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A HARNACK TYPE INEQUALITY FOR CERTAIN COMPLEX MONGE-AMPERÉ EQUATIONS

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0. Introduction

In finding a Kähler-Einstein metric on a compact Kähler manifold (M, g) with $C_1(M) > 0$, one needs to solve the following complex Monge-Amperé equations:

$$(*)_t \qquad \begin{aligned} &(\omega_g + \partial\bar{\partial}\phi)^n = e^{f - t\phi}\omega_g^n, \\ &\omega_g + \partial\bar{\partial}\phi > 0, \qquad \text{on } M, \end{aligned}$$

where ω_g is the Kähler form associated with the metric g, $\omega_g^n = \omega_g \wedge \cdots \wedge \omega_g$ is the volume form, $0 \leq t \leq 1$, $\partial\bar{\partial} f = \mathrm{Ric}(g) - \omega_g$, $\int_M e^f \omega_g^n = \int_M \omega_g^n = \mathrm{Vol}_g(M)$, and $n = \dim M$.

While the prior estimates of higher derivatives have been obtained by Yau [8] for the solutions of $(*)_t$ more than ten years ago, little is known about the supreme norms of the solutions of $(*)_t$. In [7], the author proved that $-\inf_M \phi$ is bounded from above by $n \sup_M \Phi + C$ for any solution ϕ of $(*)_t$, where C is a constant independent of ϕ. Actually, it is implied in the proof there that C depends only on t. By a completely different method, Siu also proved a slightly weaker version of the above inequality [6], i.e., for any $\varepsilon > 0$, there is a constant C_ε, depending on the metric g, t and ε, such that for any solution ϕ of $(*)_t$ $-\inf_M \phi \leq (n + \varepsilon) \sup_M \phi + C_\varepsilon$. In this note, we develop the idea in the proof of the above Harnack type inequality in [7] and prove the following theorem.

Theorem 1. *Let (M, g) be a compact Kähler manifold with $C_1(M) > 0$, $n = \dim M$. Then for any $\psi \in C^2(M, R)$ with $\omega_g + \partial\bar{\partial}\psi \geq 0$ and $\int_M e^{f - t\psi}\omega_g^n = \mathrm{Vol}_g(M)$, the solution ϕ of $(*)_t$ satisfies the Harnack type inequality*

$$(0.1) \qquad -\frac{1}{\mathrm{Vol}_g(M)} \int_M (\phi - \psi)(\omega_g + \partial\bar{\partial}\phi)^n \leq n \sup_M (\phi - \psi).$$

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Furthermore, there is a constant $C(t)$ depending only on t such that for $t > 0$, the solution ϕ of $(*)_t$ satisfies

$$-\inf_M(\phi - \psi) \leq n \sup_M(\phi - \psi) + C(t). \tag{0.2}$$

An interesting and surprising corollary of Theorem 1 is the following.

Corollary 1. *There is a universal constant C such that for any Kähler-Einstein manifold (M, g) with $C_1(M) > 0$, i.e., $\mathrm{Ric}(g) = \omega_g$, and for any C^2 function ψ with $\omega_g + \partial\bar{\partial}\psi \geq 0$, $\int_M e^{-\psi}\omega_g^n = \mathrm{Vol}_g(M)$, the following inequality holds:*

$$\sup_M \psi \leq -n \inf_M \psi + C. \tag{0.3}$$

Note that the constant C is computable and the inequality (0.3) is sharp; for example, one can consider the case $(M, g) = (CP^n$, *Fubini-Study metric*) to see the sharpness of (0.3).

1. The proof of Theorem 1

First, we assume that $\psi \in C^\infty(M, R)$, $\omega_g + \partial\bar{\partial}\psi > 0$. Then we can define a new Kähler metric $\tilde{g}$ such that the associated Kähler form is $\omega_g + \partial\bar{\partial}\psi$. Put $\tilde{f} = f + \log(\omega_g^n/\omega_{\tilde{g}}^n) - t\psi$. Then

$$\int_M e^{\tilde{f}}\omega_{\tilde{g}}^n = \int_M e^{f - t\psi}\omega_g^n = \mathrm{Vol}_g(M). \tag{1.1}$$

Rewrite $(*)_t$ in terms of $\omega_{\tilde{g}}$ and $\tilde{f}$ as follows:

$$\begin{aligned} &(\omega_{\tilde{g}} + \partial\bar{\partial}(\phi - \psi))^n = e^{\tilde{f} - t(\phi - \psi)}\omega_{\tilde{g}}^n, \\ &(\omega_{\tilde{g}} + \partial\bar{\partial}(\phi - \psi)) > 0, \text{ on } M. \end{aligned} \tag{1.2}$$

In [8], in order to show the uniqueness of Kähler-Einstein metrics on a compact Kähler manifold with positive first Chern class, Bando and Mabuchi first prove the solvability of $(*)_{t'}$ for $t' \leq t \leq 1$ under the assumption that $(*)_t$ has a solution. We will apply this idea to the following equations in the first part of our proof. Precisely, we will first prove that the following equations are always solvable for $0 \leq s \leq t$:

$$\begin{aligned} &(\omega_{\tilde{g}} + \partial\bar{\partial}\theta)^n = e^{\tilde{f} - s\theta}\omega_{\tilde{g}}^n, \\ &(\omega_{\tilde{g}} + \partial\bar{\partial}\theta) > 0, \text{ on } M. \end{aligned} \tag{$1.3)_s$$

As usual, we use the continuity method. Define $S = \{s \in [0, t] | (1.3)_{s'}$ is solvable for $s' \in [s, t]\}$. Since (1.2) has a solution ϕ, $t \in S$ and S is nonempty. It is sufficient to show that S is both open and closed. For the openness, we should estimate the first eigenvalue of the metric g_θ associated with the Kähler form $\omega_g + \partial\bar\partial\theta$ for the solution θ of $(1.3)_s$.

Lemma 1.1. *The first nonzero eigenvalue $\lambda_1(g_\theta)$ is greater than s.*

Proof. By the well-known Bochner identity (see [1]), it suffices to show that $\mathrm{Ric}(g_\theta)$ is strictly bounded from below by s. From $(1.3)_s$, we have

$$\begin{aligned}\mathrm{Ric}(g_\theta) &= \mathrm{Ric}(\tilde g) - \partial\bar\partial\tilde f + s\partial\bar\partial\theta \\ &= \mathrm{Ric}(\tilde g) - \partial\partial f - \partial\partial\log\left(\frac{\omega_g^n}{\omega_{\tilde g}^n}\right) + t\partial\bar\partial\psi + s\partial\bar\partial\theta \\ &= \mathrm{Ric}(g) - \partial\bar\partial f + t\partial\bar\partial\psi + s\partial\bar\partial\theta \\ &= (1-t)\omega_g + (t-s)\omega_{\tilde g} + s(\omega_{\tilde g} + \partial\bar\partial\theta) > s\omega_{g_\theta}.\end{aligned}$$

The first variation of $(1.3)_s$ at θ is $\Delta_s u = -su$, where Δ_s is the Laplacian of the metric g_θ. Lemma 1.1 implies that the linearized operator $\Delta_s - s$ of $(1.3)_s$ is invertible; then the openness follows from Implicit Function Theorem.

For the closedness of S, by the standard theory of elliptic equations [4] and Yau's estimates of higher derivatives for solutions of complex Monge-Ampere equations of type $(1.3)_s$, it suffices to estimate C^0-norms of the solutions of $(1.3)_s$.

Suppose that $(1.3)_s$ is solvable for $s \in (s_0, t]$ and θ_s is the solution. From the proof of the openness of S, one can actually conclude that $\{\theta_s\}_{s\in(s_0,t]}$ is a smooth family in $C^\infty(M, R)$, i.e., θ_s varies smoothly with s.

Define, as in [1],

$$I(\theta_s) = \frac{1}{\mathrm{Vol}_g(M)}\int_M \theta_s(\omega_{\tilde g}^n - (\omega_{\tilde g} + \partial\bar\partial\theta_s)^n), \qquad J(\theta_s) = \int_0^1 \frac{I(x\theta_s)}{x}\,dx.$$

Lemma 1.2 ([1], [2], [7]). (i) $(n+1)J(\theta_s)/n \le I(\theta_s) \le (n+1)J(\theta_s)$,
(ii) $d(I(\theta_s) - J(\theta_s))/ds = -(\mathrm{Vol}_g(M))^{-1}\int_M \theta_s(\Delta_s\dot\theta_s)\omega_g^n$,
where $\dot\theta_s = d\theta_{s'}/ds'|_{s'=s}$, g_s is the Kähler metric associated with $\omega_{\tilde g} + \partial\bar\partial\theta_s$ and Δ_s is its Laplacian.

As a corollary, we have the following lemma which was observed by Bando and Mabuchi [2] and the author [7].

Lemma 1.3. *$I(\theta_s) - J('\theta_s)$ is monotonically increasing.*

Proof. Differentiate $(1.3)_s$ with respect to s:

$$(1.4)_s \qquad \Delta_s\dot\theta_s = -s\dot\theta_s - \theta_s.$$

Substituting $(1.4)_s$ into the right-hand side of the formula in Lemma 1.2(ii), we obtain

$$(1.5) \qquad \frac{d}{ds}(I(\theta_s) - J(\theta_s)) = \frac{1}{\mathrm{Vol}_g(M)}\int_M (\Delta_s\dot\theta_s + s\dot\theta_s)\Delta_s\dot\theta_s\omega_{g_s}^n.$$

Write $\dot{\theta}_s$ in the eigenfunction expansion, i.e.,

$$\dot{\theta}_s = \sum_{i=0}^{\infty} a_i u_i, \tag{1.6}$$

where $-\Delta_s u_i = \lambda_i u_i$, $0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots$.

By Lemma 1.1., $\lambda_1 > s$. Hence,

$$\begin{aligned}\frac{d}{ds}(I(\theta_s) - J(\theta_s)) &= \frac{1}{\mathrm{Vol}_g(M)} \int_M \left(\sum_{i=0}^{\infty} a_i(\lambda_i - s)u_i\right)\left(\sum_{j=0}^{\infty} a_j \lambda_j u_j\right)\omega_{g_s}^n \\ &= \frac{1}{\mathrm{Vol}_g(M)} \sum_{i=0}^{\infty} |a_i|^2 \int_M (\lambda_i - s)\lambda_i |u_i|^2 \omega_{g_s}^n \ge 0,\end{aligned}$$

and the lemma is proved.

In the following, we always denote by C the constant independent of s.

Lemma 1.4. *There is a constant $C > 0$ such that for any solution θ_s of $(1.3)_s$, $0 < s \le t$, we have $\sup_M |\theta_s| \le C$.*

Proof. In [7], we define a holomorphic invariant $\alpha(M)$ on the compact Kähler manifold M with $C_1(M) > 0$. For any $\lambda < \alpha(M)$, there is a constant C_λ, which may depend on the metric $\tilde{g}$, such that

$$\int_M e^{-\lambda(u - \sup_M u)}\, dV_{\tilde{g}} \le C_\lambda \quad \text{for } u \in C^2(M, R),\ \omega_{\tilde{g}} + \partial\bar{\partial}u \ge 0. \tag{1.7}$$

In case $s \in (0, \alpha(M)/(n+2)]$, $\int_M e^{-(n+1)s(\theta_s - \sup_M \theta_s)}\, dV_{\tilde{g}} \le C$ for a constant C. For $p > 0$,

$$\begin{aligned}&\int_M e^{-p(\theta_s - \sup_M \theta_s)}(e^{\tilde{f} - s\theta_s} - 1)\, dV_{\tilde{g}} \\ &= \int_M e^{-p(\theta_s - \sup_M \theta_s)}(\omega_{g_s}^n - \omega_{\tilde{g}}^n) \\ &= \int_M e^{-p(\theta_s - \sup_M \theta_s)} \partial\bar{\partial}(\theta_s - \sup_M \theta_s)(\omega_{g_s}^{n-1} + \omega_{g_s}^{n-2} \wedge \omega_{\tilde{g}} + \cdots + \omega_{\tilde{g}}^{n-1}) \\ &= \frac{4}{p}\int_M \partial(e^{-p(\theta_s - \sup_M \theta_s)/2}) \wedge \bar{\partial}(e^{-p(\theta_s - \sup_M \theta_s)/2}) \\ &\qquad\qquad \wedge (\omega_{g_s}^{n-1} + \omega_{g_s}^{n-2} \wedge \omega_{\tilde{g}} + \cdots + \omega_{\tilde{g}}^{n-1}) \\ &\ge \frac{4}{p}\int_M |\tilde{\nabla}(e^{-p(\theta_s - \sup_M \theta_s)/2})|^2 dV_{\tilde{g}} \\ &\ge \frac{4c}{p}\left(\int_M e^{-np(\theta_s - \sup_M \theta_s)/(n-1)}\, dV_{\tilde{g}}\right)^{(n-1)/n} \\ &\quad - \frac{4C}{p}\int_M e^{-p(\theta_s - \sup_M \theta_s)}\, dV_{\tilde{g}},\end{aligned}$$

where c is the Sobolev constant, depending only on $(M,\tilde{g})$. Using Hölder inequality on the left-handed side of the above, we have

$$\begin{aligned}
&\left(\int_M e^{-np(\theta_s-\sup_M\theta_s)/(n-1)}\,dV_{\tilde{g}}\right)^{(n-1)/n}\\
\text{(1.8)}\qquad &\leq Cp\left(\int_M e^{-(n+1)p(\theta_s-\sup_M\theta_s)/n}\,dV_{\tilde{g}}\right)^{n/(n+1)}\\
&\quad\cdot\left(\left(\int_M e^{-(n+1)s\theta_s}\,dV_{\tilde{g}}\right)^{1/n+1}+1\right).
\end{aligned}$$

Now $\sup_M\theta_s\geq 0$, since

$$\begin{aligned}
\mathrm{Vol}_{\tilde{g}}(M)=\int_M\omega_{\tilde{g}}^n=\int_M\omega_{g_s}^n=\int_M e^{\tilde{f}-s\theta_s}\,dV_{\tilde{g}}&\geq e^{-s\sup_M\theta_s}\int_M e^{\tilde{f}}\,dV_{\tilde{g}}\\
&=e^{-s\sup_M\theta_s}\,\mathrm{Vol}_{\tilde{g}}(M).
\end{aligned}$$

Then

$$\begin{aligned}
\text{(1.9)}\qquad\left(\int_M e^{-(n+1)\theta_s}\,dV_{\tilde{g}}\right)^{n+1}&\leq\left(\int_M e^{-(n+1)s(\theta_s-\sup_M\theta_s)}\,dV_{\tilde{g}}\right)^{n+1}\\
&\leq C^{n+1}.
\end{aligned}$$

Substituting (1.9) into (1.8), we have

$$\text{(1.10)}\qquad \left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{np/(n-1)}\leq C^{1/p}p^{1/p}\left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{(n+1)p/n}.$$

Put $p_0=ns$ and $p_{m+1}=p_mn^2/(n^2-1)$. Then

$$\begin{aligned}
\left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{p_{m+1}}&\leq C^{(n/(n-1))(1/p_{m+1})}\left(\frac{n-1}{n}p_{m+1}\right)^{(n/(n-1))/(1/p_{m+1})}\\
&\qquad\cdot\left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{p_m}\\
&\leq (Cp_m)^{1/p_m}\left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{p_m}\\
&\quad\cdots\\
&\leq C^{1/ns}\sum_{m=0}^{\infty}\left(\frac{n^2-1}{n^2}\right)^m\left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{p_0}\\
&\qquad\cdot\exp\left(\frac{1}{np}\sum_{m=1}^{\infty}\left(\frac{n^2-1}{n^2}\right)^m\left(m\log\frac{n^2}{n^2-1}+\log(ns)\right)\right)\\
&\leq C,
\end{aligned}$$

and it follows that

$$-\inf_M(\theta_s-\sup_M\theta_s)=\log\left(\lim_{m\to\infty}\left|e^{-(\theta_s-\sup_M\theta_s)}\right|_{p_m}\right)\leq C,$$

i.e., for $s\in(0,\alpha(M)/(n+2)]$, $\sup_M|\theta_s|\leq C$.

In the case $s \geq \alpha(M)/(n+2)$, $\mathrm{Ric}(g_s) \geq s \geq \alpha(M)/(n+2)$. Then by Bochner's identity and results of Croke [3] and P. Li [5], we have both the Sobolev inequality and the Poincaré inequality with their constants uniformly bounded on (M, g_s). Since $-\Delta_s \theta_s \geq -n$, the standard Moser iteration implies that

$$-\inf_M \theta_s \leq C \int_m (-\theta_s)\omega_{g_s}^n + C, \tag{1.11}$$

(also see [7] for details).

On the other hand, by Green's formula on $(M, \tilde{g})$, it follows that

$$\sup_M \theta_s \leq \int_M \theta_s \omega_{\tilde{g}}^n + C. \tag{1.12}$$

By Lemmas 1.3 and 1.2(i),

$$I(\theta_s) \leq (n+1)(I(\theta_s) - J(\theta_s)) \leq (n+1)(I(\theta_t) - J(\theta_t)) \leq C. \tag{1.13}$$

Since $\int_{\{\theta_s > 0\}} \theta_s e^{\tilde{f} - s\theta_s}\, dV_{\tilde{g}}$ and $\int_{\{\theta_s < 0\}} (-\theta_s)\, dV_{\tilde{g}}$ are obviously bounded by a constant C independent of s, the Lemma follows from (1.11)–(1.13) and the definition of $I(\theta_s)$.

Now the closeness of S follows from the above lemma. Hence, $(1.3)_s$ is solvable for $0 \leq s \leq t$. Then there is a smooth family of $\{\theta_s\}_{0 \leq s}$ such that $\theta_t = \phi - \psi$. By Lemma 1.2(ii) and $(1.4)_s$, we have

$$\begin{aligned}
\frac{d}{ds}(I(\theta_s) - J(\theta_s)) &= -\frac{1}{\mathrm{Vol}_{\tilde{g}}(M)} \int_M \theta_s(-s\dot{\theta}_s - \theta_s)\omega_{\tilde{g}}^n, \\
&= -\frac{1}{\mathrm{Vol}_{\tilde{g}}(M)} \frac{d}{ds}\left(\int_M \theta_s \omega_{g_s}^n\right) + \frac{1}{\mathrm{Vol}_{\tilde{g}}(M)} \int_M \dot{\theta}_s \omega_{g_s}^n.
\end{aligned}$$

Differentiating $\mathrm{Vol}_{\tilde{g}}(M) = \int_M e^{\tilde{f} - s\theta_s}\, dV_{\tilde{g}}$ gives

$$\int_M (-s\dot{\theta}_s - \theta_s) e^{\tilde{f} - s\theta_s}\, dV_{\tilde{g}} = 0. \tag{1.14}$$

Hence,

$$\frac{d}{ds}(I(\theta_s) - J(\theta_s)) = \frac{1}{s\,\mathrm{Vol}_{\tilde{g}}(M)} \frac{d}{ds}\left(s \int_M (-\theta_s) e^{\tilde{f} - s\theta_s}\, dV_{\tilde{g}}\right);$$

i.e.,

$$\begin{aligned}
&\frac{d}{ds}(s(I(\theta_s) - J(\theta_s))) - (I(\theta_s) - J(\theta_s)) \\
&\quad = \frac{d}{ds}\left(\frac{s}{\mathrm{Vol}_{\tilde{g}}(M)} \int_M -\theta_s e^{\tilde{f} - s\theta_s}\, dV_{\tilde{g}}\right).
\end{aligned} \tag{1.15}$$

Note that

$$I(\theta_s) - J(\theta_s) \geq \frac{1}{n+1} I(\theta_s) = \frac{1}{(n+1)\operatorname{Vol}_{\tilde{g}}(M)} \int_M \theta_s(\omega_{\tilde{g}}^n - \omega_{g_s}^n)$$
$$= \frac{1}{(n+1)\operatorname{Vol}_{\tilde{g}}(M)} \int_M \partial\theta_s \wedge \bar{\partial}\theta_s$$
$$\wedge (\omega_{\tilde{g}}^{n-1} + \omega_{\tilde{g}}^{n-2} \wedge \omega_{g_s} + \cdots + \omega_{g_s}^{n-1}) \geq 0.$$

Then it follows from (1.15) and Lemma 1.2(i) that

$$\frac{1}{\operatorname{Vol}_{\tilde{g}}(M)} \int_M (-\theta_t)\omega_{g_s}^n \leq I(\theta_t) - J(\theta_t) \leq \frac{n}{n+1} I(\theta_t) \tag{1.16}$$
$$= \frac{n}{n+1} \frac{1}{\operatorname{Vol}_{\tilde{g}}(M)} \int_M \theta_t(\omega_{\tilde{g}}^n - \omega_{g_t})^n,$$

i.e.,

$$-\frac{1}{\operatorname{Vol}_{\tilde{g}}(M)} \int_M (\phi - \psi)(\omega_g + \partial\bar{\partial}\phi)^n \leq \frac{n}{\operatorname{Vol}_{\tilde{g}}(M)} \int_M (\phi - \psi)\omega_{\tilde{g}}^n$$
$$\leq n \sup_M(\phi - \psi),$$

which is just (0.1). The inequality (0.2) follows from Moser's iteration and the fact that $\operatorname{Ric}(g_t) \geq t > 0$. Hence Theorem 1 is proved. (We refer the reader to the proof of Lemma 1.4 for details.)

2. The proof of Corollary 1

From (0.2) in Theorem 1, for any $\psi \in C^2(M, R)$ with $\omega_g + \partial\bar{\partial}\psi \geq 0$ and $\int_M e^{-\psi}\omega_g^n = \operatorname{Vol}_g(M)$, we have

$$-\inf_M(\phi - \psi) \leq n \sup_M(\phi - \psi) + C(1), \tag{2.1}$$

where ϕ is the solution of $(*)_1$ and $C(1)$ is a universal constant. Note that here $f \equiv 0$, since g has been a Kähler-Einstein metric. This implies that $\phi \equiv 0$ is a solution of $(*)_1$. For $\phi = 0$, (2.1) becomes

$$-\inf_M(-\psi) \leq n \sup_M(-\psi) + C(1).$$

Because $\inf_M(-\psi) = -\sup_M(\psi)$ and $\sup_M(-\psi) = -\inf_M \psi$, Corollary 1 is proved.

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