)\overline{\partial}_E s, \overline{\partial}_E s)_K + \int_X (e^s\phi, \phi)_K - \|\phi\|_K^2 - \tau\int_X \operatorname{Tr} s.$$

Proof. This follows immediately since $(\phi, \phi)_H = (e^s\phi, \phi)_K$ and $\log K^{-1}H = \log e^s = s$.

3.3 Properties of $\mathbf{M}_{\phi,\tau}$. For convenience we will make the following definition.

Definition 3.3.1. Let ϕ be a prescribed section of E and let τ be a real parameter. Define $m_{\phi,\tau}\colon \mathscr{M}et \to \Omega^0(\operatorname{End} E)$ by

$$m_{\phi,\tau}(H) = \Lambda F_H - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*_H} + \frac{\sqrt{-1}}{2}\tau\mathbf{I}. \tag{3.15}$$

Lemma 3.3.2.

(i) $\displaystyle \frac{d}{dt}\mathbf{M}_{\phi,\tau}(Ke^{ts}) = 2\sqrt{-1}\int_X \operatorname{Tr}(sm_{\phi,\tau}(Ke^{ts}))$,

(ii) $\mathbf{M}_{\phi,\tau}(K, H) + \mathbf{M}_{\phi,\tau}(H, J) = \mathbf{M}_{\phi,\tau}(K, J)$,

(iii) $\displaystyle \frac{d}{dt}m_{\phi,\tau}(He^{ts})|_{t=0} = \Lambda\overline{\partial}_E D'_H(s) - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*_H}s.$

Proof. (i) Let $K_t = Ke^{ts}$. Since (cf. [18])

$$\frac{d}{dt}\mathbf{M}_D(K_t) = 2\sqrt{-1}\int_X \operatorname{Tr}(s\Lambda F_{K_t}), \tag{3.16}$$

we see that

$$\begin{aligned}\frac{d}{dt}\mathbf{M}_{\phi,\tau}(K, K_t) &= 2\sqrt{-1}\int_X \operatorname{Tr}\left(s\left(\Lambda F_{K_t} - \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*_t} + \frac{\sqrt{-1}}{2}\tau\mathbf{I}\right)\right) \\ &= 2\sqrt{-1}\int_X \operatorname{Tr}(sm_{\phi,\tau}(K_t)).\end{aligned}$$

(ii) Write

$$\mathbf{M}_{\phi,\tau)}(K, H) = \mathbf{M}_D(K, H) + \mathbf{M}_\phi(K, H) + \mathbf{M}_\tau(K, H), \tag{3.17}$$

where

$$\mathbf{M}_\phi(K, H) = \|\phi\|_H^2 - \|\phi\|_K^2, \tag{3.18a}$$

and

$$\mathbf{M}_\tau(K, H) = -\tau \int_X \operatorname{Tr}(\log K^{-1} H). \tag{3.18b}$$

The functional $\mathbf{M}_D$ satisfies (ii) (cf. [18], [3]), and $\mathbf{M}_\phi$ clearly does too. Furthermore,

$$\operatorname{Tr}(\log K^{-1}H) = \log\det(K^{-1}H) = \log\det(H) - \log\det(K).$$

Hence $\mathbf{M}_\tau$ satisfies (ii) and so does the sum $\mathbf{M}_D + \mathbf{M}_\phi + \mathbf{M}_\tau$.

(iii) If $H_t = He^{ts}$ and $F_t = F_{H_t}$, then by (1.17)

$$\Lambda F_t = \Lambda F_H + \Lambda\overline{\partial}_E(e^{-ts}D'_H(e^{ts})).$$

So

$$\frac{d}{dt}(\Lambda F_t)|_{t=0} = \Lambda\overline{\partial}_E D'_H(s). \tag{3.19}$$

Furthermore, if $\phi \otimes \phi^{*_t}$ is calculated using the metric He^{ts}, then

$$\frac{d}{dt}\phi\otimes\phi^{*_t}|_{t=0} = \frac{d}{dt}\phi\otimes\phi^{*}e^{ts}|_{t=0} = \phi\otimes\phi^{*}s. \tag{3.20}$$

Remark. The significance of (i) is clear—it is this property of $\mathbf{M}_{\phi,\tau}$ that allows us to find vortex solutions by minimizing $\mathbf{M}_{\phi,\tau}(K, -)$. The significance of (ii) is the following: If $\mathbf{M}_{\phi,\tau}(K, -)$ has a minimum at H, then $\frac{d}{dt}\mathbf{M}_{\phi,\tau}(K, He^{ts})|_{t=0} = 0$ for all $s \in L_2^p(S(H))$. By (ii),

$$\frac{d}{dt}\mathbf{M}_{\phi,\tau}(K, He^{ts})|_{t=0} = \frac{d}{dt}\mathbf{M}_{\phi,\tau}(H, He^{ts})|_{t=0}. \tag{3.21}$$

Hence by (i), $2\sqrt{-1}\int_X \operatorname{Tr}(sm_{\phi,\tau}(H)) = 0$ for all $s \in L_2^p(S(H))$. The importance of (iii) will be seen in the next subsection.

3.4. Minima of $\mathbf{M}_{\phi,\tau}$. For technical reasons which will become clear (cf. Lemma 3.7.2) we will want to minimize $\mathbf{M}_{\phi,\tau}(K, -)$ not over $\mathscr{M}et_2^p$ but over a constrained subset of such metrics. In particular, we chose a real number B such that $\|m_{\phi,\tau}(K)\|_{L^p,K}^p \le B$, where $\|m_{\phi,\tau}(K)\|_{L^p,K}^p = \int_X |m_{\phi,\tau}(K)|_K^p d\,\mathrm{vol}$. Define

$$\mathscr{M}et_2^p(B) = \{H \in \mathscr{M}et_2^p \,|\, \|m_{\phi,\tau}(H)\|_{L^p,H}^p \le B\}. \tag{3.22}$$

We need to understand the minima of $\mathbf{M}_{\phi,\tau}(K, \text{-})$ on $\mathscr{M}et_2^p(B)$, and to this end we make the following definition.

Definition 3.4.1. For any metric $H \in \mathscr{M}et_2^p(B)$ let

$$\operatorname{Ker}(H) = \{s \in L_2^p(S(H)) | \overline{\partial}_E(s) = 0, s\phi = 0\}. \tag{3.23}$$

Let

(3.24) $\quad \mathrm{Ker} = \bigcup \mathrm{Ker}(H)$, where the union is over all $H \in \mathcal{M}et_2^p(B)$.

Lemma 3.4.2. *Suppose that H minimizes $\mathbf{M}_{\phi,\tau}(K, -)$ on $\mathcal{M}et_2^p(B)$. If* $\mathrm{Ker}(H) = \{0\}$, *then* $m_{\phi,\tau}(H) = 0$.

Proof. By Lemma 3.3.2, if $s \in L_2^p(S(H))$ and $H_t = He^{ts}$, then

$$\frac{d}{dt}(m_{\phi,\tau}(H_t))|_{t=0} = L(s)$$

where

$$L(s) = \Lambda\overline{\partial}_E D'_H(s) - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H}s. \tag{3.25}$$

Hence

$$\frac{d}{dt}(-\mathrm{Tr}(m_{\phi,\tau}(H_t))^2)|_{t=0} = -2\,\mathrm{Tr}(m_{\phi,\tau}(H)L(s)) = 2(m_{\phi,\tau}(H), L(s))_H.$$

It follows, since p is chosen to be even, that

$$\frac{d}{dt}\|m_{\phi,\tau}(H_t)\|^p_{L^p, H_t}|_{t=0} = \frac{p}{2}\int_X |m_{\phi,\tau}(H)|_H^{p-2}(m_{\phi,\tau}(H), L(s))_H.$$

Suppose now that $s \in L_2^p(S(H))$ can be chosen such that

$$L(s) = -m_{\phi,\tau}(H). \tag{3.26}$$

Then, with this particular $s \in L_2^p(S(H))$, we get

$$\frac{d}{dt}\|m_{\phi,\tau}(H_t)\|^p_{L^p, H_t}|_{t=0} = -\frac{p}{2}\|m_{\phi,\tau}(H)\|^p_{L^p, H}. \tag{3.27}$$

Furthermore, with s as in (3.26), Lemma 3.3.2 implies that

$$\begin{aligned}\frac{d}{dt}\mathbf{M}_{\phi,\tau}(H, He^{ts})|_{t=0} &= -2\int_X \mathrm{Tr}(s\sqrt{-1}\Lambda\overline{\partial}_E D'_H(s)) \\ &\quad - \int_X \mathrm{Tr}(s\phi \otimes \phi^{*H}s) \\ &= -\{2\|D'_H(s)\|^2_{L^2, H} + \|s\phi\|^2_{L^2, H}\}.\end{aligned} \tag{3.28}$$

In the last line we have used the Kähler identity $\sqrt{-1}[\Lambda, \overline{\partial}_E] = D'^*_H$ (cf. Proposition 1.2) and the self-adjointness of s with respect to H. These last two equations show that a minimum of $\mathbf{M}_{\phi,\tau}(K, -)$ could not occur on the boundary of $\mathcal{M}et_2^p(B)$ unless $s = 0$ is a solution to (3.26). In fact, if $s = 0$ is a solution, then (3.26) yields $m_{\phi,\tau}(H) = 0$, while if (3.26) has a nonzero solution, then at a minimizing metric H we must have

$$\frac{d}{dt}\mathbf{M}_{\phi,\tau}(H, He^{ts})\bigg|_{t=0} = 0$$

for all $s \in L_2^p(S(H))$. But by Lemma 3.3.2

$$\frac{d}{dt}\mathbf{M}_{\phi,\tau}(H, He^{ts})\bigg|_{t=0} = 2\langle s, \sqrt{-1}m_{\phi,\tau}(H)\rangle_H.$$

The proof of the lemma is thus reduced to showing that if $\mathrm{Ker}(H) = 0$, then (3.26) does indeed have a solution in $L_2^p(S(H))$. Notice that we can write

$$\Lambda\overline{\partial}_E D'_H(s) = -\sqrt{-1}D'^*_H D'_H(s) = -\sqrt{-1}\Delta'_H(s),$$

and that

$$\Delta'_H : L_2^p(S(H)) \to L_2^p(S(H))$$

is a self-adjoint elliptic operator; so therefore is the operator

$$\Delta'_H + \tfrac{1}{2}\phi \otimes \phi'^H : L_2^p(S(H)) \to L_2^p(S(H)).$$

Hence to prove the lemma it is enough to show that $\Delta'_H + \frac{1}{2}\phi \otimes \phi^{*_H}$ has no kernel. Now

$$\begin{aligned}(\Delta'_H + \tfrac{1}{2}\phi \otimes \phi^{*_H})s = 0 \\ &\Rightarrow \langle \Delta'_H(s) + \tfrac{1}{2}\phi \otimes \phi^{*_H}s, s\rangle_H = 0 \\ &\Rightarrow \|D'_H(s)\|^2_{L^2,H} + \|s\phi\|^2_{L^2,H} = 0.\end{aligned}$$

Furthermore, if $s = s^{*_H}$ then $\overline{D'_H(s)^{\mathrm{T}}} = \overline{\partial}_E(s)$. Hence

$$(\Delta'_H + \tfrac{1}{2}\phi \otimes \phi^{*_H})s = 0 \Rightarrow \overline{\partial}_E s = 0 = s\phi.$$

So $\mathrm{Ker}(H) = \{0\}$ does imply $s = 0$, and the proof of the lemma is complete.

3.5. The condition $\mathrm{Ker} = 0$. The definition of the set Ker depends on B, the upper bound on $\|m_{\phi,\tau}(H)\|^p_{L^p,H}$ for all $H \in \mathcal{M}et^p_2(B)$. This will not be important since we are primarily interested in establishing that $\mathrm{Ker} = 0$. The next lemma shows that under the assumption of ϕ-stability, $\mathrm{Ker} = 0$ for any choice of B. We thus have complete freedom in choosing the background metric K and can let this choice dictate the selection of B.

Lemma 3.5.1. *E splits (holomorphically) as $E_\phi \oplus F$, where $\phi \in \Omega^0(X, E_\phi)$ if and only if* $\mathrm{Ker} \neq \{0\}$ *for some choice of B. In particular, if $\mu_M < \tau\,\mathrm{Vol}(X)/4\pi < \mu_m(\phi)$, then* $\mathrm{Ker} = \{0\}$ *for any choice of B.*

Proof. If $E = E_\phi \oplus F$ we can choose a background metric K which respects this splitting, i.e., with respect to which F is the orthogonal complement of E_ϕ. Projection onto F then clearly constitutes a nonzero element of $\operatorname{Ker}(K)$. Choosing B such that $\|m_{\phi,\tau}(K)\|_{L^p,K}^p \leq B$, it follows that $\operatorname{Ker} \neq \{0\}$.

Conversely, suppose that for some choice of B we can find $s \neq 0$ such that $\overline{\partial}_E s = s\phi = 0$ and $s^* = s$ with respect to some metric $H \in \mathcal{M}et_2^p(B)$. Then $D_H'(s) = 0$ and so $D(s) = 0$. It follows that E splits *holomorphically* into $\operatorname{Ker}(s) \oplus \operatorname{Im}(s)$. (Actually we can say more: the eigenvalues of s are constant, and E splits holomorphically into the eigenspaces corresponding to distinct eigenvalues.) Clearly $\phi \in \operatorname{Ker}(s)$ and so this splitting is of the type $E = E_\phi \oplus F$.

Suppose now that $\mu_M < \tau \operatorname{Vol}(X)/4\pi < \mu_m(\phi)$. Suppose further that E splits holomorphically as $E = E_\phi \oplus F$ with $\phi \in \Omega^0(X, E_\phi)$. Then

$$\frac{R\mu - R_\phi \mu(E_\phi)}{R - R_\phi} > \frac{\tau \operatorname{Vol}(X)}{4\pi} \tag{3.29}$$

and

$$\mu(F) < \frac{\tau \operatorname{Vol}(X)}{4\pi}. \tag{3.30}$$

Here R_ϕ is the rank of E_ϕ, the first relation holds as $\phi \in \Omega^0(X, E_\phi)$ and the second is true since F is a holomorphic subbundle. However $\mu(F) = (R\mu - R_\phi \mu(E_\phi))/(R - R_\phi)$ and so we get a contradiction. It thus follows that regardless of the choice of B, $\operatorname{Ker} = \{0\}$.

3.6. Statement of the main estimate (C^0 version). Thus far, we have established that under the hypotheses of the theorem we can look for solutions to $\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*H} + \frac{\sqrt{-1}}{2}\tau \mathbf{I} = 0$ by minimizing $\mathbf{M}_{\phi,\tau}$ on $\mathcal{M}et_2^p(B)$. Following Simpson we show that minima exist by proving an estimate of the form (3.3) holds for all s such that $Ke^s \in \mathcal{M}et_2^p(B)$. We do this by assuming no such estimate holds and then by deriving a contradiction.

Proposition 3.6.1. *Either positive constants C_1 and C_2 can be found such that for all $Ke^s \in \mathcal{M}et_2^p(B)$*

$$\sup |s| \leq C_1 \mathbf{M}_{\phi,\tau}(Ke^s) + C_2,$$

or $(E, \overline{\partial}_E, \phi)$ is not ϕ-stable.

The proof of this proposition occupies the next eight subsections. We remind the reader that K is a fixed (smooth) background metric on E, B is a constant chosen such that $\|m_{\phi,\tau}(K)\|_{L^p,K}^p \leq B$, and $p > 2n$.

3.7. Equivalence of C^0 and L^1 bounds. We will need to use an L^1 version of the estimate in Proposition 3.6.1 and thus need the following technical result.

Proposition 3.7.1. *Let K and H be hermitian metrics on E with $H = Ke^s$ for some $s \in L_2^p(S(K))$. Then*

$$\Delta|s| \leq 2(|m_{\phi,\tau}(K)|_K + |m_{\phi,\tau}(H)|_H) \tag{3.31}$$

where the norm on $|s|$ can be with respect to K or H.

Proof. We note first of all that

$$|s|_H^2 = (\mathrm{ad}(e^s)s, s)_K = (s, s)_K = |s|_K^2.$$

The norm on $|s|$ may thus be taken with respect to either H or K. If $D_K = \overline{\partial}_E + D_K'$ is the metric connection determined by $\overline{\partial}_E$ and K, and F_K is its curvature, then we have seen that $F_H = F_K + \overline{\partial}_E(e^{-s}D_K'(e^s))$. Hence

$$\langle\sqrt{-1}\Lambda F_H - \sqrt{-1}\Lambda F_K, s\rangle_K = \langle\sqrt{-1}\Lambda(\overline{\partial}_E(e^{-s}D_K'(e^s)), s)\rangle_K. \tag{3.32}$$

Here we are using the ad-invariant L^2 inner product on $\mathrm{End}(E)$ induced by the metric on E.

We make a local calculation by choosing an orthonormal (with respect to K) frame $\{v_i\}_{i=1}^R$ of eigenvectors of s. Let $\{\lambda_i\}_{i=1}^R$ denote the corresponding eigenvectors. If $\{\hat{v}_i\}_{i=1}^R$ is the corresponding dual basis, then s can be written as

$$s = \sum_{i=1}^R \lambda_i v_i \otimes \hat{v}_i. \tag{3.33}$$

Direct calculation yields

$$e^{-s}D_K'(e^s) = \sum_i \partial\lambda_i v_i \otimes \hat{v}_i + \sum_{k,j} A_{jk}(e^{\lambda_k - \lambda_j})v_j \otimes \hat{v}_k, \tag{3.34}$$

where

$$D_K'(v_i) = A_{ji}v_j. \tag{3.35}$$

Hence

$$(e^{-s}D_K'(e^s), s)_K = \sum_i \lambda_i\partial\lambda_i = \tfrac{1}{2}\partial|s|^2. \tag{3.36}$$

Now since D_K is compatible with K, we have

$$\overline{\partial}(e^{-s}D_K'(e^s), s)_K = (\overline{\partial}_E(e^{-s}D_K'(e^s)), s)_K - (e^{-s}D_K'(e^s), D_K's)_K, \tag{3.37}$$

and hence

(3.38)
$$\begin{aligned}(\overline{\partial}_E(e^{-s}D_K'(e^s)), s)_K &= \tfrac{1}{2}\overline{\partial}\partial|s|^2 + (e^{-s}D_K'(e^s), D_K's)_K \\ &= |s|\overline{\partial}\partial|s| + \overline{\partial}|s|\partial|s| + (e^{-s}D_K'(e^s), D_K's)_K.\end{aligned}$$

Define

$$\Delta_\partial |s| \equiv i\Lambda\bar{\partial}\partial|s|, \tag{3.39}$$

where Δ_∂ is the positive Laplacian. We then get

$$\begin{aligned} |s|\Delta_\partial|s| = (\sqrt{-1}\Lambda F_H - \sqrt{-1}\Lambda F_K, s)_K \\ - \sqrt{-1}\Lambda(e^{-s}D'_K(e^s), D'_K s)_K + \sqrt{-1}\Lambda\partial|s|\bar{\partial}|s|. \end{aligned} \tag{3.40}$$

Another local calculation establishes that

$$(e^{-s}D'_K(e^s), D'_K s)_K = \sum_i \partial\lambda_i \wedge \bar{\partial}\lambda_i + \sum_{i\neq j}(\lambda_i - \lambda_j)(e^{\lambda_i - \lambda_j} - 1)A_{ji}\wedge\overline{A}_{ji}. \tag{3.41}$$

Recall that for any $\alpha \in \Lambda^{10}(M)$ it can be shown that $\sqrt{-1}\Lambda\alpha\wedge\bar{\alpha} \geq 0$ (cf. Lemma 2.1). Thus

$$\sqrt{-1}\Lambda(e^{-s}D'_K(e^s), D'_K s)_K \geq \sum_i \sqrt{-1}\Lambda\partial\lambda_i\wedge\bar{\partial}\lambda_i. \tag{3.42}$$

Next we show that

$$\sqrt{-1}\Lambda\left(\partial|s|\bar{\partial}|s| - \sum_i \partial\lambda_i\wedge\bar{\partial}\lambda_i\right) \leq 0: \tag{3.43}$$

Using the local orthonormal frame, we can write $|s| = (\sum\lambda_i^2)^{1/2}$. Since the calculation is purely local, we can think of λ_i as

$$\lambda_i : \mathbf{C}^n \to \mathbf{C},$$

with the coordinates on $\mathbf{C}^n$ chosen such that the Kähler form on X is given by $\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$. Define

$$\tilde{\lambda} : \mathbf{C}^n \to \mathbf{C}^R \quad \text{by} \quad \tilde{\lambda} \doteq (\lambda_1, \cdots, \lambda_R). \tag{3.44}$$

Clearly $|\tilde{\lambda}| = |s|$. Furthermore $\sqrt{-1}\Lambda(\partial\tilde{\lambda}, \partial\tilde{\lambda}) = \sum_i \sqrt{-1}\Lambda\partial\lambda_i\wedge\bar{\partial}\lambda_i$, where we have used the standard inner product on $\Omega^*(\mathbf{C}^n, \mathbf{C}^R)$ and the fact that $\lambda_i = \bar{\lambda}_i$.

Using "polar coordinates" we can write $\tilde{\lambda} = (|s|, \chi_1, \cdots, \chi_{R-1})$ and see that

$$\sqrt{-1}\Lambda(\partial\tilde{\lambda}, \partial\tilde{\lambda}) = \sqrt{-1}\Lambda\partial|s|\bar{\partial}|s| + \sum_{i=1}^{R-1}\sqrt{-1}\Lambda\partial_{\chi_j}\wedge\bar{\partial}\chi_j. \tag{3.45}$$

Hence

$$\sqrt{-1}\Lambda\left(\partial|s|\bar{\partial}|s| - \sum_i \partial\lambda_i\wedge\bar{\partial}\lambda_i\right) = -\sum_{i=1}^{R-1}\sqrt{-1}\Lambda\partial\chi_j\wedge\bar{\partial}\chi_j \leq 0$$

as required. (3.40), (3.42), and (3.43) thus show that

$$|s|\Delta_{\partial}|s| \leq (\sqrt{-1}m_{\phi,\tau}(H) - \sqrt{-1}m_{\phi,\tau}(K), s)_K - \tfrac{1}{2}\sum|\phi_i|^2\lambda_i(e^{\lambda_i} - 1);$$

here $\{\phi_i\}$ are the components of ϕ with respect to the local frame. From this we obtain

$$\Delta_{\partial}|s| \leq \left|\frac{(\sqrt{-1}m_{\phi,\tau}(H), s)_K}{|s|}\right| + \left|\frac{(\sqrt{-1}m_{\phi,\tau}(K), s)_K}{|s|}\right|.$$

But

$$(\sqrt{-1}m_{\phi,\tau}(H), s)_K = (\sqrt{-1}m_{\phi,\tau}(H), \operatorname{ad}(e^{-s})s)_H = (\sqrt{-1}m_{\phi,\tau}(H), s)_H,$$

which gives

$$\Delta_{\partial}|s| \leq |m_{\phi,\tau}(K)|_K + |m_{\phi,\tau}(H)|_H.$$

The result now follows from the fact that on a Kähler manifold $\Delta_{\partial} = \frac{1}{2}\Delta$.

The importance of this proposition stems from the following result.

Lemma 3.7.2. *If X is compact n-manifold, then we can find a smooth function $a\colon [0,\infty) \to [0,\infty)$ with $a(0) = 0$ and $a(x) = x$ for $x > 1$ such that the following is true: Suppose f is a positive bounded function on X and $\Delta f \leq b$, where b is a function in $L^p(X)$ $(p > n)$ with $\|b\|_{L^p} \leq B$. Then*

$$\sup|f| \leq C(B)a(\|f\|_{L_1}),$$

where $C(B)$ is a constant which depends on B.

Proof. This is Proposition 2.1 in [19].

If we restrict our choice of metrics to $\mathcal{M}et_2^p(B)$, then we have proved

Proposition 3.7.3. *Let $H, K \in \mathcal{M}et_2^p(B)$ with $H = Ke^s$. Then there is a constant C (depending on B) and a continuous function a as in Lemma 3.7.2 such that*

$$\sup|s| \leq C(B)a(\|s\|_{L^1}), \tag{3.46}$$

where the norm can be calculated with respect to K or H.

Proof. Apply Lemma 3.7.2 with $f = |s|$ and

$$b = 2(|m_{\phi,\tau}(K)|_K + |m_{\phi,\tau}(H)|_H).$$

The result then follows from Proposition 3.7.1.

Suppose that it is *not* possible to find constants C_1 and C_2 such that for all $s \in L_2^p(S(K))$

$$\sup|s| \leq C_1\mathbf{M}_{\phi,\tau}(Ke^s) + C_2.$$

By the above results it follows that no estimate of this type on $\|s\|_{L^1}$ holds.

3.8. Construction of an unbounded sequence.

Lemma 3.8.1. *If no such estimate holds, then it is possible to find a sequence* $\{s_i\}_{i=1}^{\infty}$ *in* $L_2^p(S(K))$ *and a sequence of positive constants* $\{C_i\}_{i=1}^{\infty}$ *such that*

(i) $C_i \to \infty$,

(ii) $\|s_i\|_{L^1} \to \infty$, *and*

(iii) $\|s_i\|_{L^1} \geq C_i \mathbf{M}_{\phi,\tau}(Ke^s)$.

Proof. Suppose that for a given C_i we can find C'' and N_i such that $\|s\|_{L^1} \leq C_i \mathbf{M}_{\phi,\tau}(Ke^s) + C''$ whenever $\|s\|_{L^1} > N_i$. Let

$$S_N = \{s \in L_2^p(S(K)) | \|s\|_{L^1} \leq N_i\}.$$

We claim that $\mathbf{M}_{\phi,\tau}(K, \text{-})$ is bounded below on S_N. In fact $\mathbf{M}_{\phi,\tau}$ can be written

$$\begin{aligned} \mathbf{M}_{\phi,\tau}(Ke^s) &= \|\Psi(s)\bar{\partial}_E(s)\|^2 + 2\sqrt{-1}\,\mathrm{Tr}\int_X \Lambda s F_K \\ &\quad + \|\phi\|^2_{Ke^s} - \|\phi\|^2_K - \tau \int_X \mathrm{Tr}(s). \end{aligned} \tag{3.47}$$

Hence if $\|s\|_{L^1}$, and therefore $\sup|s|$, are bounded, then $\mathbf{M}_{\phi,\tau}$ is bounded below. Say $\mathbf{M}_{\phi,\tau} \geq -\lambda$ on S_N for some positive constant λ. Then $\|s\|_{L^1} \leq N_i + C_i(\mathbf{M}_{\phi,\tau} + \lambda)$ when $s \in S_N$. By replacing C'' (e.g. with $\max\{C'', N_i + C_i\lambda\}$) it would be possible to enforce the estimate for all $s \in L_2^p(S(K))$. The lemma is proved.

Choose sequences $\{C_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=1}^{\infty}$ as in Lemma 3.8.1 and let

$$l_i = \|s\|_{L^1}. \tag{3.48}$$

Define

$$u_i = l_i^{-1} s_i. \tag{3.49}$$

Notice that $\|u_i\|_{L^1} = 1$ and therefore $\sup|u_i| \leq C(B)$.

3.9. Construction and properties of the limiting bundle endomorphism. We now show that there is a subsequence in $\{u_i\}_{i=1}^{\infty}$ which converges weakly in $L_1^2(S(K))$ to some u_∞. By examining the properties of this limiting object we will arrive at a contradiction. Our analysis of u_∞ is based largely on the next proposition, which is modelled on Proposition 6.3.3 in [18] and to which it should be compared.

Proposition 3.9.1. $\{u_i\}_{i=1}^{\infty}$ *contains a subsequence, which we again call* $\{u_i\}_{i=1}^{\infty}$, *that converges weakly in* $L_1^2(S(K))$ *to* u_∞. u_∞ *is nontrivial and satisfies the following: Let* $\mathscr{F} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *be any smooth positive function* *which satisfies* $\mathscr{F}(x, y) \leq 1/(x-y)$ *whenever* $x > y$. *Let* $\not{f}: \mathbf{R} \to \mathbf{R}$ *be a smooth positive function such that* $\not{f}(x) = 0$ *whenever* $x \leq \varepsilon$ *for some* $\varepsilon > 0$. *Then*

$$\begin{aligned} \sqrt{-1}\int_X \operatorname{Tr} u_\infty \Lambda F_K + \int_X (\mathscr{F}(u_\infty)\overline{\partial}_E u_\infty, \overline{\partial}_E u_\infty)_K \\ + \frac{1}{2}\int_X (\not{f}(u_\infty)\phi, \phi)_K - \frac{\tau}{2}\int_X \operatorname{Tr} u_\infty \leq 0. \end{aligned} \tag{3.50}$$

Proof. We first show that for large enough i

(3.51)

$$\begin{aligned} \sqrt{-1}\int_X \operatorname{Tr} u_i \Lambda F_K + \int_X (\mathscr{F}(u_i)\overline{\partial}_E u_i, \overline{\partial}_E u_i)_K \\ + \frac{1}{2}\int_X (\not{f}(u_i)\phi, \phi)_K - \frac{\tau}{2}\int_X \operatorname{Tr} u_i \leq \frac{2}{C_i} + \frac{\|\phi\|_K^2}{1_i}, \end{aligned}$$

and then show that the expression on the left is well behaved in the limit $i \to \infty$. The key is a comparison of this expression with (3.14), i.e.,

$$\begin{aligned} \mathbf{M}_{\phi,\tau}(K, Ke^{s_i}) = 2\int_X (\Psi(s_i)\overline{\partial}_E s_i, \overline{\partial}_E s_i)_K \\ + \int_X (e^{s_i}\phi, \phi)_K + 2\sqrt{-1}\int_X \operatorname{Tr} s_i \Lambda F_K - \tau\int_X \operatorname{Tr} s_i - \|\phi\|_K^2. \end{aligned}$$

Since the $\sup |u_i|$ are uniformly bounded, we can take $\mathscr{F}$ to be compactly supported and hence (cf. Simpson [18]) for large enough l

$$\mathscr{F}(x, y) \leq l\Psi(lx, ly). \tag{3.52}$$

Furthermore $\not{f}$ can similarly be taken to be compactly supported, in which case we can show that for large enough l

$$\not{f}(x) \leq e^{lx}/l. \tag{3.53}$$

It is clear that (3.53) holds for $x \leq \varepsilon$. Now choose l_0 such that $e^{l_0\varepsilon}/l_0 > \sup(\not{f})$ and $l_0 > 1/\varepsilon$. Then for $x > \varepsilon$ and $l \geq l_0$ we have

$$\not{f}(x) \leq \sup(\not{f}) < \frac{e^{l_0\varepsilon}}{l_0} < \frac{e^{l\varepsilon}}{l} < \frac{e^{lx}}{l}.$$

We remark that the inequality $e^{l_0\varepsilon}/l_0 < e^{l\varepsilon}/l$ follows from the fact that $e^{\varepsilon t}/t$ is increasing for $t > 1/\varepsilon$. For large enough i we thus have

$$\begin{aligned} (\Psi(s_i)\overline{\partial}_E s_i, \overline{\partial}_E s_i)_K &= l_i^2(\Psi(l_i u_i)\overline{\partial}_E u_i, \overline{\partial}_E u_i)_K \\ &\geq l_i(\mathscr{F}(u_i)\overline{\partial}_E u_i, \overline{\partial}_E u_i)_K \end{aligned} \tag{3.54a}$$

(3.54b) $$(e^{s_i}\phi, \phi)_K = (e^{l_i u_i \phi}, \phi)_K \geq l_i(\not{\mathscr{F}}(u_i)\phi, \phi)_K.$$

It follows then that

$$l_i\left\{\sqrt{-1}\int_X \operatorname{Tr} u_i \Lambda F_K + \int_X (\mathscr{F}(u_i)\overline{\partial}_E u_i, \overline{\partial}_E u_i)_K + \frac{1}{2}\int_X (\not{\mathscr{F}}(u_i)\phi, \phi)_K - \frac{\tau}{2}\int_X \operatorname{Tr} u_i\right\} \leq \frac{1}{2}\mathbf{M}_{\phi,\tau}(K, Ke^{s_i}) + \frac{1}{2}\|\phi\|_K^2.$$

That is,

(3.55)
$$\sqrt{-1}\int_X \operatorname{Tr} u_i \Lambda F_K + \int_X (\mathscr{F}(u_i)\overline{\partial}_E u_i, \overline{\partial}_E u_i)_K + \frac{1}{2}\int_X (\not{\mathscr{F}}(u_i)\phi, \phi)_K - \frac{\tau}{2}\int_X \operatorname{Tr} u_i \leq \frac{1}{2C_i} + \frac{\|\phi\|_K^2}{l_i}.$$

Simpson's proof that the u_i are bounded in L_1^2 is unaffected by the terms $\int_X (\not{\mathscr{F}}(u_i)\phi, \phi)_K$ and $(\sqrt{-1}\int_X \operatorname{Tr} u_i \Lambda F_K - \tau\int_X \operatorname{Tr} u_i)$ which are both bounded. The trick is to use the fact that the $\sup|u_i|$ are uniformly bounded. This enables one to chose an $\mathscr{F}$ which satisfies the requirements of the proposition *and* such that $\mathscr{F}(u_i) = c$ for some small constant c. Equation (3.55) can then be used to show that $\overline{\partial}_E(u_i)$ is in L^2. But u_i is self-adjoint and of type $(0, 0)$. It follows that ∇u_i is in L^2 and hence that u_i is bounded in L_1^2. We may conclude therefore that $\{u_i\}_{i=1}^{\infty}$ has a subsequence which converges weakly in L_1^2 to u_∞. Furthermore since $\|u_\infty\|_{L^1} = 1$ and $\sup|u_\infty| \leq C(B)$, u_∞ is nontrivial. (We note in passing that we can see here the importance of using the L^1 rather than the C^0 norm.)

Simpson has shown that in the limit $i \to \infty$

(3.56)
$$\sqrt{-1}\int_X \operatorname{Tr} u_i \Lambda F_K + \int_X (\mathscr{F}(u_i)\overline{\partial}_E u_i, \overline{\partial}_E u_i)_K \to \sqrt{-1}\int_X \operatorname{Tr} u_\infty \Lambda F_K + \int_X (\mathscr{F}(u_\infty)\overline{\partial}_E u_\infty, \overline{\partial}_E u_\infty)_K.$$

It is the second term that is the problematic one. One uses Proposition 3.2.2 and that fact that $u_i \to u_\infty$ in $L_{0,b}^2(S(K))$ to show that $\mathscr{F}^{1/2}(u_i) \to \mathscr{F}^{1/2}(u_\infty)$ in $\operatorname{Hom}(L^2, L^q)$ when $q < 2$. Recall that

$$L_{0,b}^2(S(K)) = \{s \in L_0^2(S(K)) \,|\, |s| \leq b \text{ a.e.}\}.$$

The required convergence follows from this and the L^2 bounds on $\overline{\partial}_E u_i$.

It is clear that

$$\tau \int_X \operatorname{Tr} u_i \to \tau_X \operatorname{Tr} u_\infty, \tag{3.57}$$

and so it remains to show that

$$\int_X (\not{f}(u_i)\phi, \phi)_K \to \int_X (\not{f}(u_\infty)\phi, \phi)_K. \tag{3.58}$$

Now $|u_i|$ is uniformly bounded and $L_1^2 \subset L_{0,b}^2$ is a compact embedding. We thus have that $u_i \to u_\infty$ strongly in $L_{0,b}^2(S(K))$ for some b. But $u \to \not{f}(u)$ is continuous on $L_{0,b}^2(S(K))$ (cf. Proposition 3.2.2) and hence $\not{f}(u_i) \to \not{f}(u_\infty)$ as $i \to \infty$. Since the $\sup|u_i|$ are bounded, this is enough to ensure that $\int_X(\not{f}(u_i)\phi, \phi)_K \to \int_X(\not{f}(u_\infty)\phi, \phi)_K$. The proposition is proved.

The next two lemmas now follow. They correspond to §§6.3.4 and 6.3.5 in [18].

Lemma 3.9.2. *The eigenvalues of u_∞ are constant almost everywhere.*

Lemma 3.9.3. *Let the eigenvalues of u_∞ be $\lambda_1, \cdots, \lambda_r$. If $\mathscr{F}: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ satisfies $\mathscr{F}(\lambda_i, \lambda_j) = 0$ whenever $\lambda_i > \lambda_j$, $1 \le i, j \le r$, then*

$$\|\mathscr{F}(u_\infty)\bar{\partial}_E u_\infty\|_{L^2}^2 = 0. \tag{3.59}$$

Proofs. Both proofs use the same device, namely the following observation: Given any smooth function $\mathscr{F}: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and any $N > 0$, we can find a smooth function $\mathscr{F}_1: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that

(i) $\mathscr{F}_1(x, x) = \mathscr{F}(x, x)$,

(ii) $N(\mathscr{F}_1)^2$ satisfies the requirements of Proposition 3.9.1.

In the case of Lemma 3.9.2 we show that for any smooth function $f: \mathbf{R} \to \mathbf{R}$, $\operatorname{Tr}(f(u_\infty))$ is constant almost everywhere. In particular, the elementary symmetric functions of the eigenvalues (and hence the eigenvalues themselves) can be shown in this way to be constant almost everywhere. In fact we need only show that $\bar{\partial}\operatorname{Tr}(f(u_\infty)) = 0$, since $\operatorname{Tr}(f(u_\infty))$ is real valued. Furthermore by Proposition 3.2.2(iv),

$$\bar{\partial}\operatorname{Tr}(f(u_\infty)) = \operatorname{Tr}(df(u_\infty)\bar{\partial}_E u_\infty). \tag{3.60}$$

We now use the above device with $\mathscr{F} = df$. This gives, with suitably chosen $\mathscr{F}_1$,

$$\begin{aligned} |\bar{\partial}\operatorname{Tr}(f(u_\infty))|^2 &= |\operatorname{Tr}(\mathscr{F}_1(u_\infty)\bar{\partial}_E u_\infty)|^2 \le C|\mathscr{F}_1(u_\infty)\bar{\partial}_E u_\infty|^2 \\ &= \frac{C}{N}(N(\mathscr{F}_1)^2(u_\infty)\bar{\partial}_E u_\infty, \bar{\partial}_E u_\infty)_K. \end{aligned}$$

Hence, by (3.50),

$$\|\bar\partial \operatorname{Tr}(f(u_\infty))\|_{L^2}^2 \le C'/N. \tag{3.61}$$

Since this holds for all $N > 0$, we get the required result.

To prove Lemma 3.9.3, notice that if $\mathscr{F}(\lambda_i, \lambda_j) = 0$ whenever $\lambda_i > \lambda_j$, then for any $A \in \Omega^1(X, \operatorname{End} E)$

$$(\mathscr{F}(u_\infty)A, \mathscr{F}(u_\infty)A)_K = \sum_{i=1}^{r} \mathscr{F}^2(\lambda_i, \lambda_i)|A_{ii}|^2. \tag{3.62}$$

Here we have used a local frame for E with respect to which u_∞ is diagonal. This expression is thus valid locally. If we replace $\mathscr{F}$ by a suitably chose $\mathscr{F}_1$ (as above), we then get

$$|\mathscr{F}(u_\infty)A|^2 = (\mathscr{F}_1^2(u_\infty)A, A)_K.$$

Proposition 3.9.1 then gives

$$\|\mathscr{F}(u_\infty)\bar\partial_E u_\infty\|_{L^2}^2 \le \frac{1}{N}\int_X (N\mathscr{F}_1^2(u_\infty)\bar\partial_E u_\infty, \bar\partial_E u_\infty)_K \le \frac{C}{N}, \tag{3.63}$$

which proves the result.

Lemma 3.9.4. *If* $\not{f}$ *is as in Proposition* 3.9.1 *then* $\|\not{f}(u_\infty)\phi\|_{L^2} = 0$.

Proof. Suppose $\not{f}(x) = 0$ whenever $x \le \varepsilon$. Then the same is true of $N\not{f}^2(x)$ for any $N > 0$. Equation (3.51) from Proposition 3.9.1—with $N\not{f}^2$ in place of $\not{f}$—leads to

$$\|\not{f}(u_\infty)\phi\|_{L^2}^2 \le \frac{2}{N}\left(-\int_X (\mathscr{F}(u_\infty)\bar\partial_E u_\infty, \bar\partial_E u_\infty)_K \right.$$
$$\left. - \sqrt{-1}\int_X \operatorname{Tr} u_\infty \Lambda F_K + \frac{\tau}{2}\int_X \operatorname{Tr} u_\infty\right)$$

for some suitable $\mathscr{F}$. Since this holds for any $N > 0$, the result follows.

3.10. Construction of filtration of $\mathscr{E}$ by subsheaves. We now show that the limiting endomorphism u_∞ gives rise to a filtration of E. The components in the filtration are defined initially by projection operators in $L_1^2(S(K))$. In the next subsection we will show that the filtration is in fact by subsheaves of $\mathscr{E}$.

Definition 3.10.1. Let $\lambda_1, \cdots, \lambda_r$ denote the distinct eigenvalues of u_∞, listed in ascending order. For $j < r$ define $p_j\colon \mathbf{R} \to \mathbf{R}$ to be a smooth positive function such that

$$p_j(x) = \begin{cases} 1 & \text{if } x \le \lambda_j, \\ 0 & \text{if } x \ge \lambda_{j+1}. \end{cases} \tag{3.64}$$

Define

$$\pi_j = p_j(u_\infty). \tag{3.65}$$

Proposition 3.10.2. *Let π_j be as above for $j \le r$. Then*

(i) $\pi_j \in L_1^2(S(K))$,
(ii) $\pi_j^2 = \pi_j = \pi_j^*$, *and*
(iii) $(1-\pi_j)\overline{\partial}_E(\pi_j) = 0$ *almost everywhere.*

Furthermore, if $\lambda_p \le 0 < \lambda_{p+1}$ (i.e., λ_p is the largest nonpositive eigenvalue), then

(iv) $\|(1-\pi_p)\phi\|_{L^2}^2 = 0$ *and*
(v) *not all the eigenvalues of u_∞ are positive.*

Proof. (i)–(iii) are proved in §6.4.3 in [18]. The crucial ingrediant, viz. Lemma 3.9.3, still holds, so no modifications in the proof are required here. The proofs are as follows.

(i) The maps π_j are in L_1^2 since u_∞ is bounded in L_1^2.

(ii) The maps are projections, i.e., $\pi_j^2 = \pi_j$, since $(p_j)^2 - p_j = 0$ at all the eigenvalues of u_∞. The maps are self-adjoint since the eigenvalues of u_∞ are real.

(iii) Recall (Proposition 3.2.2) that

$$\overline{\partial}_E(\pi_j) = dp_j(u_\infty)\overline{\partial}_E(u_\infty).$$

Define

$$\mathscr{F}_j(x, y) = (1-p_j(y))dp_j(x, y). \tag{3.66}$$

Then

$$\mathscr{F}_j(u_\infty)\overline{\partial}_E(u_\infty) = (1-\pi_j)\overline{\partial}_E(\pi_j),$$

and $\mathscr{F}_j(\lambda_k, \lambda_l) = 0$ whenever $\lambda_k > \lambda_l$, $1 \le i, j \le r$. We can therefore apply Lemma 3.9.3 and prove the result.

(iv) Chose ε such that $0 < \varepsilon < \lambda_{p+1}$ and define a smooth positive function $k\colon \mathbf{R} \to \mathbf{R}$ such that

$$k(x) = \begin{cases} 1 & \text{if } x \le \varepsilon, \\ 0 & \text{if } x \ge \lambda_{p+1}. \end{cases} \tag{3.67}$$

Let $f = 1 - k$. Then $f(u_\infty) = (1-\pi_p)$ and f satisfies the criteria of Proposition 3.9.1. Hence (iv) follows by Lemma 3.10

(v) Suppose all the eigenvalues are positive. Then $\pi_p = 0$, and by (iv) $\|\phi\|_{L^2} = 0$, which is not so.

3.11. Weak subbundles.

Definition 3.11.1 [21]. An L_1^2 subsystem is a section $\pi \in L_1^2(s(K))$ such that $\pi^2 = \pi = \pi^*$ and $(1-\pi)\overline{\partial}_E(\pi) = 0$.

Proposition (3.10.2) can be interpreted as saying that the projections π_j define "weak" L_1^2 subsystems in the sense of Uhlenbeck and Yau [21]. The next theorem elevates these to the status of reflexive subsheaves. We can then get expressions for the slopes μ_j of these sheaves and show that one of them must violate the condition of ϕ-stability.

Theorem 3.11.2 [21]. *If E is a holomorphic subbundle with background metric K, and π is an L_1^2 subsystem, then there is a reflexive subsheaf $\mathscr{E}'$ such that π is projection onto $\mathscr{E}'$, defined where $\mathscr{E}'$ is a subbundle.*

Corollary 3.11.3. *With $\pi_j \in L_1^2(s(K))$ as above, there are reflexive subsheaves $\mathscr{E}_j$ such that π_j is projection onto $\mathscr{E}_j$, defined where $\mathscr{E}_j$ is a subbundle.*

Proof. By Proposition 3.10.2(i)–(iii), π_j define L_1^2 subsystems. This proves the theorem.

Let the degree and rank of the subsheaf $\mathscr{E}_j$ corresponding to the projection π_j be $\deg(\mathscr{E}_j, \omega)$ and R_j respectively. They are calculated by means of the fact that off a set of codimension 2, $\mathscr{E}_j$ is locally free. Off this codimension 2 set the calculations are those for a subbundle.

Lemma 3.11.4. *The rank and degree of $\mathscr{E}_j$ are given by*

$$R_j = \operatorname{Tr}(\pi_j), \tag{3.68a}$$

where the trace is taken at a regular point of π_j, and by

$$\deg(\mathscr{E}_j, \omega) = i\int_X \operatorname{Tr}(\pi_j \Lambda F_K) - \int_X |\overline{\partial}_E \pi_j|_K^2, \tag{3.68b}$$

where K is a metric on the bundle E.

Proof. The formula for R_j follows from the fact that π_j is a projection onto $\mathscr{E}_j$, where $\mathscr{E}_j$ is a subbundle, and the degree formula follows from Lemma 2.3.

We are now in a position to display the contradiction that will prove Proposition 3.6.1. We do this in the next two subsections by means of Lemmas 3.12.2 and 3.13.1.

3.12. Consequence of no estimate holding.

Definition 3.12.1. Let $\mu_j = \deg(\mathscr{E}_j, \omega)/R_j$ be the slope of $\mathscr{E}_j$, and for $j \le r-1$ set $a_j = \lambda_{j+1} - \lambda_j$. Define

$$Q(\hat{\tau}) = \lambda_r R(\mu - \hat{\tau}) - \sum_{j=1}^{r-1} a_j R_j(\mu_j - \hat{\tau}), \tag{3.69}$$

where R and μ are the rank and slope of E respectively, and

$$\hat{\tau} = \frac{\tau \operatorname{Vol}(X)}{4\pi} \tag{3.70}$$

Lemma 3.12.1. *Suppose that no estimate of the form in Proposition* 3.6.1 *holds. Then*

$$Q(\hat{\tau}) \leq 0. \tag{3.71}$$

Proof. By Lemma 3.11.4 we calculate

$$\begin{aligned}
2\pi Q(\hat{\tau}) &= \sqrt{-1}\lambda_r \int_X \operatorname{Tr}\Lambda F_K - \sum_{j=1}^{r-1} \int_X a_j \operatorname{Tr}(\pi_j \Lambda F_K) \\
&\quad + \sum_{j=1}^{r-1} \int_X a_j |\overline{\partial}_E \pi_j|_K^2 - \frac{\tau}{2\operatorname{Vol}(X)} \left(\lambda_r R - \sum_{j=1}^{r-1} a_j R_j \right) \\
&= \sqrt{-1} \int_X \operatorname{Tr} u_\infty \Lambda F_K + \int_X \sum_{j=1}^{r-1} (a_j \overline{\partial}_E \pi_j, \overline{\partial}_E \pi_j)_K - \frac{\tau}{2} \int_X \operatorname{Tr}(u_\infty).
\end{aligned}$$

Here we have used the fact that

$$u_\infty = \lambda_r \mathbf{I} - \sum_{j=1}^{r-1} a_j \pi_j. \tag{3.72}$$

By Proposition 3.2.2(iv), and with p_j as in (3.64), $\sum_{j=1}^{r-1}(a_j \overline{\partial}_E \pi_j, \overline{\partial}_E \pi_j)_K$ can now be written as

$$\left(\sum_{j=1}^{r-1} a_j (dp_j)^2 (u_\infty) \overline{\partial}_E(u_\infty), \overline{\partial}_E(u_\infty) \right)_K .$$

This is an expression of the form $(\mathscr{F}(u_\infty)\overline{\partial}_E u_\infty, \overline{\partial}_E u_\infty)_K$ where $\mathscr{F}$ satisfies the requirements of Proposition 3.9.1 (cf. [18, §6.4.6]). It follows by that proposition (with $\not{\ell} = 0$) that $Q(\hat{\tau}) \leq 0$.

3.13. Consequence of ϕ-stability.

Lemma 3.13.1. *If* $\mu_M < \hat{\tau} < \mu_m(\phi)$, *then* $Q > 0$.

Proof. Using all the notation of the above, let λ_p be the largest nonpositive eigenvalue of u_∞. By Proposition 3.10.2(v), $p \geq 1$. There are two cases to consider.

Case 1. Suppose $p = r$, i.e., all the eigenvalues are nonpositive. In this case $\mu - \hat{\tau} < 0$, and $\mu_j - \hat{\tau} < 0$ for $j = 1, \cdots, r-1$. It follows immediately that

$$Q(\hat{\tau}) = \lambda_r R(\mu - \hat{\tau}) - \sum_{j=1}^{r-1} a_j R_j (\mu_j - \hat{\tau}) > 0.$$

Case 2. Suppose $p < r$. Since $\mathscr{E} \subset \mathscr{E}_{r-1} \supset \cdots \supset \mathscr{E}_{p+1} \supset \mathscr{E}_p$ and $\phi \in \mathscr{E}_p$, we have $\phi \in \mathscr{E}_j$ for $j \geq p$. Hence

$$R\mu - R_j\mu_j > (R - R_j)\hat{\tau}$$

for $j = p, p+1, \cdots, r-1$. Thus

$$-\sum_{j=p}^{r-1} a_j R_j(\mu_j - \hat{\tau}) > \left(-\sum_{j=p}^{r-1} a_j\right) R(\mu - \hat{\tau}) = (-\lambda_r + \lambda_p) R(\mu - \hat{\tau}),$$

which gives

$$Q(\hat{\tau}) > \lambda_p R(\mu - \hat{\tau}) - \sum_{j=1}^{p-1} a_j R_j(\mu_j - \hat{\tau}) > 0.$$

The lemma is proved.

This concludes the proof of Proposition 3.10.2.1.

3.14. Conclusion to the proof of Theorem 3.1.1. To conclude the proof of the theorem we need the following lemma (which corresponds to the results in §6.6 in [18]).

Lemma 3.14.1. (a) *If* $\mathbf{M}_{\phi,\tau}(Ke^s)$ *is bounded above, then* $\|s\|_{L_2^p}$ *is bounded.*

(b) *Suppose* $\{s_i\}$ *is a minimizing sequence for* $\mathbf{M}_{\phi,\tau}(K, -)$. *Then by* (a) $s_i \to s$ *weakly in* L_2^p *and hence* $H = Ke^s$ *is a solution to* $\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*_H} + \frac{\sqrt{-1}}{2}\tau\mathbf{I} = 0$. *Furthermore* H *is smooth.*

Proof of 3.14.1(a). The idea is to show that a contradiction results from assuming that a sequence $\{s_i\}_{i=1}^{\infty}$ can be found with $\mathbf{M}_{\phi,\tau}(Ke^{s_i})$ bounded but $\|s_i\|_{L_2^p} \to \infty$. One proves this by means of the following two lemmas.

Lemma 3.14.2. *Let* $\{s_i\}_{i=1}^{\infty}$ *be such that for each* i, $Ke^{s_i} \in \mathscr{M}et_2^p(B)$ *and*

$$\sup |s_i| \leq C_1 \mathbf{M}_{\phi,\tau}(Ke^{s_i}) + C_2 \leq C_3.$$

Suppose that $\|s_i\|_{L_2^p} \to \infty$. *Then the* $\|s_i\|_{L_1^2}$ *are uniformly bounded and, after passing to a subsequence,* $s_i \to s$ *in* C^0 *norm.*

Proof. We show that the $\|s_i\|_{L_1^2}$ are uniformly bounded by the method used to show the same for $\|u_i\|_{L_1^2}$ in the proof of Proposition 3.9.1. By (3.14) we have

$$\int_X (\Psi(s_i)\overline{\partial}_E s_i, \overline{\partial}_E s_i)_K = \tfrac{1}{2}\mathbf{M}_{\phi,\tau}(K, Ke^{s_i}) - P(s_i),$$

where

$$(3.73)\quad P(s_i) = \frac{1}{2}\int_X (e^{s_i}\phi, \phi)_K + \sqrt{-1}\int_X \operatorname{Tr} s_i \Lambda F_K - \frac{\tau}{2}\int_X \operatorname{Tr} u_\infty - \frac{1}{2}\|\phi\|_K^2.$$

If $\sup|s_i| \le C_3$ then $|P(s_i)|$ is bounded. Furthermore, we can find $\mathscr{F}: \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ such that $\mathscr{F}(s_i) = c$, with c chosen to be positive but small enough to ensure that

$$\int_X (c\overline{\partial}_E s_i, \overline{\partial}_E s_i)_K \le \int_X (\Psi(s_i)\overline{\partial}_E s_i, \overline{\partial}_E s_i)_K,$$

which gives a bound on $\|\nabla_{s_i}\|_{L^2}$ and therefore on $\|s_i\|_{L_1^2}$.

It follows that $s_i \to s$ in L^2. Let

$$(3.74)\qquad s_{ij} = \log(e^{-s_i}e^{s_j}).$$

Then $s_{ij} \to 0$ in L^2 for $i, j \to \infty$. Also (cf. Proposition 3.7.1)

$$\Delta|s_{ij}|_{H_i} \le 2(|m_{\phi,\tau}(H_j)|_{H_j} + |m_{\phi,\tau}(H_i)|_{H_i}).$$

We can now apply Proposition 3.7.3 and conclude that $\sup|s_{ij}| \le Ca(\|s_{ij}\|_{L^1})$. Notice that since $\sup|s_i|_K$ is bounded, the norms can be taken with respect to K. It follows that $s_{ij} \to 0$ in C^0 norm. The sequence $\{s_i\}$ is therefore Cauchy in C^0 and it follows that $s_i \to s$ in C^0.

Lemma 3.14.3. *Let $\{H_i\}_{i=1}^\infty$ be a sequence of metrics in $\mathscr{M}et_2^p(B)$ such that $H_i \to H$ in C^0 norm. Then the H_i are bounded in L_2^p.*

Proof. The difference between this lemma and Lemma 6.6.3 in [18] is that here the metrics are constrained by a bound on $\|m_{\phi,\tau}(H_i)\|_{L^p,H_i}^p$, whereas the metrics considered by Simpson are constrained by a bound on $\|\Lambda F_{Hi}\|_{L^p}$. These two bounds are in fact equivalent if there is a uniform bound on $\sup|s_i|$. Recall (from (3.15)) that

$$\Lambda F_H = m_{\phi,\tau}(H) + \frac{\sqrt{-1}}{2}\phi\otimes\phi^{*_H} - \frac{\sqrt{-1}}{2}\tau\mathbf{I}.$$

Hence if $H = Ke^s$ and $\sup|s| \le B'$, then

$$(3.75)\qquad \|\Lambda F_H\|_{L^p,H} \le \|m_{\phi,\tau}(H)\|_{L^p,H} + C(B').$$

The constant $C(B')$ depends on B' and other fixed parameters like ϕ and τ, but does not depend on s or H. The bound on $\|m_{\phi,\tau}(H_i)\|_{L^p,H_i}^p$ therefore gives a bound on $\|\Lambda F_{Hi}\|_{L^p}$.

We may thus apply the proof given in [18] (which is an adaptation of Donaldson's proof in [5]) to the present case.

Proof of 3.14.1(b). If $\{Ke^{s_i}\}_{i=1}^{\infty}$ is a minimizing sequence for $\mathbf{M}_{\phi,\tau}(K, -)$, then by (a) $\|s_i\|_{L_2^p}$ are uniformly bounded. After passing to a subsequence, $\{s_i\}_{i=1}^{\infty}$ therefore converges weakly in L_2^p. Let the weak limit be s. Now $L_2^p \subset C^1$ as $p > 2n$, and $\mathbf{M}_{\phi,\tau}$ is continuous on $C^1(S)$. Hence $\mathbf{M}_{\phi,\tau}$ is continuous in the weak topology on $\mathcal{M}et_2^p(B)$, and we can conclude that $\mathbf{M}_{\phi,\tau}(Ke^{s_i}) \to \mathbf{M}_{\phi,\tau}(Ke^{s})$. So $H = Ke^s$ minimizes $\mathbf{M}_{\phi,\tau}(K, -)$ on $\mathcal{M}et_2^p(B)$. By Lemma 3.5.1, Ker $= \{0\}$ and therefore $m_{\phi,\tau}(H) = 0$. That is,

$$\sqrt{-1}\Lambda F_H + \tfrac{1}{2}\phi \otimes \phi^{*H} = \tfrac{\tau}{2}\mathbf{I}.$$

To see that the solution is smooth we use the formula

$$\Delta_K'(h) = \sqrt{-1}h(\Lambda F_H - \Lambda F_K) + \sqrt{-1}\Lambda\overline{\partial}_E(h)h^{-1}D_K'(h), \tag{3.76}$$

where $h = e^s$. This equation follows from (1.17) and the expression (1.22a) for Δ_K'.

Hence if $m_{\phi,\tau}(H) = 0$, then h satisfies

$$\Delta_K'(h) = \sqrt{-1}\Lambda\overline{\partial}_E(h)h^{-1}D_K'(h) - \sqrt{-1}h\Lambda F_K - \tfrac{1}{2}h\phi \otimes \phi^{*K}h + \tfrac{\tau}{2}\mathbf{I}. \tag{3.77}$$

Elliptic regularity can thus be used to show that h—and therefore H—is smooth. Lemma 3.14 is proved.

The uniqueness of the solution follows from the form (and in particular the convexity) of the functional $\mathbf{M}_{\phi,\tau}(K, -)$. Suppose that $H' = He^s$ also minimizes $\mathbf{M}_{\phi,\tau}(K, -)$ for some smooth self-adjoint $s \in S(H)$. From (3.10.2) and (3.13) we can calculate that

$$\frac{d^2}{dt^2}\mathbf{M}_{\phi,\tau}(H, He^{ts}) = 2\left(\int_X |\overline{\partial}_E s|_{H_t}^2 + \int_X |s\phi|_{H_t}^2\right). \tag{3.78}$$

It follows that $\overline{\partial}_E s = 0$ and $s\phi = 0$. Hence $s \in \mathrm{Ker}(H)$. But Ker $= \{0\}$, and therefore $H' = H$. This concludes the proof of Theorem 3.1.1.

4. Properties of ϕ-stability

In this section we discuss some properties of ϕ-stable bundles. We start by giving a source of examples of such structures.

Proposition 4.1. *Let E be a holomorphic bundle over a closed Kähler manifold X. Suppose that X is algebraic and that E is stable. Then the Kähler form on X can be chosen such that given any holomorphic section ϕ of E of the condition $\mu_M < \mu_m(\phi)$ is satisfied, i.e., the pair (E, ϕ) is ϕ-stable.*

The proof of this will follow from the following lemma.

Lemma 4.1. *Suppose that X is algebraic. Let $\mathscr{E}' \subset \mathscr{E}$ be a reflexive subsheaf with slope $\mu(\mathscr{E}')$. If $\mu(\mathscr{E}') < \mu$ (where μ is the slope of E), then*

$$\mu(\mathscr{E}') \leq \mu - 1/R(R-1), \tag{4.1}$$

where R is the rank of E.

Proof of Lemma 4.1. Since X is algebraic, the Kähler form can be chosen to represent a class in the integral cohomology $H^2(X, Z)$. With the Kähler form thus chosen, μ and $\mu(\mathscr{E}')$ can be written $\mu = p/R$ and $\mu(\mathscr{E}') = p'/R'$, where p and p' are integers. Since E is stable, $\mu(\mathscr{E}')$ is less than μ and therefore $R'p - Rp' > 0$. Now suppose that $\mu - \mu(\mathscr{E}') < 1/(RR')$. Then $R'p - Rp' < 1$. That is, we get $0 < R'p - Rp' < 1$, which is impossible. Hence $\mu - \mu(\mathscr{E}') \geq 1/(RR') \geq 1/[R(R-1)]$ as required.

Proof of Proposition 4.1. E is stable and therefore $\mu(\mathscr{E}') < \mu$ for every reflexive subsheaf $\mathscr{E}' \subset \mathscr{E}$. Thus it follows from Lemma 4.1 that

$$\frac{R\mu - R'\mu(\mathscr{E}')}{R - R'} \geq \mu + \frac{R'}{R(R-1)}$$

for *every* reflexive subsheaf $\mathscr{E}' \subset \mathscr{E}$. Hence if ϕ is any holomorphic section of E, then

$$\mu_m(\phi) \geq \mu + \frac{1}{R(R-1)}. \tag{4.2}$$

But if E is stable, then $\mu_M = \mu$ and hence $\mu_M < \mu_m(\phi)$. The proposition is proved.

Finally, we look at restrictions on the characteristic classes of ϕ-stable bundles. It is well known that in a *stable* bundle over a compact Kähler manifold the first and second Chern classes are constrained by the Bogomolov-Gieseker inequality [1], [6]. In fact, if $[\omega] \in H^2(X, \mathbf{R})$ represents the cohomology class of the Kähler form on the base manifold X, and the rank of the bundle is R, then

$$\left(c_2(E) - \left(\frac{R-1}{2R}\right) c_1(E)^2\right) \cup [\omega]^{n-2} \geq 0. \tag{4.3}$$

The Chern classes of a ϕ-stable structure are constrained by a similar inequality.

Definition 4.1. Let $c_1(E) \in H^2(X, \mathbf{R})$ and $ch_2(E) \in H^4(X, \mathbf{R})$ be the first Chern class and second Chern character of E respectively. Define

$$C_1(E, \omega) = \int_X c_1(E) \wedge \omega^{[n-1]} \tag{4.4a}$$

and

$$Ch_2(E, \omega) = \int_X ch_2(E) \wedge \omega^{[n-2]}, \tag{4.4b}$$

where $\omega^{[m]} = \omega^m/m!$.

Theorem 4.1. *Let $(E, \overline{\partial}_E, \phi)$ be a ϕ-stable structure over X. Let τ be any real number satisfying $\mu_M < \tau \operatorname{Vol}(X)/4\pi < \mu_m(\phi)$. Then*

$$\frac{\tau}{4\pi} C_1(E, \omega) - Ch_2(E, \omega) \geq 0. \tag{4.5}$$

This result follows from the fact that the pair of equations

$$\overline{\partial}_E \phi = 0,$$
$$\Lambda F_H - \frac{\sqrt{-1}}{2}\phi \otimes \phi^{*_H} + \frac{\sqrt{-1}}{2}\tau \mathbf{I} = 0$$

come from a variational problem. Before proving the theorem we therefore need to describe the functional with which these equations are associated. As before, we let $\mathscr{A}$ and $\Omega^0(X, E)$ denote the space of connections on E and the space of sections of E respectively.

Suppose that we fix a hermitian bundle metric K on E. The Kähler metric on the base X induces an inner product on $\Omega^*(X, \mathbf{C})$. This inner product, combined with the metric K on E give rise to metrics on $\Omega^*(X, E)$ and $\Omega^*(X, \operatorname{End} E)$ (cf. §1) The metric K gives an identification $E \approx E^*$ and also $E \otimes E^* \approx \operatorname{End} E$. Using these metric constructions we can thus define the following real valued functional.

Definition 4.2. Define $YMH_\tau : \mathscr{A} \times \Omega^0(X, E) \to \mathbf{R}$ by

$$YMH_\tau(D, \phi) = \|F_D\|^2_{L^2} + \|D\phi\|^2_{L^2} + \tfrac{1}{4}\|\phi \otimes \phi^* - \tau \mathbf{I}\|^2_{L^2}. \tag{4.6}$$

Here $F_D \in \Omega^2(X, \operatorname{End} E)$ is the curvature of the connection D, $D\phi \in \Omega^1(X, E)$ is the covariant derivative, $\mathbf{I} \in \Omega^0(X, \operatorname{End} E) \approx \Omega^0(X, E \otimes E^*)$ is the identity section, and τ is a real parameter.

This same functional, when defined over $\mathbf{R}^d$ and with $\tau = 1$, is known as the Yang-Mills-Higgs functional (cf. [11]). Notice that when ϕ is the zero section this functional is essentially the Yang-Mills functional. In the special case where $d = 2$ and E is a line bundle, the critical points are called *vortices*. The equations which define the vortices turn out to be a special case of (2.1) and (2.2). It is for this reason that we have called (2.2) the vortex equation.

Clearly $YMH_\tau(\, , \,)$ is nonnegative. Furthermore, when restricted to a suitable subspace of $\mathscr{A} \times \Omega^0(X, E)$, the functional has a lower bound coming from the topology of E. To see this we restrict this functional to the set of connections on E that are both integrable and unitary with respect to K. We denote this subset of $\mathscr{A}$ by $\mathscr{A}^{1,1}(K)$. We then get the following result.

Theorem 4.2. *If* $(D, \phi) \in \mathscr{A}^{1,1}(K) \times \Omega^0(X, E)$, *then*

$$\begin{aligned} YMH_\tau(D, \phi) = 2\|D''\phi\|^2_{L^2} + \|\sqrt{-1}\Lambda F_D + \tfrac{1}{2}\phi\otimes\phi^* - \tfrac{\tau}{2}\mathbf{I}\|^2_{L^2} \\ + \tau\int_X \operatorname{Tr}\sqrt{-1}\Lambda F_D + \int_X \operatorname{Tr}(F_D\wedge F_D)\wedge\omega^{[n-2]}. \end{aligned} \tag{4.7}$$

Here D'' *is the* $(0, 1)$ *part of the connection* D.

Proof. This follows from the Kähler identities (1.17) for unitary connections and the fact that for a connection $D \in \mathscr{A}^{1,1}(K)$,

$$((F_D, F_D))\omega^{[n]} = |\Lambda F_D|^2\omega^{[n]} + \operatorname{Tr} F_D\wedge F_D\wedge\omega^{[n-2]}.$$

Here $((\ ,\ ))$ is the inner product defined in §1, and ω is the Kähler form on X. The theorem is proved.

The first two terms in (4.7) are nonnegative, and the last two are *independent of the connection* D. In fact the Chern-Weil homomorphism gives the following formulas:

Lemma 4.2. *Let* D *be a connection on* E *with curvature* F_D. *Let* $c_1(E) \in H^2(X, \mathbf{R})$ *and* $ch_2(E) \in H^4(X, \mathbf{R})$ *be the first Chern class and second Chern character of* E *respectively. Then*

(i) $[\sqrt{-1}\operatorname{Tr}(F_D)/2\pi] \in H^2(X, \mathbf{R})$ *represents* $c_1(E)$,

(ii) $[-\operatorname{Tr}(F_D\wedge F_E)/8\pi^2] \in H^4(X, \mathbf{R})$ *represents* $ch_2(E)$.

Proof. See, for example, [8].

On $\mathscr{A}^{1,1}(K)\times\Omega^0(X, E)$ the functional can thus be written as

$$\begin{aligned} YMH_\tau(D, \phi) = 2\|D''\phi\|^2_{L^2} + \|\sqrt{-1}\Lambda F_D + \tfrac{1}{2}\phi\otimes\phi^* - \tfrac{\tau}{2}\mathbf{I}\|^2_{L^2} \\ + 2\pi\tau C_1(E, \omega) - 8\pi^2 Ch_2(E, \omega). \end{aligned} \tag{4.8}$$

Here we have made use of Definitions 4.1. Notice that this formula displays both the topological lower bound of the functional, and the conditions required to achieve it, viz.

$$D''\phi = 0, \tag{4.9a}$$

$$\sqrt{-1}\Lambda F_D + \tfrac{1}{2}\phi\otimes\phi^* - \tfrac{\tau}{2}\mathbf{I} = 0. \tag{4.9b}$$

If D is the metric connection compatible with K and the holomorphic structure on E, then (4.9a) says that ϕ is a holomorphic section. Equation (4.9b) becomes the vortex equation.

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Suppose that we fix $\tau \in (\mu_M, \mu_m(\phi))$. By Theorem 3.1 we can find a smooth hermitian metric H on E such that with the given τ, $\overline{\partial}_E$, and ϕ, H satisfies the vortex equation

$$\sqrt{-1}\Lambda F_H + \tfrac{1}{2}\phi \otimes \phi^{*_H} - \tfrac{1}{2}\tau \mathbf{I} = 0.$$

We can use H and the Kähler form ω on X to define the functional

$$YMH_\tau : \mathscr{A} \times \Omega^0(X, E) \to \mathbf{R}$$

as above. The metric connection $D_H = D_{\overline{\partial}_E, H}$ is a unitary, integrable connection on E, i.e., is in $\mathscr{A}^{1,1}(H)$. The expression for $YMH_\tau(D, \phi)$ given in (4.8) thus applies. But ϕ is holomorphic, and H is a solution to the vortex equation. The first two terms are thus both zero and we get

$$YMH_\tau(D_H, \phi) = 2\pi\tau C_1(E, \omega) - 8\pi^2(Ch_2(E, \omega).$$

Inequality (4.5) now follows directly from the nonnegativity of YMH_τ. The theorem is proved.

Discussion. It is natural to ask how the inequality (4.5) compares to the Bogomolov-Gieseker inequality (4.3). Of course the only case in which a direct comparison is possible is the case in which $(E, \overline{\partial}_E)$ is a stable bundle, and $\phi \in \Omega^0(X, E)$ is a holomorphic section. By Proposition 4.1, $(E, \overline{\partial}_E, \phi)$ is then ϕ-stable, and thus both inequalities are applicable to the Chern classes of E. Suppose then that we are in this situation, i.e., that $(E, \overline{\partial}_E, \phi)$ represents a stable bundle with a prescribed holomorphic section. In order to compare the two inequalities, we use the relation (cf. [8])

(4.10) $$Ch_2(E) = \tfrac{1}{2}c_1^2(E) - c_2(E).$$

We can then write (4.5) as

(4.11)
$$C_2(E, \omega) - \left(\frac{R-1}{2R}\right) C_1^2(E, \omega) \geq \frac{1}{R}\left(\frac{C_1^2(E, \omega)}{2} - \frac{R\tau}{4\pi} C_1(E, \omega)\right).$$

This must hold for all $\tau \operatorname{Vol}(X)/4\pi$ in the interval $(\mu_M, \mu_m(\phi))$. In particular it will be true for $\tau \operatorname{Vol}(X)/4\pi = \mu_M$. In general $\mu_M \geq \mu(E)$, but since $(E, \overline{\partial}_E)$ is stable, we have equality between μ_M and $\mu(E)$. Rewriting (4.11) with $\tau = 4\pi C_1(E, \omega)/[R \operatorname{Vol}(X)]$ gives

(4.12) $$C_2(E, \omega) - \left(\frac{R-1}{2R}\right) C_1^2(E, \omega) \geq \frac{1}{R}\left(\frac{C_1^2(E, \omega)}{2} - \frac{C_1(E, \omega)^2}{\operatorname{Vol}(X)}\right).$$

The left-hand side of (4.12) is the same as that in (4.3). Hence if

$$\frac{C_1^2(E, \omega)}{2} - \frac{C_1(E, \omega)^2}{\mathrm{Vol}(X)} \geq 0, \tag{4.13}$$

then inequality (4.12) is stronger than the Bogomolov-Gieseker inequality. The quantity on the left-hand side of (4.13) is however always negative. The calculation is as follows.

Using the Lefschetz decomposition of $H^2(X, \mathbf{C})$ with respect to the Kähler form ω we may write $c_1(E) = a\omega + \beta$, where β is a (real) primitive $(1, 1)$ form. So $\int_X \beta \wedge \omega^{n-1} = 0$, which gives

$$C_1(E, \omega) = \int_X a \frac{\omega^n}{(n-1)!} = na\,\mathrm{Vol}(X). \tag{4.14}$$

Notice that $C_1(E, \omega)$ is nonnegative (since E is assumed to have a holomorphic section) and that therefore $a > 0$. Furthermore, using the fact that β is primitive we get

$$\begin{aligned} C_1^2(E, \omega) &= \int_X (a\omega + \beta)^2 \wedge \frac{\omega^{n-2}}{(n-2)!} \\ &= n(n-1)a^2\,\mathrm{Vol}(X) + \int_X \beta \wedge \beta \wedge \frac{\omega^{n-2}}{(n-2)!}. \end{aligned} \tag{4.15}$$

Hence

(4.16)

$$\frac{C_1^2(E, \omega)}{2} - \frac{C_1(E, \omega)^2}{\mathrm{Vol}(X)} = -\frac{n(n+1)}{2} a^2\,\mathrm{Vol}(X) + \int_X \beta \wedge \beta \wedge \frac{\omega^{n-2}}{(n-2)!}.$$

But, by the Hodge-Riemann bilinear relations for real primitive $(1, 1)$ forms, $\int_X \beta \wedge \beta \wedge \omega^{n-2}/(n-2)! < 0$ (cf. [22, p. 207]). Hence $C_1^2(E, \omega)/2 - C_1(E, \omega)^2/\mathrm{Vol}(X) < 0$.

At first sight this result might appear puzzling. One feels that as the parameter τ approaches the value $(4\pi/\mathrm{Vol}(X))\mu(E)$ one ought to recover results for stable systems. This is because when τ has this value the only possible solutions to (4.9a) and (4.9b) consist of the zero section and a Hermitian-Yang-Mills metric. It is in this sense that, as τ approaches $(4\pi/\mathrm{Vol}(X))\mu(E)$, the notion of ϕ-stability degenerates into that of ordinary stability. Why then do we not recover the Bogomolov-Gieseker inequality when $\tau = (4\pi/\mathrm{Vol}(X))\mu(E)$? The reason, roughly speaking, is that the functional $YMH_\tau(D, \phi)$ is not perfectly adapted to the task at hand.

We obtained the inequality (4.5) from the observation that $YMH_\tau(D, \phi) \geq 0$. This would yield the best possible inequality if the infimum of the functional were in fact zero. However to achieve $YMH_\tau(D, \phi) = 0$ requires $F_D = 0$, $D\phi = 0$, and $\phi \otimes \phi^* = \tau\mathbf{I}$. These three equations cannot be satisfied unless $\tau = 0$ and $C_1(E, \omega) = 0$. Indeed in this special case our inequality (4.5) does reduce to the one for stable bundles. In all other cases the functional $YMH_\tau(D, \phi)$ is strictly positive. It is thus impossible to achieve equality in the inequality for ϕ-stable bundles.

This is in contrast to the case of the Bogomolov-Gieseker inequality where equality is achieved precisely when the bundle is projectively flat (cf. [12, p. 114]). In fact the Bogomolov-Gieseker inequality can be derived by considering the nonnegative functional given by $\|F^\#\|^2_{L^2}$. Here $F^\#$ denotes the trace free part of the curvature. One shows that for connections whose curvature is of type $(1, 1)$,

$$\text{(4.17)} \quad \|F^\#\|^2_{L^2} = \|\Lambda F^\#\|^2_{L^2} + 4\pi^2 \int_X \left(2c_2(E) - \left(\frac{R-1}{R}\right) c_1^2(E)\right) \wedge \omega^{[n-2]}.$$

It follows from this that a stable bundle satisfies

$$C_2(E, \omega) - \left(\frac{R-1}{2R}\right) C_1^2(E, \omega) = 0,$$

when it supports a metric connection for which $F^\# = 0$. This last condition *is* attainable.

Since zero is merely a lower bound for $YMH_\tau(D, \phi)$ but is an infimum for $\|F^\#\|^2_{L^2}$, we cannot expect the inequality arising from the positivity of the former to be as sharp as that arising from the nonnegativity of the latter. This raises the question as to whether it is possible to "improve" the functional $YMH_\tau(D, \phi)$ and thereby to get a better inequality, perhaps even one that will merge correctly with the Bogomolov-Gieseker inequality. We hope to pursue such questions in future research.

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