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ON THE EVOLUTION OF HARMONIC MAPS IN HIGHER DIMENSIONS

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Abstract

We establish partial regularity results and the existence of global regular solutions to the evolution problem for harmonic maps with small data. The key ingredient is a decay estimate analogous to the well-known monotonicity formula for energy minimizing harmonic maps.

1. Let $\mathscr{M},\mathscr{N}$ be (compact) Riemannian manifolds of dimensions m,n with metrics γ, g respectively. In local coordinates $x = (x^1,\cdots,x^m)$ and $u = (u^1,\cdots,u^n)$ we denote

$$\gamma = (\gamma_{\alpha\beta})_{1\le\alpha,\beta\le m}, g = (g_{ij})_{1\le i,j\le n} \text{ and } (\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}.$$

For a C^1-map $u\colon \mathscr{M} \to \mathscr{N}$ the energy of u is given by the intrinsic Dirichlet integral

$$E(u) = \int_{\mathscr{M}} e(u)\, d\mathscr{M}$$

with density

$$e(u;x) = \frac{1}{2}\gamma^{\alpha\beta}(x) g_{ij}(u)\frac{\partial}{\partial x^\alpha}u^i \cdot \frac{\partial}{\partial x^\beta}u^i$$

in local coordinates. A summation convention is used. Since $\mathscr{N}$ is compact, $\mathscr{N}$ may be isometrically embedded into $\mathbf{R}^N$ for some N, and E becomes the standard Dirichlet integral of maps $u\colon \mathscr{M} \to \mathscr{N} \subset \mathbf{R}^N$.

u is harmonic iff E is stationary at u; in particular

$$\begin{aligned}(1.1)\qquad &\frac{d}{d\varepsilon}E(u+\varepsilon\phi)|_{\varepsilon=0}\\ &\quad = \int_U (-\Delta_{\mathscr{M}}u + \Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}})^i g_{ij}(u)\phi^j\, dx = 0\end{aligned}$$

for any smooth variation ϕ with support in a coordinate neighborhood $U \subset \mathbf{R}^m$ and such that $(u+\varepsilon\phi)(U)$ is contained in a coordinate chart V in the target space, where

$$\Delta_{\mathscr{M}} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\gamma^{\alpha\beta}\frac{\partial}{\partial x^\beta}\cdot\right)$$

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is the Laplace-Beltrami operator on $\mathscr{M}$ and the term

$$(\Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}})^k = \gamma^{\alpha\beta}\Gamma_{ij}^k(u)\frac{\partial}{\partial x^\alpha}u^i\frac{\partial}{\partial x^\beta}u^i, \qquad 1 \le k \le n,$$

involves the Christoffel symbols of the metric g, i.e., u is harmonic iff u satisfies

$$-\Delta_{\mathscr{M}}u + \Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}} = 0. \tag{1.2}$$

Regarding u as a map $u\colon \mathscr{M} \to \mathscr{N} \subset \mathbf{R}^N$ and $E(u)$ as the ordinary Dirichlet integral, u is harmonic iff

$$\int_{\mathscr{M}} -\Delta_{\mathscr{M}}u \cdot \phi \, d\mathscr{M} = 0$$

for all smooth $\phi\colon \mathscr{M} \to \mathbf{R}^N$ tangent to $\mathscr{N}$ at u, i.e., such that $\phi(x) \in T_{u(x)}\mathscr{N}$, the tangent space to $\mathscr{N}$ at $u(x)$, $x \in \mathscr{M}$. (Note that

$$\int_{\mathscr{M}} \Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}} \cdot \phi \, d\mathscr{M} = 0 \tag{1.3}$$

for all such ϕ, i.e., $\Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}}$ is orthogonal to $T_u\mathscr{N}$; cf. Schoen [8, §1].)

Harmonic maps—in particular smooth E-minimizing maps—are distinguished representants of maps $\mathscr{M} \to \mathscr{N}$. In order to understand how much of the topological structure of a space $\mathscr{N}$ is captured by harmonic maps $\mathscr{M} \to \mathscr{N}$ it is natural to study the following problems.

Problem 1. Given a (smooth) map $u_0\colon \mathscr{M} \to \mathscr{N}$, is there a harmonic map homotopic to u_0?

In particular, we may ask for representations of the fundamental groups of $\mathscr{N}$ by harmonic maps:

Problem 2. Given a (smooth) map $u_0\colon S^m \to \mathscr{N}$, is there a harmonic map homotopic to u_0?

In dimensions $m = 2$ Sacks and Uhlenbeck [7] have given an (essentially) affirmative answer to Problem 2. Moreover, the existence of harmonic 2-spheres turns out to be precisely the obstruction for solving Problem 1 in general.

In dimensions $m > 2$—apart from certain particular cases—essentially no significant progress has been made since the fundamental result by Eells and Sampson [2] in 1964:

Theorem 1.1. *Suppose the sectional curvature $\kappa_{\mathscr{N}}$ of $\mathscr{N}$ is not positive. Then for any (smooth) map $u_0\colon \mathscr{M} \to \mathscr{N}$ there is a (smooth) E-minimizing map $u\colon \mathscr{M} \to \mathscr{N}$ homotopic to u_0.*

Their method is based on an analysis of the evolution problem

$$(1.4) \qquad \partial_t u - \Delta_{\mathscr{M}} u + \Gamma_{\mathscr{N}}(u)(\nabla u, \nabla u)_{\mathscr{M}} = 0, \qquad u|_{t=0} = u_0,$$

which by (1.1) may be regarded as the L^2-gradient flow for E with respect to the metric $g(u)$. Eells and Sampson prove that under the above curvature restriction on the target, (1.4) possesses a global regular solution $u(t)$, which converges to a harmonic map as $t \to \infty$.

In [11] the latter result was generalized to arbitrary target manifolds in the case $m = 2$:

Theorem 1.2. *Suppose $m = 2$. For any (smooth) map $u_0 \colon \mathscr{M} \to \mathscr{N}$ there exists a (unique) global distribution solution to (1.4) which is regular on $\mathscr{M} \times [0, \infty[$ with exception of at most finitely many points $(x_k, t_k)_{1 \le k \le K}$, $t_k \le \infty$. At a singularity $(\bar{x}, \bar{t})$ a nonconstant, smooth harmonic map $\bar{u} \colon \overline{\mathbf{R}^2} \cong S^2 \to \mathscr{N}$ separates in the sense that for sequences*

$$R_m \searrow 0, \quad x_m \to \bar{x}, \quad t_m \nearrow \bar{t}$$

as $m \to \infty$

$$u_m(x) \equiv u(\exp_{x_m}(R_m x), t_m) \to \bar{u} \quad in\ H^{1,2}_{\mathrm{loc}}(\mathbf{R}^2; \mathscr{N}).$$

Moreover, $u(t)$ converges weakly in $H^{1,2}(\mathscr{M}, \mathscr{N})$ to a smooth harmonic map $\mathscr{M} \to \mathscr{N}$ as $t \to \infty$ (strongly, if $t = \infty$ is regular).

Here $\exp_q \colon T_q\mathscr{M} \to \mathscr{M}$ denotes the exponential map,

$$H^{1,2}(\mathscr{M}; \mathscr{N}) = \{u \in H^{1,2}(\mathscr{M}; \mathbf{R}^N) | u(\mathscr{M}) \subset \mathscr{N} \text{ a.e.}\}$$

and $H^{1,2}(\mathscr{M}; \mathbf{R}^N)$ is the standard Sobolev space of square-integrable (L^2-) functions $u \colon \mathscr{M} \to \mathbf{R}^N$ with distributional derivative $\nabla u \in L^2$. Remark that if $m = 2$ the space $H^{1,2}(\mathscr{M}; \mathscr{N})$ coincides with the closure of the space $C^\infty(\mathscr{M}; \mathscr{N})$ of smooth functions $u \colon \mathscr{M} \to \mathscr{N}$ in the $H^{1,2}$-norm.

For $m > 2$ this is no longer true ([10, Example, p. 267]; cf., however, Proposition 7.2 below).

The purpose of this note is to partially extend Theorem 1.2 to the case $m > 2$. In this case no existence and regularity results for (1.4) and arbitrary target manifolds are known unless certain a-priori restrictions relating the size of the image $u(\mathscr{M} \times \mathbf{R}_+)$ to a bound for the sectional curvature of $\mathscr{N}$ are satisfied (cf. e.g. [4]). However, unless $\mathscr{M}$ is a manifold with boundary $\partial\mathscr{M}$ and boundary conditions are posed on $\partial\mathscr{M}$ such conditions seem unnatural.

Imposing no a-priori restrictions on $\mathscr{N}$ or the range of u we prove partial regularity results (Theorem 6.1) and global existence and regularity results for smooth initial data with small energies (Theorem 7.1).

The basic ingredients are a monotonicity estimate, cf. Proposition 3.3, and the ε-regularity theorem, cf. Theorem 5.1, which are reminiscent of the well-known monotonicity formula and ε-regularity theorem for minimizing harmonic maps in high dimensions (cf. Schoen-Uhlenbeck [9], Schoen [8]).

For simplicity we restrict ourselves to the case $\mathscr{M} = \mathbf{R}^m$. However, our results seem to carry over to compact manifolds $\mathscr{M}$.

2. Notations

Let $z = (x,t)$ denote points in $\mathbf{R}^m \times \mathbf{R}$. For a distinguished point $z_0 = (x_0, t_0), R > 0$ let $B_R(x_0) = \{x \mid |x - x_0| < R\}$ be a Euclidean ball centered at x_0, and let $P_R(z_0) = \{z = (x,t) \mid |x - x_0| < R,\ |t - t_0| < R^2\}$ be a parabolic cylinder of radius R centered at z_0. Also let $S_R(t_0) = \{z = (x,t) \mid t = t_0 - R^2\}$ and $T_R(t_0) = \{z = (x,t) \mid t_0 - 4R^2 < t < t_0 - R^2\}$ be respectively horizontal sections and horizontal layers in $\mathbf{R}^m \times \mathbf{R}$. Note that (1.4) is invariant under scaling,

$$u \mapsto u_R(x,t) = u(Rx, R^2 t),$$

and translation, $x \mapsto x - x_0$, $t \to t - t_0$. Using this invariance property we will often shift the center of attention to the origin $z_0 = 0$. In this case we simply write $P_R(0) = P_R$, etc.

Weighted estimates will involve the fundamental solution

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^{m/2}} \exp\left(-\frac{(x - x_0)^2}{4(t_0 - t)}\right), \qquad t < t_0,$$

to the (backward) heat equation with singularity at z_0. (Again $G_0(z) = G(z)$, for simplicity.)

δ denotes the parabolic distance function

$$\delta((x,t),(y,s)) = \max\{|x - y|, \sqrt{|s - t|}\}.$$

the letters c, C denote generic constants.

A map $u\colon \mathbf{R}^m \times [t_0, t_1] \to \mathbf{R}^N$ is regular iff u and ∇u are uniformly bounded and $\partial_t u, \nabla^2 u \in L^p_{\mathrm{loc}}$ for all $p < \infty$.

Remark 2.1. With this definition, by [5, Theorem IV.9.1, p.341 f.] any regular solution u to (1.4) on an interval $[0, t_0]$ may be extended to a regular solution of an equation $(\partial_t - \Delta)u \in L^\infty_{\mathrm{loc}}$ on $\mathbf{R}_+$ by letting u solve $(\partial_t - \Delta)u = 0$ for $t > t_0$.

Lemmas 3.1 and 3.2, resp. Propositions 3.3 and 4.1 below, will also apply to the extended function u.

Moreover, for a regular solution u of (1.4), also $\partial_t u$, $\nabla^2 u$, etc. will be uniformly bounded, if the initial data u_0 are smooth.

3. Energy estimates and monotonicity formula

Let $u\colon \mathbf{R}^m \times [0,T] \to \mathcal{N}$ be a regular solution to (1.4) with $E(u(t)) < \infty$ for $t \in [0,T]$. The following estimate is well known:

Lemma 3.1.

$$\sup_{0\le t\le T} E(u(t)) + \int_0^T \int_{\mathbf{R}^m} |\partial_t u|^2 \, dx\, dt \le E(u_0).$$

Proof. Simply multiply (1.4) by $\partial_t u$ and integrate by parts. By (1.3) and since $E(u(t)) < \infty$ for all t the nonlinear term and boundary integrals vanish. q.e.d.

We also need a weighted decay estimate analogous to Lemma 3.1. This is our key result.

Lemma 3.2. *Let $u\colon \mathbf{R}^m \times [0,T] \to \mathcal{N}$ be a regular solution to* (1.4) *with $|\nabla u(x,t)| \le c < \infty$ uniformly. Then for any point $z_0 = (x_0, t_0) \in \mathbf{R}^m \times]0,T[$ the function*

$$\Phi(R;u) = \frac{1}{2} R^2 \int_{S_R(t_0)} |\nabla u|^2 G_{z_0}\, dx$$

is nondecreasing in R for $0 < R \le R_0 = \sqrt{t_0}$.

Proof. By translation we may achieve that $z_0 = 0$. We establish that

$$\frac{d}{dR}\Phi(R;u)\big|_{R=R_1} \ge 0.$$

By scale invariance $\Phi(R;u) = \Phi(1;u_R)$, where $u_R(x,t) = u(Rx, R^2 t)$; also it suffices to consider $R_1 = 1$.

By the exponential decay of G and regularity of u we may differentiate under the integral sign:

$$\begin{aligned}
\frac{d}{dR}\Phi(R;u) = \frac{d}{dR}\Phi(1;u_R)\big|_{R=1} &= \int_{S_1} \nabla u \nabla \left(\frac{d}{dR} u_R\big|_{R=1}\right) G\, dx \\
&= \int_{S_1} \nabla u \nabla (x\cdot \nabla u + 2t\partial_t u) G\, dx \\
&= -\int_{S_1} \Delta u (x\cdot \nabla u + 2t\partial_t u) G\, dx \\
&\quad - \int_{S_1} \nabla u (x\cdot \nabla u + 2t\partial_t u) \nabla G\, dx.
\end{aligned}$$

The vector $x\cdot\nabla u + 2t\partial_t u$ is tangent to $\mathcal{N}$ at u; hence by (1.3)–(1.4) and using that $\nabla G = -\frac{x}{2}G$, $t = -1$ on S_1:

$$\frac{d}{dR}\Phi(R;u)\big|_{R=1} = -\int_{S_1}\left(\partial_t u - \frac{x}{2}\cdot\nabla u\right)(x\cdot\nabla u - 2\partial_t u)G\,dx$$
$$= \frac{1}{2}\int_{S_1}(2\partial_t u - x\cdot\nabla u)^2 G\,dx \geq 0. \quad \text{q.e.d.}$$

In particular, Lemma 3.2 implies the following monotonicity formula for solutions to (1.4).

Proposition 3.3. *Suppose $u\colon \mathbf{R}^m \times [0, t_0 = 4R_0^2] \to \mathcal{N}$ is a regular solution to* (1.4) *with $|\nabla u(x,t)| \leq c < \infty$ uniformly. Then for any point $z_0 = (x_0, t_0)$ the function*

$$\Psi(R;u)\colon = \int_{T_R(z_0)} |\nabla u|^2 G_{z_0}\,dx\,dt$$

is nondecreasing for $0 < R < R_0$.

Proof. Shift $z_0 = 0$ and compute for $0 < R < R_1 < R_0$ (with $r'/r = R_1/R =: \lambda$):

$$\Psi(R;u) = \int_{-4R^2}^{-R^2}\int_{\mathbf{R}^m}|\nabla u|^2 G\,dx\,dt = 4\int_R^{2R} r^{-1}\Phi(r;u)\,dr$$
$$= 4\int_{R_1}^{2R_1}\frac{\Phi(r'/\lambda;u)}{\Phi(r';u)}r'^{-1}\Phi(r';u)\,dr' \leq \Psi(R_1;u)$$

by Lemma 3.2.

4. A Bochner-type estimate

Suppose $u\colon Q \to \mathcal{N}$ is a regular solution of (1.4) in an open space-time region $Q \subset \mathbf{R}^m \times \mathbf{R}$. Taking the gradient of both sides of (1.4) and multiplying by ∇u we obtain

$$\partial_t\nabla u\cdot\nabla u - \Delta\nabla u\cdot\nabla u = (\partial_t - \Delta)\left(\frac{|\nabla u|^2}{2}\right) + |\nabla^2 u|^2$$
$$= -\nabla(\Gamma_{\mathcal{N}}(u)(\nabla u, \nabla u))\cdot\nabla u$$
$$\leq \varepsilon|\nabla^2 u|^2 + C(\varepsilon)|\nabla u|^4.$$

Choosing $\varepsilon = 1$ yields the following differential inequality for the energy density $e(u) = \frac{1}{2}|\nabla u|^2$ of u:

Proposition 4.1. *Let $u\colon Q \to \mathcal{N}$ be a regular solution to* (1.4) *in Q with energy density $e(u)$. Then there holds*

$$(\partial_t - \Delta)e(u) \leq ce(u)^2$$

with a constant c depending only on $\mathcal{N}$ and m.

Remark 4.2. By the maximum principle for the heat equation, Proposition 4.1 implies an a-priori estimate for $|\nabla u|$ on a small time interval for any regular solution u of (1.4) with regular initial data u_0. This guarantees the existence of solutions to (1.4), locally. If $E(u_0) < \infty$, by Lemma 3.2 and a covering argument we can also see that $E(u(t)) \leq c < \infty$ uniformly, locally near $t = 0$, and the energy inequality of Lemma 3.1 will hold.

5. The ε-regularity theorem

Our monotonicity formula Proposition 3.3 allows us to use ideas of Schoen-Uhlenbeck [9] and Schoen [8] to prove the following:

Theorem 5.1. *There exists a constant $\varepsilon_0 > 0$ depending only on $\mathscr{N}$ and m such that for any regular solution $u\colon \mathbf{R}^m \times [-4R_0^2, 0] \to \mathscr{N}$ of (1.4) with $E(u(t)) \leq E_0 < \infty$, uniformly in t, the following is true:*

If for some $R \in]0, R_0[$ there holds

$$\Psi(R) := \Psi(R; u) = \int_{T_R} |\nabla u|^2 G \, dx \, dt < \varepsilon_0,$$

then

$$\sup_{P_{\delta R}} |\nabla u|^2 \leq c(\delta R)^{-2}$$

with constants $\delta > 0$ depending on $\mathscr{N}, m, E_0$, and $\inf\{R, 1\}$, and c depending on $\mathscr{N}$ and m, only.

Proof. We closely follow Schoen's proof [8, Theorem 2.2] for the analogous result in the stationary case.

Let $r_1 = 2\delta R$, $\delta \in]0, \frac{1}{4}[$ to be determined in the sequel. For $r, \sigma \in [0, r_1[$, $r + \sigma < r_1$, and $z_0 = (x_0, t_0) \in P_r$ our monotonicity formula (for the extended function u: cf. Remark 2.1) implies

$$\begin{aligned} \sigma^{-n} \int_{P_\sigma(z_0)} |\nabla u|^2 \, dx \, dt &\leq c \int_{P_\sigma(z_0)} |\nabla u|^2 G_{(x_0, t_0 + 2\sigma^2)} \, dx \, dt \\ &\leq c \int_{T_\sigma(t_0 + 2\sigma^2)} |\nabla u|^2 G_{(x_0, t_0 + 2\sigma^2)} \, dx \, dt \\ &\leq c \int_{T_R} |\nabla u|^2 G_{(x_0, t_0 + 2\sigma^2)} \, dx \, dt. \end{aligned} \tag{5.1}$$

But on T_R, given $\varepsilon > 0$, if $\delta > 0$ is small enough:

$$\begin{aligned}
G_{(x_0,t_0+2\sigma^2)}(x,t) &\le \frac{C}{(4\pi|t|)^{m/2}} \exp\left(-\frac{|x-x_0|^2}{4(t_0+2\sigma^2-t)}\right) \\
&\le C\exp\left(\frac{|x|^2}{4|t|} - \frac{|x-x_0|^2}{4|t_0+2\sigma^2-t|}\right) G(x,t) \\
&\le C\exp\left(c\delta^2\frac{|x|^2}{R^2}\right) G(x,t) \qquad (5.2)\\
&\le \begin{cases} C\,G(x,t) & \text{if } |x| \le \frac{R}{\delta}, \\ C\,R^{-m}\exp(-c\delta^{-2}) & \text{if } |x| \ge \frac{R}{\delta} \end{cases} \\
&\le C\,G(x,t) + C\,R^{-2}\exp\left((2-m)\log R - c\delta^{-2}\right) \\
&\le C\,G(x,t) + \varepsilon R^{-2}.
\end{aligned}$$

Remark that $\delta \sim (|\ln R| + |\ln \varepsilon|)^{-1/2}$ for small R and may be chosen independent of R if $R \ge 1$. Hence

$$\sigma^{-n}\int_{P_\sigma(z_0)} |\nabla u|^2\,dx\,dt \le c\Psi(R) + c\varepsilon E_0 \le c(\varepsilon_0 + \varepsilon E_0). \tag{5.3}$$

There exists $\sigma_0 \in [0, r_1[$ such that

$$(r_1-\sigma_0)^2 \sup_{\overline{P}_{\sigma_0}} e(u) = \max_{0\le\sigma\le r_1} (r_1-\sigma)^2 \sup_{\overline{P}_\sigma} e(u);$$

moreover, there exists $(x_0,t_0) \in \overline{P}_{\sigma_0}$ such that

$$\sup_{\overline{P}_{\sigma_0}} e(u) = e(u)(x_0,t_0) =: e_0.$$

Set $\rho_0 = \frac{1}{2}(r_1-\sigma_0)$. By choice of σ_0 and (x_0,t_0)

$$\sup_{P_{\rho_0}(x_0,t_0)} e(u) \le \sup_{P_{\sigma_0+\rho_0}} e(u) \le 4e_0.$$

Introduce $r_0 = \sqrt{e_0}\cdot\rho_0$ and define a map $v\colon P_{r_0} \to \mathcal{N}$ by letting

$$v(x,t) = u\left(\frac{x-x_0}{\sqrt{e_0}}, \frac{t-t_0}{e_0}\right).$$

v is a (regular extension of a) solution to (1.4) in P_{r_0}; moreover, v satisfies $e(v)(0,0) = 1$ and $\sup_{P_{r_0}} e(v) \le 4$. By our Bochner-type estimate Proposition 4.1 therefore $e(v)$ satisfies

$$(\partial_t - \Delta)e(v) \le c_1 e(v)$$

with a constant c_1 depending only on m and $\mathcal{N}$. Thus, if instead of $e(v)$ we consider the function $f(x,t) = \exp(-c_1t)e(v)$ in P_{r_0} and if $r_0 \ge 1$, Moser's Harnack inequality [6, Theorem 1, p. 102] implies the estimate

$$1 = e(v)(0,0) \le c\int_{P_1} e(v)\,dx\,dt.$$

But, scaling back, by (5.3) and since $1/\sqrt{e_0} + \sigma_0 \le \rho_0 + \sigma_0 < r_1$

$$\int_{P_1} e(v)\,dx\,dt = (\sqrt{e_0})^n \int_{P_{1/\sqrt{e_0}}(x_0,t_0)} e(u)\,dx\,dt \le c(\varepsilon_0 + \varepsilon E_0)$$

and we obtain a contradiction for small $\varepsilon_0, \varepsilon > 0$. Hence we may assume $r_0 \le 1$. But then the Harnack-inequality gives

$$\begin{aligned} 1 = e(v)(0,0) &\le c r_0^{-n-2} \int_{P_{r_0}} e(v)\,dx\,dt \\ &= c r_0^{-2} \rho_0^{-n} \int_{P_{\rho_0}(x_0,t_0)} e(u)\,dx\,dt. \end{aligned}$$

I.e., by (5.3) and since $\rho_0 + \sigma_0 = \frac{1}{2}(r_1 + \sigma_0) < r_1$:

$$\rho_0^2 e_0 = r_0^2 \le c\Psi(R) + c\varepsilon E_0 \le C.$$

Finally, by choice of σ_0 this implies

$$\max_{0 \le \sigma \le r_1} (r_1 - \sigma)^2 \sup_{P_\sigma} e(u) \le 4\rho_0^2 e_0 \le 4C.$$

Hence, we may choose $\sigma = \frac{1}{2} r_1 = \delta R$ and divide by σ^2 to complete the proof.

Remark 5.2. Instead of (5.2) we may estimate for $K > 0$, $R > 0$, uniformly on T_R:

$$G_{(x_0,t_0+2\sigma^2)}(x,t) \le \frac{C}{R^m} \le c(K) G(x,t) \quad \text{if } |x| \le KR, \tag{5.4'}$$

resp.

$$\begin{aligned} G_{(x_0,t_0+2\sigma^2)}(x,t) &\le c \cdot \exp\left(\frac{|x|^2}{4|t - R^2|} - \frac{|x - x_0|^2}{4|t - t_0 - 2\sigma^2|} \right) G_{(0,R^2)}(x,t) \\ &\le c \cdot \exp(-C^{-1} K^2) G_{(0,R^2)}(x,t) \quad \text{if } |x| \ge KR, \end{aligned} \tag{5.4''}$$

provided $(x_0, t_0) \in P_\sigma$, $\sigma < R/2$. Hence we obtain that for any $\varepsilon > 0$ there holds

$$G_{(x_0,t_0+2\sigma^2)} \le C(\varepsilon) G + \varepsilon G_{(0,R^2)},$$

uniformly on T_R, uniformly in $R > 0$. Thus, instead of (5.3) we obtain

$$\sigma^{-n} \int_{P_\sigma(z_0)} |\nabla u|^2\,dx\,dt \le C(\varepsilon)\Psi(R) + c\varepsilon \int_{T_R} |\nabla u|^2 G_{(0,R^2)}\,dx\,dt.$$

If now $|\nabla u| \le C < \infty$, we may apply Proposition 3.3 to the term on the far right and deduce that

$$\begin{aligned} \sigma^{-n} \int_{P_\sigma(z_0)} |\nabla u|^2\,dx\,dt &\le C(\varepsilon)\varepsilon_0 + c\varepsilon \int_{T_{R_0}} |\nabla u|^2 G_{(0,R^2)}\,dx\,dt \\ &\le C(\varepsilon)\varepsilon_0 + c\varepsilon R_0^{2-m} E_0, \end{aligned}$$

where $E_0 = E(\mu_0) \geq E(u(t))$ by Lemma 3.1 and Remark 4.2. With this modification, setting $\delta = \frac{1}{4}$ and leaving the remainder of the proof of Theorem 5.1 unchanged we obtain the following variants of this result.

Theorem 5.3. *For any $R_0 > 0$ and E_0 there exists a constant $\varepsilon_0 > 0$ depending on $R_0, E_0, \mathcal{N}$, and m such that for any regular solution $u\colon \mathbf{R}^m \times [-4R_0^2, 0] \to \mathcal{N}$ of* (1.4) *with $E(u(t)) \leq E_0 < \infty$ the following is true:*

If for some $R \in]0, R_0[$ there holds

$$\Psi(R; u) = \int_{T_R} |\nabla u|^2 G\, dx\, dt < \varepsilon_0,$$

then

$$\sup_{P_{R/4}} |\nabla u|^2 \leq CR^{-2}$$

with a constant C depending on $\mathcal{N}$ and m only.

Theorem 5.4. *For any $C_0 > 0$ there exists a constant $\varepsilon_0 > 0$ depending on $C_0, \mathcal{N}$, and m such that for any regular solution $u\colon \mathbf{R}^m \times [-4R_0^2, 0] \to \mathcal{N}$ of* (1.4) *with $|\nabla u| \leq c < \infty$ uniformly the following is true:*

If for some $R \in]0, R_0[$ there holds

$$\psi(R; u) = \int_{T_R} |\nabla u|^2 G\, dx\, dt < \varepsilon_0$$

while $\int_{T_R} |\nabla u|^2 G_{(0,R^2)}\, dx\, dt \leq C_0$, then

$$\sup_{P_{R/4}} |\nabla u|^2 \leq CR^{-2}$$

with a constant C depending only on $\mathcal{N}$ and m.

6. Partial regularity

Using the a-priori estimate obtained previously we can prove the partial regularity of weak solutions u to (1.4) with finite energy and which can be weakly approximated by smooth global solutions to (1.4):

Theorem 6.1. *Suppose $u\colon \mathbf{R}^m \times \mathbf{R}_+ \to \mathcal{N}$ is the limit of a sequence $\{u_k\}$ of regular solutions u_k to* (1.4) *with uniformly finite energy*

$$E(u_k(t)) \leq E_0 < \infty \quad \forall k \in \mathbf{N},\ t > 0$$

in the sense that $E(u(t)) \leq E_0$ almost everywhere and $\nabla u_k \to \nabla u$ weakly in $L^2(Q)$ for any compact $Q \subset \mathbf{R}^m \times \mathbf{R}_+$. Then u solves (1.4) *in the classical sense and is regular on a dense open set $Q_0 \subset \mathbf{R}^m \times \mathbf{R}_+$ whose complement Σ has locally finite m-dimensional Hausdorff-measure* (*with respect to the parabolic metric δ*)*. Moreover, there exists $t_0 > 0$* (*depending on $\mathcal{N}, m$*

and E_0) *such that* $\Sigma \cap (\mathbf{R}^m \times [t_0, \infty[\,) = \emptyset$. *Finally,* $u(t) \to u_\infty \equiv p \in \mathscr{N}$ *in* C^1_{loc} *as* $t \to \infty$, *where* $u_\infty \equiv p$ *is a constant map.*

Proof. This proof is modelled on [8, proof of Corollary 2.3]. Define

$$\Sigma = \bigcap_{R>0} \left\{ z_0 \in \mathbf{R}^m \times \mathbf{R}_+ \mid \liminf_{k\to\infty} \int_{T_R(z_0)} |\nabla u_k|^2 G_{z_0}\, dx\, dt \geq \varepsilon_0 \right\},$$

where $\varepsilon_0 > 0$ is the constant determined in Theorem 5.1. Σ is closed. Indeed, if $\{z_l\}$ is a sequence of points in Σ converging to $z_0 \in \mathbf{R}^m \times \mathbf{R}_+$, then for any $R > 0$ and $l \in \mathbf{N}$ we have

$$\liminf_{k\to\infty} \int_{T_R(z_l)} |\nabla u_k|^2 G_{z_l}\, dx\, dt \geq \varepsilon_0.$$

Since $G_{z_l} \to G_{z_0}$ uniformly away from $z_0 = (x_0, t_0)$ and $E(u_k) \leq E_0$ uniformly, this implies that for any $\delta > 0$

$$\liminf_{k\to\infty} \int_{t_0-\delta-4R^2}^{t_0+\delta-R^2} \int_{\mathbf{R}^m} |\nabla u_k|^2 G_{z_0}\, dx\, dt \geq \varepsilon_0.$$

Since $R, \delta > 0$ were arbitrary, by Proposition 3.3 this implies that

$$\liminf_{k\to\infty} \int_{T_R(z_0)} |\nabla u_k|^2 G_{z_0}\, dx\, dt \geq \varepsilon_0$$

for all $R > 0$, whence $z_0 \in \Sigma$ as claimed. Σ has locally finite m-dimensional Hausdorff-measure with respect to the metric δ, given by

$$m\text{-meas}(\Sigma) = \lim_{R\to 0} \inf \left\{ c(m) \sum_i R_i^m \right\}.$$

The infimum here is taken with respect to all covers $\mathscr{J}$ of Σ by cylinders $P_{R_i}(z_i)$ of radius $R_i \leq R$. It will suffice to show that

$$m\text{-meas}(\Sigma \cap Q) \leq c(Q, E_0)$$

for all compact regions $Q \subset \mathbf{R}^m \times \mathbf{R}_+$. Let $R > 0$ be given and let $\mathscr{J} = \{P_{R_i}(z_i)\}$ be a cover of $\Sigma \cap Q$ with $R_i \leq R$. We may assume $z_i \in \Sigma$: By Vitali's covering lemma (cf. Caffarelli-Kohn-Nirenberg [1, Lemma 6.1, p. 806] for a parabolic version) there exists a subfamily $\mathscr{J}' = \{P_i = P_{R_i}(z_i')\}$ of $\mathscr{J}$ such that $P_i \cap P_j = \emptyset$ for $i \neq j$ and such that the collection $\{P_{5R_i}(z_i')\}$ covers $\Sigma \cap Q$. Note that for arbitrary $z_0 = (x_0, t_0)$, $k \in \mathbf{N}$, $\varepsilon > 0$, by (5.4′), (5.4″) there is a constant $C(\varepsilon)$ such that:

$$\int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0}\, dx\, dt \leq cR^{-m} \int_{P_{C(\varepsilon)R}(z_0)} |\nabla u_k|^2\, dx\, dt$$
$$+ \varepsilon \int_{T_R(t_0)} |\nabla u_k|^2 G_{(z_0+(0,R^2))}\, dx\, dt.$$

Applying Lemma 3.2 the last term may be dominated for sufficiently small $\varepsilon > 0$:

$$\begin{aligned}\varepsilon \int_{T_R(t_0)} &|\nabla u_k|^2 G_{(z_0+(0,R^2))}\,dx\,dt \\ &\le \varepsilon c(t_0+R^2)\int_{\mathbf{R}^m} |\nabla u_k|^2 G_{(z_0+(0,R^2))}\,dx\Big|_{t=0} \\ &\le \varepsilon c(Q)E_0 \le \tfrac{1}{2}\varepsilon_0.\end{aligned}$$

Thus for $z_0 \in \Sigma \cap Q$, $0 < R$ we can choose a cylinder $P_{R_0}(z_0)$ of radius $R_0 < R$ such that for sufficiently large k

$$\int_{P_{R_0}(z_0)} |\nabla u_k|^2\,dx\,dt \ge c(Q,E_0)^{-1}R_0^m\varepsilon_0. \tag{6.1}$$

Since Σ is closed, we may cover $\Sigma \cap Q$ by finitely many such cylinders $P_{R_i}(z_i)$ from which we extract a disjoint finite subfamily $\mathscr{J}' = \{P_i = P_{R_i}(z_i')\}$ as above. We choose $k \in \mathbf{N}$ such that (6.1) is satisfied on each cylinder P_i. By summation over i,

$$\begin{aligned}\sum_i R_i^m &\le c(Q,E_0)\varepsilon_0^{-1}\sum_i \int_{P_i} |\nabla u_k|^2\,dx\,dt \\ &= c(Q,E_0)\varepsilon_0^{-1}\int_{\bigcup_i P_i} |\nabla u_k|^2\,dx\,dt \le c(Q,E_0) < \infty.\end{aligned}$$

Moreover, the collection $\{P_{5R_i}(z_i')\}$ covers $\Sigma \cap Q$ with $\sup_i R_i < R$. Hence

$$m\text{-meas}(\Sigma \cap Q) \le \lim_{R\to 0}\left\{\inf_{\mathscr{J}}\left\{c(m)\sum_i R_i^m\right\}\right\} \le c(Q,E_0),$$

as was to be shown.

Next, for $z_0 = (x_0,t_0) \notin \Sigma$ there exists $R > 0$ such that

$$\int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0}\,dx\,dt \le \varepsilon_0$$

for infinitely many $k \in \mathbf{N}$. By Theorem 5.1 then also $|\nabla u_k| \le C$ uniformly in a uniform neighborhood of z_0, and a-priori bounds for higher derivatives may be derived from (1.4). It follows that a subsequence $u_k \to u$ in $C^2_{\mathrm{loc}}(\mathbf{R}^m \times \mathbf{R}_+ \backslash \Sigma; \mathscr{N})$ and u is a regular solution of (1.4) off Σ.

Finally, using Proposition 3.3 for large t_0 such that $4R^2 \le t_0$, we may estimate

$$\int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0}\,dx\,dt \le \int_0^{t_0/4}\int_{\mathbf{R}^m} |\nabla u_k|^2 G_{z_0}\,dx\,dt \le Ct_0^{(2-m)/2}E_0 < \varepsilon_0$$

uniformly in k, and we obtain full regularity for $t_0 > C(E_0/\varepsilon_0)^{2/(m-2)}$. Moreover, choosing R as large as possible and applying Theorem 5.1 we infer the uniform decay $|\nabla u(x,t)|^2 \le C/t$ for large t, and $u(t) \to u_\infty \equiv \text{const}$ $(t \to \infty)$.

7. Small initial data

In particular, Theorem 6.1 can be turned into a global existence and regularity result for smooth initial data with small energy:

Theorem 7.1. *There exists a constant $\varepsilon_1 > 0$ depending on $C_1, \mathscr{N}$ and m such that for initial data $u_0 \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^m; \mathscr{N})$ with $\nabla u_0 \in L^\infty$ and $\|\nabla u_0\|_\infty \le C_1$, $E(u_0) < \varepsilon_1$, there exists a unique smooth solution u of* (1.4) *which as $t \to \infty$ converges to a constant map $u_\infty \equiv p \in \mathscr{N}$.*

The proof is a consequence of Theorem 6.1 and the following approximation result for functions $u_0 \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^m; \mathscr{N})$ with finite energy and satisfying (7.1) below. (This result is analogous to an approximation result of Schoen-Uhlenbeck [10, Proposition, p. 267] in the case $m = 2$.)

Proposition 7.2. *There exists $\varepsilon_2 > 0$ such that any map $u \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^m; \mathscr{N})$ satisfying the condition*

$$\sup_{R<R_0} R^{2-m} \int_{B_R(x_0)} |\nabla u|^2 \, dx \le \varepsilon_2, \tag{7.1}$$

uniformly for all $x_0 \in \mathbf{R}^m$ and for some $R_0 > 0$, can be approximated in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^m; \mathscr{N})$ by smooth maps $u_k \in C^\infty(\mathbf{R}^m; \mathscr{N})$.

Moreover, if u has finite energy, resp. $|\nabla u| \in L^\infty$, we may choose u_k with finite energy and $E(u_k) \le cE(u)$ with a constant c depending only on $\mathscr{N}$, resp. $\|\nabla u_k\|_{L^\infty} \le c\|\nabla u\|_{L^\infty}$.

Proof. There is $\delta_0 > 0$ such that any point $q \in \mathbf{R}^N$ at distance $> \delta_0$ from $\mathscr{N}$ has a unique nearest neighbor $\pi(q) \in \mathscr{N}$. Moreover, this projection π from the δ_0-neighborhood $U_{\delta_0}(\mathscr{N})$ of $\mathscr{N}$ onto $\mathscr{N}$ is smooth.

For $R < R_0$ let $\phi = \phi_R$ be a mollifier,

$$\phi_R \in C_0^\infty(B_R), \quad 0 \le \phi_R \le CR^{-m}, \quad \int_{B_R} \phi_R \, dw = 1.$$

For u as above and $R < R_0$ let

$$u_R(\bar{x}) = (u * \phi_R)(\bar{x}) = \int_{B_R(\bar{x})} u(x)\phi_R(\bar{x} - x) \, dx.$$

It is well known that $u_R \in C^\infty$ and $u_R \to u$ in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^m, \mathbf{R}^N)$ as $R \to 0$. Hence if we show that $u_R \colon \mathbf{R}^m \to U_{\delta_0}(\mathscr{N})$ for sufficiently small R, the functions $v_R = \pi \circ u_R$, $0 < R < R_0$, will lie in $C^\infty(\mathbf{R}^m; \mathscr{N})$ and will converge to u in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^m, \mathscr{N})$, as required. As in [10], for any $\bar{x} \in \mathbf{R}^m$ we estimate

$$\begin{aligned}\operatorname{dist}(u_R(\bar{x}), \mathscr{N})^2 &\leq CR^{-m} \int_{B_R(\bar{x})} |u_R(\bar{x}) - u(x)|^2\,dx \\ &\leq CR^{2-m} \int_{B_R(\bar{x})} |\nabla u|^2\,dx \leq C\varepsilon_2 \quad \text{if } R < R_0,\end{aligned}$$

which will be $< \delta_0^2$ if $\varepsilon_2 > 0$ is small enough.

Finally, by smoothness of π and Fubini's theorem

$$\begin{aligned}\int_{\mathbf{R}^m} |\nabla v_R|^2\,dx \leq C \int_{\mathbf{R}^m} |\nabla u_R|^2\,dx &= C \int_{\mathbf{R}^m} \left| \int_{\mathbf{R}^m} \nabla u(y) \phi_R(x-y)\,dy \right|^2 dx \\ &\leq C \int_{\mathbf{R}^m} \left(\int_{\mathbf{R}^m} \phi_R(x-y)\,dy \right) \left(\int_{\mathbf{R}^m} |\nabla u|^2(y) \phi_R(x-y)\,dy \right) dx \\ &= C \int_{\mathbf{R}^m} |\nabla u|^2\,dx,\end{aligned}$$

where $C = \|\nabla \pi\|_\infty = C(\mathscr{N})$. The estimate for $\|\nabla u_k\|_{L^\infty}$ is obtained in a similar way.

Proof of Theorem 7.1. If $\varepsilon_1 > 0$ is sufficiently small, by Proposition 7.2 there is a sequence $u_{k0} \in C^\infty(\mathbf{R}^m; \mathscr{N})$ of smooth functions approximating u_0 in $H^{1,2}_{\mathrm{loc}}$ and with$E(u_{k0}) \leq CE(u_0) = E_0$, $\|\nabla u_{k0}\|_\infty \leq C\|\nabla u_0\|_\infty$. We will show that for $\varepsilon_1 > 0$ sufficiently small

$$\sup_{x_0, R>0} R^2 \int_{\mathbf{R}^m} |\nabla u_{k0}|^2 G_{(x_0,R^2)}\,dx < \varepsilon_0,$$

which by Theorem 5.1 and Lemma 3.2 will imply the existence of smooth global solutions u_k to (1.4) with initial data u_{k0}.

But using the explicit formula for G, for $0 < R < e^{-m}$

$$\begin{aligned}\int_{\mathbf{R}^m} |\nabla u_{k_0}|^2 G_{(x_0,R^2)}\,dx &\leq CR^{-m} \int_{B_{2R|\ln R|}(x_0)} |\nabla u_{k_0}|^2\,dx + CR^{-m+|\ln R|} E_0 \\ &\leq C|\ln R|^m \|\nabla u_0\|_\infty^2 + CE_0\end{aligned}$$

and this is $< R^{-2}\varepsilon_0$ if $R < R_1 = R_1(\|\nabla u_0\|_\infty, E_0)$; while for $R > R_1$ we can achieve

$$\int_{\mathbf{R}^m} |\nabla u_{k0}|^2 G_{(x_0,R^2)}\,dx \leq CR^{-m} E_0 < R^{-2}\varepsilon_0,$$

if $E_0 < C^{-1} R_1^{m-2} \varepsilon_0$.

Hence Theorem 5.1, Lemma 3.2 and our monotonicity formula Proposition 3.3 guarantee uniform global a-priori bounds $|\nabla u_k(x,t)|^2 \leq C/t$. Since by Remark 4.2 (cf. also [3]) (1.4) for smooth initial data u_{k0} admits smooth solutions locally, we thus obtain global smooth solutions u_k to (1.4) with data

u_{k0}. Moreover, $\{u_k\}$ is uniformly bounded in C^1, hence relatively compact in C^0 locally in $\mathbf{R}^m \times [0,\infty[$ with uniform limit u solving (1.4) with initial data u_0. Since u is continuous, u is also regular. (This follows from standard results in regularity theory for parabolic systems; cf. [5].)

Remark 7.3. Inspection of the proof shows that $\nabla u_0 \in L^{m+\mu}_{\mathrm{loc}}$ and uniform local boundedness

$$\sup_{x_0} \int_{B_1(x_0)} |\nabla u_0|^{m+\mu}\, dx \le C$$

for some $\mu > 0$ would suffice instead of $\nabla u_0 \in L^\infty$.

8. Tangent maps

The appearance of singularities can be related to nonconstant harmonic mappings of $(m-1)$-dimensional spheres, as in the case of locally minimizing weakly harmonic maps (cf. Schoen-Uhlenbeck [9, Theorem III, p. 310]):

Theorem 8.1. *Suppose* $u\colon \mathbf{R}^m \times [0,t_0[\to \mathscr{N}$ *with uniformly finite energy* $E(u(t)) < E_0 < \infty$ *is a locally regular solution to* (1.4), *which develops a singularity as* $t \nearrow t_0$. *Then there exist sequences* $R_k \to 0$, $\overline{R}_l \to \infty$, $x_k \in \mathbf{R}^m$ *and* $t_k \nearrow t_0$ *such that*

$$\left.\begin{aligned} u_k(x,t) &\equiv u(x_k + R_k x, t_k + R_k^2 t) \to u_\infty \\ \bar{u}_l(x,t) &\equiv u_\infty(\overline{R}_l x, \overline{R}_l^2 t) \to \bar{u}_\infty \end{aligned}\right\} \quad \text{in } C^1_{\mathrm{loc}}(\mathbf{R}^m \times]-\infty, 0[; \mathscr{N})$$

if first $k \to \infty$ *and then* $l \to \infty$, *where either*

$$\bar{u}_\infty(x,t) \equiv \bar{v}_\infty(x/|x|) \tag{8.1}$$

is induced by a nonconstant harmonic map $\bar{v}_\infty\colon S^{m-1} \to \mathscr{N}$, *or*

$$u_\infty(x,t) \equiv \bar{w}_\infty(x/\sqrt{|t|}), \tag{8.2}$$

u_∞ *being a nonconstant solution to* (1.4) *in the half-space* $\{t < 0\}$ *and homogeneous on curves* $t = cx^2$.

Proof. Suppose there exists R_0 satisfying $4R_0^2 < t_0$ such that for all $z_0 = (x_0, t_0)$ there holds

$$\int_{T_{R_0}(z_0)} |\nabla u|^2 G_{z_0}\, dx\, dt < \varepsilon_0.$$

Then by Theorem 5.1 ∇u remains uniformly bounded as $t \nearrow t_0$, contradicting the hypothesis.

Thus, given a sequence of radii R_k such that $R_k \to 0$ as $k \to \infty$, there exist points $z_k = (x_k, t_k)$, $t_k < t_0$, such that

$$\int_{T_{R_k}(z_k)} |\nabla u|^2 G_{z_k}\, dx\, dt = \sup_{\substack{\bar{z}=(\bar{x},\bar{t}) \\ \bar{t}\le t_k, \overline{R}\le R_k \\ \bar{t}-4\overline{R}^2 \ge 0}} \int_{T_{\overline{R}}(\bar{z})} |\nabla u|^2 G_{\bar{z}}\, dx\, dt = \varepsilon_0.$$

Moreover, since $|\nabla u| \le C$ uniformly for $t \le \bar{t} < t_0$, it follows that $t_k \nearrow t_0$.

Rescale, letting

$$u_k(x,t) = u(x_k + R_k x, t_k + R_k^2 t).$$

Then $u_k \colon \mathbf{R}^m \times]-t_k/R_k^2, 0[\to \mathscr{N}$ solves (1.4) and satisfies

$$\sup_{\substack{\bar{z}=(\bar{x},\bar{t}) \\ \bar{t}\le 0, \overline{R}\le 1 \\ \bar{t}\ge 4\overline{R}^2 - t_k/R_k^2}} \int_{T_{\overline{R}}(\bar{z})} |\nabla u_k|^2 G_{\bar{z}}\, dx\, dt = \int_{T_1} |\nabla u|^2 G\, dx\, dt = \varepsilon_0.$$

By Theorem 5.4 the family $\{u_k\}$ is uniformly bounded in C^1_{loc}. Passing to a subsequence we may assume that $u_k \to \bar{u}$ uniformly locally (and in C^1; cf. the proof of Theorem 6.1), where $\bar{u} \colon \mathbf{R}^m \times]-\infty, 0] \to \mathscr{N}$ is a nonconstant, regular solution of (1.4) such that

$$\sup_{\substack{\bar{z}=(\bar{x},\bar{t}) \\ \bar{t}\le 0, \overline{R}\le 1}} \int_{T_{\overline{R}}(\bar{z})} |\nabla \bar{u}|^2 G_{\bar{z}}\, dx\, dt = \int_{T_1} |\nabla \bar{u}|^2 G\, dx\, dt = \varepsilon_0.$$

Moreover, by Proposition 3.3 for any $\bar{z} = (\bar{x}, \bar{t})$, $\bar{t} \le 0$, and $\overline{R} > 0$

$$\begin{aligned}
&\int_{T_{\overline{R}}(\bar{z})} |\nabla \bar{u}|^2 G_{\bar{z}}\, dx\, dt \\
&\quad = \lim_{k\to\infty} \int_{T_{\overline{R}}(\bar{z})} |\nabla u_k|^2 G_{\bar{z}}\, dx\, dt \\
&\quad = \lim_{k\to\infty} \int_{T_{\overline{R}R_k}(x_k + R_k\bar{x}, t_k + R_k^2\bar{t})} |\nabla u|^2 G_{(x_k + R_k\bar{x}, t_k + R_k^2\bar{t})}\, dx\, dt \\
&\quad \le \sup_{x_0 \in \mathbf{R}^m} \int_{T_1(x_0,t_0)} |\nabla u|^2 G_{(x_0,t_0)}\, dx\, dt \le CE_0
\end{aligned}$$

uniformly in $\overline{R}$ and $\bar{z}$.

Letting $\Psi(R, \bar{u}) = \int_{T_R} |\nabla \bar{u}|^2 G\, dx\, dt$ as above, we have by Proposition 3.3

$$\int_0^\infty \frac{d}{dR} \Psi(R; \bar{u})\, dR = \int_0^\infty \left| \frac{d}{dR} \Psi(R, \bar{u}) \right| dR < \infty;$$

and there exists a sequence $\overline{R}_l \geq 0$ such that

$$\frac{d}{dR}\Psi(\overline{R}_l; \bar{u}) \to 0 \qquad (l \to \infty).$$

Let $\bar{u}_l = \bar{u}_{\overline{R}_l} \equiv \bar{u}(\overline{R}_l x, \overline{R}_l^2 t)$; then as in the proof of Lemma 3.2

$$\begin{aligned}\frac{d}{dR}\overline{\Psi}(\overline{R}_l, \bar{u}) &= \frac{d}{dR}\overline{\Psi}(1, \bar{u}_l)\\ &= 2\int_{T_1} \nabla\bar{u}_l \nabla(x\cdot\nabla\bar{u}_l + 2t\cdot\partial_t\bar{u}_l)G\,dx\,dt\\ &= -2\int_{T_1} \partial_t\bar{u}_l(x\cdot\nabla\bar{u}_l + 2t\cdot\partial_t\bar{u}_l)G\,dx\,dt\\ &\quad -2\int_{T_1}\frac{x\cdot\nabla\bar{u}_l}{2t}(x\cdot\nabla\bar{u}_l + 2t\partial_t\bar{u}_l)G\,dx\,dt\\ &= \int_{T_1}\frac{1}{|t|}(x\cdot\nabla\bar{u}_l + 2t\partial_t\bar{u}_l)^2 G\,dx\,dt.\end{aligned}$$

It follows that either $\partial_t\bar{u}_l, x\cdot\nabla\bar{u}_l \to 0$ in L^2_{loc} in which case (using Theorem 5.4 again)

$$\bar{u}_l \to \bar{u}_\infty(x,t) \equiv \bar{v}_\infty(x/|x|)$$

converges to a map $\bar{u}_\infty$ induced by a nonconstant harmonic map $\bar{v}_\infty\colon S^{m-1} \to \mathscr{N}$; or $\bar{u}_l \to \bar{u}_\infty$ where $\bar{u}_\infty$ is a nonconstant solution to (1.4) on $\mathbf{R}^m\times]-\infty, 0[$ with $\partial_t\bar{u}_\infty = x\cdot\nabla\bar{u}_\infty/2|t|$, i.e.,

$$\bar{u}_\infty(x,t) = \bar{w}_\infty(x/\sqrt{|t|}). \quad \text{q.e.d.}$$

Note that by Theorem 6.1 if a solution u of (1.4) behaves irregularly as $t \to \bar{t} \leq \infty$, necessarily a singularity must be encountered in finite time.

A natural question is whether homogeneous solutions of the kind (8.2) may appear.

Added in proof. J. Eells has kindly pointed out a result of J. C. Mitteau (J. Differential Geometry 9 (1974) 41–54) related to my Theorem 7.1. Recently, F. Bethuel and X. Zheng (Preprint, Univ. Paris VI, Analyse Numérique) have studied the density of $C^\infty(\mathscr{M}, \mathscr{N})$ in $H^{1,p}(\mathscr{M}, \mathscr{N})$ and obtained necessary and sufficient conditions for the density to hold. This is related to my Theorem 7.2. R. S. Hamilton (private communication) has observed how the monotonicity formula of Lemma 3.2 can be extended to solutions of (1.4) on an arbitrary compact manifold.

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