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ON THE AVERAGE INDICES OF CLOSED GEODESICS

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Introduction

A nonconstant closed curve $c\colon S^1 = \mathbf{R}/\mathbf{Z} \to M$ on a compact Riemannian manifold M with metric g is a closed geodesic on M iff it is a critical point of the energy functional $E\colon \Lambda M \to \mathbf{R}$, $E(c) = \frac{1}{2}\int_{S^1} g(\dot{c},\dot{c})$ on the Hilbert manifold ΛM of closed curves (cf. [11, Chapter 1]). Due to a theorem of Lusternik and Fet there always exists a closed geodesic on a compact Riemannian manifold.

ΛM carries a canonical O(2)-action leaving E invariant. With a closed geodesic c all iterates c^m, $m \in \mathbf{N}$, with $c^m(t) = c(mt)$ are closed geodesics too. Two closed geodesics $c_1, c_2\colon S^1 \to M$ are geometrically distinct if their images $c_1(S^1)$ and $c_2(S^1)$ are distinct. D. Gromoll and W. Meyer prove in [6] that on a compact Riemannian manifold there are infinitely many geometrically distinct closed geodesics if the sequence $b_i(\Lambda M; F)$ of Betti numbers of ΛM w.r.t. a field F is unbounded. In [21] M. Vigue-Poirrier and D. Sullivan prove that for a compact simply-connected manifold the sequence $b_i(\Lambda M; \mathbf{Q})$ of rational Betti numbers of ΛM is bounded iff the cohomology algebra $H^*(M; \mathbf{Q})$ of M is a truncated polynomial algebra $T_{d,n+1}(x)$ with the generator x of degree d and height $n+1$.

If M is a compact rank-one symmetric space ("CROSS") then the sequence $b_i(\Lambda M; F)$ is bounded for any field F. In this case one can use the following result of W. Klingenberg and F. Takens (cf. [13], [11, 3.3]): For a C^4-generic metric on a compact manifold either there exists a nonhyperbolic closed geodesic of twist type (then a version of the Birkhoff-Lewis fixed point theorem due to J. Moser [18] implies the existence of infinitely many geometrically distinct closed geodesics) or all closed geodesics are hyperbolic. So far there is no example of a simply-connected compact Riemannian manifold with only hyperbolic closed geodesics. If M is a compact simply-connected manifold rational homotopy equivalent to a CROSS with a metric all of whose

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closed geodesics are hyperbolic then N. Hingston shows in [9] that

$$\liminf n(l)\frac{\log(l)}{l} > 0,$$

where $n(l)$ is the number of geometrically distinct closed geodesics of length $\leq l$. Due to D. Sullivan [19] there are infinitely many rational homotopy types of simply-connected compact manifolds M with $H^*(M;\mathbf{Q}) \cong T_{d,n+1}(x)$ besides the rational homotopy types of CROSS's.

1. Theorem. *If M is a compact simply-connected manifold with $H^*(M;\mathbf{Q}) \cong T_{d,n+1}(x)$, where d is even endowed with a Riemannian metric all of whose closed geodesics are hyperbolic, then there are infinitely many geometrically distinct ones.*

Together with the above quoted theorems we get

2. Corollary. *For a C^4-generic metric on a compact Riemannian manifold with finite fundamental group there are infinitely many geometrically distinct closed geodesics.*

We say a Riemannian metric is *admissible*, if the set of closed geodesics as a subset of ΛM is the disjoint union of nondegenerate critical submanifolds B_k^m, $m \in \mathbf{N}$, $k \in \{1, \cdots, r\}$, with $B_k^m = \{c^m | c \in B_k\}$, and the quotient spaces B_k/S^1 are simply connected. The CROSS's provide examples of admissible metrics; bumpy metrics with only finitely many geometrically distinct closed geodesics are other examples if they exist. The sequence $\mathrm{ind}(c^m)$, $m \in \mathbf{N}$, of the indices of the iterates c^m of a closed geodesic is described by a theorem of R. Bott (cf. [4] or Theorem 1.1) from which the existence of the *average index*

$$\alpha_c = \lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m}$$

follows. For an admissible metric we get for any $k = 1, \cdots, r$ the positive average index $\alpha_k = \alpha_c$, $c \in B_k$, the invariant $\gamma_k = \gamma_c \in \{\pm 1/2, \pm 1\}$, $c \in B_k$, defined by $2\gamma_c \equiv \mathrm{ind}(c^2) - \mathrm{ind}(c) \pmod 2$, $\gamma_c(-1)^{\mathrm{ind}(c)} > 0$ and the Euler characteristic χ_k of B_k/S^1. Then we prove in 3.1(a) using Morse inequalities the following relation between the average indices α_k:

3. Theorem. *If M is a compact simply-connected manifold endowed with an admissible metric, then $H^*(M;\mathbf{Q}) \cong T_{d,n+1}(x)$ and*

$$B(d,n) = \lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m}(-1)^i b_i(\Lambda M/S^1;\mathbf{Q}) = \sum_{k=1}^{r}\frac{\gamma_k}{\alpha_k}\chi_k,$$

where the rational number $B(d,n)$ is an invariant of the rational homotopy type of M.

In 2.5 and 2.6, we compute $B(d,n) = -n(n+1)\,d/(2d(n+1)-4)$ for even d and $B(d,1) = (d+1)/(2d-2)$ for odd d respectively. Hence for an admissible metric the set $\{1, 1/\alpha_1, \cdots, 1/\alpha_r\}$ is linearly dependent over $\mathbf{Q}$. As an application we can estimate the number of geometrically distinct closed geodesics under certain pinching assumptions for the sectional curvature.

4. Corollary. *For a bumpy Riemannian metric on the n-dimensional complex projective space $P^n\mathbf{C}$ with sectional curvature K satisfying $4/(n+1)^2 \leq K \leq 1$ $(n \geq 5)$ and with only finitely many geometrically distinct closed geodesics, there are at least $2n$ geometrically distinct ones of which at least $n(n+1)/(n+7)$ are nonhyperbolic.*

In §4 we show that Theorems 1 and 3 remain valid for admissible Finsler metrics. While there is no example of a bumpy Riemannian metric with only finitely many geometrically distinct closed geodesics, there are such examples of bumpy nonsymmetric Finsler metrics due to A. Katok [10]; the geometry of those metrics is studied by W. Ziller in [23]. We consider these examples on the 2-sphere.

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1. Invariants of closed geodesics

The general references for this chapter are [11, Chapters 1, 2.4 and 3.2] and [2, Chapters 1 and 2]. Let M be a compact Riemannian manifold with metric g. Then

$$\Lambda M = \left\{ c \colon S^1 = \mathbf{R}/\mathbf{Z} \to M | c \text{ absolutely continuous, } \int_{S^1} g(\dot{c}, \dot{c}) < \infty \right\}$$

is the *Hilbert manifold of closed curves* on M. ΛM carries a metric g_1 induced by g and an $\mathrm{O}(2)$-action

$$\mathrm{O}(2) \times \Lambda M \to \Lambda M, \qquad (z, c) \to z \cdot c$$

of isometries since $\mathrm{O}(2)$ acts on S^1. We identify $S^1 = \mathrm{SO}(2) \subset \mathrm{O}(2)$ and we will use only the S^1-action in the following. Let $I(c) = \{z \in S^1 | z \cdot c = c\}$ be the *isotropy group* of $c \in \Lambda M$ with respect to the S^1-action. If $c \in \Lambda M$ is not a fixed point, its *multiplicity* $\mathrm{mul}(c)$ is the order of its finite isotropy group $I(c)$. A curve c with $\mathrm{mul}(c) = 1$ is called *prime*. The differentiable energy functional

$$E \colon \Lambda M \to R, \qquad E(c) = \frac{1}{2} \int_{S^1} g(\dot{c}, \dot{c})$$

is $\mathrm{O}(2)$-invariant and satisfies the condition C of Palais-Smale. The fixed points of the S^1-action are the point curves $\Lambda^0 M = E^{-1}(0)$. The critical points of E are the point curves and the closed geodesics on M (closed geodesics are always assumed to be nonconstant). For a closed geodesic c the *index* $\mathrm{ind}(c)$ is defined to be the index of the Hessian $D^2E(c)$ of the energy functional E at c. Considering the presence of the S^1-action we always have that the nullity of $D^2E(c)$ at a closed geodesic c is at least 1. Therefore the nullity $\mathrm{null}(c)$ of a closed geodesic c is defined to be the nullity of $D^2E(c)$ minus 1. The index and the nullity are constant along an $\mathrm{O}(2)$-orbit $\mathrm{O}(2)\cdot c$ of a closed geodesic c. A closed geodesic c is *nondegenerate* if $\mathrm{null}(c) = 0$. For any $m \in \mathbf{N}$ we define

$$m\colon \Lambda M \to \Lambda M, \qquad c^m(t) = c(mt).$$

If c is a closed geodesic, then c^m is also with $\mathrm{mul}(c^m) = m \cdot \mathrm{mul}(c)$.

Now we will derive estimates for the sequence $\mathrm{ind}(c^m)$, $m \in \mathbf{N}$. Since the tangent vector field $\dot{c}$ of a closed geodesic on M can be viewed as a periodic orbit of the geodesic flow on the tangent bundle TM, we can associate to c the linearized Poincaré map P_c. P_c is a linear endomorphism of $E \oplus E$ where E is the $(n-1)$-dimensional orthogonal complement of $\dot{c}(0)$ in the tangent space $T_{c(0)}M$ at $c(0)$, and P_c is symplectic with respect to the standard symplectic structure on $E \oplus E$. Let $\tilde{P}_c$ be the complexification of P_c, $\tilde{E}$ the complexification of E and $S^1 = \{z \in \mathbf{C} | z\bar{z} = 1\}$ the unit circle in $\mathbf{C}$. Then we have the following index theorem of R. Bott.

1.1. Theorem [4, Theorems A, B]. *Let c be a closed geodesic on a Riemannian manifold M with linearized Poincaré map P_c and let* $N(z) = \dim\ker(\tilde{P}_c - z\mathrm{id})$ *for $z \in S^1$. Then* $\mathrm{null}(c^m) = \sum_{z^m=1} N(z)$, *and the conjugacy class of P_c in the group of linear symplectic maps in $E \oplus E$ determines a function $I\colon S^1 \to \mathbf{N}_0$ up to a constant with the following properties*:

(a) $I(z) = I(\bar{z})$.

(b) *If $N(z) = 0$ (i.e., z is not an eigenvalue of $\tilde{P}_c$), then I is constant nearby z.*

(c) *The splitting numbers $S^{\pm}(z) = lim_{\theta\to\pm 0} I(e^{i\theta}z) - I(z)$ are nonnegative and bounded by $N(z)$.*

(d) $\mathrm{ind}(c^m) = \sum_{z^m=1} I(z)$.

1.2. Now let $(z_j, \bar{z}_j) = (e^{2\pi i a_j}, e^{-2\pi i a_j})$ with $1 \le j \le l-1$, $l \le n$ be the eigenvalues of $\tilde{P}_c$ of modulus 1 with $0 = a_0 < a_1 < \cdots < a_{l-1} \le a_l = \frac{1}{2}$. Set $I_j = I(e^{2\pi i a})$ for $a \in (a_{j-1}, a_j)$, and suppose that if $a_1 = 0$ then $I_1 = 0$, and that if $a_{l-1} = \frac{1}{2}$ then $I_l = 0$. From the definition of the Riemann integral one gets immediately the

1.3. Corollary [4, Corollary 1]. *The average index*

$$\alpha_c = \lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m}$$

is well defined and satisfies

$$\alpha_c = \int_0^1 I(e^{2\pi it})\,dt = 2\sum_{j=1}^{l} I_j(a_j - a_{j-1}).$$

If $\alpha_c = 0$, then $\operatorname{ind}(c^m) = 0$ for all $m \in \mathbf{N}$. Now we estimate the difference $(\operatorname{ind}(c^m) - m\alpha_c)$.

1.4. Theorem. *Let c be a closed geodesic on a Riemannian manifold of dimension n, $S^{\pm}(z)$ the splitting numbers defined in* 1.1, *and* $L(z) = \dim\ker(\tilde{P}_c - z\mathrm{id})^{n-1}$ *the dimension of the generalized eigenspace of the eigenvalue z. Then*

$$\sum_{|z|=1} L(z) = 2L \le 2(n-1), \qquad L \in \mathbf{N},$$

and for all $m \in \mathbf{N}$ we have

$$|\operatorname{ind}(c^m) - m\alpha_c| \le S \le L \le n-1,$$

with

$$S = S^+(1) + \sum_{\substack{|z|=1\\ \operatorname{Im}(z)>0}} \{S^+(z) + S^-(z)\} + S^-(-1).$$

Proof. Let $f(x) = I(e^{2\pi ix})$, and x_i, $1 \le i \le m$, be defined by $x_1 = 0$, $x_{2i} = x_{2i+1} = i/m$ and $x_m = \frac{1}{2}$. If m is even, and $y_i = \frac{i}{2m}$ for $1 \le i \le m$, then

$$\begin{aligned} |\operatorname{ind}(c^m) - m\alpha_c| &= \left|\sum_{i=1}^{m} f(x_i) - 2m\int_0^{1/2} f(x)\,dx\right| \\ &\le 2m\sum_{i=1}^{m}\int_{y_{i-1}}^{y_i} |f(x_i) - f(x)|\,dx \le S. \end{aligned}$$

From [2, 2.13 and the remark at the end of §1] it follows that $S^+(z) = S^-(z) \le L(z)/2 \in \mathbf{N}_0$ if $z = \pm 1$, and $S^+(z) + S^-(z) \le L(z)$ if $z \ne \pm 1$.

1.5. Remarks. (a) Let c be a closed geodesic on M, and $p = c(0)$. Then the loop space $\Omega_p M = \{c \in \Lambda M | c(0) = p\}$ with fixed initial point p is a submanifold of ΛM. Let $E' = E|\Omega_p M$ be the restriction of the energy functional, such that its critical points are the geodesic loops with initial point p. So for a closed geodesic the Ω-index $\operatorname{ind}_\Omega(c)$ is defined as the index of the Hessian $D^2E'(c)$. The Ω-index is constant along the orbit $\mathrm{O}(2)\cdot c$. From the index theorem of M. Morse (cf. [12, 2.5.9]) we get that $\operatorname{ind}_\Omega(c)$ equals the number of conjugate points $c(t_0)$, $0 < t_0 < 1$, of $c(0)$ along $c|[0,1)$ where we count with multiplicities. Since the *concavity* $\operatorname{con}(c)$ satisfies $\operatorname{con}(c) = \operatorname{ind}(c) - \operatorname{ind}_\Omega(c)$ and $0 \le \operatorname{con}(c) \le n-1$ (cf. [2, Chapter 1]), we also have for the average index $\alpha_c = \lim_{m\to\infty}((\mathrm{ind}_\Omega(c^m))/m)$. From [2, 2.7, remark b] it follows that

$$0 \le I(z) - \mathrm{ind}_\Omega(c) \le n-1.$$

Hence $0 \le \alpha_c - \mathrm{ind}_\Omega(c) \le n-1$, and also, in consequence of $\alpha_{c^m} = m\alpha_c$

$$0 \le m\alpha_c - \mathrm{ind}_\Omega(c^m) \le n-1,$$

which implies that

$$-\mathrm{con}(c^m) \le m\alpha_c - \mathrm{ind}(c^m) \le n-1-\mathrm{con}(c^m).$$

(b) If c is a closed geodesic on M and $\dim M = n$ with $P_c = \mathrm{id}$ (e.g. a closed geodesic on a CROSS) then $\mathrm{ind}_\Omega(c^m) = \mathrm{ind}(c^m) = m\alpha_c - (n-1)$ and $\alpha_c \in \mathbf{N}$ since $S^+(1) = S^-(1) = n-1$ (cf. [2, 2.13]). If the symplectic normal form of P_c (using the convention of [2, Chapter 1]) is given by

$$\begin{pmatrix} J_R(z,1,1) & 0 \\ 0 & J_R(z,1,1) \end{pmatrix} \quad \text{with } J_R(z,1,1) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

$z = e^{i\phi}$ and $\phi = 2\pi a$, $a \in (0, \frac{1}{2}) \cap \mathbf{R}\backslash\mathbf{Q}$, then

$$\sup_{m\in\mathbf{N}} (\mathrm{ind}(c^m) - m\alpha_c) = \sup_{m\in\mathbf{N}} (m\alpha_c - \mathrm{ind}(c^m)) = n-1,$$

since $\mathrm{ind}(c^m) - m\alpha_c = (1+[2am]-2am)(n-1)$ where $[x]$ is the largest integer $\le x$. Hence $(n-1)$ is the optimal universal bound for $|\mathrm{ind}(c^m) - m\alpha_c|$, and $m\alpha_c - \mathrm{ind}_\Omega(c^m)$, on an n-dimensional Riemannian manifold.

(c) Since $S \le 2\sum_{j=1}^{l} I_j$ we also get $|\mathrm{ind}(c^m) - m\alpha_c| \le 2\sum_{j=1}^{l} I_j$ which was shown in [22]. The bound $S \le n-1$ depends only on the symplectic normal form of P_c whereas $2\sum_{j=1}^{l} I_j \ge 2\,\mathrm{ind}(c)$.

1.6. Definition. For a closed geodesic c with average index α_c we define the invariants β_c, γ_c by

$$\beta_c = \sup_{m\in\mathbf{N}} |\mathrm{ind}(c^m) - m\alpha_c|, \qquad \gamma_c \in \{\pm\tfrac{1}{2}, \pm 1\},$$

with $\gamma_c(-1)^{\mathrm{ind}(c)} > 0$ and $2\gamma_c \equiv I(-1) = \mathrm{ind}(c^2) - \mathrm{ind}(c) \pmod 2$. Then

$$\mathrm{ind}(c^m) \equiv \frac{1}{2}\left(1 - \frac{\gamma_c}{|\gamma_c|}\right) + 2|\gamma_c|m \pmod 2,$$

and $\beta_c \le \dim M - 1$. A closed geodesic c is *hyperbolic* if none of the eigenvalues of its linearized Poincaré map has modulus 1. Then by 1.1 all iterates c^m, $m \in \mathbf{N}$, are nondegenerate and $\mathrm{ind}(c^m) = m\,\mathrm{ind}(c)$. So the average index $\alpha_c = \mathrm{ind}(c)$ of a hyperbolic closed geodesic c is a nonnegative integer, $\beta_c = 0$, and $\gamma_c = 1$ if $\mathrm{ind}(c)$ is even and $\gamma_c = -\frac{1}{2}$ if $\mathrm{ind}(c)$ is odd. A closed geodesic c is said to be *elliptic* if all eigenvalues of its linearized Poincaré map have modulus 1.

Now we consider closed geodesics on a surface (i.e., $\dim M = 2$). c is *orientable* iff the normal bundle of the immersion $c\colon S^1 \to M$ is orientable. Define $\lambda_c \in \{\pm 1\}$ to be $+1$ iff c is orientable; then $\lambda_{c^m} = \lambda_c^m$. If M is orientable then all closed geodesics are orientable. From [12, 3.4], for an elliptic closed geodesic c on a surface with $\mathrm{null}(c) \neq 1$, it follows that $\mathrm{ind}(c) \equiv (\lambda_c + 1)/2 \pmod 2$. Therefore from 1.4 we get

1.7. Corollary. *Let c be an elliptic closed geodesic on a surface. The average index α_c is irrational iff* $\mathrm{null}(c^m) = 0$ *for all* $m \in \mathbf{N}$. *If* $\mathrm{null}(c) \neq 1$ *and* $\mathrm{null}(c^2) \neq 1$, *then the average index α_c and $\lambda_c \in \{\pm 1\}$ determine the sequence* $\mathrm{ind}(c^m)$, $m \in \mathbf{N}$, *completely. If $2m\alpha_c \notin \mathbf{N}$ we get*

$$\mathrm{ind}(c^m) = 2\left[\frac{[m\alpha_c]}{2} + \frac{1-\lambda_c^m}{4}\right] + \frac{1+\lambda_c^m}{2},$$

and $\mathrm{ind}_\Omega(c^m) = [m\alpha_c]$ *($[x]$ is the largest integer $\leq x$).*

Proof. Using the convention of [2] we get as possible symplectic normal forms for P_c with eigenvalues $z_1 = e^{2\pi i a}$, $a_1 \in [0, \frac{1}{2}]$,

$$z_1 = \pm 1, \qquad \begin{pmatrix} z_1 & 0 \\ \sigma & z_1 \end{pmatrix}, \qquad \sigma \in \{0, \pm 1\},$$

$$z_1 \neq \pm 1, \qquad \begin{pmatrix} \cos 2\pi a_1 & -\sigma \sin 2\pi a_1 \\ \sigma \sin 2\pi a_1 & \cos 2\pi a_1 \end{pmatrix}, \qquad \sigma \in \{\pm 1\}.$$

If $z_1 = \pm 1$, then

$$S^+(z_1) = S^-(z_1) = \begin{cases} 1, & \sigma = 0, 1, \\ 0, & \sigma = -1. \end{cases}$$

Since $\mathrm{null}(c)$, $\mathrm{null}(c^2) \neq 1$, we have $\sigma = 0$ and therefore

$$\mathrm{ind}(c^m) = I(1) + (m-1)I_2 = m\alpha_c - 1,$$

for $z_1 = 1$, and

$$\mathrm{ind}(c^{2m}) = 2m\alpha_c - 1, \quad \mathrm{ind}(c^{2m+1}) = (2m+1)\alpha_c,$$

for $z_1 = -1$. Now assume $z_1 = e^{2\pi i a_1}$, $a_1 \in (0, \frac{1}{2})$. Then

$$I_1 - I_2 = S^-(z_1) - S^+(z_1) = -\sigma, \qquad \alpha_c = I_2 - 2a_1\sigma.$$

From 1.1 we get that $\mathrm{null}\, c^m = 0$ for all $m \in \mathbf{N}$ iff $a_1 \in \mathbf{R}\backslash\mathbf{Q}$. If $2ma_1 \in \mathbf{N}$, then the symplectic normal form of P_{c^m} is $\pm\mathrm{id}$, and therefore $\mathrm{ind}(c^m)$ is determined by $\alpha_{c^m} = m\alpha_c$ as shown above. If $2ma_1 \notin \mathbf{N}$, from 1.4 it follows that $\mathrm{ind}(c^m)$ is determined by the conditions $|\mathrm{ind}(c^m) - m\alpha_c| \leq 1$ and $\mathrm{ind}(c^m) \equiv (\lambda_c^m + 1)/2 \pmod 2$ since $m\alpha_c \notin \mathbf{N}$. From 1.5 we also get $\mathrm{ind}_\Omega(c^m) = [m\alpha_c]$ in this case.

1.8. Remark. G. A. Hedlund [8] proved this result for the case $\mathrm{null}(c^m) = 0$ for all $m \in \mathbf{N}$. Then $m\alpha_c \notin \mathbf{N}$, so $\mathrm{ind}(c^m)$ is uniquely determined by $|\mathrm{ind}(c^m) - m\alpha_c| \leq 1$ and $\mathrm{ind}(c^m) \equiv (\lambda_c^m + 1)/2 \pmod 2$.

2. The Morse inequalities and the space $\Lambda M/S^1$

2.1. In the following we want to apply Morse theory to the quotient space $\Lambda M/S^1$ using the S^1-invariant energy functional $E\colon \Lambda M \to \mathbf{R}$ which is defined on the S^1-Hilbert manifold of closed curves introduced in §1. Therefore we need some generic assumptions on the metric g on M:

A connected submanifold B (without boundary) of ΛM is a *nondegenerate critical submanifold of constant multiplicity* if all points of B are critical points of E, $E(B) = a \in \mathbf{R}$, the index, nullity and multiplicity are constant along B and $\mathrm{null}(c) = \dim B - 1$ (so we can write $\mathrm{ind}(B)$, $\mathrm{null}(B)$ and $\mathrm{mul}(B)$). Since $\mathrm{null}(B) = \mathrm{null}(B^m)$ the linearized Poincaré map P_c of $c \in B$ can only have 1 or $e^{2\pi i a}$, $a \in \mathbf{R}\backslash\mathbf{Q}$, as an eigenvalue of modulus 1. If c is a nondegenerate closed geodesic (i.e., $\mathrm{null}(c) = 0$), then the orbit $\mathrm{O}(2) \cdot c$ consists of two critical circles of the same index and multiplicity. A metric g is *bumpy* if all closed geodesics are nondegenerate. As a generalization of the case of a bumpy metric with only finitely many geometric distinct closed geodesics (which may not exist), which includes the CROSS's, we introduce the following notion. We say a Riemannian metric g is *admissible* if the set of closed geodesics as a subset of ΛM is the union of disjoint nondegenerate critical submanifolds B_k^m, $k = 1, \cdots, r$; $m \in \mathbf{N}$, of constant multiplicity with $B_k^m = \{c^m | c \in B_k\}$, $B_k^1 = B_k$, where the quotient spaces B_k/S^1 are simply-connected. Then for each $k = 1, \cdots, r$ the invariants $\alpha_k = \alpha_c$, $\beta_k = \beta_c$, $\gamma_k = \gamma_c$ are defined for any $c \in B_k$. Since the Palais-Smale condition holds, the submanifolds B_k^m are compact. If M is a simply-connected manifold with an admissible metric, then it follows as in the proof of the theorem of Gromoll-Meyer that $\alpha_k > 0$ for all $k = 1, \cdots, r$ and that the sequence $b_i(\Lambda M; F)$ of Betti numbers is bounded for any field F. Hence the rational cohomology algebra $H^*(M; \mathbf{Q})$ has exactly one generator, i.e., is isomorphic to a truncated polynomial algebra $T_{d,n+1}(x)$ with a generator x of degree d and height $(n+1)$, i.e., $\dim M = nd$ ($T_{d,n+1}(x)$ is the quotient of the polynomial algebra $\mathbf{Q}[x]$ by the ideal (x^{n+1})) as shown in [21].

For a S^1-space X we denote by $\overline{X}$ the quotient space X/S^1. For each $a \in \mathbf{R}$ let $\Lambda^a M = \{c \in \Lambda M | E(c) \leq a\}$. Let (X, Y) be a space pair and F be a field, such that the Betti numbers $b_i = b_i(X, Y; F) = \dim H_i(X, Y; F)$ are finite for all $i \in \mathbf{N}_0$. We call the (formal power) series $P(X, Y; F)(t) = \sum_{i=0}^{\infty} b_i t^i$ the

Poincaré series of (X, Y) with respect to F. We call the set

$$V = \{B_k^m | k = 1, \cdots, r,\ m \in \mathbf{N},\ m \equiv 1 (\operatorname{mod} 2) \text{ or } |\gamma_k| = 1\}$$

(i.e., $B_k^m \in V$ iff $\operatorname{ind}(B_k^m) \equiv \operatorname{ind}(B_k) \pmod 2$) the set of *homologically visible critical submanifolds* since the following holds.

2.2. Proposition. *Let $a_1 < a_2$ be two regular values of the energy functional and let a be the only critical value in (a_1, a_2). Then*

$$P(\bar{\Lambda}^{a_2} M, \bar{\Lambda}^{a_1} M; \mathbf{Q})(t) = \sum_{\substack{B \in V \\ E(B) = a}} t^{\operatorname{ind}(B)} P(\overline{B}; \mathbf{Q})(t).$$

Proof. Let $B = B_k^m$, $k = 1, \cdots, r$, be any critical submanifold with $E(B) = a$, and $N(B)$ the negative normal bundle of B which is a S^1-Riemannian vector bundle of dimension $\operatorname{ind}(B)$. On each fiber the S^1-action induces an orthogonal $\mathbf{Z}_m$-action. Let $DN(B)$ (resp. $SN(B)$) be the associated disc (resp. sphere) bundle. Then

$$P(\bar{\Lambda}^{a_2} M, \bar{\Lambda}^{a_1} M; \mathbf{Q})(t) = \sum_{E(B) = a} P(\overline{D}N(B), \overline{S}N(B); \mathbf{Q})(t),$$

(cf. [11, Chapter 2.4]). $\overline{D}N(B)$ (resp. $\overline{S}N(B)$) is a bundle over $\overline{B}$ with fiber $D^i/\mathbf{Z}_m$ (resp. $S^{i-1}/\mathbf{Z}_m$), where $D^i = \{x \in \mathbf{R}^i | \|x\| \le 1\}$, $S^{i-1} = \{x \in \mathbf{R}^i | \|x\| = 1\}$, $i = \operatorname{ind}(B)$. Let T be a generator of $\mathbf{Z}_m$. Then $\mathbf{Z}_m$ acts on a fiber D^i of the S^1-disc bundle $DN(B)$ over B. The dimension of the subspace of D^i on which T acts as $-$identity is odd, iff m is even and $|\gamma_k| = \frac{1}{2}$ since this dimension is given by $I(-1) = \operatorname{ind}(B_k^2) - \operatorname{ind}(B_k)$ (cf. [20], [11, 4.1]). Therefore

$$P(D^i/\mathbf{Z}_m, S^{i-1}/\mathbf{Z}_m; \mathbf{Q})(t) = \begin{cases} t^i, & \text{if } m \equiv 1 \pmod 2 \text{ or } |\gamma_k| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and hence

$$P(\overline{D}N(B), \overline{S}N(B); \mathbf{Q})(t) = \begin{cases} t^i P(\overline{B}; \mathbf{Q})(t), & \text{if } m \equiv 1 \pmod 2 \text{ or } |\gamma_k| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

using the Thom isomorphism in the first case ($\overline{B}$ is simply-connected by definition). q.e.d.

Let M be a compact simply-connected Riemannian manifold with an admissible metric. From 1.4 and 2.1 it follows that for any $N \in \mathbf{N}$ there are only finitely many m with $\operatorname{ind}(B_k^m) \le N$ for each k. Let $(c_l)_{l \ge 0}$ be the sequence of positive critical values of the energy functional with $c_l < c_{l+1}$, and let $(a_l)_{l \ge 0}$ be a sequence with $a_0 = 0$, $a_l < c_l < a_{l+1}$ for all $l \ge 0$. Then the *Morse series*

$M_{E,\mathbf{Q}}(t) = M(t)$ of the energy functional E of the space $\bar{\Lambda}M$ for rational coefficients is defined by

$$M(t) = \sum_{l=0}^{\infty} P(\bar{\Lambda}^{a_{l+1}}M, \bar{\Lambda}^{a_l}M; \mathbf{Q})(t).$$

Using 2.2 we get

$$M(t) = \sum_{B\in V} t^{\operatorname{ind}(B)} P(\overline{B}; \mathbf{Q})(t).$$

Then there is a series $Q(t) = \sum_{i=0}^{\infty} q_i t^i$ with nonnegative integer coefficients q_i such that

$$M(t) = P(\bar{\Lambda}M, \bar{\Lambda}^0 M; \mathbf{Q})(t) + (1+t)Q(t).$$

This is a version of the "Morse inequalities" (cf. 2.3(a)) which follows from the exactness of long homology sequences of the filtration $(\bar{\Lambda}^{a_l}M)_{l\geq 0}$.

2.3. Remarks. (a) If $R(t) = \sum_{i=0}^{\infty} r_i t^i$ is a (formal power) series and $m \in \mathbf{N}$, then we define the polynomial $R^m(t) = \sum_{i=0}^{m} r_i t^i$. If $M(t) = \sum_{i=0}^{\infty} w_i t^i$, and $P(t) = P(\bar{\Lambda}M, \bar{\Lambda}^0 M; \mathbf{Q})(t) = \sum_{i=0}^{\infty} b_i t^i$, then we can write the Morse equality also in the form:

$$w_i = b_i + q_i + q_{i-1}, \qquad i \in \mathbf{N}_0,$$

or

$$(*) \qquad (-1)^m q_m = M^m(-1) - P^m(-1),$$

which is equivalent to the usual form of the Morse inequalities:

$$\sum_{i=0}^{m} (-1)^{m-i} w_i \geq \sum_{i=0}^{m} (-1)^{m-i} b_i.$$

(b) A series $R(t) = \sum_{i=0}^{\infty} r_i t^i$ is said to be *lacaunary* if either $r_{2i} = 0$ for all $i \in \mathbf{N}_0$ or $r_{2i+1} = 0$ for all $i \in \mathbf{N}_0$. Let for $|\gamma_k| = 1$

$$M_k'(t) = \sum_{m=1}^{\infty} t^{\operatorname{ind}(B_k^m)},$$

and for $|\gamma_k| = \frac{1}{2}$

$$M_k'(t) = \sum_{m=1}^{\infty} t^{\operatorname{ind}(B_k^{2m-1})},$$

i.e., $M_k'(t) = \sum_{B_k^m \in V} t^{\operatorname{ind}(B_k^m)}$, and hence $M_k'(t)$ is lacaunary. Set $M_k(t) = P(\overline{B}_k; \mathbf{Q})(t) M_k'(t)$. Then the Morse series $M(t)$ is given by $M(t) = \sum_{k=1}^{r} M_k(t)$.

The energy functional E is *perfect* if $M_{E,\mathbf{Q}}(t) = P(t)$, i.e., if $Q(t) = 0$. If the Morse series is lacaunary, then E is perfect.

(c) Let $M_k(t) = \sum_{i=0}^{\infty} w_{k,i} t^i$. Then from the estimate $|\operatorname{ind}(B_k^m) - m\alpha_k| \leq \beta_k \leq \dim M - 1$ (cf. 1.6) we get

$$w_{k,i} \leq \frac{2\beta_k + \dim \overline{B}_k}{\alpha_k} + 1.$$

Since $\sum_{k=1}^{r} w_{k,i} = b_i + q_i + q_{i-1}$ and $b_i, q_i \geq 0$, the sequence, $(q_i)_{i \geq 0}$ is bounded.

(d) In our main theorem 3.1 we use 2.3(a) $(*)$, hence we need estimates for $M_k^m(-1)$. Since $|\operatorname{ind}(B_k^m) - m\alpha_k| \leq \beta_k$ and

$$M_k'^m(-1) = (-1)^{\operatorname{ind}(B_k)} \#\{l \mid \operatorname{ind}(B_k^l) \leq m,\ l \text{ is odd or } |\gamma_k| = 1\},$$

we get

$$\left| M_k'^m(-1) - m\frac{\gamma_k}{\alpha_k} \right| \leq |\gamma_k| \left(\frac{\beta_k}{\alpha_k} - 1 \right) + 2,$$

$$\left| M_k^m(-1) - m\frac{\gamma_k \chi_k}{\alpha_k} \right| \leq \left\{ |\gamma_k| \left(\frac{\beta_k + \dim \overline{B}_k}{\alpha_k} - 1 \right) + 2 \right\} P(\overline{B}_K; \mathbf{Q})(1),$$

where $\chi_k = P(\overline{B}_k; \mathbf{Q})(-1)$ is the Euler characteristic of $\overline{B}_k$.

For the study of admissible metrics we need the homology of $\bar{\Lambda} M$:

2.4. Theorem. *If M is a compact simply-connected manifold with $H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x)$, and d is even, then the Poincaré series of $(\bar{\Lambda} M, \bar{\Lambda}^0 M)$ (for homology with rational coefficients) is given by*

$$P(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q})(t) = t^{d-1} \left(\frac{1}{1 - t^2} + \frac{t^{d(n+1)-2}}{1 - t^{d(n+1)-2}} \right) \frac{1 - t^{dn}}{1 - t^d}.$$

Proof. At first we remark that $H^*(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q}) \cong H^*_{S^1}(\Lambda M, \Lambda^0 M; \mathbf{Q})$ where H_{S^1} is the S^1-equivariant cohomology (cf. [9]), since $\Lambda^a M$ for any $a > 0$ is S^1-homotopy equivalent to a S^1-space X where the multiplicities of the points which are not fixed points are bounded.

Let $E(x_1, \cdots, x_l)$ denote the free algebra over $\mathbf{Q}$ generated by the elements $x_1, \cdots, x_l$, i.e., $E(x_1, \cdots, x_l)$ is the tensor product of the polynomial algebra generated by the elements x_k, $1 \leq k \leq l$, of even degree and the exterior algebra generated by the elements x_k, $1 \leq k \leq l$, of odd degree. The minimal model for M is given by $E(x, y)$ with $\deg x = d$, $\deg y = d(n+1) - 1$ and the differential d_0 with $d_0 x = 0$, $d_0 y = x^{n+1}$ (cf. [21, add.]). Using [7, Example 2, Chapter 5] we get (E, d_1) as the model for the homotopy quotient ΛM_{S^1}: $E = E(e, x, \bar{x}, y, \bar{y})$ with $\deg e = 2$, $\deg x = \deg \bar{x} + 1 = d$; $\deg y = \deg \bar{y} + 1 = d(n+1) - 1$ and the differential d_1: $d_1 e = 0$; $d_1 x = -e\bar{x}$; $d_1 y = x^{n+1} - e\bar{y}$; $d_1 \bar{x} = 0$; $d_1 \bar{y} = -(n+1)x^n \bar{x}$. Let F be the ideal of E generated by the exterior generators $\bar{x}$ and y. Then the image of $d_1 y$ in E/F is nonzero, and therefore $H^*(E, d_1) \cong H^*(E', d_1)$ with $E' = E/(y, dy)E$ (see Proposition 2 of [21]).

So we can set $E' = E(e,x,\bar{x},\bar{y})/(x^{n+1} = e\bar{y})$ with the differential $d_1 e = 0$; $d_1 x = -e\bar{x}$; $d_1\bar{x} = 0$; $d_1\bar{y} = -(n+1)x^n\bar{x}$. Hence $\{e^r | r \geq 0\} \cup \{x^p\bar{x}\bar{y}^q | 0 \leq p \leq n-1, q \geq 0\}$ is a set of additive generators of $H^*(E', d_1) \cong H^*_{S^1}(\Lambda M; \mathbf{Q})$ with Poincaré series

$$P_{S^1}(\Lambda M; \mathbf{Q})(t) = \frac{1}{1-t^2} + \frac{t^{d-1}}{1-t^{d(n+1)-2}} \frac{1-t^{dn}}{1-t^d}.$$

Since $\Lambda^0 M$ is the fixed point set of the S^1-action on ΛM, we have $H^*_{S^1}(\Lambda^0 M; \mathbf{Q}) \cong \mathbf{Q}[e] \otimes T_{d,n+1}(x)$ with $\deg e = 2$, and the homomorphism $H^{2k}_{S^1}(\Lambda M; \mathbf{Q}) \to H^{2k}_{S^1}(\Lambda^0 M; \mathbf{Q})$ induced by the inclusion is injective for all $k \geq 0$. Therefore the claim follows from the exact long cohomology sequence of $(\Lambda M, \Lambda^0 M)$.

2.5. Remarks. (a) Let M be a simply-connected compact manifold. The Poincaré series of $H^*(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q})$ are computed for M rational homotopy equivalent to a sphere or a product of odd-dimensional sphere in [20, p. 32] and by using equivariant Morse theory for the standard metrics for M rational homotopy equivalent to a sphere or a projective space in [9, p. 104]. For $H^*(M; \mathbf{Q}) \cong H^*(S^d; \mathbf{Q}) \cong T_{d,2}(x)$ with d odd one gets

$$P(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q})(t) = t^{d-1}\left(\frac{1}{1-t^2} + \frac{t^{d-1}}{1-t^{d-1}}\right).$$

(b) For each $d', n \in \mathbf{N}$ with $d'n \equiv 1 \pmod 2$ there is a simply connected compact manifold M with $H^*(M; \mathbf{Q}) \cong T_{2d',n+1}(x)$ (cf. [19, Theorem 13.2]), so there are infinitely many rational homotopy types of compact simply-connected manifolds with only one generator for $H^*(M; \mathbf{Q})$ besides the rational homotopy types of a sphere or a projective space. Therefore for these homotopy types there is a prime field $\mathbf{Z}_p$ such that $H^*(M; \mathbf{Z}_p)$ has more than one generator. So far there is no analogue for prime fields of the theorem of Vigue-Poirrier and Sullivan [21]. Hence we cannot conclude that the sequence of Betti numbers $b_i(\Lambda M; \mathbf{Z}_p)$ is unbounded. This would be necessary to apply the theorem of Gromoll-Meyer [6] on the existence of infinitely many geometrically distinct closed geodesics for any metric on M. The Betti numbers $b_i(\Omega M; \mathbf{Z}_p)$ of the loop space ΩM are in this case unbounded as shown by McCleary [15].

2.6. Corollary. *Let M be a simply-connected compact manifold with $H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x)$.*

(a) *For*

$$\begin{aligned} B(d,n) &= \lim_{m\to\infty} \frac{1}{m} P^m(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q})(-1) \\ &= \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m} (-1)^i b_i(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q}), \end{aligned}$$

we get

$$B(d,n) = \begin{cases} \frac{n(n+1)\,d}{2d(n+1)-4}, & d \text{ even},\\ \frac{d+1}{2(d-1)}, & d \text{ odd (then } n=1). \end{cases}$$

(b) *If d is even we get, for $j \in \mathbf{N}$,*

$$\begin{aligned} P^{bj}(\bar\Lambda M, \bar\Lambda^0 M; \mathbf{Q})(-1) &= \sum_{i=0}^{bj} (-1)^i b_i(\bar\Lambda M, \bar\Lambda^0 M; \mathbf{Q}) \\ &= bjB(d,n) + \frac{1}{4} n(n+1)\,d, \end{aligned}$$

with $b = d(n+1) - 2$.

2.7. Let M be a simply-connected Riemannian manifold with a bumpy metric and $H^*(M;\mathbf{Q}) \cong T_{d,n+1}(x)$. Since the metric is bumpy, all coefficients of the Morse series are even. Since $b_{d-1}(\bar\Lambda M, \bar\Lambda^0 M; \mathbf{Q}) = 1$, it follows from the Morse inequalities that there are two prime closed geodesics c_1, c_2 with $\operatorname{ind}(c_1) = \operatorname{ind}(c_2) - 1 \le d - 1$; this is a special case of a theorem of Fet [5]. Hence the energy functional in this case is not perfect.

3. Admissible metrics

For an admissible metric as defined in 2.1 the set of prime closed geodesics is the union of finitely many disjoint compact manifolds B_k, $k = 1, \cdots, r$. Since the invariants $\alpha_c, \beta_c, \gamma_c$ are the same for any $c \in B_k$, we can assign to each k the positive average index α_k and the invariants $\beta_k \ge 0$ and $\gamma_k \in \{\pm\frac{1}{2}, \pm 1\}$. Let χ_k be the Euler characteristic of $\overline{B}_k = B_k/S^1$.

Theorems 1 and 3 in the introduction are then included in the following main theorem.

3.1. Theorem. *If M is a simply-connected compact Riemannian manifold with an admissible metric, then $H^*(M;\mathbf{Q}) \cong T_{d,n+1}(x)$ and the following hold.*

(a) *Let $B(d,n)$ be the topological invariant introduced in* 2.6 (*depending only on d, n*). *Then*

$$B(d,n) = \sum_{k=1}^{r} \frac{\gamma_k \chi_k}{\alpha_k}.$$

(b) *If d is even, then*

$$\sum_{\substack{\beta_k > 0 \\ \text{or } \dim \overline{B}_k > 0}} \left\{ |\gamma_k| \left(\frac{\beta_k + \dim \overline{B}_k}{\alpha_k} - 1 \right) + 2 \right\} P(\overline{B}_k; \mathbf{Q})(1) \ge \frac{1}{4} n(n+1)\,d,$$

in particular there is a nonhyperbolic closed geodesic.

Proof. (a) Assume that $B_k \cup B_{k+s} = \mathrm{O}(2) \cdot c_k$ for $k = 1, \cdots, s$ with c_k a prime closed geodesic with $\beta_k = 0$. Then we have $\operatorname{ind}(c_k^m) = m\alpha_k$ for $k = 1, \cdots, s$, and $M_k^{jN}(-1) = \gamma_k jN/\alpha_k$ for the number $M_k^{jN}(-1)$ with $N = 2\prod_{k=1}^{s} \alpha_k$, where $M_k(t)$ is the series introduced in 2.3(b) (giving the contribution of B_k and its iterates B_k^m, $m \geq 1$, to the Morse series $M(t) = \sum_{k=1}^{r} M_k(t)$). For $k = 2s+1, \cdots, r$, from 2.3(d) for all $j \geq 1$ with

$$\varepsilon_k = \left\{ |\gamma_k| \left(\frac{\beta_k + \dim \overline{B}_k}{\alpha_k} - 1 \right) + 2 \right\} P(\overline{B}_k; \mathbf{Q})(1),$$

it follows that

$$|M_k^{jN}(-1) - jN \frac{\gamma_k \chi_k}{\alpha_k}| \leq \varepsilon_k.$$

Let $P(t) = P(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q})(t)$. Then from 2.3(a) (*) we get

$$q_{jN} = M^{jN}(-1) - P^{jN}(-1),$$

and hence

$$\left| q_{jN} - jN \left(\sum_{k=1}^{r} \frac{\gamma_k \chi_k}{\alpha_k} - \frac{P^{jN}(-1)}{jN} \right) \right| \leq \sum_{k=2s+1}^{r} \varepsilon_k.$$

Since $(q_j)_{j \geq 0}$ is bounded (cf. 2.3(c)), and $B(d,n) = \lim_{j \to \infty} (P^j(-1)/j)$, we get (a) for $j \to \infty$.

(b) If d is even with $b = d(n+1) - 2$, we have

$$P^{bN}(-1) = bNB(d,n) + \frac{n(n+1)\,d}{4}$$

from 2.6, and

$$q_{bN} \leq bN \left(\sum_{k=1}^{r} \frac{\gamma_k \chi_k}{\alpha_k} - B(d,n) \right) + \sum_{k=2s+1}^{r} \varepsilon_k - \frac{n(n+1)\,d}{4}$$

from (a). Hence $q_{bN} \geq 0$ yields

$$\sum_{k=2s+1}^{r} \varepsilon_k \geq \frac{n(n+1)\,d}{4}.$$

3.2. Remarks. (a) In 3.1(b) we need the additional assumption that d is even, (i.e., that M is not rational homotopy equivalent to an odd-dimensional sphere, since for d odd $-P^{2(d-1)N}(-1) \geq B(d,1)2(d-1)N$, and we do not get an estimate for the number $n(l)$ of geometric distinct closed geodesics with length $\leq l$ as N. Hingston gives in [9, 6.2].

(b) One can generalize 3.1(a) as follows: If M is a compact simply-connected Riemannian manifold with a bumpy metric where $\bigcup_{k \in \mathbf{N}} \mathrm{O}(2) \cdot c_k$ is the set of prime closed geodesics such that

$$a(N) = \#\{k | \operatorname{ind}(c_k) \leq N\}$$

is finite for all N. Hence for $k \in \mathbf{N}$, $\alpha_k = \alpha_{c_k} > 0$ which is satisfied for metrics of positive sectional curvature. Moreover, if $\gamma_k = \gamma_{c_k}$, then

$$\left| 2 \sum_{\operatorname{ind}(c_k) \leq N} \frac{\gamma_k}{\alpha_k} - \frac{1}{N} \sum_{i=0}^{N} (-1)^i b_i(\bar{\Lambda} M, \bar{\Lambda}^0 M; \mathbf{Q}) \right| \leq \frac{c a_N}{N},$$

with $c = 4(\dim(M) - 1)/\min(\alpha_k) + 6$.

3.3. Example. If M is a simply-connected CROSS with $H^*(M; \mathbf{Q}) = T_{d,n+1}(x)$, i.e., a sphere or a projective space with the standard metric, then the metric is admissible, and the set of prime closed geodesics can be identified with the unit tangent bundle (i.e., $r = 1$). The Morse series is lacaunary, hence it equals the Poincaré series (cf. 2.3(b)). We get $\alpha_1 = d(n+1)-2$, $\beta_1 = nd-1$, $\gamma_1 = (-1)^{d-1}$, $\chi_1 = n(n+1)\,d/2$ for even d and $\chi_1 = d+1$ for odd d (cf. e.g. [9, p. 104]); hence we can compute $B(d, n)$ using 3.1(a).

Given a lower bound for the average index we can use 3.1 to estimate the number of geometric distinct closed geodesics:

3.4. Corollary. *Let M be a simply-connected compact manifold with $H^*(M; \mathbf{Q}) \cong T_{d,n+1}(x)$ endowed with a bumpy metric with sectional curvature K satisfying $0 < \delta^2 \leq K \leq 1$ for $\delta \in \mathbf{Q}$ and $\delta \geq \frac{1}{2}$ if d is odd. Then we get $\alpha_c \geq 2\delta(nd - 1)$ for the average index α_c of any closed geodesic c, and the following hold.*

(a) *There are at least $[|B(d,n)|\delta(nd-1)]'$ geometric distinct prime closed geodesics, which have index congruent to $(d-1)$ modulo 2, and one prime closed geodesic c with $d \geq \operatorname{ind}(c) \equiv d \pmod 2$ ($[x]'$ is the smallest integer $\geq x$).*

(b) *If d is even, and there are only finitely many closed geodesics, then there are at least*

$$\left[\frac{1}{3 + \delta^{-1}} \frac{n(n+1)\,d}{4} \right]'$$

nonhyperbolic geometric distinct closed geodesics.

Proof. From [12, 2.6.9 and 2.6.10] we get $L(c) \geq 2\pi$ for the length $L(c)$ of a closed geodesic c on M. Therefore if $\delta = p/q$, then $L(c^{qj}) \geq 2\pi qj = \pi 2jp/\delta$, and hence the comparison theorem of Morse-Schoenberg [12, 2.6.2] gives $\operatorname{ind}(c^{qj}) \geq 2jp(nd-1)$ which implies $\alpha_c \geq 2\delta(nd-1)$.

(a) Let $B_k \cup B_{k+r} = \mathrm{O}(2) \cdot c_k$, $k = 1, \cdots, r$, be the prime critical submanifolds. Then from 3.1(a) we get

$$0 < (-1)^{d+1} B(d,n) = 2(-1)^{d+1} \sum_{k=1}^{r} \frac{\gamma_k}{\alpha_k}$$
$$\leq \sum_{\substack{k=1 \\ \operatorname{ind} c_k \equiv d-1 \ (\mathrm{mod}\, 2)}}^{r} \frac{1}{\delta(nd-1)} = \frac{r}{\delta(nd-1)}.$$

The existence of a prime closed geodesic c with $\operatorname{ind}(c) \equiv d \pmod 2$ and $\operatorname{ind}(c) \leq d$ follows from 2.7.

(b) From 3.1(b) we get that if $\mathrm{O}(2) \cdot c_k$, $k = 1, \cdots, s$, are the prime critical submanifolds with $\beta_k > 0$, then

$$3s + 2\sum_{k=1}^{s} \frac{\beta_k}{\alpha_k} \geq 3s + 2\frac{nd-1}{2\delta(nd-1)}$$
$$= s(3+\delta^{-1}) \geq \frac{n(n+1)\,d}{4}. \quad \text{q.e.d.}$$

Corollary 4 in the introduction is a special case of 3.4. Using Lusternik-Schnirelmann theory in [3] there are estimates for the number of closed geodesics and their length under certain curvature assumptions without generic assumptions on the metric.

4. Finsler metrics

Let M be a compact (differentiable) manifold, and TM its tangent bundle. A function $F\colon TM \to \mathbf{R}$ which is differentiable outside the zero section such that the second derivative of F^2 in the direction of the fiber is positive definite and $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$, and $x \in TM$ is called a *Finsler metric* on M. A Finsler metric is *symmetric* if $F(x) = F(-x)$ also holds for all $x \in TM$. If g is a Riemannian metric on M, then $F(x) = \sqrt{g(x,x)}$ defines a symmetric Finsler metric. Conversely if F is a Finsler metric which is C^2 at the zero section, then there is such a Riemannian metric. For a piecewise differentiable curve $c\colon [a,b] \to M$ the length $L(c) = \int_a^b F(\dot c(t))\,dt$ is defined, and $\delta\colon M \times M \to \mathbf{R}$, $\delta(p,q) = \inf\{L(c) | c(a) = p,\ c(b) = q\}$ defines a pseudodistance on M, which is a distance if F is symmetric (for a nonsymmetric Finsler metric the length of a curve depends on the orientation). Let g be any Riemannian metric on M, and ΛM the Hilbert manifold of closed curves on M introduced in §1 with the metric g_1 induced by g. On ΛM we have the canonical $\mathrm{O}(2)$-action by isometries, but for a nonsymmetric Finsler metric F

only the S^1-action leaves the energy functional

$$E\colon \Lambda M \to \mathbf{R}, \qquad E(c) = \frac{1}{2}\int_{S^1} F^2(\dot{c}(t))\,dt$$

invariant. E is a C^1-function with locally Lipschitzian differential (cf. [16]), and the critical points of E are the point curves $\Lambda^0 M$ and the closed geodesics of the Finsler metric F on M. As in the Riemannian case there is an $\varepsilon > 0$ such that for points p, q with $\delta(p,q) < \varepsilon$, there is a unique minimal geodesic $c\colon [0,1] \to M$ with $c(0) = p$, $c(1) = q$. Since the energy functional is twice differentiable at its critical points, the index and nullity of a closed geodesic are defined as in the Riemannian case. The index theorem of Bott (1.1) applies, the invariants $\alpha_c, \beta_c, \gamma_c$ are also defined, and the results of §1 remain valid for Finsler metrics. Since E is C^2 at its critical points, we can use Morse theory as shown in [17], and so 2.1–2.3 as well as 3.1 remain valid for Finsler metrics.

Two closed geodesics $c_1, c_2\colon S^1 \to M$ of a Finsler metric F on M are *geometrically distinct* if $c_1(S^1) \neq c_2(S^1)$ or if F is not symmetric, and $c_1(S^1) = c_2(S^2)$, but the orientations of c_1 and c_2 are different. There are examples of bumpy nonsymmetric Finsler metrics with only finitely many geometrically distinct closed geodesic on spheres and projective spaces; these examples are due to A. Katok [10] and are studied in detail in [23]:

4.1. Example (Metrics on the two-sphere S^2). Let $\lambda \in (0,1) \cap \mathbf{R}\backslash\mathbf{Q}$. Then there is on S^2 a bumpy nonsymmetric Finsler metric F_λ with only two closed geodesics c_1 and c_2 (which differ only by orientation) with lengths $L(c_1) = 2\pi/(1+\lambda)$ and $L(c_2) = 2\pi/(1-\lambda)$, and the conjugate points of $c_1(0)$ and $c_2(0)$ respectively along $c_1, c_2\colon \mathbf{R} \to M$ occur at $t = k\pi$, $k \in \mathbf{N}$. Therefore we get $\mathrm{ind}_\Omega(c_1^k) = [2k/(1+\lambda)]$ and $\mathrm{ind}_\Omega(c_2^k) = [2k/(1-\lambda)]$, and hence the average indices (cf. 1.5)

$$\alpha_1 = \alpha_{c_1} = \frac{2}{1+\lambda}, \qquad \alpha_2 = \alpha_{c_2} = \frac{2}{1-\lambda}.$$

Since $\alpha_1, \alpha_2 \in \mathbf{R}\backslash\mathbf{Q}$, c_1 and c_2 are elliptic, and $\mathrm{ind}(c_1^m)$ and $\mathrm{ind}(c_2^m)$ are odd for all m (cf. 1.7). Hence from 3.1 we get, in consequence of $\gamma_1 = \gamma_2 = -1$,

$$B(2,1) = \frac{-1}{\alpha_1} + \frac{-1}{\alpha_2} = -1.$$

Since the Morse polynomial in this case is lacaunary, the energy functional is perfect. From 1.7 and 3.1 it follows that a bumpy Finsler metric on S^2 with only finitely many closed geodesics has at least two elliptic closed geodesics. By 3.1(b) we see that there is one elliptic closed geodesic, and since its average index is irrational (cf. 1.7) it follows from 3.1(a) that there is another elliptic closed geodesic. Since 2.7 remains valid for symmetric Finsler metrics, a bumpy symmetric Finsler metric on S^2 has at least three closed geodesics. For the nonsymmetric Finsler metrics F_λ on S^2 we get $\operatorname{ind}(c_2) \to \infty$ for $\lambda \to 1$. Therefore the method of the proof of the theorem of Lusternik and Schnirelmann (cf. [14], [1]) cannot be used to prove the existence of two closed geodesics for a nonsymmetric Finsler metric since one does not know a priori which homology class remains hanging at c_2. For a bumpy nonsymmetric (resp., symmetric) Finsler metric on S^2 with only two (resp., three) prime closed geodesics c_1, c_2 (resp., c_1, c_2, c_3 and $\operatorname{ind} c_3 = 2$), the average index $\alpha_1 = \alpha_{c_1} \in (1,2) \cap \mathbf{R}\backslash\mathbf{Q}$ determines $\alpha_2 = \alpha_{c_2} = \alpha_1/(\alpha_1 - 1) \in (2,\infty)$ and hence the sequences $\operatorname{ind}(c_1^m)$ and $\operatorname{ind}(c_2^m)$ (cf. 1.7). If there is a bumpy Riemannian metric on S^2 with only three prime closed geodesics c_1, c_2, c_3, where c_3 is hyperbolic with $\operatorname{ind}(c_3) = 2$, and c_1, c_2 are elliptic, then $\operatorname{ind}(c_1) = 1$ and $\operatorname{ind}(c_2) = 3$, due to the theorem of Lusternik-Schnirelmann. Thus, as in the Finsler case, $1/\alpha_1 + 1/\alpha_2 = 1$, but $\alpha_2 \in (2,4) \cap \mathbf{R}\backslash\mathbf{Q}$ and $\alpha_1 \in (\frac{4}{3}, 2) \cap \mathbf{R}\backslash\mathbf{Q}$ since $\operatorname{ind}(c_2) = 3$.

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