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# HERMITIAN FINSLER METRICS AND THE KOBAYASHI METRIC

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**Abstract**

The problem of local equivalence of Hermitian Finsler metrics under holomorphic changes of coordinates is solved. On such a Finsler metric we find some differential conditions which imply that the Finsler metric is the Kobayashi metric of the underlying manifold (these conditions are satisfied if the metric is the Kobayashi metric on a bounded, strictly convex domain in $\mathbb{C}^n$ with smooth boundary).

## 0. Introduction

The infinitesimal Kobayashi metric is a real-valued function $F_M$ on the tangent bundle of a complex manifold $M$. For $p \in M$ and $v \in T_pM$,

$$F_M(p, v) = \inf\{1/r\colon \text{there is a holomorphic } f\colon \Delta_r \to M \text{ with } f(0) = p,\ f'(0) - v\},$$

where $\Delta_r = \{z \in \mathbb{C}\colon |z| < r\}$. $F_M$ is clearly an invariant of the complex structure on $M$, and, indeed, information about $F_M$ can yield information about the complex function theoretic aspects of $M$ (see, e.g., [3], [4]). One can think of $F_M(p, v)$ as being the length of the tangent vector $v$. One can then define the length of a curve by integrating the length of the tangent vector to the curve, and define a metric $d_M$ by considering the infimum of the lengths of all curves joining two points.

It is natural to try to understand the geometry of this metric. However, the techniques of differential geometry can, in general, be applied only indirectly, because $F_M$ need not have any smoothness, even away from the zero section of the tangent bundle. (For example, the Kobayashi metric on the polydisk in $\mathbb{C}^n$ is not $C^1$.) However, Lempert [5] has shown that when the complex manifold is a bounded domain $D \subset \mathbb{C}^n$ with smooth, strictly convex boundary, then the Kobayashi metric is smooth. (By smooth we shall always mean $C^\infty$. The results of this paper hold when considering less generous regularity assumptions, but, for what needs to be done here,

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working in such greater generality would only serve to obscure.) Moreover, Lempert, in the process of obtaining this regularity result, also showed that, in this particular case, the Kobayashi metric had a number of other nice properties. We will restate and summarize these as:

**Theorem** (*Lempert* [5]). *Let $D \subset \mathbb{C}^n$ be a bounded domain with smooth, strongly convex boundary. Then $F_D$ is smooth away from $\{v = 0\}$ and satisfies the following three conditions*:

(i) *In every complex tangent direction there is a totally geodesic complex curve.*

(ii) *The restriction of $F_D$ to such a totally geodesic curve is a Hermitian metric with curvature $-4$.*

(iii) *$d_D$ is complete.*

By totally geodesic complex curve, we mean a complex curve with the property that any two points of the curve may be joined by a real path in the curve with length equal to the distance between the two points.

The goal of this paper is to provide a type of converse to the above theorem.

A Hermitian Finsler metric on a complex manifold $M$ is a function $F: TM \to \mathbb{R}$ which satisfies

(1) $F(p, v) > 0$ if $v \in T_pM$ is nonzero,

(2) $F(p, \lambda v) = |\lambda| F(p, v)$ for all $\lambda \in \mathbb{C}$.

The Kobayashi metric on a smoothly bounded domain $D \subset \mathbb{C}^n$ is an example of a Hermitian Finsler metric. The bulk of this paper is dedicated to a study of the differential geometry of smooth Hermitian Finsler metrics. §§1–3 deal with the local equivalence problem: Given two smooth Hermitian Finsler metrics, when can we obtain one from the other by a local biholomorphic change of coordinates? The "solution" to this problem is given in Theorem 1, and may be interpreted as giving an intrinsically defined connection in an intrinsically defined principal bundle. §4 attempts to interpret some of the local differential invariants of smooth Hermitian Finsler metrics. §5 calculates the structure equations of the connection. §6 returns to geodesics, and calculates the Euler-Lagrange equations of the length functional in terms of the connection. §7 interprets conditions (i) and (ii) of the theorem above in terms of the local invariants, and contains the converse of that theorem (Theorem 2).

It should be noted that in studying real Finsler metrics, most authors assume that the unit ball (in the tangent space) is strictly convex. (See, for example, Chern's solution of the equivalence problem.) Here, no such assumption is made. For the equivalence method in §§1–3 to succeed all that is needed is strict pseudoconvexity. One result of this weakened hypothesis is that the notion of geodesic becomes very strange. This problem is sidestepped, however, because the final theorem—the converse mentioned above—makes no mention of geodesics. Both conditions (i) and (ii) are stated in terms of the vanishing of certain invariants, and the notion of geodesic is used only as motivation.

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## 1. The equivalence problem: The overdetermined algorithm and reduction to the space line elements

We wish to consider a smooth Hermitian Finsler metric on a complex manifold. Since we will be working locally, we consider such a metric to be a $C^\infty$ function $F\colon U \times \mathbb{C}^n \to \mathbb{R}$, where $U$ is an open set in $\mathbb{C}^n$, satisfying the following two hypotheses:

(i) $F(z, w) \geq 0$, with equality if and only if $w = 0$.

(ii) $F(z, \lambda w) = |\lambda| F(z, w)$ for all $\lambda \in \mathbb{C}$.

Here we have identified $TU$ with $U \times \mathbb{C}^n$. The problem we wish to study is the following: Given two such functions $F\colon U \times \mathbb{C}^n \to \mathbb{R}$ and $F'\colon U' \times \mathbb{C}^n \to \mathbb{R}$, when are they locally equivalent, i.e., given $z_0 \in U$ and $z'_0 \in U'$, when is there a biholomorphic change of variables $\varphi$ from a neighborhood $N$ of $z_0$ to a neighborhood of $z'_0$ taking $F$ into $F'$, i.e., such that

$$F'(\varphi(z), \varphi'(z)w) = F(z, w)$$

for all $(z, w) \in N \times \mathbb{C}^n$? To solve this problem, we shall apply the Method of Equivalence due to É. Cartan, a modern description of which the reader will find in R. Gardner's monograph [2]. The first step is to describe the problem as one dealing with the preservation of certain coframes. To do this, we approach the problem slightly differently.

Since $F$ satisfies condition (ii), given any (real) curve $\gamma\colon [a, b] \to U$, we can define the length of $\gamma$ to be

$$L(\gamma) = \int_a^b F(\gamma(t), \gamma'(t))\, dt,$$

and this is independent of the parametrization of the curve. An equivalent statement of our problem is to determine when there is a biholomorphic change of variables $\varphi$ from a neighborhood $N$ of $z_0$ to a neighborhood of $z'_0$ which preserves lengths of curves. Thus it is natural to consider the space $\mathbb{R} \times U \times \mathbb{C}^n = \{(t, z, w)\}$ (which may be thought of as the space of one-jets of maps from $\mathbb{R}$ to $U$), and consider changes of variables $(t, z, w) \mapsto (t', z', w')$ satisfying the following conditions:

(1) $t' = t'(t)$, $z' = z'(z)$, $w' = w'(z, w)$. Equivalently,

$$\begin{aligned} dt' &\equiv 0 \quad \operatorname{mod}(dt), \\ dz' &\equiv 0 \quad \operatorname{mod}(dz), \\ dw' &\equiv 0 \quad \operatorname{mod}(dz, dw), \end{aligned} \tag{1.1}$$

where we have used the notation, e.g., $\operatorname{mod}(dz)$ for $\operatorname{mod}(dz^1, \cdots, dz^n)$. Thus the coframe $(dt, dz, dw, d\bar{z}, d\bar{w})$ is determined up to transformation

$$\begin{bmatrix} dt \\ dz \\ dw \\ d\bar{z} \\ d\bar{w} \end{bmatrix} \mapsto \omega = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ 0 & B & C & 0 & 0 \\ 0 & 0 & 0 & \overline{A} & 0 \\ 0 & 0 & 0 & \overline{B} & \overline{C} \end{bmatrix} \begin{bmatrix} dt \\ dz \\ dw \\ d\bar{z} \\ d\bar{w} \end{bmatrix}$$

where $a \in \mathbb{R}$, $a \neq 0$, $A, C \in \mathrm{Gl}(n, \mathbb{C})$, $B \in M_n(\mathbb{C})$, and we think of $dz$, $dw$, $d\bar{z}$, $d\bar{w}$ as column vectors.

(2) Given a curve $\gamma: \mathbb{R} \to U$ we can consider its graph $\{(t, \gamma(t), \gamma'(t))\} \subset \mathbb{R} \times U \times \mathbb{C}^n$. We make the change of variables to take the graph of any curve to the graph of a curve. Thus:

$$dz' - w'\,dt' \equiv 0 \quad \operatorname{mod}(dz - w\,dt). \tag{1.2}$$

(3) To preserve the lengths of curves, we want the element of arc length, $F(z, w)\,dt$, to be preserved along curves; thus

$$F'(z', w')\,dt' \equiv F(z, w)\,dt \quad \operatorname{mod}(dz - w\,dt). \tag{1.3}$$

These last two conditions give us that the coframe $(F\,dt, dz - w\,dt, dw, d\bar{z} - \bar{w}\,dt, d\bar{w})$ is determined up to a transformation,

$$\begin{bmatrix} F\,dt \\ dz - w\,dt \\ dw \\ d\bar{z} - \bar{w}\,dt \\ d\bar{w} \end{bmatrix} \mapsto \theta = \begin{bmatrix} 1 & t_v & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 \\ u & H & G & 0 & 0 \\ 0 & 0 & 0 & \overline{E} & 0 \\ 0 & 0 & 0 & \overline{H} & \overline{G} \end{bmatrix} \begin{bmatrix} F\,dt \\ dz - w\,dt \\ dw \\ d\bar{z} - \bar{w}\,dt \\ d\bar{w} \end{bmatrix},$$

where $v$, $u$ are column vectors of length $n$, $E, G \in \mathrm{Gl}(n, \mathbb{C})$, and $H \in M_n(\mathbb{C})$.

Thus we are looking for mappings which will preserve both the coframes $\omega$ and the coframes $\theta$. In this case we apply the overdetermined algorithm (see [2]): We express the coframe $\theta$ in terms of the coframe $\omega$, $\theta = m\omega$, where $m$ is the matrix

$$\begin{bmatrix} (F - {}^t vw)/a & {}^t vA^{-1} & 0 & 0 & 0 \\ -Ew/a & EA^{-1} & 0 & 0 & 0 \\ (uF - Hw)/a & HA^{-1} - GC^{-1}BA^{-1} & GC^{-1} & 0 & 0 \\ -\overline{E}\overline{w}/a & 0 & 0 & \overline{E}\overline{A}^{-1} & 0 \\ (uF - Hw)/a & 0 & 0 & \overline{H}\overline{A}^{-1} - \overline{G}\overline{C}^{-1}\overline{B}\overline{A}^{-1} & \overline{G}\overline{C}^{-1} \end{bmatrix}$$

We can now normalize $m$ to be the identity matrix except for the $(2, 1)$-entry and the $(2n+2, 1)$-entry which we normalize to 1. This is done by taking $G = C$, $E = A$, $H = B$, $v = 0$, $a = F$, $u = Hw/a$, and

$$-Ew/a = {}^t(1, 0, \cdots, 0). \tag{1.4}$$

By the homogeneity of $F$,

$$\frac{\partial F}{\partial w^j} w^j = \frac{1}{2} F. \tag{1.5}$$

Here $j = 1, \cdots, n$ and we use here, as in what follows, the summation convention—repeated indices are to be summed. In general, the indices $j, k, l$, etc. will range from 1 to $n$, while the indices $\alpha$, $\beta$, $\gamma$, etc. will range from 2 to $n$. Therefore we can obtain (1.4) by taking $A = A_0$, where

$$A_0 = \begin{bmatrix} -2\frac{\partial F}{\partial w} \\ \widetilde{A}_0 \end{bmatrix}, \tag{1.6}$$

and $\widetilde{A}_0$ is an $(n-1) \times n$ matrix of rank $n-1$ satisfying $\widetilde{A}_0 w = 0$. If we let $v^\alpha{}_j$ be the entries of $\widetilde{A}_0$, we can define

$$\begin{aligned} \theta_0 &= F\,dt, \\ \omega_0^1 &= -2\frac{\partial F}{\partial w^j}\,dz^j, \\ \omega_0^\alpha &= v^\alpha{}_j\,dz^j, \\ \dot{\omega}_0^k &= dw^k. \end{aligned} \tag{1.7}$$

Then $(\theta_0 \omega_0^k, \dot{\omega}_0^k)$ is a coframe on $\mathbb{R} \times U \times \mathbb{C}^n$ and is intrinsically (i.e., independent of choices of coordinates, $\widetilde{A}_0$, etc.) defined up to a transformation

$$
(1.8) \qquad \begin{aligned}
\theta_0{}^{*} &= \theta_0, \\
\omega_0^{1*} &= \omega_0^1 + v_\gamma \omega_0^\gamma, \\
\omega_0^{\alpha*} &= A^\alpha{}_\gamma \omega_0^\gamma, \\
\dot{\omega}_0^{k*} &= u^k \omega_0^1 + B^k{}_\gamma \omega_0^\gamma + C^k{}_j \omega_0^j.
\end{aligned}
$$

Now consider transformations $\Phi$ of $\mathbb{R} \times U \times \mathbb{C}^n$ given by

$$
(1.9) \qquad \Phi(t, z, w) = (t/|\lambda| + \tau, z, \lambda w),
$$

$\lambda \in \mathbb{C}, \lambda \neq 0, \tau \in \mathbb{R}$. Then

$$
(1.10) \qquad \begin{aligned}
\Phi^*(\theta_0) &= \theta_0, \\
\Phi^*(\omega_0^1) &= \frac{\lambda}{|\lambda|} \omega_0^1, \\
\Phi^*(\omega_0^\alpha) &= \omega_0^\alpha, \\
\Phi^*(\dot{\omega}_0^k) &= \lambda \dot{\omega}_0^k.
\end{aligned}
$$

So if we quotient by the action of such transformations $\Phi$, we get a $G$-structure on $U \times \mathbb{P}^{n-1}$: the coframe $(\omega_0^1, \omega_0^\alpha, \dot{\omega}_0^\alpha)$ is well defined up to a transformation,

$$
(1.11) \qquad \begin{aligned}
\omega_0^{1*} &= \mu \omega_0^1 + v_\gamma \omega_0^\gamma, \\
\omega_0^{\alpha*} &= A^\alpha{}_\gamma \omega_0^\gamma, \\
\dot{\omega}_0^{\alpha*} &= u^\alpha \omega_0^1 + B^\alpha{}_\gamma \omega_0^\gamma + C^\alpha{}_\gamma \dot{\omega}_0^\gamma,
\end{aligned}
$$

where $\mu \in \mathbb{C}$, $|\mu| = 1$, $A, C \in \mathrm{Gl}(n-1, \mathbb{C})$, $u, v \in \mathbb{C}^{n-1}$, and $B \in M_{n-1}(\mathbb{C})$. Moreover, any map preserving this $G$-structure lifts to an equivalence of the original structure, and any equivalence of the original structure descends to an equivalence of the quotients. We will therefore only concern ourselves with the equivalence problem on the quotient $U \times \mathbb{P}^{n-1}$, the space of complex line elements. Of course, any explicit calculations we do will be done in homogeneous coordinates, i.e., on $U \times \mathbb{C}^n$.

## 2. First order normalizations

We now have, on $U \times \mathbb{P}^{n-1}$, that the coframe $(\omega_0^1, \omega_0^\alpha, \dot{\omega}_0^\alpha)$ is well defined up to a transformation (1.11). Let $\mathscr{P}$ be the principle bundle of all such coframes. The general coframe of $\mathscr{P}$ will be given by

$$(2.1)\qquad \begin{aligned} \omega^1 &= \mu\omega_0^1 + v_a\omega_0^\alpha, \\ \omega^\alpha &= A^\alpha{}_\gamma\omega_0^\gamma, \\ \dot\omega^\alpha &= u^\alpha\omega_0^1 + B^\alpha{}_\gamma\omega_0^\gamma + C^\alpha{}_\gamma\dot\omega_0^\gamma. \end{aligned}$$

Taking $\mu$, $v^\alpha$, $A^\alpha{}_\gamma$, $u^\alpha$, $B^\alpha{}_\gamma$, $C^\alpha{}_\gamma$ as fiber coordinates of $\mathscr{P}$, equations (2.1) also define $\omega^1$, $\omega^\alpha$, $\dot\omega^\alpha$ as forms on $\mathscr{P}$. Any equivalence $\varphi\colon U\times\mathbb{P}^{n-1}\to U'\times\mathbb{P}^{n-1}$ will lift to a map of principle bundles $\Phi\colon\mathscr{P}\to\mathscr{P}$ which preserves the forms $\omega^1$, $\omega^\alpha$, $\dot\omega^\alpha$. The next step in the Equivalence Method is to use the exterior derivatives of $\omega^1$, $\omega^\alpha$, $\dot\omega^\alpha$ to reduce the group of $\mathscr{P}$. We can write

$$(2.2)\qquad \begin{aligned} d\omega^1 &= \psi\wedge\omega^1 + \beta_a\wedge\omega^\alpha + \text{torsion}, \\ d\omega^\alpha &= \pi^\alpha{}_\gamma\wedge\omega^\gamma + \text{torsion}, \\ d\dot\omega^\alpha &= \gamma^\alpha\wedge\omega^1 + \mu^\alpha{}_\gamma\omega^\gamma + \eta^\alpha{}_\gamma\wedge\dot\omega^\gamma + \text{torsion}, \end{aligned}$$

where the torsion terms are quadratic in the base forms $\omega^1$, $\omega^\alpha$, $\dot\omega^\alpha$, and the forms $\psi=\bar\psi$, $\beta_a$, $\pi^\alpha{}_\gamma$, $\gamma^\alpha$, $\mu^\alpha{}_\gamma$, $\eta^\alpha{}_\gamma$ correspond to Maurer-Cartan forms of the group of $\mathscr{P}$. We have the integrability conditions

$$d\omega^1\equiv d\omega^\alpha\equiv 0 \mod(\omega^1,\omega^\alpha), \qquad d\dot\omega\equiv 0 \mod(\omega^1,\omega^\gamma,\dot\omega^\gamma).$$

Therefore all the torsion terms can be absorbed into the forms $\psi$, $\beta_a$, $\pi^\alpha{}_\gamma$, $\gamma^\alpha$, $\mu^\alpha{}_\gamma$, and $\eta^\alpha_\gamma$ except the following:

$$(2.3)\qquad \begin{aligned} d\omega^1 &= \psi\wedge\omega^1 + \beta_\alpha\wedge\omega^a + a_\alpha\omega^1\wedge\dot\omega^\alpha, \\ d\omega^\alpha &= \pi^\alpha{}_\gamma\wedge\omega^\gamma + \omega^1\wedge(b^\alpha\omega^{\bar 1} + c^\alpha{}_{\bar\beta}\omega^{\bar\beta} + e^\alpha{}_{\bar\beta}\dot\omega^{\bar\beta} + f^\alpha{}_\gamma\dot\omega^\gamma), \\ d\dot\omega^\alpha &= \gamma^\alpha\wedge\omega^1 + \mu^\alpha{}_\gamma\wedge\omega^\gamma + \eta^\alpha{}_\gamma\wedge\dot\omega^\gamma. \end{aligned}$$

The idea is now to use the group of $\mathscr{P}$ to normalize the torsion coefficients $a_\alpha$, $b^\alpha$, $c^\alpha{}_{\bar\beta}$, $f^\alpha{}_\gamma$. We shall see how the group acts by calculating its infinitesimal action, which can be done by taking the exterior derivatives of equations (2.3). First, calculating $\mod(\omega^\alpha,\omega^{\bar\beta})$ and $\mod(\bigwedge^3(\text{base}))$,

$$\begin{aligned} 0\equiv d^2\omega^1 &\equiv \omega^1\wedge\dot\omega^\alpha\wedge(da_\alpha + a_\gamma\eta^\gamma{}_\alpha - f^\gamma{}_\alpha\beta_\gamma) \\ &\quad + \omega^1\wedge(id\psi + b^\alpha\beta_\alpha\wedge\omega^{\bar 1} + e^\alpha{}_{\bar\beta}\beta_\alpha\wedge\dot\omega^{\bar\beta}). \end{aligned}$$

Add this to its conjugate and evaluate modulo $\omega^\alpha$ , $\omega^{\bar\beta}$ , $\omega^1-\omega^{\bar 1}$ , $\bigwedge^3$(base) :

$$\begin{aligned} 0 \equiv\, & \omega^1 \wedge \dot\omega^\alpha \wedge (da_\alpha + a_\gamma \eta^\gamma{}_\alpha - f^\gamma{}_\alpha \beta_\gamma) \\ & + \omega^{\bar 1} \wedge \dot\omega^{\bar\beta} \wedge (da_{\bar\beta} + a_{\bar\sigma} \eta^{\bar\sigma}{}_{\bar\beta} - f^{\bar\sigma}{}_{\bar\beta} \beta_{\bar\sigma}) \\ & + \omega^1 \wedge e^\alpha{}_{\bar\beta} \beta_\alpha \wedge \dot\omega^{\bar\beta} + \omega^{\bar 1} \wedge e^{\bar\beta}{}_\gamma \beta_{\bar\beta} \wedge \dot\omega^\gamma . \end{aligned}$$

It follows from this that we must have $e^\alpha{}_{\bar\beta} = 0$ and

$$da_\alpha \equiv -a_\gamma \eta^\gamma{}_\alpha + f^\gamma{}_\alpha \beta_\gamma \quad \text{mod(base)}.$$

From this we see that we will be able to choose the frame so that $a_\alpha = 0$ provided that the matrix $(f^\gamma{}_\alpha)$ is invertible. To see this, we need to calculate this matrix in coordinates.

**Parametric calculations I.** First, note that we can take

$$\omega_0^1 = -2 \frac{\partial F}{\partial w^j} dz^j , \tag{2.5}$$

$$\omega_0^\alpha = v^\alpha{}_j dz^j , \tag{2.6}$$

where the $(n-1) \times n$ matrix $v^\alpha{}_j$ has rank $n-1$ and satisfies

$$v^\alpha{}_j w^j = 0. \tag{2.7}$$

Then

$$\omega^1 = -2\mu \frac{\partial F}{\partial w^j} dz^j + v_\alpha v^\alpha{}_j dz^j , \tag{2.8}$$

$$\omega^\alpha = A^\alpha{}_\gamma v^\gamma{}_j dz^j . \tag{2.9}$$

The forms $\omega_0^1$ and $\omega_0^\alpha$ form a basis for $T^* U$, so we may write

$$dz^j = a^j \omega_0^1 + u^j{}_\gamma \omega_0^\gamma \tag{2.10}$$

for some $a^j$ , $u^j{}_\gamma$ . We then must have

$$\delta_k^j = -2a^j \frac{\partial F}{\partial w^k} + u^j{}_\gamma v^\gamma{}_k , \tag{2.11}$$

$$-2 \frac{\partial F}{\partial w^j} a^j = 1 , \tag{2.12}$$

$$\frac{\partial F}{\partial w^j} u^j{}_\gamma = 0 , \tag{2.13}$$

$$v^\alpha{}_j a^j = 0 , \tag{2.14}$$

$$v^\alpha{}_j u^j{}_\gamma = \delta_\gamma^\alpha . \tag{2.15}$$

(2.14) implies that $a^j = \lambda w^j$ for some $\lambda \in \mathbb{C}$. From (2.12) we see that $\lambda = -1/F$. Thus

$$dz^j = -\frac{w^j}{F}\omega_0^1 + u^j{}_\gamma \omega_0^\gamma. \tag{2.16}$$

Also,

$$\omega_0^\alpha = A'^\alpha{}_\gamma \omega^\gamma, \tag{2.17}$$

where

$$A'^\alpha{}_\gamma A^\gamma{}_\mu = A^\alpha{}_\gamma A'^\gamma{}_\mu = \delta_\mu^\alpha, \tag{2.18}$$

$$\omega_0^1 = \frac{1}{\mu}\omega^1 - \frac{1}{\mu} v_\alpha A'^\alpha{}_\gamma \omega^\gamma. \tag{2.19}$$

Therefore,

$$dz^j = -\frac{w^j}{\mu F}\omega^1 + \left(\frac{w^j}{\mu F} v_\gamma + u^j{}_\gamma\right) A'^\gamma{}_\mu \omega^\mu. \tag{2.20}$$

We can now calculate that

$$d\omega^\alpha \equiv \frac{1}{\mu F} A^\alpha{}_\gamma v^\gamma{}_j \, dw^j \wedge \omega^1 \quad \operatorname{mod}(\omega^\gamma). \tag{2.21}$$

So let

$$\dot{\omega}_0^\alpha = \frac{1}{F} v^\alpha{}_j \, dw^j, \tag{2.22}$$

$$\dot{\omega}^\alpha = \frac{1}{F} C^\alpha{}_\gamma v^\gamma{}_j \, dw^j + B^\alpha{}_\gamma \omega_0^\gamma + u^\alpha \omega_0^1. \tag{2.23}$$

Then

$$d\omega^\alpha \equiv \frac{1}{\mu} A^\alpha{}_\gamma C'^\gamma{}_\mu \dot{\omega}^\mu \wedge \omega^1 \quad \operatorname{mod}(\omega^\gamma), \tag{2.24}$$

where

$$C'^\alpha{}_\gamma C^\gamma{}_\mu = C^\alpha{}_\gamma C'^\gamma{}_\mu = \delta_\mu^\alpha, \tag{2.25}$$

so that

$$f^\alpha{}_\mu = \frac{1}{\mu} A^\alpha{}_\gamma C'^\gamma{}_\mu. \tag{2.26}$$

Thus, the matrix is always invertible.

Note also, the calculations above show that

$$b^\alpha = c^\alpha{}_{\bar\beta} = e^\alpha{}_{\bar\beta} = 0, \tag{2.27}$$

so that

$$d\omega^{\alpha} = \pi^{\alpha}{}_{\gamma} \wedge \omega^{\gamma} + f^{\alpha}{}_{\gamma}\omega^{1} \wedge \dot{\omega}^{\gamma}. \tag{2.28}$$

Then by calculating $\operatorname{mod}(\omega^{\gamma}, \bigwedge^{3}(\text{base}))$ we obtain

$$0 \equiv d^{2}\omega^{\alpha} \equiv (df^{\alpha}{}_{\gamma} - \pi^{\alpha}{}_{\mu}f^{\mu}{}_{\gamma} + if^{\alpha}{}_{\gamma}\psi + f^{\alpha}{}_{\mu}\eta^{\mu}{}_{\gamma}) \wedge \omega^{1} / \dot{\omega}^{\gamma},$$

which implies that

$$df^{\alpha}{}_{\gamma} \equiv \pi^{\alpha}{}_{\mu}f^{\mu}{}_{\gamma} - if^{\alpha}{}_{\gamma}\psi - f^{\alpha}\mu\eta^{\mu}{}_{\gamma} \quad \operatorname{mod}\text{base}. \tag{2.30}$$

Thus we can normalize

$$a_{\alpha} = 0, \tag{2.31}$$

$$f^{\alpha}{}_{\gamma} = -\delta^{\alpha}_{\gamma} \tag{2.32}$$

and restrict ourselves to the bundle $\mathscr{P}_1$ of coframes satisfying (2.31) and (2.32). This bundle will be preserved by equivalences.

## 3. Second order normalizations

From (2.4) and (2.30) we see that now on the bundle $\mathscr{P}_1$

$$\beta_{\gamma} = 0, \tag{3.1}$$

$$\eta^{\alpha}{}_{\gamma} = \pi^{\alpha}{}_{\gamma} - i\delta^{\alpha}_{\gamma}\psi. \tag{3.2}$$

So we now have

$$\begin{aligned} d\omega^{1} &= \psi \wedge \omega^{1} + \omega^{\alpha} \wedge \eta_{\alpha}, \\ d\omega^{\alpha} &= \pi^{\alpha}{}_{\gamma} \wedge \omega^{\gamma} + \dot{\omega}^{\alpha} \wedge \omega^{1}, \\ d\dot{\omega}^{\alpha} &= \gamma^{\alpha} \wedge \omega^{1} + \mu^{\alpha}{}_{\gamma} \wedge \omega^{\gamma} + (\pi^{\alpha}{}_{\gamma} - i\delta^{\alpha}_{\gamma}\psi) \wedge \dot{\omega}^{\gamma} + \dot{\omega}^{\mu} \wedge \varphi_{\mu}{}^{\alpha}, \end{aligned} \tag{3.3}$$

where $\eta^{\alpha}$ are $\varphi_{\mu}{}^{\alpha}$ are new torsions:

$$\eta_{\alpha} = a_{\alpha}\omega^{1} + b_{\alpha}\omega^{\bar{1}} + C_{\alpha\gamma}\omega^{\gamma} + e_{\alpha\bar{\beta}}\omega^{\bar{\beta}} + f_{\alpha\gamma}\omega^{\gamma} + g_{\alpha\bar{\beta}}\omega^{\bar{\beta}}, \tag{3.4}$$

$$\varphi_{\mu}{}^{\alpha} = h_{\mu}{}^{\alpha}\omega^{\bar{1}} + k_{\mu}{}^{\alpha}{}_{\bar{\beta}}\omega^{\bar{\beta}} + l_{\mu}{}^{\alpha}{}_{\gamma}\dot{\omega}^{\gamma} + m_{\mu}{}^{\alpha}{}_{\bar{\beta}}\omega^{\bar{\beta}}, \tag{3.5}$$

and we may assume

$$C_{\alpha\gamma} = -C_{\gamma\alpha}, \tag{3.6}$$

$$l_{\mu}{}^{\alpha}{}_{\gamma} = -l_{\gamma}{}^{\alpha}{}_{\mu}. \tag{3.7}$$

Next, let us calculate the relations which we get among these torsion coefficients by using $d^2 = 0$. First by calculating $\mathrm{mod}(\omega^\alpha)$ we find

$$(3.8) \qquad 0 \equiv d^2\omega^1 \equiv (id\psi - \eta_\alpha) \wedge \omega^1 \wedge \dot{\omega}^\alpha \quad \mathrm{mod}(\omega^\alpha).$$

Hence,

$$(3.9) \qquad 0 \equiv (-id\psi - \eta_{\bar\beta} \wedge \dot{\omega}^{\bar\beta}) \wedge \omega^{\bar 1} \quad \mathrm{mod}(\omega^{\bar\beta}).$$

Adding these two equations yields

$$(3.10) \qquad 0 \equiv (-\eta_\alpha \wedge \dot{\omega}^\alpha - \eta_{\bar\beta} \wedge \dot{\omega}^{\bar\beta}) \wedge \omega^1 \quad \mathrm{mod}(\omega^\alpha, \omega^{\bar\beta}, \omega^1 - \omega^{\bar 1})$$

so that

$$(3.11) \qquad 0 \equiv \eta_\alpha \wedge \dot{\omega}^\alpha + \eta_{\bar\beta} \wedge \dot{\omega}^{\bar\beta} \quad \mathrm{mod}(\omega^\alpha, \omega^{\bar\beta}, \omega^1, \omega^{\bar 1}).$$

It now follows from (3.4) that

$$(3.12) \qquad f_{\alpha\gamma} = f_{\gamma\alpha},$$

$$(3.13) \qquad g_{\alpha\bar\beta} = \overline{g_{\beta\bar\alpha}} = g_{\bar\beta\alpha}.$$

Second, again calculating $\mathrm{mod}(\omega^\gamma)$ gives

$$(3.14) \qquad 0 \equiv d^2\omega^\alpha \equiv \dot{\omega}^\mu \wedge \varphi_\mu{}^\alpha \wedge \omega^1 \quad \mathrm{mod}(\omega^\gamma).$$

From (3.5) it follows that $\varphi_\mu{}^\alpha = 0$:

$$(3.15) \qquad h_\mu{}^\alpha = k_{\mu\bar\beta}^{\alpha} = l_{\mu\gamma}^{\alpha} = m_\mu{}^\alpha{}_{\bar\beta} = 0.$$

Now, having eliminated most of the torsion in this fashion, we can calculate the infinitesimal action of the group on the torsion by calculating $d^2\omega^1 = 0$, as we did in the last section. A straightforward calculation gives

$$(3.16) \qquad da_\alpha = a_\gamma \pi^\gamma{}_\alpha - f_{\alpha\gamma}\gamma^\gamma \quad \mathrm{mod\,base},$$

$$(3.17) \qquad db_\alpha \equiv 2ib_\alpha\psi - b_\gamma\pi^\gamma{}_\alpha - g_{\alpha\bar\beta}\gamma^{\bar\beta} \quad \mathrm{mod\,base},$$

$$(3.18) \qquad \begin{aligned} dc_{\alpha\gamma} \equiv & -ic_{\alpha\gamma}\psi - c_{\mu\gamma}\pi^\mu{}_\alpha - c_{\alpha\mu}\pi^\mu{}_\gamma \\ & - \tfrac{1}{2}f_{\alpha\mu}\mu^\mu{}_\gamma + \tfrac{1}{2}f_{\gamma\mu}\mu^\mu{}_\alpha \quad \mathrm{mod\,base}, \end{aligned}$$

$$(3.19) \qquad de_{\alpha\bar\beta} \equiv -ie_{\alpha\bar\beta}\psi - e_{\gamma\bar\beta}\pi^\gamma{}_\gamma - e_{\alpha\bar\beta}\pi^{\bar\sigma}{}_{\bar\beta} - g_{\alpha\bar\sigma}\mu^{\bar\sigma}{}_{\bar\beta} \quad \mathrm{mod\,base},$$

$$(3.20) \qquad df_{\alpha\gamma} \equiv -2if_{\alpha\gamma}\psi - f_{\mu\gamma}\pi^\mu{}_\alpha - f_{\alpha\mu}\pi^\mu{}_\gamma \quad \mathrm{mod\,base},$$

$$dg_{\alpha\bar{\beta}} \equiv -g_{\gamma\bar{\beta}}\pi^{ga}{}_{\alpha} - g_{\alpha\bar{\sigma}}\pi^{\bar{\sigma}}_{\bar{\beta}} \quad \text{mod base.} \tag{3.21}$$

**Assumption.** The Hermitian matrix $g_{\alpha\bar{\beta}}$ is positive definite (in particular, invertible).

It follows that we can normalize

$$b_{\alpha} = e_{\alpha\bar{\beta}} = 0. \tag{3.22}$$

In order to make calculations easier, we do not normalize $g_{\alpha\bar{\beta}}$ nor do we normalize any of the other torsion. The normalizations (3.22) then give that

$$\gamma^{\alpha} = \mu^{\alpha}{}_{\gamma} = 0, \tag{3.23}$$

and we now write

$$\begin{aligned} d\omega^1 = \psi \wedge \omega^1 + a_{\alpha}\omega^{\alpha} \wedge \omega^1 + c_{\alpha\gamma}\omega^{\alpha} \wedge \omega^{\gamma} \\ + f_{\alpha\gamma}\omega^{\alpha} \wedge \omega^{\gamma} + g_{\alpha\bar{\beta}}\omega^{\alpha} \wedge \dot{\omega}^{\bar{\beta}}, \end{aligned} \tag{3.24}$$

$$d\omega^{\alpha} = \pi^{\alpha}{}_{\gamma} \wedge \omega^{\gamma} + \dot{\omega} \wedge \omega^1, \tag{3.25}$$

$$\begin{aligned} d\dot{\omega}^{\alpha} = (\pi^{\alpha}\gamma^{\alpha}{}_{\gamma} - i\delta^{\alpha}_{\gamma}\psi) \wedge \dot{\omega}^{\gamma} + E^{\alpha}\omega^1 \wedge \omega^{\bar{1}} + F^{\alpha}{}_{\gamma}\omega^1 \wedge \omega^{\gamma} \\ + G^{\alpha}{}_{\bar{\beta}}\omega^1 \wedge \omega^{\bar{\beta}} + H^{\alpha}{}_{\gamma}\omega^1 \wedge \dot{\omega}^{\gamma} + J^{\alpha}{}_{\bar{\beta}}\omega^1 \wedge \dot{\omega}^{\bar{\beta}} \\ + K^{\alpha}{}_{\gamma}\omega^{\gamma} \wedge \omega^{\bar{1}} + L^{\alpha}{}_{\gamma\mu}\omega^{\gamma} \wedge \omega^{\mu} + M^{\alpha}{}_{\gamma\bar{\beta}}\omega^{\gamma} \wedge \omega^{\bar{\beta}} \\ + N^{\alpha}_{\gamma\mu}\omega^{\gamma} \wedge \dot{\omega}^{\mu} + P^{\alpha}{}_{\gamma\bar{\beta}}\omega^{\gamma} \wedge \dot{\omega}^{\bar{\beta}}, \end{aligned} \tag{3.26}$$

where

$$L^{\alpha}{}_{\gamma\mu} = -L^{\alpha}{}_{\mu\gamma}. \tag{3.27}$$

The equivalence method now calls for us to normalize $g_{\alpha\bar{\beta}}$ and then require that the matrix $\pi^{\alpha}{}_{\gamma}$ be a skew-hermitian with respect to $g_{\alpha\bar{\beta}}$. This, however, would make certain calculations more difficult. To achieve the same result we do as follows.

Calculating $\mathrm{mod}(\omega^1, \omega^{\alpha} \wedge \omega^{\gamma}, \omega^{\alpha} \wedge \dot{\omega}^{\gamma})$ gives

$$\begin{aligned} 0 \equiv d^2\omega^1 \equiv (dg_{\alpha\bar{\beta}} + \pi_{\alpha\bar{\beta}} + \pi_{\bar{\beta}\alpha} + H_{\bar{\beta}\alpha}\omega^{\bar{1}} + N_{\alpha\bar{\sigma}\bar{\beta}}\omega^{\bar{\sigma}}) \wedge \omega^{\alpha} \wedge \dot{\omega}^{\bar{\beta}} \\ + F_{\alpha\bar{\sigma}}\omega^{\alpha} \wedge \omega^{\bar{1}} \wedge \omega^{\bar{\sigma}} - L_{\alpha\bar{\sigma}\bar{\nu}}\omega^{\alpha} \wedge \omega^{\bar{\sigma}} \wedge \dot{\omega}^{\bar{\nu}}. \end{aligned} \tag{3.28}$$

Here, and in what follows, we shall use $g_{\alpha\bar{\beta}}$ to lower indices and $g^{\alpha\bar{\beta}}$, the inverse of $g_{\alpha\bar{\beta}}$, to raise indices. Thus

$$F_{\alpha\bar{\sigma}} = L_{\alpha\bar{\sigma}\bar{\nu}} = 0, \tag{3.29}$$

$$dg_{\alpha\bar{\beta}} + \pi_{\alpha\bar{\beta}} + \pi_{\bar{\beta}\alpha} = A_{\alpha\bar{\beta}\gamma}\dot{\omega}^{\gamma} + A_{\bar{\beta}\alpha\bar{\sigma}}\dot{\omega}^{\bar{\sigma}} - N_{\bar{\beta}\gamma\alpha}\omega^{\gamma} - N_{\alpha\bar{\sigma}\bar{\beta}}\omega^{\bar{\sigma}} - H_{\bar{\beta}\alpha}\omega^{1} - H_{\alpha\bar{\beta}\beta}\omega^{\bar{1}}, \tag{3.30}$$

with

$$A_{\bar{\beta}\alpha\bar{\sigma}} = A_{\bar{\sigma}\alpha\bar{\beta}}. \tag{3.31}$$

So let us define

$$\varphi_{\bar{\beta}\alpha} = \pi_{\bar{\beta}\alpha} - A_{\alpha\bar{\beta}\gamma}\dot{\omega}^{\gamma} + N_{\bar{\beta}\gamma\alpha}\omega^{\gamma} + H_{\bar{\beta}\alpha}\omega^{1}. \tag{3.32}$$

Then

$$dg_{\alpha\bar{\beta}} + \varphi_{\bar{\beta}\alpha} + \varphi_{\alpha\bar{\beta}} = 0. \tag{3.33}$$

Also,

$$d\omega^{\alpha} = \varphi^{\alpha}{}_{\gamma} \wedge \omega^{\gamma} + \dot{\omega}^{\alpha} \wedge \omega^{1} + A_{\gamma}{}^{\alpha}{}_{\mu}\dot{\omega}^{\mu} \wedge \omega^{\gamma} - N^{\alpha}{}_{\mu\gamma}\omega^{\mu} \wedge \omega^{\gamma} - H^{\alpha}{}_{\gamma}\omega^{1} \wedge \omega^{\gamma}, \tag{3.34}$$

$$\begin{aligned} d\dot{\omega}^{\alpha} = {} & (\pi^{\alpha}{}_{\gamma} - i\delta^{\alpha}_{\gamma}\psi) \wedge \dot{\omega}^{\gamma} + E^{\alpha}\omega^{1} \wedge \omega^{\bar{1}} + G^{\alpha}{}_{\bar{\beta}}\omega^{1} \wedge \omega^{\bar{\beta}} + J^{\alpha}{}_{\bar{\beta}}\omega^{1} \wedge \dot{\omega}^{\bar{\beta}} \\ & + K^{\alpha}{}_{\gamma}\omega^{\gamma} \wedge \omega^{\bar{1}} + M^{\alpha}{}_{\gamma\bar{\beta}}\omega^{\gamma} \wedge \omega^{\bar{\beta}} + P^{\alpha}{}_{\gamma\bar{\beta}}\omega^{\gamma} \wedge \dot{\omega}^{\bar{\beta}}. \end{aligned} \tag{3.35}$$

We may now assume that

$$N^{\alpha}{}_{\mu\gamma} = -N^{\alpha}{}_{\gamma\mu}. \tag{3.36}$$

**Theorem 1.** *There are unique forms $\psi = \bar{\psi}$ and $\varphi^{\alpha}{}_{\gamma}$ satisfying* (3.24), (3.33), (3.34).

*Proof.* Since we have constructed such forms, existence is clear. To prove uniqueness, suppose $\psi^*$ and $\varphi^{\alpha}{}_{\gamma}{}^*$ are other such forms. Then, using (3.24) we obtain

$$\begin{aligned} 0 = {} & i(\psi^* - \psi) \wedge \omega^{1} + (a_{\alpha}{}^* - a_{\alpha})\omega^{\alpha} \wedge \omega^{1} + (c_{\alpha\gamma}{}^* - c_{\alpha\gamma})\omega^{\alpha} \wedge \omega^{\gamma} \\ & + (f_{\alpha\gamma}{}^* - f_{\alpha\gamma})\omega^{\alpha} \wedge \dot{\omega}^{\gamma} + (g_{\alpha\bar{\beta}}{}^* - g_{\alpha\bar{\beta}})\omega^{\alpha} \wedge \dot{\omega}^{\bar{\beta}}. \end{aligned} \tag{3.37}$$

This implies that

$$0 = (i(\psi^* - \psi) + (a_{\alpha}{}^* - a_{\alpha})\omega^{\alpha}) \wedge \omega^{1}, \tag{3.38}$$

and hence that

$$i(\psi^* - \psi) + (a_{\alpha}{}^* - a_{\alpha})\omega^{\alpha} \equiv 0 \quad \mathrm{mod}(\omega^{1}), \tag{3.39}$$

which is impossible (since $\psi$, $\psi^*$ are real) unless $a_{\alpha}{}^* = a_{\alpha}$ and $\psi^* = \psi$. The proof that $\varphi^{\alpha}{}_{\gamma}{}^* = \varphi^{\alpha}{}_{\gamma}$ is similar, by using (3.34) instead of (3.24).

Thus, on the bundle $\mathscr{P}_1$, we have an $e$-structure—the forms $\omega^1$, $\omega^\alpha$, $\dot\omega^\alpha$, $\psi$, $\varphi^\alpha{}_\gamma$ are intrinsically defined on $\mathscr{P}_1$ and form a basis for $T^*\mathscr{P}_1$. This completes the formal solution of the equivalence problem. The functions $a_\alpha$, $c_{\alpha\gamma}$, $f_{\alpha\gamma}$, $g_{\alpha\bar\beta}$ are second order invariants, and the functions $A_\gamma{}^\alpha{}_\mu$, $n^\alpha{}_{\mu\gamma}$, $H^\alpha{}_\gamma$, $E^\alpha$, $G^\alpha{}_{\bar\beta}$, $J^\alpha{}_{\bar\beta}$, $K^\alpha{}_\gamma$, $M^\alpha{}_{\gamma\bar\beta}$, $P^\alpha_{\gamma\bar\beta}$ are third order invariants.

## 4. Interpretations of the second order invariants, or, parametric calculations II

In this section we wish to present some geometric interpretations of the second order invariants $a_\alpha$, $c_{\alpha\gamma}$, $f_{\alpha\gamma}$, $g_{\alpha\bar\beta}$. In particular, we would like an interpretation of the definiteness of $g_{\alpha\bar\beta}$. To this end, we continue the parametric calculations carried out in §1. Recall that we may take

$$\omega^1 = -2\frac{\partial F}{\partial w^j}\,dz^j\,, \tag{4.1}$$

$$\omega^\alpha = v^\alpha{}_j\,dz^j. \tag{4.2}$$

(These are written as forms on $U \times \mathbb{C}^n$, and present forms on $U \times \mathbb{P}^{n-1}$. Thus these are not the intrinsic forms on $\mathscr{P}_1$, but rather the pull-back to $U \times \mathbb{P}^{n-1}$ of the intrinsic forms. Since we can determine the intrinsic forms from these using the group action—cf. (2.1)—we can work with the forms (4.1) and (4.2) so that we do not have to carry around the excess baggage of the group parameters.) We then have

$$d\omega^\alpha \equiv \frac{1}{F}v^\gamma{}_j\,dw^j \wedge \omega^1 \quad \operatorname{mod}(\omega^\gamma). \tag{4.3}$$

So we let

$$\dot\omega^\alpha_0 = \frac{1}{F}v^\gamma{}_j\,dw^j. \tag{4.4}$$

Then

$$u^k{}_\alpha\dot\omega^\alpha_0 = \frac{1}{F}\,dw^k - \frac{2}{F^2}w^k\frac{\partial F}{\partial w^j}\,dw^j\,, \tag{4.5}$$

so

$$dw^k = Fu^k{}_\alpha\dot\omega^\alpha_0 - \frac{2}{F}w^k\frac{\partial F}{\partial w^j}\,dw^j. \tag{4.6}$$

We shall want to define

$$\dot\omega^\alpha = \dot\omega^\alpha_0 + p^\alpha{}_\gamma\omega^\gamma + q^\alpha\omega^1\,, \tag{4.7}$$

by choosing $p^{\alpha}{}_{\gamma}$ and $q^{\alpha}$ so that the normalizations (3.22) are satisfied. Calculation gives

(4.8)
$$\begin{aligned}
d\omega^1 = {} & 2F\frac{\partial^2 F}{\partial w^j \partial w^{\bar k}} u^j{}_\alpha u^{\bar k}{}_{\bar\beta}\,\omega^\alpha \wedge \dot\omega_0^{\bar\beta} + 2F\frac{\partial^2 F}{\partial w^j \partial w^k} u^j{}_\alpha u^k{}_\gamma\,\omega^\alpha \wedge \dot\omega_0^\gamma \\
& + \omega^\gamma \wedge \left( -2\frac{\partial^2 F}{\partial w^j \partial w^{\bar k}} \frac{w^{\bar k}}{F} u^j{}_\gamma \omega^{\bar 1} + 2\frac{\partial^2 F}{\partial w^j \partial z^{\bar k}} u^{\bar k}{}_{\bar\beta} u^j{}_\gamma \omega^{\bar\beta} \right) \\
& + \omega^1 \wedge \left( \frac{1}{F}\frac{\partial F}{\partial z^k} u^k{}_\alpha \omega^\alpha - \frac{1}{F}\frac{\partial F}{\partial z^{\bar k}} u^{\bar k}{}_{\bar\beta}\omega^{\bar\beta} + \frac{1}{F^2}\frac{\partial F}{\partial z^{\bar k}} w^{\bar k}\omega^{\bar 1} \right. \\
& \qquad \left. - \frac{1}{F^2}\frac{\partial F}{\partial z^k} w^k \omega^1 + \frac{1}{F}\frac{\partial F}{\partial w^k} dw^k - \frac{1}{F}\frac{\partial F}{\partial w^{\bar k}} dw^{\bar k} \right) \\
& + \left( -\frac{2}{F}\frac{\partial F}{\partial z^k} u^k{}_\alpha + \frac{2}{F}\frac{\partial^2 F}{\partial w^j \partial z^k} w^k u^j{}_\alpha \right) \omega^1 \wedge \omega^\alpha \\
& - 2\frac{\partial^2 F}{\partial w^j \partial z^k} u^k{}_\alpha u^j{}_\gamma\,\omega^\alpha \wedge \omega^\gamma.
\end{aligned}$$

This gives us the following:

$$g_{\alpha\bar\beta} = 2F\frac{\partial^2 F}{\partial w^j \partial w^{\bar k}} u^j{}_\alpha u^{\bar k}{}_{\bar\beta} = \frac{\partial^2 (F^2)}{\partial w^j \partial w^{\bar k}} u^j{}_\alpha u^{\bar k}{}_{\bar k}, \tag{4.9}$$

$$f_{\alpha\gamma} = 2F\frac{\partial^2 F}{\partial w^j \partial w^k} u^j{}_\alpha u^k{}_\gamma = \frac{\partial^2 (F^2)}{\partial w^j \partial w^k} u^j{}_\alpha u^k{}_\gamma, \tag{4.10}$$

the final equalities here using (2.13).

**Definition.** $I_z = \{w : F(z, w) = 1\}$ is the *indicatrix* of $F$.

**Proposition 1.** (1) *The matrix* $g_{\alpha\bar\beta}$ *is positive definite if and only if the indicatrix is a strictly pseudoconvex real hypersurface.*

(2) *The matrix* $f_{\alpha\gamma}$ *vanishes identically if and only if the Finsler metric is Riemannian.*

*Proof.* (1) Note that (2.13) implies that the vectors $(u^1{}_\alpha, \cdots, u^n{}_\alpha)$, $\alpha = 2, \cdots, n$, are in the maximal complex tangent space of $I_z$. Since $u^j{}_\alpha$ has rank $n-1$, they span the maximal complex tangent space. Equation (4.9) thus displays $g_{\alpha\bar\beta}$ as the Levi form of the real hypersurface $I_z$.

(2) We calculate, using the homogeneity of $F^2$ and (2.13),

$$\frac{\partial^2 (F^2)}{\partial w^j \partial w^k} u^j{}_\alpha w^k = \frac{\partial (F^2)}{\partial w^j} u^j{}_\alpha = 2F\frac{\partial F}{\partial w^j} u^j{}_\alpha = 0. \tag{4.11}$$

Also we have

$$\frac{\partial^2(F^2)}{\partial w^j \partial w^k} w^j w^k = \frac{\partial(F^2)}{\partial w^j} w^j = 0. \tag{4.12}$$

Since the vectors $(u^1{}_\alpha, \cdots, u^n{}_\alpha)$, $\alpha = 2, \cdots, n$, together with $(w^1, \cdots, w^n)$ span $\mathbb{C}^n$ by (4.10), the vanishing of $f_{\alpha\gamma}$ is equivalent to the vanishing of the quadratic form

$$\frac{\partial^2(F^2)}{\partial w^j \partial w^k}.$$

However, if this form vanishes, then $F^2$ must be linear in $w$, and since $F$ is real, linear in $\bar{w}$ as well. It follows that

$$F^2 = h_{j\bar{k}}(z) w^j w^{\bar{k}},$$

and the metric is Riemannian.

Returning now to (4.8), we make the final normalizations by taking

$$p^\gamma{}_\alpha = 2 g^{\gamma\bar{\beta}} \frac{\partial^2 F}{\partial w^{\bar{j}} \partial z^k} u^k{}_\alpha u^{\bar{j}}{}_{\bar{\beta}}, \tag{4.13}$$

$$q^\gamma = -\frac{2}{F} g^{\gamma\bar{\beta}} \frac{\partial^2 F}{\partial w^{\bar{j}} \partial z^k} w^k u^{\bar{j}}{}_{\bar{\beta}}. \tag{4.14}$$

It then follows that

$$a_\alpha = \frac{2}{F}\frac{\partial F}{\partial z^k} u^k{}_\alpha - \frac{2}{F}\frac{\partial^2 F}{\partial w^j \partial z^k} w^k u^j{}_\alpha + \frac{2}{F} f_{\alpha\gamma} g^{\gamma\bar{\beta}} \frac{\partial^2 F}{\partial w^{\bar{j}} \partial z^k} w^k u^{\bar{j}}{}_{\bar{\beta}}, \tag{4.15}$$

$$\begin{aligned} c_{\alpha\gamma} = & -\frac{\partial^2 F}{\partial w^j \partial z^k} u^k{}_\alpha u^j{}_\gamma + \frac{\partial^2 F}{\partial w^j \partial z^k} w^k{}_\gamma u^j{}_\alpha \\ & - f_{\alpha\mu} g^{\mu\bar{\beta}} \frac{\partial^2 F}{\partial w^{\bar{j}} \partial z^k} u^k{}_\gamma u^{\bar{j}}{}_{\bar{\beta}} + f_{\gamma\mu} g^{\mu\bar{\beta}} \frac{\partial^2 F}{\partial w^{\bar{j}} \partial z^k} u^k{}_\alpha u^{\bar{j}}{}_{\bar{\beta}}. \end{aligned} \tag{4.16}$$

**Proposition 2.** *If the metric is Riemannian, then $a_\alpha$ and $c_{\alpha\gamma}$ both vanish identically if and only if the metric is Kähler.*

*Proof.* We suppose $F^2 = h_{j\bar{k}}(z) w^j w^{\bar{k}}$, and then calculate that

$$c_{\alpha\gamma} = \frac{1}{2F}\left(\frac{\partial H_{k\bar{l}}}{\partial z^j} - \frac{\partial h_{j\bar{l}}}{\partial z^k}\right) w^{\bar{l}} u^k{}_\alpha u^j{}_\gamma,$$

$$a_\alpha = \frac{1}{F^2}\left(\frac{\partial h_{k\bar{l}}}{\partial z^j} - \frac{\partial H_{j\bar{l}}}{\partial z^k}\right) w^{\bar{l}} w^k u^j{}_\alpha.$$

It follows that $a_\alpha$ and $c_{\alpha\gamma}$ vanish identically if and only if

$$\frac{\partial h_{k\bar{l}}}{\partial z^j} = \frac{\partial h_{j\bar{l}}}{\partial z^k},$$

which is the Kähler condition.

## 5. Structure equations

In this section we derive the formulas for the exterior derivatives of the forms $\psi$ and $\varphi^\alpha{}_\gamma$. We do this by using $d^2 = 0$. First, differentiating (3.24) we obtain

$$(5.1)\quad 0 = \omega^\alpha \wedge \omega^1 \wedge Da_\alpha + i\omega^1 \wedge \Psi + \omega^\alpha \wedge \omega^\gamma \wedge Dc_{\alpha\gamma} + \omega^\alpha \wedge \dot{\omega}^\gamma \wedge Df_{\alpha\gamma},$$

where

$$(5.2)\quad \begin{aligned} Da_\alpha = da_\alpha + a_\gamma \varphi^\gamma{}_\alpha + (E^\alpha - f\alpha\mu E^\mu)\omega^{\bar{1}} + (K_{\alpha\bar{\beta}} - f_{\alpha\gamma} G^\gamma{}_{\bar{\beta}})\omega^{\bar{\beta}} \\ + (2c_{\alpha\gamma} + a_\mu A_\alpha{}^\mu{}_\gamma + f_{\mu\gamma} H^\mu{}_\alpha)\dot{\omega}^\gamma + (H_{\bar{\beta}\alpha} - f_{\alpha\gamma} J^\gamma{}_{\bar{\beta}})\dot{\omega}^{\bar{\beta}}, \end{aligned}$$

$$(5.3)\quad \Psi = d\psi = i g_{\alpha\bar{\beta}} \dot{\omega}^\alpha \wedge \dot{\omega}^{\bar{\beta}},$$

$$(5.4)\quad \begin{aligned} Dc_{\alpha\gamma} = dc_{\alpha\gamma} - ic_{\alpha\gamma}\psi + c_{\mu\gamma}\varphi^\mu{}_\alpha + c_{\alpha\mu}\varphi^\mu{}_\gamma \\ + (-a_\mu N^\mu{}_{\alpha\gamma} - c_{\mu\gamma} H^\mu{}_\alpha - c_{\alpha\mu} H^\mu{}_\gamma)\omega^1 \\ + \tfrac{1}{2}(G_{\alpha\gamma} - G_{\gamma\alpha} - f_{\alpha\mu} K^\mu{}_\gamma + f_{\gamma\mu} K^\mu{}_\alpha)\omega^{\bar{1}} - a_\mu c_{\alpha\gamma}\omega^\mu \\ + \tfrac{1}{2}(M_{\bar{\beta}\alpha\gamma} - M_{\bar{\beta}\gamma\alpha} - f_{\alpha\mu} M_\gamma{}^\mu{}_{\bar{\beta}} + f_{\gamma\mu} M_\alpha{}^\mu{}_{\bar{\beta}})\omega^{\bar{\beta}} - f_{\mu\rho} N^\mu{}_{\alpha\gamma}\dot{\omega}^\rho \\ - \frac{1}{2}(a_\alpha g_{\gamma\bar{\beta}} - a_\gamma g_{\alpha\bar{\beta}} - f_{\alpha\mu} P^\mu{}_{\gamma\bar{\beta}} + f_{\gamma\mu} P^\mu_{\alpha\bar{\beta}} - 2N_{\bar{\beta}\alpha\gamma})\dot{\omega}^{\bar{\beta}}, \end{aligned}$$

(5.5)

$$\begin{aligned} Df_{\alpha\gamma} = df_{\alpha\gamma} - 2if_{\alpha\gamma}\psi + f_{\mu\gamma}\varphi^\mu{}_\alpha + f_{\alpha\mu}\varphi^\mu{}_\gamma + J_{\alpha\gamma}\omega^{\bar{1}} \\ + (2C_{\rho\mu} A_\alpha{}^\mu{}_\gamma - a_\rho f_{\alpha\gamma})\omega^\rho + P_{\alpha\bar{\sigma}\gamma}\omega^{\bar{\sigma}} - f_{\mu\rho} A_\alpha{}^\mu{}_\gamma \dot{\omega}^\rho - A_{\alpha\bar{\beta}\gamma}\dot{\omega}^{\bar{\beta}}. \end{aligned}$$

Therefore, by Cartan's lemma we have

$$(5.6)\quad Dc_{\alpha\gamma} = c_{\alpha\gamma,\mu}\omega^\mu + c_{\alpha\gamma,\dot{\mu}}\dot{\omega}^\mu + c_{\alpha\gamma,1}\omega^1,$$

$$(5.7)\quad Df_{\alpha\gamma} = f_{\alpha\gamma,\mu}\omega^\mu + f_{\alpha\gamma,\dot{\mu}}\dot{\omega}^\mu + f_{\alpha\gamma,1}\omega^1,$$

where

$$f_{\alpha\gamma,\dot{\mu}} = f_{\alpha\mu,\dot{\gamma}}, \tag{5.8}$$

$$c_{\alpha\gamma,\mu} + c_{\gamma\mu,\alpha} + c_{\mu\alpha,\gamma} = 0, \tag{5.9}$$

$$c_{\alpha\gamma,\dot{\mu}} - f_{\alpha\mu,\gamma} = c_{\gamma\alpha,\dot{\mu}} - f_{\gamma\mu,\alpha}. \tag{5.10}$$

Further, from (3.6) and (3.12) (differentiated) it follows that

$$c_{\alpha\gamma,1} = -c_{\gamma\alpha,1}, \tag{5.11}$$

$$c_{\alpha\gamma,\mu} = -c_{\gamma\alpha,\mu}, \tag{5.12}$$

$$c_{\alpha\gamma,\dot{\mu}} = -c_{\gamma\alpha,\dot{\mu}}, \tag{5.13}$$

$$f_{\alpha\gamma,1} = +f_{\gamma\alpha,1}, \tag{5.14}$$

$$f_{\alpha\gamma,\mu} = +f_{\gamma\alpha,\mu}, \tag{5.15}$$

$$f_{\alpha\gamma,\dot{\mu}} = +f_{\gamma\alpha,\dot{\mu}}, \tag{5.16}$$

$$J_{\alpha\gamma} = J_{\gamma\alpha}, \tag{5.17}$$

$$P_{\alpha\bar{\sigma}\gamma} = P_{\gamma\bar{\sigma}\alpha}. \tag{5.18}$$

Substituting (5.6) and (5.7) into (5.1) gives

$$0 = (i\Psi + Da_\alpha \wedge \omega^\alpha + c_{\alpha\gamma,1}\omega^\alpha \wedge \omega^\gamma + f_{\alpha\gamma,1}\omega^\alpha \wedge \dot{\omega}^\gamma) \wedge \omega^1. \tag{5.19}$$

Therefore,

$$i\Psi + Da_\alpha \wedge \omega^\alpha + c_{\alpha\gamma,1}\omega^\alpha \wedge \omega^\gamma + f_{\alpha\gamma,1}\omega^\alpha \wedge \dot{\omega}^\gamma = \lambda \wedge \omega^1 \tag{5.20}$$

for some one-form $\lambda$. Taking conjugates and adding we obtain

$$\begin{aligned} 0 = {} & Da_\alpha \wedge \omega^\alpha + c_{\alpha\gamma,1}\omega^\alpha \wedge \omega^\gamma + f_{\alpha\gamma,1}\omega^\alpha \wedge \dot{\omega}^\gamma - \lambda \wedge \omega^1 \\ & + Da_{\bar{\beta}}\omega^{\bar{\beta}} + c_{\bar{\beta}\bar{\sigma},\bar{1}}\omega^{\bar{\beta}} \wedge \omega^{\bar{\sigma}} + f_{\bar{\beta}\bar{\sigma},\bar{1}}\omega^{\bar{\beta}} \wedge \dot{\omega}^{\bar{\sigma}} - \bar{\lambda} \wedge \omega^{\bar{1}}. \end{aligned} \tag{5.21}$$

It now follows from Cartan's lemma that

$$Da_\alpha = a_{\alpha,\gamma}\omega^\gamma + R_{\alpha\bar{\sigma}}\omega^{\bar{\sigma}} + f_{\alpha\gamma,1}\dot{\omega}^\gamma + a_{\alpha,1}\omega^1 + a_{\alpha,\bar{1}}\omega^{\bar{1}}, \tag{5.22}$$

$$\lambda = -a_{\alpha,1}\omega^\alpha - a_{\bar{\eta},1}\omega^{\bar{\beta}} + Q\omega^{\bar{1}} + b\omega^1 \tag{5.23}$$

with

$$a_{\alpha,\gamma} - a_{\gamma,\alpha} = 2c_{\alpha\gamma}, \tag{5.24}$$

$$R_{\alpha\bar{\sigma}} = R_{\bar{\sigma}\alpha}. \tag{5.25}$$

We then have, from (5.20),

$$d\psi = -ig_{\alpha\bar{\beta}}\dot{\omega}^\alpha \wedge \dot{\omega}^{\bar{\beta}} - iR_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}} - ia_{\alpha,\bar{1}}\omega^\alpha \wedge \omega^{\bar{1}} + ia_{\bar{\beta},1}\omega^{\bar{\beta}} \wedge \omega^1 + iQ\omega^1 \wedge \omega^{\bar{1}}.$$

Second, differentiating (3.34), we obtain

(5.27)
$$0 = \dot{\omega}^{\mu} \wedge \omega^{\gamma} \wedge DA_{\gamma}{}^{\alpha}{}_{\mu} - \omega^{\mu} \wedge \omega^{\gamma} \wedge DN^{\alpha}{}_{\mu\gamma} - \omega^{1} \wedge \omega^{\gamma} \wedge DH^{\alpha}{}_{\gamma} + \Phi^{\alpha}{}_{\gamma} \wedge \omega^{\gamma},$$

where

(5.28)
$$\begin{aligned} DA_{\gamma}{}^{\alpha}{}_{\mu} = {} & dA_{\gamma}{}^{\alpha}{}_{\mu} - A_{\gamma}{}^{\rho}{}_{\mu}\varphi^{\alpha}{}_{\rho} + A_{\gamma}{}^{\alpha}\rho\varphi^{\rho}{}_{\mu} + A_{\rho}{}^{\alpha}{}_{\mu}\varphi^{\rho}{}_{\gamma} - iA_{\gamma}{}^{\alpha}{}_{\mu}\psi \\ & + H^{\alpha}{}_{\rho}A_{\gamma}{}^{\rho}{}_{\mu}\omega^{1} - A_{\rho}{}^{\alpha}{}_{\tau}A_{\gamma}{}^{\rho}{}_{\mu}\dot{\omega}^{\tau} + (H^{\alpha}{}_{\tau}f_{\gamma\mu} - 2N_{\rho}{}^{\alpha}{}_{\tau}A_{\gamma}{}^{\rho}{}_{\mu})\omega^{\tau}, \end{aligned}$$

(5.29)
$$\begin{aligned} DN^{\alpha}{}_{\mu\gamma} = {} & dN^{\alpha}{}_{\mu\gamma} - N^{\rho}{}_{\mu\gamma}\varphi^{\alpha}{}_{\rho} + N^{\alpha}{}_{\rho\gamma}\varphi^{\rho}{}_{\mu} + N^{\alpha}{}_{\mu\rho}\varphi^{\rho}{}_{\gamma} \\ & + \tfrac{1}{2}(A_{\gamma}{}^{\alpha}{}_{\rho}K^{\rho}{}_{\mu} - A_{\mu}{}^{\alpha}{}_{\rho}K^{\rho}{}_{\gamma})\omega^{\bar{1}} + (H^{\alpha}{}_{\tau}c_{\mu\gamma} - 2N^{\alpha}{}_{\rho\tau}N^{\rho}{}_{\mu\gamma})\omega^{\tau} \\ & + \tfrac{1}{2}(A_{\gamma}{}^{\alpha}{}_{\rho}M^{\rho}{}_{\mu\bar{\beta}} - A_{\mu}{}^{\alpha}{}_{\rho}M^{\rho}{}_{\gamma\bar{\beta}})\omega^{\bar{\beta}} + (-A_{\rho}{}^{\alpha}{}_{\tau}N^{\rho}{}_{\mu\gamma} - c_{\mu\gamma}\delta^{\alpha}_{\tau})\dot{\omega}^{\tau} \\ & + (A_{\gamma}{}^{\alpha}{}_{\rho}P^{\rho}{}_{\mu\bar{\beta}} - A_{\mu}{}^{\alpha}{}_{\rho}P^{\rho}{}_{\gamma\bar{\beta}})\dot{\omega}^{\bar{\beta}}, \end{aligned}$$

(5.30)
$$\begin{aligned} DH^{\alpha}{}_{\gamma} = {} & dH^{\alpha}{}_{\gamma} - H^{\mu}{}_{\gamma}\varphi^{\alpha}{}_{\mu} + H^{\alpha}{}_{\mu}\varphi^{\mu}{}_{\gamma} + iH^{\alpha}{}_{\gamma}\psi \\ & + (K^{\alpha}{}_{\gamma} + A_{\gamma}{}^{\alpha}{}_{\mu}E^{\mu})\omega^{\bar{1}} + (-2N^{\alpha}{}_{\rho\mu}H^{\rho}{}_{\gamma} + H^{\alpha}{}_{\gamma}a_{\mu})\omega^{\mu} \\ & + (M^{\alpha}{}_{\gamma\bar{\beta}} - A_{\gamma}{}^{\alpha}{}_{\mu}G^{\mu}{}_{\bar{\beta}})\omega^{\bar{\beta}} + (a_{\gamma}\delta^{\mu}_{\alpha} - A_{\rho}{}^{\alpha}{}_{\mu}H^{\rho}{}_{\gamma} + N^{\alpha}{}_{\gamma\mu})\dot{\mu}^{\mu} \\ & + (P^{\alpha}{}_{\gamma\bar{\beta}} - A_{\gamma}{}^{\alpha}\mu J^{\mu}{}_{\bar{\beta}})\dot{\omega}^{\bar{\beta}}, \end{aligned}$$

(5.31)
$$\Phi^{\alpha}{}_{\gamma} = d\varphi^{\alpha}{}_{\gamma} - \varphi^{\alpha}{}_{\mu} \wedge \varphi^{\mu}{}_{\gamma} - g_{\gamma\bar{\beta}}\dot{\omega}^{\alpha} \wedge \dot{\omega}^{\bar{\beta}}.$$

Note that

(5.32)
$$DA_{\gamma}{}^{\alpha}{}_{\mu} = DA_{\mu}{}^{\alpha}{}_{\gamma},$$

(5.33)
$$DN^{\alpha}{}_{\gamma\mu} = -DN^{\alpha}{}_{\mu\gamma}.$$

Equation (5.27) may be written

(5.34)
$$0 = (\Phi^{\alpha}{}_{\gamma} + DA_{\gamma}{}^{\alpha}{}_{\mu} \wedge \dot{\omega}^{\mu} +_{D} N^{\alpha}{}_{\gamma\mu} \wedge \omega^{\mu} - DH^{\alpha}{}_{\gamma} \wedge \omega^{1}) \wedge \omega^{\gamma},$$

It follows that

(5.35)
$$\Phi^{\alpha}{}_{\gamma} + DA_{\gamma}{}^{\alpha}{}_{\mu} \wedge \dot{\mu} + DN^{\alpha}{}_{\gamma\mu} \wedge \omega^{\mu} - DH^{\alpha}{}_{\gamma} \wedge \omega^{1} = \lambda^{\alpha}{}_{\gamma\mu}$$

for some one-form $\lambda^{\alpha}{}_{\gamma\mu} = \lambda^{\alpha}_{\mu\gamma}$. By differentiating (3.33) we can calculate that

(5.36)
$$0 = \Phi_{\bar{\beta}\gamma} + \Phi_{\gamma\bar{\beta}}.$$

Together with (5.35) this gives

$$0 = DA_{\gamma\bar{\beta}\mu} \wedge \dot{\omega}^{\mu} + (DN_{\bar{\beta}\gamma\mu} - \lambda_{\bar{\beta}\gamma\mu}) \wedge \omega^{\mu} - DH_{\bar{\beta}\gamma} \wedge \omega^{1} \tag{5.37}$$
$$+ DA_{\bar{\beta}\gamma\bar{\sigma}} \wedge \dot{\omega}^{\bar{\sigma}} + (DN_{\gamma\bar{\beta}\bar{\sigma}} - \lambda_{\gamma\bar{\beta}\bar{\sigma}}) \wedge \omega^{\bar{\sigma}} - DH_{\gamma\bar{\beta}} \wedge \omega^{\bar{1}}.$$

The Cartan Lemma then yields

$$DA_{\gamma\bar{\beta}\mu} = A_{\gamma\bar{\beta}\mu,\rho}\omega^{\rho} + A_{\gamma\bar{\beta}\mu,\bar{\sigma}}\omega^{\bar{\sigma}} + A_{\gamma\bar{\beta}\mu,\dot{\rho}}\dot{\omega}^{\rho} + A_{\gamma\bar{\beta}\mu,\dot{\bar{\sigma}}}\dot{\omega}^{\bar{\sigma}} \tag{5.38}$$
$$+A_{\gamma\bar{\beta}\mu,1}\omega^{1} + A_{\gamma\bar{\beta}\mu\bar{1}}\omega^{\bar{1}},$$

$$DN_{\bar{\beta}\gamma\mu} = N_{\bar{\beta}\gamma\mu,\rho}\omega^{\rho} + N_{\bar{\beta}\gamma\mu,\bar{\sigma}}\omega^{\bar{\sigma}} + N_{\bar{\beta}\gamma\mu,\dot{\rho}}\dot{\omega}^{\rho} + N_{\bar{\beta}\gamma\mu,\dot{\bar{\sigma}}}\dot{\omega}^{\bar{\sigma}} \tag{5.39}$$
$$+N_{\bar{\beta}\gamma\mu,1}\omega^{1} + N_{\bar{\beta}\gamma\mu,\bar{1}}\omega^{\bar{1}},$$

$$\lambda_{\bar{\beta}\gamma\mu} = L_{\bar{\beta}\gamma\mu\rho}\omega^{\rho} + L_{\bar{\beta}\gamma\mu\bar{\sigma}}\omega^{\bar{\sigma}} + L_{\bar{\beta}\gamma\mu\dot{\rho}}\dot{\omega}^{\rho} + L_{\bar{\beta}\gamma\mu\dot{\bar{\sigma}}}\dot{\omega}^{\bar{\sigma}} \tag{5.40}$$
$$+L_{\bar{\beta}\gamma\mu,1}\omega^{1} + L_{\bar{\beta}\gamma\mu,\bar{1}}\omega^{\bar{1}},$$

(5.41)
$$DH_{\bar{\beta}\gamma} = H_{\bar{\gamma},\rho}\omega^{\rho} + H_{\bar{\beta}\gamma,\bar{\sigma}}\omega^{\bar{\sigma}} H_{\bar{\beta}\gamma,\dot{\rho}}\dot{\omega}^{\rho} + H_{\bar{\beta}\gamma,\dot{\bar{\sigma}}}\dot{\omega}^{\bar{\sigma}} + H_{\bar{\beta}\gamma,1}\omega^{1} + H_{\bar{\beta}\gamma,\bar{1}}\omega^{\bar{1}},$$

where

$$A_{\gamma\bar{\beta}\mu,\dot{\rho}} = A_{\gamma\bar{\beta}\rho,\dot{\mu}}, \tag{5.42}$$
$$A_{\gamma\bar{\beta}\mu,\dot{\bar{\sigma}}} = A_{\bar{\beta}\gamma\bar{\sigma},\dot{\mu}}, \tag{5.43}$$
$$A_{\gamma\bar{\beta}\mu,\rho} - N_{\bar{\beta}\gamma\rho,\dot{\mu}} + L_{\bar{\beta}\rho,\dot{\mu}} = 0, \tag{5.44}$$
$$A_{\gamma\bar{\beta}\mu,\bar{\sigma}} - N_{\gamma\bar{\beta}\bar{\sigma},\dot{\mu}} + L_{\gamma\bar{\beta}\bar{\sigma},\dot{\mu}} = 0, \tag{5.45}$$
$$A_{\gamma\bar{\beta}\mu,1} = H_{\bar{\beta}\gamma,\dot{\mu}}, \tag{5.46}$$
$$A_{\gamma\bar{\beta}\mu,\bar{1}} = H_{\gamma\bar{\beta},\dot{\mu}}, \tag{5.47}$$
$$N_{\bar{\beta}\gamma\mu,\rho} - L_{\bar{\beta}\gamma,\mu,\rho} = N_{\bar{\beta}\gamma\rho,\mu} - L_{\bar{\beta}\gamma\rho,\mu}, \tag{5.48}$$
$$N_{\bar{\beta}\gamma\mu,\bar{\sigma}} - L_{\bar{\beta}\gamma,\mu,\bar{\sigma}} = N_{\gamma\bar{\beta}\bar{\sigma},\mu} - L_{\gamma\bar{\eta}\bar{\sigma},\mu}, \tag{5.49}$$
$$N_{\bar{\beta}\gamma\mu,1} - L_{\bar{\beta}\gamma\mu,1} + H_{\bar{\beta}\gamma,\mu} = 0, \tag{5.50}$$
$$N_{\bar{\beta}\gamma\mu,\bar{1}} - L_{\bar{\beta}\gamma\mu,\bar{1}} + H_{\gamma\bar{\beta},\mu} = 0, \tag{5.51}$$
$$H_{\gamma\bar{\beta},1} - H_{\bar{\beta}\gamma,1} = 0. \tag{5.52}$$

Hence we have

$$
\begin{aligned}
d\varphi^{\alpha}{}_{\gamma} = {} & \varphi^{\alpha}{}_{\mu} \wedge \varphi^{\mu}{}_{\gamma} + g_{\gamma\bar{\beta}}\delta^{\alpha}_{\mu}\dot{\omega}^{\mu} \wedge \dot{\omega}^{\bar{\beta}} \\
& + A_{\gamma}{}^{\alpha}{}_{\mu,\bar{\sigma}}(\dot{\omega}^{\mu} \wedge \omega^{\bar{\sigma}} + A_{\gamma}{}^{\alpha}{}_{\mu,\dot{\bar{\sigma}}}\dot{\omega}^{\mu} \wedge \dot{\omega}^{\bar{\sigma}} + A_{\gamma}{}^{\alpha}{}_{\mu,\bar{1}}\dot{\mu}^{\mu} \wedge \omega^{\bar{1}} \\
& + N^{\alpha}{}_{\gamma\mu,\bar{\sigma}}\check{\omega}^{\mu} \wedge \omega^{\bar{\sigma}} + N^{\alpha}{}_{\gamma\mu,\dot{\bar{\sigma}}}\check{\omega}^{\mu} \wedge \dot{\omega}^{\bar{\sigma}} + N^{\alpha}{}_{\gamma\mu,\bar{1}}\check{\omega}^{\mu} \wedge \omega^{\bar{1}} \\
& + L^{\alpha}{}_{\gamma\mu,\bar{\sigma}}\check{\omega}^{\mu} \wedge \omega^{\bar{\sigma}} + L^{\alpha}{}_{\gamma\mu,\dot{\bar{\sigma}}}\check{\omega}^{\mu} \wedge \dot{\omega}^{\bar{\sigma}} + L^{\alpha}{}_{\gamma\mu,\bar{1}}\check{\omega}^{\mu} \wedge \omega^{\bar{1}} \\
& + H^{\alpha}{}_{\gamma,\bar{\sigma}}\omega^{1} \wedge \omega^{\bar{\sigma}} + H^{\alpha}{}_{\gamma,\dot{\bar{\sigma}}}\omega^{1} \wedge \dot{\omega}^{\bar{\sigma}} + H^{\alpha}{}_{\gamma,\bar{1}}\omega^{1} \wedge \omega^{\bar{1}}.
\end{aligned}
\tag{5.53}
$$

## 6. Geodesics

In this section we wish to derive the Euler-Lagrange equations of the geodesics of our Finsler metric. So consider a curve $\gamma\colon [0, 1] \to M$. If $\gamma$ is regular, i.e., if $\gamma'(t)$ is never vanishing, $\gamma$ has a natural lift to a map into $\mathbb{P}TM$, $t \mapsto (\gamma(t), [\gamma'(t)])$, where $[\gamma'(t)]$ is the complex line spanned by $\gamma'(t)$. We can then lift this curve in $\mathbb{P}TM$ to a curve $\Gamma\colon [0, 1] \to \mathscr{P}_1$ satisfying

$$\Gamma^{*}\omega^{\alpha} = 0, \tag{6.1}$$

$$\Gamma^{*}\omega^{1} = \lambda\, dt, \qquad \lambda > 0. \tag{6.2}$$

The curve $\Gamma$ may be thought of as a coframe adapted to $\gamma$. Recall that

$$\omega^{1} = -2\mu \frac{\partial F}{\partial w^{j}}\, dz^{j}.$$

If $\gamma$ is given by $t \mapsto z(t)$, the lift of $\gamma$ to $\mathbb{P}TM$ will be given by $t \mapsto (z(t), z'(t))$. Then

$$\Gamma^{*}\omega^{1} = -2\mu \frac{\partial F}{\partial w^{j}}(z(t), z'(t))z^{j\prime}(t) = -\mu F(z(t), z'(t))$$

by the homogeneity of $F$, so $\mu = -1$, and $\lambda = F(z(t)z'(t))$ in (6.2). It follows that the length of the curve $\gamma$

$$L(\gamma) = \int_{0}^{1} \Gamma^{*}\omega^{1}. \tag{6.3}$$

Now consider a smooth variation $\gamma\colon (-\varepsilon, \varepsilon) \times [0, 1] \to M$, $\gamma_S(t) = \gamma(s, t)$. We adapt a coframe $\Gamma_s$ to each $\gamma_s$ in a smooth fashion, obtaining a smooth lift $\Gamma\colon (-\varepsilon, \varepsilon) \times [0, 1] \to \mathscr{P}_1$ satisfying

$$\Gamma^{*}\omega^{\alpha} \equiv \Gamma^{*}(\omega^{1} - \omega^{\bar{1}}) \equiv 0 \quad \mathrm{mod}(ds). \tag{6.4}$$

Let $\Gamma_s(t) = \Gamma(s, t)$, and define

$$f(s) = \int_0^1 \Gamma_s^* \omega^1. \tag{6.5}$$

Then, letting $C$ be the curve $t \mapsto (0, t)$ in $(-\varepsilon, \varepsilon) \times [0, 1]$, we have

$$\begin{aligned} f'(0) &= \int_C L_{\partial/\partial t} \Gamma^* \omega^1 = \int_C \left( \frac{\partial}{\partial t} \lrcorner d(\Gamma^* \omega^1) + d\left( \frac{\partial}{\partial t} \lrcorner \Gamma^* \omega^1 \right) \right) \\ &= \int_C \left( \frac{\partial}{\partial t} \lrcorner d(\Gamma^* \omega^1) \right) + \frac{\partial}{\partial t} \lrcorner \Gamma^* \omega^1 \Big|_{s=0}^{1}. \end{aligned} \tag{6.6}$$

Suppose

$$\Gamma^* \omega^1 = A\, ds + B\, dt, \tag{6.7}$$

$$\Gamma^* \omega^\alpha = C^\alpha\, ds, \tag{6.8}$$

$$\Gamma^* \dot{\omega}^\alpha = F^\alpha\, ds + G^\alpha\, dt, \tag{6.9}$$

$$\Gamma^* \psi = H\, ds + J\, dt. \tag{6.10}$$

Assume that the variation $\gamma$ fixed endpoints: $\gamma(s, 0) = \gamma(0, 0)$, $\gamma(s, 1) = \gamma(0, 1)$. It then follows that $B(s, 0) = B(s, 1) = 0$, and the boundary term in (6.6) vanishes:

$$\begin{aligned} f'(0) &= \int_C \frac{\partial}{\partial t} \lrcorner d(\Gamma^* \omega^1) \\ &= \int_C \frac{\partial}{\partial t} \lrcorner \Gamma^* (i\psi \wedge \omega^1 + a_\alpha \omega^\alpha \wedge \omega^1 + c_{\alpha\gamma} \omega^\alpha \wedge \omega^\gamma \\ &\qquad + f_{\alpha\gamma} \omega^\alpha \wedge \dot{\omega}^\gamma + g_{\alpha\bar{\beta}} \omega^\alpha \wedge \dot{\omega}^{\bar{\beta}}) \\ &= \int_C [iHB - iAJ + C^\alpha (A_\alpha B + f_{\alpha\gamma} G^\gamma + g_{\alpha\bar{\beta}} G^{\bar{\beta}})]\, ds \\ &= \int_C \mathrm{Re}[-iAJ + C^\alpha (a_\alpha B + f_{\alpha\gamma} G^\gamma + g_{\alpha\bar{\beta}} G^{\bar{\beta}})]\, ds, \end{aligned} \tag{6.11}$$

since $f'(0)$ is real, as are $H$ and $B$. Since $A$ and $C^\alpha$ are arbitrary (they correspond to the variation of the curve in $M$), if $f(0)$ is minimal, we must have

$$J = a_\alpha B + f_{\alpha\gamma} G^\gamma + g_{\alpha\bar{\beta}} G^{\bar{\beta}} = 0. \tag{6.12}$$

In this fashion we obtain the following:

**Proposition 3.** *Suppose $\gamma\colon [0, 1] \to M$ is a geodesic. Then when we choose a frame along $\gamma$ so that $\omega^\alpha = \omega^1 - \omega^{\bar{1}} = 0$, we must also have $\psi = a_\alpha \omega^1 + f_{\alpha\gamma} \dot{\omega}^\gamma + g_{\alpha\bar{\beta}} \dot{\omega}^{\bar{\beta}} = 0$.*

Note that this does not say that if we have a curve $\gamma$ and a coframe along $\gamma$ with $\omega^\alpha = \omega^1 - \omega^{\bar 1} = \psi = a_\alpha \omega^1 + f_{\alpha\gamma}\dot\omega^\gamma + g_{\alpha\bar\beta}\dot\omega^{\bar\beta} = 0$, the $\gamma$ is therefore a geodesic. To obtain such a statement we would need further assumptions, for example, that the indicatrix is strongly convex. The result above, however, is sufficient for our present purpose.

## 7. Characterization of the Kobayashi metric

Having a description of the geodesics of our metric, we return now to Lempert's results, and consider totally geodesic complex curves.

Let $\gamma : \zeta \mapsto \gamma(\zeta)$ be a holomorphic curve in $M$. Such a curve has a natural lift $\tilde\gamma : \zeta \mapsto (h\gamma(\zeta), [\gamma'(\zeta)])$ to a curve in $\mathbb{P}TM$ which satisfies $\tilde\gamma^*\omega^\alpha = 0$. Note that

(7.1)
$$d(a_\alpha\omega^1 + f_{\alpha\gamma}\dot\omega^\gamma + g_{\alpha\bar\beta}\dot\omega^{\bar\beta}) \equiv (\varphi^\gamma{}_\alpha + i\delta^\gamma_\alpha\psi)\wedge(a_\gamma\omega^1 + f_{\gamma\mu}\dot\omega^\mu + g_{\gamma\bar\beta}\dot\omega^{\bar\beta}) \quad \operatorname{mod}(\Lambda^2(\text{base})).$$

This describes how the group of the bundle $\mathscr{P}_1$ acts on $a_\alpha\omega^1 + f_{\alpha\gamma}\dot\omega^\gamma + g_{\alpha\bar\beta}\dot\omega^{\bar\beta}$. It follows that the vanishing of all the $a_\alpha\omega^1 + f_{\alpha\gamma}\dot\omega^\gamma + g_{\alpha\bar\beta}\dot\omega^{\bar\beta}$ is independent of the choice of coframe. Now,

$$\tilde\gamma^*(a_\alpha\omega^1 + f_{\alpha\gamma}\dot\omega^\gamma + g_{\alpha\bar\beta}\dot\omega^{\bar\beta}) = A_\alpha\tilde\gamma^*(\omega^1) + B_\alpha\tilde\gamma^*(\omega^{\bar 1}). \tag{7.2}$$

So if there are geodesics in $\gamma$, they occur only in the directions

$$A_\alpha\omega^1 + B_\alpha\omega^{\bar 1} = 0. \tag{7.3}$$

Further if $\gamma$ is totally geodesic, i.e., if $\gamma$ is tangent to every real direction along $\gamma$ there is a geodesic which remains in $\gamma$, so $A_\alpha = B_\alpha = 0$, and

$$\tilde\gamma^*(a_\alpha\omega^1 + f_{\alpha\gamma}\dot\omega^\gamma + g_{\alpha\bar\beta}\dot\omega^{\bar\beta}) = 0. \tag{7.4}$$

However, along any holomorphic $\tilde\gamma$, $\omega^\alpha = 0$ which implies

$$0 = d\omega^\alpha = \dot\omega^\alpha \wedge \omega^1 \tag{7.5}$$

along $\gamma$. Therefore,

$$\dot\omega^\alpha = l^\alpha\omega^1 \tag{7.6}$$

for some functions $l^\alpha$. So if $\gamma$ is totally geodesic, then

$$a_\alpha\omega^1 + f_{\alpha\gamma}l^\gamma\omega^1 + g_{\alpha\bar\beta}l^{\bar\beta}\omega^{\bar 1} = 0, \tag{7.7}$$

from which it follows that $l^{\bar{\beta}} = 0$, and hence $a_\alpha = 0$. Thus for $\gamma$ to be totally geodesic we must have

$$a_\alpha = 0, \tag{7.8}$$

$$\dot{\omega}^\alpha = 0 \tag{7.9}$$

along $\gamma$. This last equation implies, in consequence of (3.35), that

$$E^\alpha = 0. \tag{7.10}$$

Thus we have shown

**Proposition 4.** *If through every point of $M$ and tangent to every complex direction there is a totally geodesic complex curve, then*

$$a_\alpha = E^\alpha = 0.$$

*Moreover, the totally geodesic curves are the integrals of $\omega^\alpha = \dot{\omega}^\alpha = 0$.*

Along such a totally geodesic curve, the restriction of the Finsler metric is given by $ds^2 = \omega^1 \omega^{\bar{1}}$, which is a Hermitian Riemannian metric, and $\varphi = i\psi$ is the corresponding Kähler connection form. Moreover,

$$d\varphi = -Q\omega^1 \wedge \omega^{\bar{1}}. \tag{7.11}$$

Thus the curve will have curvature $2Q$. Along an arbitrary holomorphic curve, $\dot{\omega}^\alpha = l^\alpha \omega^1$, and

$$d\varphi = -(Q - g_{\alpha\bar{\beta}} l^\alpha l^{\bar{\beta}})\omega^1 \wedge \omega^{\bar{1}}, \tag{7.12}$$

so such a curve has curvature

$$2Q - 2g_{\alpha\bar{\beta}} l^\alpha l^{\bar{\beta}} \leq 2Q. \tag{7.13}$$

**Theorem 5.** *Let $M$ be a complex manifold with a smooth complete Finsler metric $F$ with strictly pseudoconvex indicatrix satisfying*

$$a_\alpha = E^\alpha = 0, \qquad Q = -2.$$

*Then $F$ is the Kobayashi metric of $M$.*

**Remark.** Notice that while this theorem was motivated by a study of geodesics, it (and its proof, below) contains no mention of geodesics.

*Proof.* Pick $p \in M$, $v \in T_pM$, $v \neq 0$, and $\gamma: \Delta \to M$ holomorphic with $\gamma(0) = p$, $\gamma_*(\partial/\partial z) = \lambda v$, $\lambda > 0$. Then the restriction of $F$ to the image of $\gamma$ is a Hermitian metric with curvature bounded above by $-4$ by (7.13). So, by [3, Theorem 2.1, Chapter 1],

$$F(p, v) \leq \left\| \frac{1}{\lambda} \frac{\partial}{\partial z} \right\| \leq \frac{1}{\lambda}.$$

Thus $F(p, v) \leq F_M(p, v)$.

Moreover, $\{\omega^\alpha = \dot{\omega}^\alpha = 0\}$ is a Frobenius system on $\mathbb{P}TM$, so the leaf through $(p, [v])$ is a well-defined submanifold $N$, and $ds^2 = \omega^1\omega^{\bar{1}}$ defines a complete metric on $N$ with curvature $-4$. Therefore $N$ may be isometrically covered by the unit disk. Let $\Gamma\colon \Delta \to N$ be the covering map with

$$\pi \circ \Gamma(0) = p, \qquad \pi \circ \gamma'(0) = \lambda v, \quad \lambda > 0,$$

where $\pi\colon \mathbb{P}TM \to M$ is the projection. Then consideration of $\gamma = \pi \circ \gamma\colon \Delta \to M$ yields the other inequality: $F(p, v) \geq F_M(p, v)$.

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