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COMPACTIFYING COVERINGS OF CLOSED 3-MANIFOLDS

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Let M be a closed P^2-irreducible 3-manifold with infinite fundamental group. It is a longstanding conjecture that the universal cover of M must be homeomorphic to $\mathbf{R}^3$. Waldhausen [17] proved that this is the case where M is Haken. (See Heil's paper [5] for the nonorientable case.) The first result of this paper is the following generalization of Waldhausen's result.

Theorem 1.1. *Let M be a closed P^2-irreducible 3-manifold. If $\pi_1(M)$ contains the fundamental group of a closed surface other than S^2 or P^2, then the universal cover of M is homeomorphic to $\mathbf{R}^3$.*

We will say that a 3-manifold is *almost compact* if it can be obtained from a compact manifold N by removing a closed subset of ∂N. Then Theorem 1.1 is equivalent to the assertion that the universal covering of M is almost compact. A natural way in which to attempt to generalize Theorem 1.1 is to show that other coverings of M are almost compact. It was conjectured by Simon [14] that if M is any compact P^2-irreducible 3-manifold, and M_1 is a covering of M with finitely generated fundamental group, then M_1 must be almost compact. Simon verified this conjecture for the case where $\pi_1(M_1)$ is the fundamental group of a boundary component of M. Jaco [6] generalized this to the case where $\pi_1(M_1)$ is a finitely generated peripheral subgroup of $\pi_1(M)$. More recently, Thurston [15] showed that if M admits a geometrically finite complete hyperbolic structure of infinite volume, then Simon's conjecture is true. Finally, Bonahon [1] showed that any hyperbolic 3-manifold M with finitely generated fundamental group is almost compact provided $\pi_1(M)$ is not a free product.

The second result of this paper is the following.

Theorem 2.1. *Let M be a closed P^2-irreducible 3-manifold such that $\pi_1(M)$ contains a subgroup A isomorphic to $\mathbf{Z}\times\mathbf{Z}$. Then the covering of M with fundamental group A is almost compact.*

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If M is Haken then the conclusion of the theorem follows immediately from Simon's work [14]. The crucial condition needed in [14] is that if H is any finitely generated subgroup of $\pi_1(M)$, then $H \cap A$ is also finitely generated; this obviously holds, as A is isomorphic to $\mathbf{Z} \times \mathbf{Z}$. If M is not Haken, then the version of the Torus Theorem proved by Scott [13] shows that some infinite cyclic subgroup of A must be normal in $\pi_1(M)$. Now the following is another long standing conjecture.

Conjecture. *If M is a closed P^2-irreducible 3-manifold such that $\pi_1(M)$ contains an infinite cyclic normal subgroup, then M is a Seifert fiber space.*

If M is a Seifert fiber space, then Theorem 2.1 is easily proved. Thus the interest of Theorem 2.1 is that we do not assume that M is either Haken or a Seifert fiber space. We hope that our result will be a step towards proving the Conjecture.

1. The universal cover

In this section,we prove the following theorem.

Theorem 1.1. *Let M be a closed P^2-irreducible 3-manifold. If $\pi_1(M)$ contains the fundamental group of a closed surface other than S^2 or P^2 then the universal cover of M is homeomorphic to $\mathbf{R}^3$.*

We start by noting that if M is Haken, then the result was proved by Waldhausen [17] and Heil [5]. Thus throughout this section we will assume that M is not Haken. In particular, M must be orientable, as a nonorientable closed 3-manifold must have infinite first homology group and so be Haken [5].

Let F be a closed surface not S^2 or P^2 such that $\pi_1(M)$ contains $\pi_1(F)$. Let$f: F \to M$ be a map inducing the inclusion of $\pi_1(F)$ in $\pi_1(M)$. We can assume that f is 2-sided, i.e., has trivial normal bundle, by replacing F by its orientable double cover if necessary. We want to choose f to be "least area". This can be done in the smooth category by picking a Riemannian metric on M and applying the theorem of Schoen and Yau [12] which asserts the existence of a map homotopic to f of least area. Alternatively, one can triangulate M and take a normal map homotopic to f of least possible weight as in the paper [7] of Jaco and Rubinstein. We will take the first approach, but the second approach will yield the same results by essentially identical arguments.

Now let $f: F \to M$ be of least area in its homotopy class. Then f is a smooth immersion, but it need not be in general position. Let $\tilde{M}$ denote the universal covering of M. Results of [3] and [4] show that the preimage in $\tilde{M}$ of $f(F)$ consists of area minimizing embedded planes. (A surface is area minimizing if any compact subsurface is least area in its homotopy class rel boundary.) Further, if f is in general position, then these planes are also in general position and the intersection of any pair of planes has no circle components. A family of planes embedded in $\tilde{M}$ with these two properties will be called *simple.* If f is not in general position, then, of course, the preimage planes in $\tilde{M}$ are also not in general position, but the methods of Lemma 1.6 of [3] show that f can be perturbed to a general position map f' for which the preimage in $\tilde{M}$ is again a simple union of embedded planes. We conclude that in all cases there is a smooth immersion $f\colon F \to M$ such that the preimage in $\tilde{M}$ of $f(F)$ is a simple union of embedded planes. This is the only property of f which we will use, so that, from now on, it will be irrelevant whether or not f is of least area.

Let Σ denote any simple union of embedded planes in $\tilde{M}$. We will need some preliminary results about Σ.

Lemma 1.2. *Each component of $\tilde{M} - \Sigma$ is simply connected.*

Remark. When f is a least area map and Σ is the preimage of $f(F)$, then the planes of Σ are area minimizing by results of [4]. It is natural to suppose that if λ is a loop in $\tilde{M} - \Sigma$, then a least area disc spanning λ would have to be disjoint from Σ. We are unable to prove this, so we use a more combinatorial argument, which applies to any simple union of embedded planes in $\tilde{M}$.

Proof. Let λ be a loop in $\tilde{M} - \Sigma$, and let $g\colon D^2 \to \tilde{M}$ be a (possibly singular) 2-disc in $\tilde{M}$ spanning λ. Then g meets only a finite number of planes of Σ and we can assume that it meets them transversely. Thus $g^{-1}(\Sigma)$ is a union (not necessarily disjoint) of simple closed curves. Let Π denote a plane of Σ such that $g^{-1}(\Pi)$ is nonempty. Let γ denote a component of $g^{-1}(\Pi)$, and let B denote the subdisc of D bounded by γ. We will replace g by a map g' which agrees with g on $D - B$ and maps B into Π. Further if U denotes the unbounded component of $\Pi - g(\gamma)$, we ensure that $g'(B)$ lies in $\Pi - U$. This can be done, as U with the interior of a regular neighborhood of $g(\gamma)$ removed must be a half open annulus, so that Π deformation retracts onto $\Pi - U$. We repeat this procedure for each component of $g^{-1}(\Pi)$ and then perform a small homotopy supported on a small neighborhood of the discs bounded by $g^{-1}(\Pi)$ in order to to obtain $g_1\colon D^2 \to \tilde{M}$ such that $g_1^{-1}(\Pi)$ is empty. If the 2-discs spanning the components of $g^{-1}(\Pi)$ are nested, we use only the outermost components.

We claim that if g did not meet a particular plane Π', then neither does g_1. For suppose that our homotopy of g to g_1 introduces intersection with Π' in the subdisc B of D bounded by γ. Then Π' must intersect Π in the compact region $\Pi - U$. As each component of $\Pi \cap \Pi'$ is noncompact, it follows that $\Pi \cap \Pi'$ crosses the frontier of $\Pi - U$, which is contained in $g(\gamma)$. Thus g must meet Π', contradicting our assumption. Now, by repeating the above argument for each of the planes of Σ which meets g, we see that we can homotop g, fixed on λ, to a map which does not meet Σ. It follows that λ is null-homotopic in $\tilde{M} - \Sigma$, so that each component of $\tilde{M} - \Sigma$ is simply connected, as required.

Let Π be a plane of Σ and let L denote the union of the double lines $\Pi \cap \Pi'$, where Π' varies over all planes of Σ except Π itself.

Lemma 1.3. *Each component of $\Pi - L$ is simply connected.*

Proof. This can be proved in the same way as Lemma 1.2, but a simpler argument is as follows. Let R be a component of $\Pi - L$. It suffices to show that each simple closed curve λ in R is null-homotopic in R. Now λ bounds a 2-disc D in Π. As the components of L are properly embedded lines not meeting λ, they must also not meet D. Thus D lies in R and the lemma is proved.

Now let $N(F)$ denote a regular neighborhood of $f(F)$ in M.

Lemma 1.4. *Let X denote the closure of a component of $M - N(F)$. Then*

(i) *the natural map $\pi_1(X) \to \pi_1(M)$ is injective, and*

(ii) *X is a handlebody.*

Proof. Conclusion (i) follows at once from Lemma 1.2.

Now let S denote a component of ∂X. If S is a sphere, then the irreducibility of M implies that X or $\overline{M - X}$ is a 3-ball. As $f(F)$ lies in $M - X$, we deduce that X must be a 3-ball, in this case. Suppose that S is not a sphere. As M is not Haken, the natural map $\pi_1(S) \to \pi_1(M)$ cannot be injective. Thus, by (i), the natural map $\pi_1(S) \to \pi_1(X)$ also cannot be injective, so that the Loop Theorem yields a 2-disc D_1 embedded in X whose boundary is an essential circle on S. Let X_1 be obtained from X by removing the interior of a regular neighborhood of D_1. If X_1 is connected, then $\pi_1(X)$ is isomorphic to $\pi_1(X_1) \star \mathbf{Z}$, so that $\pi_1(X_1)$ injects into $\pi_1(M)$. If X_1 has components Y and Y', then $\pi_1(X)$ is isomorphic to $\pi_1(Y) \star \pi_1(Y')$, so that $\pi_1(Y)$ and $\pi_1(Y')$ inject into $\pi_1(M)$. Thus, as for X, either each component of X_1 is a 3-ball or we can find a disc D_2 embedded in X_1 with ∂D_2 essential in ∂X_1. By repeating this argument until we reach X_n which is a union of 3-balls, we see that X must be a handlebody, as claimed.

Now we come to the proof of Theorem 1.1. In order to prove that $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$, it suffices to prove that any compact subset of $\tilde{M}$

lies in the interior of a 3-ball. The following technical result will prove this. From now on, we assume that Σ is the preimage in $\tilde{M}$ of $f(F)$.

Lemma 1.5. *Let $\Pi_1, \cdots, \Pi_n$ be a collection of planes of Σ, and let $N(\Sigma_n)$ denote a regular neighborhood of $\Sigma_n = \Sigma - \bigcup_{i=1}^n \Pi_i$. Then each component of $\tilde{M} - N(\Sigma_n)$ is irreducible and simply connected, and its closure is almost compact.*

Proof of Theorem 1.1 *from Lemma* 1.5. Let C be a compact subset of $\tilde{M}$. By enlarging C, if necessary, we can assume that C is connected. As C meets only finitely many planes $\Pi_1, \cdots, \Pi_n$ of Σ, it lies in $\tilde{M} - N(\Sigma_n)$, for some regular neighborhood $N(\Sigma_n)$ of Σ_n. Let X denote the closure of the component of $\tilde{M} - N(\Sigma_n)$ which contains C. The compactification $\overline{X}$ of X given by the conclusion of Lemma 1.5 is irreducible and simply connected, so it must be a 3-ball. Hence C lies in the interior of a 3-ball in the interior of $\overline{X}$, so that C lies in the interior of a 3-ball in $\tilde{M}$, as required.

Note that this proof does not use the result of Meeks, Simon and Yau [8] that the universal covering of a P^2-irreducible 3-manifold is irreducible. The irreducibility of $\tilde{M}$ comes naturally, out of our arguments.

Proof of Lemma 1.5. We prove this by induction on n, starting with $n = 0$, when Σ_0 equals Σ. Lemma 1.3 shows that the closure of each component of $\tilde{M} - N(\Sigma)$ is the universal covering of a handlebody. Thus it is irreducible and almost compact. This can be proved easily, and also is a special case of results of Waldhausen [17].

Now we suppose that Lemma 1.5 holds when $n = k - 1$ and will show it holds when $n = k$. Let X denote the closure of a component of $\tilde{M} - N(\Sigma_k)$. Then X is the union of manifolds Y_i, each being the closure of a component of $\tilde{M} - N(\Sigma_{k-1})$, together with manifolds $R_j \times I$, each R_j being the closure of a component of $\Pi_k - N(L)$, where L consists of all double lines in Π_k. Note that there may be infinitely many Y_i's and R_j's. Our induction assumption tells us that each Y_i is irreducible, simply connected and almost compact. Lemma 1.3 implies that each R_j is simply connected as this result applies to any simple family of planes. Finally, Lemma 1.2 shows that X is simply connected. Again, this is because the proof of Lemma 1.2 works for any simple family of planes. It follows easily that X is irreducible, and it follows from the result of Simon [14] given below that X is almost compact, thus completing the proof of Lemma 1.5.

Simon's result is the following.

Theorem 1.6 [14]. *Let X be a 3-manifold formed from the disjoint union of 3-manifolds Y_i by gluing them in pairs along disjoint subsurfaces R_j of their boundary. If $\pi_1(X)$ is finitely generated, each Y_i is almost compact,* *and each R_j is incompressible in the two Y_i's in whose boundary it lies, then X is almost compact.*

2. The torus cover

In this section, we prove the following.

Theorem 2.1. *Let M be a closed P^2-irreducible 3-manifold such that $\pi_1(M)$ contains a subgroup A isomorphic to $\mathbf{Z} \times \mathbf{Z}$. Then the covering M_T of M with $\pi_1(M_T) = A$ is almost compact.*

Remark. M_T is P^2-irreducible, because $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$, so this result implies that M_T is a line bundle over the torus.

We pointed out in the introduction that this result follows from [14] when M is Haken, and that if M is not Haken, then [13] shows that some infinite cyclic subgroup of A is normal in $\pi_1(M)$. In what follows, we will always assume that M is not Haken.

Our aim is to argue very much as in §1. We start by choosing a least area map of the torus T to M such that f_* induces an isomorphism of $\pi_1(T)$ to A. As M is not Haken, it must be orientable, so that f is automatically 2-sided. As in §1, the preimage in $\tilde{M}$ of $f(T)$ consists of area-minimizing embedded planes, and the preimage in M_T of $f(T)$ consists of the images of these planes. For a given plane Π, the elements of A which stabilize Π form a subgroup $S(\Pi)$. As $S(\Pi)$ can only be isomorphic to $\{1\}$, $\mathbf{Z}$ or $\mathbf{Z}\times\mathbf{Z}$, the image in M_T of Π is a possibly singular plane, annulus or torus. We let Σ denote the collection of all these surfaces in M_T. Also as in §1, we can regularly homotop f to a general position map f'. For the purposes of this section, the map f' is all we need, but in §3 we will use the least area map f and its special properties.

Now assume that f is in general position. If each surface in Σ is embedded in M_T, we can argue almost exactly as in §1. Unfortunately, this need not be the case, but we will get around this problem by passing to a suitable finite cover of M_T. First, we handle the case when each surface of Σ is embedded in M_T. The following is the precise analogue of Lemma 1.5.

Lemma 2.2. *Suppose that each surface of Σ is embedded in M_T. Let $\Pi_1, \cdots, \Pi_n$ be a collection of surfaces of Σ, and let $N(\Sigma_n)$ denote a regular neighborhood of $\Sigma_n = \Sigma - \bigcup_{i=1}^n \Pi_i$. Then each component of $M_T - N(\Sigma_n)$ is irreducible and has fundamental group injecting into A, and its closure is almost compact.*

Proof. As in the proof of Lemma 1.5, we prove this by induction on n, starting with $n = 0$, when Σ_0 equals Σ. Lemma 1.3 implies that the closure X of a component of $M_T - N(\Sigma_0)$ is a covering of a handlebody and that $\pi_1(X)$ injects into A. As any subgroup of A is finitely generated, it can be shown easily and it follows from [14] that X is almost compact.

Now we suppose that Lemma 2.2 holds when $n = k - 1$ and will show that it holds when $n = k$. Let X denote the closure of a component of $M_T - N(\Sigma_k)$. As in the proof of Lemma 1.5, X is the union of manifolds Y_i, each being the closure of a component of $M_T - N(\Sigma_{k-1})$, together with manifolds $R_j \times I$, each R_j being the closure of a component of $\Pi_k - N(L)$, where L consists of all double lines in Π_k. Lemma 1.3 shows that $\pi_1(R_j)$ injects into $\pi_1(\Pi_k)$ and hence injects into A. Thus R_j is incompressible in the two Y_i's in whose boundary it lies. Our induction assumption implies that each Y_i is irreducible and almost compact. Thus X is irreducible. In addition, Theorem 1.6 will show that X is almost compact, so long as we know that $\pi_1(X)$ is finitely generated. But any component $\tilde{X}$ of the preimage of X in $\tilde{M}$ is simply connected, by Lemma 1.2, as $\tilde{X}$ is a component of the complement of a simple union of planes. (Note that the family of planes involved consists of the preimage of $f(F)$ with infinitely many planes removed.) Thus $\pi_1(X)$ injects into A and so is certainly finitely generated. This completes the proof of Lemma 2.2.

Proof of Theorem 2.1 *in the case when all surfaces of* Σ *are embedded in* M_T. We will show that M_T must be homeomorphic to $T \times \mathbf{R}$ by showing that any compact subset C of M_T lies in the interior of a compact submanifold of M_T which is homeomorphic to $T \times I$ and whose fundamental group maps isomorphically to A. First we enlarge C so that it is connected and so that the natural map $\pi_1(C) \to \pi_1(M_T)$ is surjective. Let $\Pi_1, \cdots, \Pi_n$ be the surfaces of Σ which meet C, and let X be the closure of the component of $M_T - N(\Sigma_n)$ which contains C. As X contains C, the natural map $\pi_1(X) \to \pi_1(M_T)$ is also surjective. Now Lemma 2.2 shows that X is irreducible and almost compact, and that the natural map $\pi_1(X) \to \pi_1(M_T)$ is an isomorphism. It follows that the compactification of X is homeomorphic to $T \times I$. As C lies in the interior of X, we can find a compact submanifold of X which contains C in its interior, is homeomorphic to $T \times I$ and whose fundamental group maps isomorphically to $\pi_1(M_T)$. This completes the proof of Theorem 2.1 in this case.

We deal with the general case by using the result of Theorem 3.1 which tells us that there is a finite covering of M_T in which the preimage of $f(F)$ consists entirely of embedded surfaces.

Proof of Theorem 2.1 *assuming Theorem* 3.1. Let M_1 denote a finite covering of M_T in which the preimage of $f(F)$ consists entirely of embedded surfaces. The special case of Theorem 2.1 which we proved above shows that M_1 is almost compact. Now it follows from Tucker's criterion for almost compactness [16] that if a manifold has a finite cover which is almost compact, it must also be almost compact. Thus M_T must be almost compact, as required.

We end this section by briefly discussing the problems which arise if one tries to prove the analogue of Theorem 2.1 for other surface groups, i.e., if F is a closed surface not S^2 or P^2, if $\pi_1(M)$ contains $\pi_1(F)$, and if M_F denotes the cover of M with $\pi_1(M_F) = \pi_1(F)$, then M_F should be almost compact. Two problems arise if one tries to argue as in the proof of Theorem 2.1. First, the surfaces which form the preimage Σ of $f(F)$ in M_F need not be embedded, and, when F is not the torus, we do not know whether there must be a finite covering with all surfaces embedded. Second, there seems to be no reason why the components of $M_F - \Sigma$ should have finitely generated fundamental group. All we know is that the fundamental group is a subgroup of $\pi_1(M_F)$, but, when F is not the torus, $\pi_1(F)$ has many infinitely generated subgroups. Further, even if the groups are finitely generated, there seems no reason why the components of $M_F - \Sigma_n$ should continue to have this property when $n \geq 1$.

3. Finite covers of M_T

In this section, we prove

Theorem 3.1. *Let M be a closed orientable irreducible 3-manifold such that $\pi_1(M)$ contains a subgroup A isomorphic to $\mathbf{Z}\times\mathbf{Z}$ and A has an infinite cyclic subgroup which is normal in $\pi_1(M)$. Let $f\colon T \to M$ be a least area map such that f_* induces an isomorphism of $\pi_1(T)$ with A. Then there is a finite cover M_1 of M_T such that the preimage of M_1 of $f(T)$ consists of embedded surfaces.*

Our proof of this theorem stems from the observation that it is easy to prove in the case when M is a Seifert fiber space. In this case, M can be Seifert fibered so that the cyclic subgroup of A which is normal in $\pi_1(M)$ is carried by a fiber and f is then homotopic to a vertical map. Let P denote the hyperbolic or Euclidean plane, as appropriate. Then there is a discrete group Γ of isometries of P such that the base orbifold of M can be identified with P/Γ. We can assume that Γ acts orientation preservingly on P by replacing M by a double cover, if necessary. In this case, Theorem 3.1 is an immediate consequence of the following result.

Lemma 3.2. *Let Γ be a discrete cocompact group of orientation preserving isometries of P, where P denotes E^2 or H^2. Let β denote an element of*

Γ of infinite order and let l denote a geodesic in P which is invariant under β. Then the following hold:

(i) *If $P = E^2$, then every translate of l by an element of Γ has simple image in $E^2/\langle\beta\rangle$.*

(ii) *If $P = H^2$, then there exists an integer $K \geq 1$ such that every translate of l by an element of Γ has simple image in $H^2/\langle\beta^K\rangle$.*

Proof. (i) Let m be a translate of l. Then $\beta^r m$ is parallel to m for all r, so that $\beta^r m$ equals m or is disjoint from m. Thus the image of m in $E^2/\langle\beta\rangle$ is simple, as required.

(ii) First, we note that if m is a translate of l which crosses l, then the translates of m by powers of β must be disjoint. For m and $\beta^r m$ cross l at the same angle, and if they intersected, we would obtain a triangle in H^2 with the sum of two angles being π. If m is disjoint from l, it is possible that βm crosses m. However, because the effect of β (and of β^{-1}) on the circle at infinity is to move all points towards one end of l, it is clear that there is an integer N such that $\beta^n m$ is disjoint from m, whenever $|n| \geq N$. (Note that as Γ is cocompact it contains no parabolic elements so that l and m cannot have a common endpoint.) Of course, N will depend on m, but if we choose m_0 with minimal distance from l among all translates of l which are disjoint from l, then the integer $K = N(m_0)$ has the property that $\beta^n m$ is disjoint from m or coincides with m, whenever $|n| \geq K$ and for all translates m of l. Thus, in particular, every translate of l has simple image in $H^2/\langle\beta^K\rangle$, as required.

The above geometric proof of Lemma 3.2 does not generalize easily to the case of a torus in a 3-manifold. However, the following more combinatorial argument can be generalized. We give this argument to make clear the simple ideas behind our proof of Theorem 3.1.

Lemma 3.3. *Let P denote the plane with some Riemannian metric, and let Γ denote a discrete group of orientation preserving isometries of P. Let β be an element of Γ of infinite order which leaves invariant a geodesic l whose image in P/Γ is a shortest closed loop representing the conjugacy class of β. Then there exists an integer $K \geq 1$ such that every translate of l by an element of Γ has simple image in $P/\langle\beta^K\rangle$.*

Proof. First, note that it follows from Lemma 1.4 of [2] that l is length minimizing, i.e., that any compact subarc is shortest rel boundary. Now we consider a translate m of l which crosses l. Then $l \cap m$ must be a single point by Lemma 3.1 of [2]. Let y denote $l \cap m$. We claim that m has simple image in $P/\langle\beta\rangle$. Suppose that m crosses $\beta^k m$ for some $k \neq 0$. Consider all the points on m and on one side of l in which translates of m by powers of β cross m, and choose the one which is nearest to y. We denote this

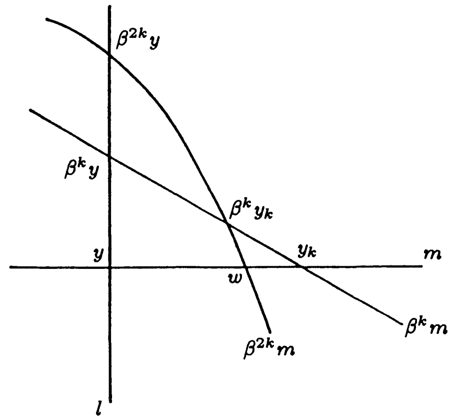

FIGURE 3.4

by $y_k = m \cap \beta^k m$. Note that y_k cannot equal y, for this would imply that β^k fixes y. Now consider the point $\beta^k y_k = \beta^k m \cap \beta^{2k} m$ in $\beta^k m$. As we cannot have $\beta^k y_k$ equal to y_k, we must have $\beta^k y_k$ strictly closer than y_k to $\beta^k y$. As $\beta^k y$ cannot equal y, we have the situation shown in Figure 3.4. The line $\beta^{2k} m$ enters the triangle formed by y_k, y and $\beta^k y$ at the point $\beta^k y_k$. As it cannot cross l or $\beta^k m$ again, it must leave the triangle by crossing m at a point w strictly closer than y_k to y, which contradicts our choice of y_k.

We have just shown that those translates of l which cross l project to embedded lines in $P/\langle\beta\rangle$. We let N denote the number of such lines. We will show that if m is any translate of l disjoint from l, then either $\beta^k m$ is disjoint from m for all $k > N$, or βm equals m. It will then follow that every translate of l has simple image in $P/\langle\beta^K\rangle$, where $K = N + 1$, thus proving Lemma 3.3.

Let m be a translate of l disjoint from l and suppose that $\beta^k m$ crosses m for some $k > N$. It follows from Lemma 3.5 below that $\beta m, \beta^2 m, \cdots, \beta^N m$ and $\beta^{N+1} m$ all cross m.

If $m = gl$ and γ denotes $g^{-1}\beta g$, then we have γ in Γ such that $\gamma l, \gamma^2 l, \cdots, \gamma^{N+1} l$ all cross l. The definition of N shows that there are distinct integers r and s, such that $1 \leq r, s \leq N + 1$ and a nonzero integer t such that $\beta^t(\gamma^r l) = \gamma^s l$. Now one of $r - s$ and $s - r$ lies between 1 and N, so that $\gamma^{r-s} l$ crosses l. Hence $\gamma^r l$ crosses $\gamma^s l$. Thus we have found a translate $\gamma^r l$ of l such that $\gamma^r l$ crosses l and crosses the translate of itself by β^t, which contradicts the first part of our proof of Lemma 3.3.

We conclude from this contradiction that if m is a translate of l which is disjoint from l then $\beta^k m$ cannot cross m for any $k > N$. Thus $\beta^k m$ is disjoint from m or equal to m for all $k > N$. But if $\beta^k m$ equals m, then the image of m in $P/\langle\beta^k\rangle$ is a shortest closed loop representing the conjugacy class of β^k, because of the length minimizing property of m. The image of this loop in $P/\langle\beta\rangle$ must be a shortest closed loop representing the conjugacy class of β (by results of [2]), so it follows that βm equals m in this case. Thus either $\beta^k m$ is disjoint from m for all $k > N$, or βm equals m, as claimed. This completes the proof of Lemma 3.3 apart from the following result which we quoted.

Lemma 3.5. *In the situation of Lemma* 3.3, *let m be a translate of l disjoint from l such that $\beta^k m$ crosses m for some $k \geq 1$. Then $\beta^r m$ crosses m when $1 \leq r \leq k$.*

Proof. This is equivalent to showing that if $r \geq 1$ and $\beta^r m \cap m$ is empty, then $\beta^k m \cap m$ is empty whenever $k \geq r$. For, if $\beta^r m$ equals m, then βm must equal m, as in the proof of Lemma 3.3, and so $\beta^k m$ must also equal m, contradicting our hypothesis. This, in turn, is equivalent to showing that if r and s are positive integers and $m \cap \beta^{-r} m$ is empty, then $\beta^s m \cap \beta^{-r} m$ must also be empty. This is what we will prove.

First, suppose that $\beta^s m \cap m$ is empty. Choose a shortest geodesic λ from l to m and denote the endpoints by x and y. Note that λ exists because l and m project to a closed loop in P/Γ, and there is a shortest path in P/Γ with endpoints on this loop in each homotopy class of such paths. Let X denote $\lambda \cup m$. Now $\beta^{-r}\lambda$ must be disjoint from m, as otherwise a subarc of $\beta^{-r}\lambda$ would form a path from l to m which is shorter than λ. Also $\beta^{-r}\lambda$ is disjoint from λ, as their endpoints are disjoint and any interior intersections would contradict the facts that λ is a shortest path from l to m, and $\beta^{-r}\lambda$ is a shortest path from l to $\beta^{-r} m$. Hence $\beta^{-r} X$ is disjoint from X. Similarly, $\beta^s X$ is disjoint from X. Now X separates the component of $\overline{P - l}$ in which it lies, and $\beta^{-r} X$ and $\beta^s X$ must lie in different components of the complement of X, because $\beta^{-r} x$ and $\beta^s x$ do. Thus $\beta^{-r} X$ and $\beta^s X$ must be disjoint. In particular, $\beta^{-r} m$ and $\beta^s m$ must be disjoint, as required.

Now consider the case where $\beta^s m$ meets m in a point which we denote by z. Again, let λ denote a shortest geodesic from l to m with endpoints x and y. (See Figure 3.6.) This time, we let X denote the union of λ and the component of $m - \{y\}$ which does not contain z. (It is important to note that y and z are distinct points. If y and z coincided, then y would simultaneously be a point of m and of $\beta^s m$ closest to x. Thus m and $\beta^s m$ would meet λ orthogonally at y, and so be tangent at y and hence

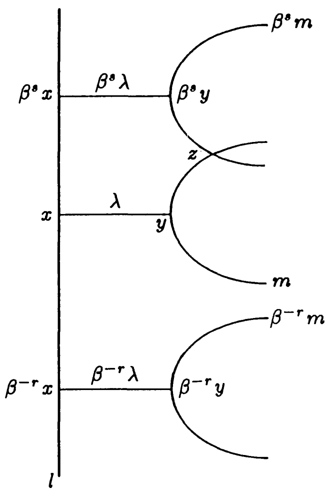

FIGURE 3.6

coincident. This contradicts our assumption that m and $\beta^s m$ cross at z.) As before, $\beta^{-r}X$ is disjoint from X. Similarly, $\beta^s X$ is disjoint from X. As before, it follows that $\beta^{-r}m$ and $\beta^s m$ must be disjoint, completing the proof of Lemma 3.5.

Now we come to the proof of Theorem 3.1, which we model after the proof of Lemma 3.3. Let α generate an infinite cyclic subgroup of $\pi_1(T)$ which is normal in $\pi_1(M)$, and let β be an element of $\pi_1(T)$ such that α and β generate a rank-two subgroup of $\pi_1(T)$. Without loss of generality, we can suppose that α and β generate $\pi_1(T)$. Let M_α denote the covering of M with $\pi_1(M_\alpha)$ generated by α. Recall that the least area map $f\colon T \to M$ lifts to an embedding of T in M_T, whose preimage in M_α is an embedded annulus A. As M_α is a regular covering of M, the complete preimage in M_α of $f(F)$ consists of translates of A by the group $\Gamma = \pi_1(M)/\langle\alpha\rangle$.

Lemma 3.7. *There is a finite cover M_0 of M_T such that all the double curves of the preimage of $f(T)$ in M_0 are simple.*

Proof. In M_α, distinct translates of the annulus A are disjoint or intersect transversely in a single essential (and embedded) circle, by Lemma 6.5 of [3]. As these double curves project to a finite number of double curves in M, there is an upper bound d on the diameters of these curves in M_α. As M is compact, there is a power β^k of β which moves all points of M_α further than distance d. Clearly, all double curves in $M_\alpha/\langle\beta^k\rangle$ will be simple, as required.

By replacing M_T by this finite cover M_0, we can now assume that α and β generate $\pi_1(T)$ and that all double curves in M_T are simple.

Lemma 3.8. *Let B be a translate of A which crosses A. Then B projects to a simple annulus in M_T.*

Proof. Suppose the result is false. Then there exists $k \neq 0$ such that $\beta^k B$ equals B or crosses B. If $\beta^k B$ equals B, then the stabilizers of A and B intersect in a group isomorphic to $\mathbf{Z} \times \mathbf{Z}$, so that Lemma 6.4(ii) of [3] implies that A and B are disjoint or coincide. Since by our hypothesis B crosses A, we see that $\beta^k B$ cannot equal B.

Let C denote the circle $B \cap A$ and, if $\beta^r B$ crosses B, let C_r denote $B \cap \beta^r B$.

Note that C_r cannot meet A, for this would imply that $\beta^r C$ meets C, contradicting our assumption that all double curves in M_T are simple. As in the proof of Lemma 3.3, we want to choose C_r "nearest" to A. Out of all the C_r's on B, and on one side of A, we choose one, C_k, which minimizes the area of the annulus B' in B bounded by C and C_k. (This may not be unique.) The circles C and $\beta^k C$ bound a compact annulus A' in A, and the circles $\beta^k C$ and C_k bound a compact annulus B'' in $\beta^k B$. The union of these annuli is an embedded torus X in M_α. As the annulus $B' \cup B''$ must be parallel to A' in M_α, we see that X bounds a solid torus.

Consider the circle $\beta^k C_k = \beta^k B \cap \beta^{2k} B$. We know that $\beta^k C_k$ does not meet A, as C_k does not meet A. Also $\beta^k C_k$ cannot meet C_k since all double curves in M_T are simple. As the area of the annulus $\beta^k B'$ is less than or equal to the area of B'', we deduce that $\beta^k C_k$ lies in the interior of B''. We conclude that $\beta^{2k} B \cap X$ contains a component, namely $\beta^k C_k$, which is essential in M_α. As X bounds a solid torus, it is clear that $\beta^{2k} B \cap X$ must have another component S which is also essential in M_α. Now $\beta^{2k} B \cap A = \beta^{2k} C$, so that $\beta^{2k} B$ cannot meet the annulus A'. Since $\beta^{2k} B \cap \beta^k B = \beta^k C_k$, the circle S must lie in the interior of the annulus B'. But this implies that the circle S equals $C_{2k} = \beta^{2k} B \cap B$, a contradiction to our choice of C_k. Hence the proof of Lemma 3.8 is complete.

Proof of Theorem 3.1. Lemma 3.8 shows that any translate of A which crosses A projects to a simple annulus in M_T. As in our proof of Lemma 3.3, we let N denote the number of such simple annuli in M_T. Let B be a translate of A disjoint from A, and suppose that $\beta^k B$ crosses B for some $k > N$. Lemma 3.9 below shows that $\beta B, \beta^2 B, \cdots, \beta^{N+1} B$ must all cross B, and this yields a contradiction exactly as in our proof of Lemma 3.3. As in that proof, we conclude that either $\beta^k B$ is disjoint from B for all $k > N$, or that βB equals B. Theorem 3.1 follows by taking M_1 to be the covering of M with fundamental group generated by α and β^{N+1}.

Lemma 3.9. *Assume the hypotheses of Theorem* 3.1 *and that α and β generate $\pi_1(T)$ and all double curves in M_T are simple. Let B be a translate* *of A disjoint from A such that $\beta^k B$ crosses B for some $k \geq 1$. Then $\beta^r B$ crosses B for $1 \leq r \leq k$.*

Proof. As in Lemma 3.5, this is equivalent to showing that if r and s are positive and $B \cap \beta^{-r} B$ is empty, then $\beta^s B \cap \beta^{-r} B$ must also be empty. This is what we will prove.

First suppose that $\beta^s B \cap B$ is empty. We can find an annulus L properly embedded in the closure N of a component of $M_\alpha - (A \cup B)$, and with boundary components on A and B, both essential in M_α. Further, we can choose L to be least area among all such annuli. Such a least area annulus exists by applying the arguments of Theorem 1 of [9]. Their result does not apply immediately as ∂N is not convex. However, ∂N is a union of minimal surfaces and this is sufficient as discussed in [11]. Also N is not compact. This too is not a problem, for given any minimizing sequence of maps of the annulus, we can apply appropriate powers of β to ensure that the end of the annulus on A always meets some fixed compact subset of A. Also N has bounded curvatures, as M_α covers the closed manifold M, and this now allows the arguments of [9] to go ahead. Let C and D denote the components of ∂L on A and B respectively. Let X denote $L \cup B$. Now $\beta^{-r} D$ is disjoint from B, so that the intersection of $\beta^{-r} L$ with B must consist of essential circles. This uses the area minimizing properties of B and $\beta^{-r} L$. If $\beta^{-r} L \cap B$ is nonempty, then a subannulus of $\beta^{-r} L$ forms an annulus joining A to B with area less than that of L. This contradiction shows that $\beta^{-r} L \cap B$ must be empty. Also $\beta^{-r} L$ is disjoint from L. For $\beta^{-r} C$ and C are distinct, and $\beta^{-r} D$ and D are disjoint so that any intersection of $\beta^{-r} L$ and L would contradict their least area properties. Thus $\beta^{-r} X$ and X are disjoint. Similarly $\beta^s X$ and X are disjoint. As in Lemma 3.5, X separates the component of $M_\alpha - A$ in which it lies, and $\beta^s C$ and $\beta^{-r} C$ lie in different components of the complement of X. It follows that $\beta^s X$ and $\beta^{-r} X$ are disjoint and hence that $\beta^s B$ and $\beta^{-r} B$ are disjoint, as required.

If $\beta^s B$ meets B, we let Z denote their circle of intersection. Note that Z and $\beta^{-s} Z$ are disjoint, as all double curves in M_T are simple. Choose an essential circle D on B disjoint from Z and $\beta^{-s} Z$, so that we can find an annulus L' properly embedded in the closure of a component of $M_\alpha - (A \cup B)$, with one boundary component on A and the other equal to D and such that L' is disjoint from $\beta^s B$ and $\beta^{-s} B$. (It seems natural to assume that D must lie between Z and $\beta^{-s} Z$ on B, but we will not assume this.) Next, we choose an annulus L to have least possible area among all annuli properly embedded in the closure of a component of $M_\alpha - (A \cup B)$ and with one boundary component on A and the other equal to D. Such a least area annulus exists by Theorem 3 of [10]. As in the previous case, the noncompactness of the closure of $M_\alpha - (A \cup B)$ causes no problems. It follows that L is disjoint from $\beta^{-s}B$. For $L \cap \beta^{-s}B$ must consist of an even number of essential circles, and this would contradict the area minimizing properties of L and $\beta^{-s}B$ unless $L \cap \beta^{-s}B$ is empty. Now define X to be the union of L and of the component Y of $B - D$ not containing Z.

As $\beta^{-r}D$ is disjoint from B, the intersection of $\beta^{-r}L$ and B must consist of essential circles. Further, the number of such circles is even, as A and $\beta^{-r}B$ lie on the same side of B. But if $\beta^{-r}L \cap B$ contained two essential circles, this would contradict the area minimizing properties of $\beta^{-r}L$ and of B. We conclude that $\beta^{-r}L \cap B$ is empty. Now $\beta^{-r}L \cap L$ must consist of essential circles and arcs with endpoints on A. But any such intersection curve would contradict the area minimizing properties of $\beta^{-r}L$ and L. We conclude that $\beta^{-r}L \cap L$ is empty. Thus $\beta^{-r}X$ and X are disjoint.

It remains to show that $\beta^s X$ and X are disjoint. This will imply that $\beta^s B$ and $\beta^{-r}B$ are disjoint, as usual. As L is disjoint from $\beta^{-s}B$, it follows that $\beta^s L$ is disjoint from B. Now $\beta^s L \cap L$ must consist of essential circles and arcs with endpoints on A, so, as before, we conclude that $\beta^s L$ is disjoint from L. Thus $\beta^s X \cap X$ is empty, as required. This completes the proof of Lemma 3.9.

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