

CONVERGENCE OF THE RICCI FLOW FOR METRICS WITH INDEFINITE RICCI CURVATURE

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Abstract

Hamilton's Ricci flow convergence theorems generally deal with metrics whose Ricci curvature is positive semidefinite. Here, we exhibit a non-trivial class of three-dimensional Riemannian metrics with Ricci curvature of indefinite sign for which the Ricci flow converges.

Recent work of Hamilton [5], [6] shows that for all 3-dimensional Riemannian geometries (Σ^3, g) with positive Ricci curvature, the "Ricci flow" generated by the (heat-like) equation

$$(1) \quad \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{3} g_{ij} r$$

(for $r := \int_{\Sigma^3} R d\mu / \int_{\Sigma^3} d\mu$, the average of the scalar curvature R over Σ^3) converges asymptotically in parameter time t to a metric of constant positive curvature. The nature of the proof of this result has led to speculation that for Riemannian geometries with Ricci curvature of indefinite sign, the Ricci flow would generally not converge. The product geometry $S^2 \times S^1$, whose Ricci flow approaches a singular "pinching" geometry, provides an example of the expected behavior.

Here, we consider a class of 3-dimensional geometries, all with non-definite or negative Ricci curvature, for which the Ricci flow does indeed always converge. This class, which we shall label \mathcal{F} , is fairly specialized. All its members are topologically T^3 (3-torus) and all are invariant under a freely acting T^2 isometry group. (There are a few other orthogonality conditions which characterize \mathcal{F} ; we spell them out in §1.) The class is nontrivial, however, and as we shall prove in §2, all metrics in \mathcal{F} converge (under Ricci flow) to a flat metric on T^3 .

The convergence proof, though fairly intricate in detail, has a single theme: showing that the scalar curvature R decays sufficiently rapidly as

$t \rightarrow \infty$. We work with the scalar curvature because, in \mathcal{F} -metrics, it essentially controls all of the curvature. The proof (as detailed in §2) proceeds by first showing that R decays uniformly at the rate $C_1/(C_2 + t)$ (here and elsewhere, C_{integer} denotes some positive constant). This guarantees that the metric goes flat, but we need faster decay to prove convergence. To get this more rapid decay, we establish that the areas of the T^2 -orbits (of the isometry group) converge as $t \rightarrow \infty$ and thence the lengths of the orthogonal S^1 paths converge as well. This allows us to do the remaining analysis on S^1 , with a finite (converging) total perimeter length for all time. Using various estimates on the circle to simplify the evolution equation for R , we are able to show that the volume average of R decays exponentially and then with a few more estimates we show that R uniformly decays to zero exponentially. Smooth convergence of the metric then follows.

An interesting feature of the Ricci flow of the \mathcal{F} metrics is that for them—unlike the Ricci-positive metrics studied by Hamilton—the limit metric is not easily predicted. The Ricci flow generated by (1), we recall, is “normalized” so that it preserves total volume of (Σ^3, g) along the flow. Since for a given compact manifold Σ^3 there is generally at most one constant positive curvature metric of a given fixed volume, the limit of the Ricci flow of a Ricci-positive metric on such a manifold is fixed. There are, however, many flat metrics of a given volume on T^3 , and the Ricci flow of a \mathcal{F} metric may converge to any one of them.

One possible consequence of this (continuous) family of possible Ricci flow limits is that for some metrics on T^3 (or on other topologies which allow flat metrics) the Ricci flow may not converge to a single flat metric but instead may “quasiconverge” to a one-parameter family of them. Bits of our proof, and some results of Grayson and Hamilton [4], support this possibility. We comment on it and other issues related to attempts to extend our results in §3.

While our results should be useful in furthering the general understanding of Ricci flow, the motivation for our work comes from general relativity. We are interested in setting up a program for approximating the evolution of cosmological spacetime solutions of Einstein’s equations via the development of a procedure for “smoothing” sets of initial data for such spacetimes [1]. The Ricci flow could be an important part of this smoothing. As a testing laboratory for this study, we are examining the fairly well-understood “Gowdy” [3] spacetime models. The initial data for the “polarized” Gowdy models involve metrics of the class \mathcal{F} . Hence it is important to our program that the Ricci flow for \mathcal{F} metrics converges.

1. The \mathcal{F} -metrics and their Ricci flow equations

Here, we define the class of metrics (“ \mathcal{F} metrics”) with which we shall be working, set up coordinates in which to study them, calculate their curvatures, and write out the equations which govern their Ricci flow.

Definition (\mathcal{F} -metrics). For each $(\Sigma^3, g) \in \mathcal{F}$, we require the following.

- (a) $\Sigma^3 = T^3$.
- (b) g is invariant under a free T^2 action on T^3 .
- (c) The killing vector fields generating the T^2 action may be chosen to be (globally) orthogonal and independent.
- (d) The areas of all the T^2 orbits are the same.
- (e) Each line in the one-dimensional congruence of paths orthogonal to the T^2 orbits is closed (a circle).

All of these properties together allow us to pick periodic coordinates (x, y, θ) so that¹ ∂_x and ∂_y are the killing fields generating the T^2 isometry, and the metric takes the form

$$(2) \quad g = e^{2a} d\theta^2 + e^f [e^W dx^2 + e^{-W} dy^2],$$

where a is a function of θ , W is a function of θ , and f is a constant (all three will also be functions of t when we consider the Ricci flow of g).

Since θ is the only noncyclic coordinate, all of our analysis is done on the circle parametrized by θ . Hence it is useful to define the arc-length parameter s on this circle,

$$(3) \quad s(\theta) = \int_0^\theta e^a d\theta,$$

with the corresponding arc-length one-form

$$(4) \quad ds = e^a d\theta$$

and arc length vector field

$$(5) \quad \partial_s = e^{-a} \partial_\theta.$$

In terms of these, we have simple expressions for the vector calculus quantities on Σ^3 ; e.g. for any scalar Ψ on Σ^3 we find

$$(6) \quad \nabla\Psi \cdot \nabla\Psi = (\partial_s\Psi)^2 \quad \text{and} \quad \nabla\Psi = \partial_s^2\Psi.$$

Note also that, so long as we choose $x \in [0, 1)$ and $y \in [0, 1)$, the volume 3-form $d\mu$ on Σ^3 may be effectively treated as a 1-form

$$(7) \quad d\mu = e^{a+f} d\theta$$

¹We use the notation $\partial_x = \partial/\partial x$ etc., here and throughout the paper.

with integration over x and y understood. For convenience in doing Fourier series below, we shall let the range of θ be 0 to 2π .

From (2), we readily calculate the curvature of the \mathcal{F} metrics. Since the Ricci curvature of a 3-dimensional Riemannian manifold determines all of its curvature, we list only that (in coordinate basis):

$$(8a) \quad R_{\theta\theta} = -\frac{1}{2}e^{2a}(\partial_s W)^2,$$

$$(8b) \quad R_{xx} = -\frac{1}{2}e^{f+W}\partial_s^2 W,$$

$$(8c) \quad R_{yy} = \frac{1}{2}e^{f-W}\partial_s^2 W.$$

The other components are zero. Contracting, we get the scalar curvature

$$(9) \quad R = -\frac{1}{2}(\partial_s W)^2,$$

and its spatial average

$$(10) \quad r := \int R d\mu \Big/ \int d\mu = e^f \int -\frac{1}{2}(\partial_s W)^2 ds \Big/ \text{Vol}[\Sigma^3].$$

Note that R is negative semidefinite. If, for some g , R vanishes everywhere on the circle, and W is smooth, then one finds that all components of the curvature (8) vanish, and the metric is flat. (This is a familiar result for Riemannian metrics on T^3 .)

We now apply the Ricci flow equation (1) to the \mathcal{F} metrics. First, we note that the flow preserves the \mathcal{F} class. Specifically it preserves the killing fields, the off-diagonal components of the metric remain zero under the flow, and the function f remains spatially constant. The Ricci flow then manifests itself as evolution equations for the three functions:

$$(11a) \quad \partial_t f = \frac{2}{3}r,$$

$$(11b) \quad \partial_t a = \frac{1}{2}(\partial_s W)^2 + \frac{1}{3}r,$$

$$(11c) \quad \partial_t W = \partial_s^2 W.$$

These equations and the asymptotic behavior of their solutions are the focus of the rest of this work.

Before discussing our main result and how its proof goes, we note a commutation relation which we will use quite often: Let Ψ be any smooth scalar functional of the metric components. Then using (5) and (11b) we find

$$(12) \quad \begin{aligned} \partial_t \partial_s \Psi &= \partial_s (\partial_t \Psi) - (\partial_t a) \partial_s \Psi \\ &= \partial_s (\partial_t \Psi) - \left[\frac{1}{2}(\partial_s W)^2 + \frac{1}{3}r \right] \partial_s \Psi. \end{aligned}$$

2. The main result and its proof

Our main result is the following

Theorem. *Let g be any \mathcal{F} metric which is everywhere C^2 . Then the Ricci flow $g(t)$ of g converges (as $t \rightarrow \infty$) to a flat metric g_∞ .*

Proof. The short and long time existence results of Hamilton [5] apply to the \mathcal{F} metrics (as well as to any 3-dimensional Riemannian metric); so for any chosen \mathcal{F} metric the Ricci flow exists, and continues, as long as the curvature is well behaved. Hence we focus on controlling the curvature—particularly the scalar curvature—and using it to prove convergence for f , a , and W . We proceed (as outlined in the introduction) in five steps:

(A) $\frac{1}{t}$ **decay of R .** Here, we show that there exist positive constants C_1 and C_2 such that

$$(13) \quad \frac{-C_1}{1 + C_2 t} \leq R(\theta, t) \leq 0,$$

so the scalar curvature decays uniformly to zero.

(1) To avoid unnecessary signs and factors of $1/2$, we shall work with $(\partial_s W)^2$ rather than R itself. Using the commutation relation (12), we find that the evolution of $(\partial_s W)^2$ is given by

$$(14) \quad \partial_t [(\partial_s W)^2] = \Delta(\partial_s W)^2 - 2(\Delta W)^2 - [(\partial_s W)^2 + \frac{2}{3}r](\partial_s W)^2.$$

Recall that $\Delta = \partial_s^2$; here and below, we shall use whichever is useful in clarifying the argument at hand.

(2) To obtain from (14) an inequality controlling the evolution of $\text{Max}_{S^1} [(\partial_s W)^2]$, we rely on the following result, adapted from Hamilton (see [6, Chapter 3])

Lemma 1. *Let D be a compact set, and let the function $\Psi: D \times R \rightarrow R$ be a smooth solution of a partial differential equation of the form*

$$(15) \quad \partial_t \Psi = F[\Psi, x, t],$$

where F is a functional of Ψ , of $x \in D$, and of $t \in R$. Let $m(t)$ be a solution of the ordinary differential equation

$$(16) \quad \frac{d}{dt} m = E[m, t],$$

where E is a function of m and of $t \in R$ which satisfies the condition

$$(17) \quad E[m, t] \geq \text{Sup}_{(\text{Max}_D[\Psi])} F[\Psi, x, t] \text{ for } m \geq \text{Max}_D[\Psi].$$

(Here “ $\text{Sup}_{(\text{Max}_D[\Psi])} F$ ” means that we are to find the least upper bound of F evaluated at all x and $\Psi(x)$ for which $\Psi(x) = \text{Max}_D[\Psi]$.) Then if $m(0) \geq \text{Max}_D[\Psi(x, 0)]$, we have

$$(18) \quad m(t) \geq \text{Sup}_D[\Psi(x, t)]$$

for all time t .

(3) Applying this lemma to (14), with $\Psi = (\partial_s W)^2$, we are led to consider the quantity

$$(19) \quad H := \text{Sup}_{\text{Max}_{S^1}[(\partial_s W)^2]} \left\{ \Delta(\partial_s W)^2 - 2(\Delta W)^2 - \left[(\partial_s W)^2 + \frac{2}{3}r \right] (\partial_s W)^2 \right\}.$$

At any maximum point s_{max} of $(\partial_s W(s))^2$, we have

$$(a) \quad \Delta(\partial_s W(s_{\text{max}}))^2 \leq 0,$$

$$(b) \quad (\partial_s W(s_{\text{max}}))^2 + \frac{2}{3}r = (\partial_s W(s_{\text{max}}))^2 + \frac{2}{3} \frac{\int[-\frac{1}{2}(\partial_s W(s))^2]}{\int d\mu} d\mu$$

$$\geq (\partial_s W(s_{\text{max}}))^2 - \frac{1}{3}(\partial_s W(s_{\text{max}}))^2$$

$$\geq \frac{2}{3}(\partial_s W(s_{\text{max}}))^2.$$

Using these, we find that the quantity in (19) satisfies the inequality

$$(20) \quad H \leq -\frac{2}{3} \left[\text{Max}_{S^1}(\partial_s W)^2 \right]^2.$$

Thus we are led to integrate the ordinary differential equation

$$(21) \quad \frac{d}{dt} m = -\frac{2}{3} m^2,$$

which has the solution

$$(22) \quad m(t) = \frac{m_0}{1 + \frac{2}{3}m_0 t}, \quad \text{for } m_0 = m(0).$$

(4) The conclusion of the lemma produces the result

$$(23) \quad (\partial_s W)^2(\theta, t) \leq \frac{m_0}{1 + \frac{2}{3}m_0 t},$$

so long as we choose $m_0 \geq \text{Max}_{S^1}[(\partial_s W)^2(\theta, 0)]$. Hence $(\partial_s W)^2$ uniformly decays to zero, and consequently $R = -\frac{1}{2}(\partial_s W)^2$ does as well.

(5) Note that besides giving us this asymptotic decay result, this analysis (especially (20)) shows that $\text{Max}_{S^1}(\partial_s W)^2$ monotonically decreases in time. This will be useful in step (D) below.

(B) Convergence of the T^2 orbit area and the S^1 perimeter length. It follows readily from (11a) and (13) that $f(t)$, which monotonically decreases in time, is bounded from below by $f(0) - C_3 \ln(1 + C_4 t)$ for constants $C_3 > 0$ and $C_4 > 0$. This allows for the possibility of a logarithmic blow-up. Here, however, we show that in fact $f(t)$ converges to a finite value. Since the orbits of the T^2 isometry group (parametrized by x and y) have area e^{+f} , and since the perimeter lengths of the orthogonal paths are given by

$$(24) \quad \int_{S^1} ds = e^{-f} V,$$

where $V = \int d\mu$ is the volume of the whole space, a constant under Ricci flow, convergence of $f(t)$ implies convergence of these quantities. The convergence of $\int_{S^1} ds$ will be useful in steps (C) and (D) below.

(1) Long time existence for $a(\theta, t)$ allows us to infer that, for finite values of t , $\Delta = \partial_s^2$ is strictly (and uniformly) elliptic. Hence we may apply the maximum principle to (11c) and thereby deduce that $W(\theta, t)$ is uniformly bounded, for all time, above and below:

$$(25) \quad \text{Min}_{S^1}[W(\theta, 0)] \leq W(\theta, t) \leq \text{Max}_{S^1}[W(\theta, 0)].$$

For reasons which will become apparent below, we wish to temporarily replace $W(\theta, t)$ by

$$(26) \quad \omega(\theta, t) := W(\theta, t) - W_m + \eta,$$

where $W_m := \text{Min}_{S^1}[W(\theta, 0)]$, and η is some positive constant. We see that $\omega(\theta, t)$ is a positive quantity, bounded from below by η and from above by $W_M - W_m + \eta$, where $W_M := \text{Max}_{S^1}[W(\theta, 0)]$. We also note that $\{f, a, \omega\}$ satisfies the Ricci flow equations (11) iff $\{f, a, W\}$ does also.

(2) We choose an integer n such that

$$(27) \quad \frac{n(n-1)}{2} \geq (W_M - W_m + \eta)^2.$$

We then calculate

$$(28) \quad \partial_t \omega^n = \Delta \omega^n - n(n-1)\omega^{n-2}(\partial_s \omega)^2$$

from which we obtain

$$(29) \quad \begin{aligned} \partial_t \left[\int_{S^1} \omega^n d\mu \right] &= \int_{S^1} \left[\Delta(\omega^n) - n(n-1)\omega^{n-2}(\partial_s \omega)^2 \right. \\ &\quad \left. + \omega^n \left(\frac{1}{2}(\partial_s \omega)^2 - \frac{1}{2} \int_{S^1} (\partial_s \omega)^2 d\mu / V \right) \right] d\mu \\ &\leq -\frac{1}{2} \int_{S^1} (\partial_s \omega)^2 d\mu / V \int_{S^1} \omega^n d\mu. \end{aligned}$$

Achievement of this inequality motivates the choice of n . Integrating (29), we find

$$(30) \quad \int_{S^1} \omega^n(\theta, t) d\mu \leq \left[\int_{S^1} \omega^n(\theta, 0) d\mu \right] \times \exp \left[-\frac{1}{2V} \int_0^t \left(\int_{S^1} (\partial_s \omega(\theta, \alpha))^2 d\mu \right) d\alpha \right].$$

(3) We know that $\omega^n(\theta, t)$ is bounded from below by $\eta > 0$ for all t . It follows that the argument of the exponential function must be bounded; specifically, we must have

$$(31) \quad \int_0^t \left(\int_{S^1} (\partial_s \omega)^2 d\mu \right) d\alpha \leq 2V \ln \left[\frac{\int_{S^1} \omega^n(\theta, 0) d\mu}{\eta^n V} \right]$$

for all values of t . Since the left-hand side of (31) is a monotonically increasing function of t , we conclude that

$$(32) \quad \lim_{t \rightarrow \infty} \int_0^t \left[\int_{S^1} (\partial_s \omega)^2 d\mu \right] = L$$

for some finite number L .

(4) The evolution equation for f , written in terms of ω , is

$$(33) \quad \partial_t f = -\frac{1}{3V} \int_{S^1} (\partial_s \omega)^2 d\mu.$$

From this, we obtain

$$(34) \quad \lim_{t \rightarrow \infty} f(t) = f(0) - \frac{1}{3V} L,$$

so f converges. The convergence result for $\int_{S^1} ds$ immediately follows.

(C) **Exponential decay of r .** In this step, we show that r , the spatial average of the scalar curvature, decays exponentially to zero. For convenience, we shall work with

$$(35) \quad I(t) := \int_{S^1} (\partial_s W)^2 ds$$

rather than r . Note that

$$(36) \quad r = -\frac{1}{2V} e^f I.$$

Since e^f is bounded above and below for all time, exponential decay of I guarantees the same for r .

(1) We calculate

$$(37) \quad \frac{d}{dt} I = -2 \int_{S^1} (\Delta W)^2 ds - \frac{1}{2} \int_{S^1} (\partial_s W)^2 ds - \frac{1}{3} \int_{S^1} (\partial_s W)^2 r ds.$$

Our aim is to be able to deduce from (37) that I satisfies the inequality $\frac{d}{dt}I \leq C_5 I$ for some $C_5 > 0$. The key step in this deduction follows from part (a) of the following.

Lemma 2. *Let $h: S^1 \rightarrow R$ be smooth and bounded. Assume that $\int_{S^1} h ds = 0$. Then there exists a pair of constants $C_6 > 0$ and $C_7 > 0$, depending only upon the length $L = \int_{S^1} ds$, such that*

$$(38) \quad (a) \quad \int_{S^1} h^2 ds \leq C_6 \int_{S^1} (\partial_s h)^2 ds,$$

$$(39) \quad (b) \quad \text{Max}_{S^1} |h| \leq C_7 \left[\int_{S^1} (\partial_s h)^2 ds \right]^{1/2}.$$

Proof. Both (a) and (b) are easily proven using Fourier series expansions for h and $\partial_s h$ on the circle. It is important, in comparing the expansions so as to obtain the inequalities, to note that the “zeroth” term in the expansion vanishes since it is proportional to $\int h ds$. So, for example, we have

$$h = \sum_{n \neq 0} a_n \exp \left[in \frac{2\pi}{L} s \right],$$

$$\partial_s h = \frac{2\pi i}{L} \sum_{n \neq 0} n a_n \exp \left[in \frac{2\pi}{L} s \right].$$

From Parseval’s, we get

$$\int_{S^1} h^2 ds = \frac{L}{2\pi} \sum_{n \neq 0} |a_n|^2 \quad \text{and} \quad \int_{S^1} |\partial_s h|^2 ds = \frac{2\pi}{L} \sum_{n \neq 0} n^2 |a_n|^2.$$

Result (a) clearly follows. Similarly we get (b). q.e.d.

(2) We wish to apply part (a) of this lemma to the function $h = (\partial_s W)$. (Clearly the necessary condition $\int_{S^1} h ds = 0$ is satisfied.) With some bit of rearrangement, and using the boundedness above and below of $L(t) = \int_{S^1} ds$, we get

$$(40) \quad -2 \int_{S^1} (\Delta W)^2 ds \leq -C_8 \int_{S^1} (\partial_s W)^2 ds = -C_8 I$$

for some $C_8 > 0$.

(3) If we plug result (40) into (37), and use the negativity of the term $-\frac{1}{2} \int_{S^1} (\partial_s W)^4 ds$, we find that I satisfies

$$(41) \quad \frac{d}{dt} I \leq -C_8 I - \frac{1}{3} r I.$$

Since e^f is bounded below for all time, we deduce from (36) that there exists $C_9 > 0$ such that $-\frac{1}{3}r \leq C_9 I$ for all time. Hence (41) becomes

$$(42) \quad \frac{d}{dt} I \leq -C_8 I + C_9 I^2.$$

(4) Generally, equation (42) admits nondecaying solutions. However, since we know from step (A) that $I(t) \leq C_{10}/(1 + C_{11}t)$, it follows that for some t_0 , we have

$$(43) \quad C_8 - C_9 I(t_0) > 0.$$

Hence, at t_0 ,

$$(44) \quad \frac{d}{dt} I(t_0) \leq -(C_8 - C_9 I(t_0))I(t_0) < 0,$$

so for all $t > t_0$, $I(t)$ is decreasing. It follows that for some $C_{12} > 0$,

$$(45) \quad \frac{d}{dt} I \leq -C_{12} I.$$

So $I(t)$ exponentially decreases to zero. Exponential decay of r is then a consequence, as argued above.

(D) Exponential decay of R . Here, we use the exponential decay of r (and I) to show that R uniformly converges to zero exponentially. For convenience, we continue work with $(\partial_s W)^2$ rather than $R = -\frac{1}{2}(\partial_s W)^2$.

(1) Equation (37) may be rewritten as

$$(46) \quad \begin{aligned} \frac{d}{dt} \int_{S^1} (\partial_s W)^2 ds + \int_{S^1} (\Delta W)^2 ds \\ = -\frac{1}{2} \int_{S^1} (\partial_s W)^4 ds - \int_{S^1} (\Delta W)^2 ds - \frac{1}{3} r I. \end{aligned}$$

If we apply part (b) of Lemma 2 to $h = \partial_s W$ in the second term of the left-hand side of this equation, and use $-\frac{1}{2} \int_{S^1} (\partial_s W)^4 ds \leq 0$, then (46) takes the form

$$(47) \quad \begin{aligned} \frac{d}{dt} \left[\int_{S^1} (\partial_s W)^2 ds \right] + C_{13} \left[\text{Max}_{S^1}(\partial_s W) \right]^2 \\ \leq - \int_{S^1} (\partial_s^2 W)^2 ds - \frac{1}{3} r \int_{S^1} (\partial_s W)^2 ds. \end{aligned}$$

We may now use part (a) of Lemma 2, together with the knowledge that r decays exponentially, to argue that, for sufficiently large time t_0 , the right-hand side of (47) is negative (or zero). We then have, for $t \geq t_0$,

$$(48) \quad \frac{d}{dt} \int_{S^1} (\partial_s W)^2 ds + C_{13} \left[\text{Max}_{S^1}(\partial_s W) \right]^2 \leq 0.$$

(2) Let us choose any $T \geq t_0$. If we integrate (48) over t , from $t = T$ to $t \rightarrow \infty$, we get

$$\begin{aligned}
 C_{13} \int_T^\infty \left[\text{Max}_{S^1}(\partial_s W)(t) \right]^2 dt &\leq \int_\infty^T \left(\frac{d}{dt} \int_{S^1} (\partial_s W)^2 \right) dt \\
 (49) \qquad \qquad \qquad &\leq \int_{S^1} (\partial_s W)^2(T) ds - \lim_{t \rightarrow \infty} \int_{S^1} (\partial_s W)(t) ds \\
 &\leq C_{14} e^{-C_{15}T}.
 \end{aligned}$$

(3) While (49) seems to indicate uniform exponential decay of $(\partial_s W)^2$, in principle it allows for high amplitude blips so long as they are sufficiently short in duration. These blips are prevented, however, by the monotonic decay result of step (A). To show that, we argue as follows:

(4) Let us use the notation $M(t) := \text{Max}_{S^1}(\partial_s W)^2(t)$, so that (49) reads

$$(50) \qquad \int_T^\infty M(t) dt \leq C_{16} e^{-C_{15}T}.$$

Now pick any $\tau \geq T + 1$. Using (50) we find

$$\begin{aligned}
 (51) \qquad \int_{\tau-1}^\tau M(t) dt &= \int_{\tau-1}^\infty M(t) dt - \int_\tau^\infty M(t) dt \\
 &\leq C_{16} e^{-C_{15}(\tau-1)} + C_{16} e^{-C_{15}\tau} \leq C_{17} e^{-C_{15}\tau}.
 \end{aligned}$$

By the mean value theorem, there must be some $\Upsilon \in [\tau - 1, \tau]$ such that

$$(52) \qquad M(\Upsilon) \leq C_{17} e^{-C_{15}\tau}.$$

Since, by step (A), we know that $M(t)$ monotonically decreases, we have

$$(53) \qquad M(\tau) \leq M(\Upsilon) \leq C_{17} e^{-C_{15}\tau}.$$

Hence, for sufficiently large T , we have exponential decay of $\text{Max}_{S^1}(\partial_s W)^2$.

(5) Since $(\partial_s W)^2$ is finite for all finite t , we may always find a constant C_{18} such that for all $t \geq 0$ and all $\theta \in S^1$,

$$(54) \qquad (\partial_s W)^2(\theta, t) \leq C_{18} e^{-C_{15}t},$$

which implies uniform exponential convergence of $(\partial_s W)^2$ to zero. Uniform exponential convergence of R to zero immediately follows.

(E) Convergence of the metric. In this final step, we show that $\lim_{t \rightarrow \infty} a(\theta, t)$ and $\lim_{t \rightarrow \infty} W(\theta, t)$ exist, and are smooth on S^1 . To do this, we first need to establish the boundedness (for all time) of $\Delta W = \partial_s^2 W$ and $\partial_s a$.

(1) Using the commutation relation (12), we calculate

$$(55) \qquad \partial_t(\Delta W) = \Delta(\Delta W) - [2(\partial_s W)^2 + \frac{2}{3}r]\Delta W$$

and also

$$(56) \quad \partial_t(\partial_s a) = -[\frac{1}{2}(\partial_s W)^2 + \frac{1}{3}r]\partial_s a + \Delta W \partial_s W.$$

(2) We wish to apply Lemma 1 to (55) for $\Psi = \Delta W$. Allowing $\frac{2}{3}r$ to perhaps dominate $2(\partial_s W)^2$ at maxima of (ΔW) , but noting that both decay exponentially, we find that

$$(57) \quad \text{Sup}_{\text{Max}_{S^1}[\Delta W]} \{ \Delta(\Delta W) - [2(\partial_s W)^2 + \frac{2}{3}r]\Delta W \} \leq C_{19}e^{-C_{20}t} \left(\text{Max}_{S^1}[\Delta W] \right).$$

Then, as discussed in Lemma 1, we are led to integrate

$$(58) \quad \partial_t n = C_{19}e^{-C_{20}t} n;$$

we get

$$(59) \quad n(t) = A \exp \left[\frac{-C_{19}}{C_{20}} e^{-C_{20}t} \right],$$

where A is a constant determined by initial conditions. From this we deduce that $\Delta W(\theta, t)$ is uniformly bounded from above, for all time, by some constant B_1 . A similar argument shows that $\Delta W(\theta, t)$ is uniformly bounded from below by a constant B_2 . So we have

$$(60) \quad B_2 \leq \Delta W(\theta, t) \leq B_1 \quad \text{for all } \theta, t.$$

(3) By means of the boundedness of ΔW just established, we now apply Lemma 1 to (56), with $\Psi = \partial_s a$. Using the result

$$(61) \quad \begin{aligned} & \text{Sup}_{\text{Max}_{S^1}[\partial_s a]} \{ -[\frac{1}{2}(\partial_s W)^2 + \frac{1}{3}r]\partial_s a + \Delta W \partial_s W \} \\ & \leq C_{21}e^{-C_{22}t} \left(\text{Max}_{S^1}(\partial_s a) \right) + C_{23}e^{-C_{24}t}, \end{aligned}$$

we are led to integrate

$$(62) \quad \partial_t P = C_{21}e^{-C_{22}t} P + C_{23}e^{-C_{24}t}.$$

This has a bounded solution, and eventually we find that

$$(63) \quad B_4 \leq \partial_s a(\theta, t) \leq B_3 \quad \text{for all } \theta, t,$$

for certain constants B_3 and B_4 .

(4) The right-hand side of the evolution equation (11b) for $a(\theta, t)$ uniformly converges to zero. Convergence for $a(\theta, t)$ is then a consequence. More explicitly, we argue as follows: From steps (A) and (C), we have

$$(64) \quad -C_{25}e^{-C_{20}t} \leq \frac{\partial}{\partial t} a(\theta, t) \leq C_{25}e^{-C_{26}t}.$$

Choose $\varepsilon > 0$. Then if we pick t_1 and t_2 such that

$$(65) \quad t_1, t_2 > T := \frac{1}{C_{26}} \ln \left(\frac{2C_{25}}{\varepsilon} \right),$$

we have

$$(66) \quad |a(\theta, t_1) - a(\theta, t_2)| < \varepsilon$$

for all $\theta \in S^1$. Hence by the Cauchy criteria, there exist $a_\infty(\theta)$ such that

$$(67) \quad \lim_{t \rightarrow \infty} a(t, \theta) = a_\infty(\theta)$$

with the convergence being uniform.

Since $a(\theta, t)$ is continuous on S^1 for all t , $a_\infty(\theta)$ is also continuous.

To show that $a_\infty(\theta)$ is differentiable, it is sufficient to show that $\partial_s a(\theta, t)$ converges uniformly. Now, from (63) we know that $\partial_s a(\theta, t)$ is uniformly bounded. It then follows from (56) that for some $C_{27} > 0$ and $C_{28} > 0$,

$$(68) \quad -C_{27}e^{-28t} \leq \partial_t(\partial_s a) \leq C_{27}e^{-C_{28}t}.$$

An argument like that just done for $a(\theta, t)$ shows that $\lim_{t \rightarrow \infty} \partial_s a(t, \theta)$ exists. It follows that $\partial_s a_\infty(\theta)$ exists.

Similar arguments may be made for higher derivatives of $a(t, \theta)$; so we conclude that $a_\infty(\theta)$ is smooth.

(5) The equation of evolution for $W(\theta, t)$ is

$$(11c) \quad \partial_t W(\theta, t) = \Delta W(\theta, t),$$

where

$$(69) \quad \Delta W = \partial_s^2 W = e^{-2a}[\partial_\theta^2 W - \partial_\theta a \partial_\theta W].$$

Since $a(\theta, t)$ and $\partial_s a(\theta, t)$ are well behaved for all t , and have uniform limits as $t \rightarrow \infty$, the laplacian Δ of (69) is uniformly and strictly elliptic for all t , with limit Δ_∞ .

Standard theorems for the heat equation [2] guarantee that if the laplacian is as well behaved as this, then for suitable critical data, (11c) has a unique smooth solution $W(\theta, t)$ with

$$(70) \quad \lim_{t \rightarrow \infty} W(\theta, t) = W_\infty(\theta),$$

where $W_\infty(\theta)$ is a solution of the laplace equation

$$(71) \quad \Delta_\infty W_\infty(\theta) = 0.$$

The only solutions of (71) on the circle are

$$(72) \quad W_\infty(\theta) = \text{constant},$$

so the limit metric is smooth.

One easily verifies that the various derivatives of $W(t, \theta)$ converge uniformly to zero, thereby agreeing with (72). This completes the proof of our theorem.

3. Conclusion

It would be nice if we could extend these results to a larger class of metrics. We have made a small step in that direction by relaxing condition (c) in the definition of the \mathcal{F} -metrics, allowing the killing vector fields to be nonorthogonal (but still necessarily independent). The metrics of this new class—the most general that we need for our study of Gowdy T^3 spacetimes and their smoothing—all may be cast into the form

$$(73) \quad g = e^{2a} d\theta^2 + e^f [(\cosh W + \cos V \sinh W) dx^2 + 2(\sin V \sinh W) dx dy + (\cosh W - \cos V \sinh W) dy^2],$$

where f , a , and W are as in the \mathcal{F} class, and V is a function of θ (if $V = 0$, then the metric of (73) lies in class \mathcal{F}). We find that step (A), at least, of our proof readily extends to this larger class. Hence the curvature of the Ricci flows of these metrics necessarily decays to zero. We have not yet determined whether we have convergence, however.

The possibility that these Ricci flows fail to converge is stimulated by the study of the (homogeneous) nil geometrics by Grayson and Hamilton [4]. The Ricci flows of these, they find, do not converge but rather approach one-parameter families of flat metrics (we call this quasi-convergence). Perhaps the Ricci flows of some of the metrics (73) do the same thing. This question is presently under investigation.

A more general class we plan to study consists of all Riemannian manifolds (Σ^3, g) such that (a) Σ^3 is a two-torus bundle over the circle S^1 , and (b) g is invariant under the T^2 action along the fibers. Our conjecture is that for all metrics in this larger class we have either convergence or quasi-convergence. We hope that some of the techniques used here will apply in studying this problem.

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