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AN EXAMPLE OF A COMPACT CALIBRATED MANIFOLD ASSOCIATED WITH THE EXCEPTIONAL LIE GROUP G_2

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1. Introduction

Recently Harvey and Lawson have introduced the concept of calibrated geometries [10], [11]. In [11] they study manifolds which have a distinguished closed differential form, which allows them to generalize Wirtinger's inequality [15]. The primary example of a calibrated geometry is a Kähler manifold, the distinguished form being the Kähler form. However there are other interesting calibrated geometries, for example those defined in [8] and [10]. To say that a 7-dimensional manifold has holonomy group a subgroup of G_2 is the same thing as saying that it has a 2-fold parallel vector cross product; that is, there exists a 2-fold vector cross product such that the associated 3-form ϕ is both closed and coclosed. (If ϕ is closed the manifold is called an "associative 7-manifold" while if ϕ is coclosed the manifold is called a "coassociative 7-manifold" in the terminology of [10], [11].)

Of particular interest are compact calibrated geometries. In addition to compact Kähler manifolds, there is the compact symplectic manifold of Kodaira-Thurston [12], [14]. In the present note we shall give an example of a compact 7-dimensional manifold V^7 with a 2-fold vector cross product such that the associated 3-form is closed; thus V^7 satisfies a natural weakening from "Hol $\subset G_2$" to "associative." This manifold is a G_2 analog of a symplectic manifold, and like Kodaira-Thurston's manifold it can be realized as the quotient of a nilpotent Lie group by a discrete subgroup.

The Kodaira-Thurston manifold T cannot be Kählerian for topological reasons. In fact there are two ways to prove that T cannot be Kählerian. The easiest way is to observe that the first Betti number of T is 3, whereas any compact Kähler manifold has even first Betti number. A second method

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(exploited in [4]) is to prove that the minimal model of T is not formal; consequently T cannot be Kähler by the results of [6].

We show that with respect to the natural metric, V^7 does not have a parallel vector cross product, so the holonomy group is not a subgroup of G_2. It is probable that V^7 possesses no metric with holonomy group a subgroup of G_2. There are topological obstructions to the existence of parallel vector cross products on a 7-dimensional manifold U^7 [1]. For example $b_3(U^7) \geqslant 1$ and $b_3(U^7) \geqslant b_1(U^7)$. But the manifold V^7 which we construct has both of these properties. R. Bryant has recently shown that locally there are many 7-dimensional Riemannian manifolds with "Hol $\subset G_2$," but a compact example is still conjectural [3]. We show that the minimal model of V^7 is not formal. It would be nice if the results of [6] could be extended to compact 7-dimensional manifolds having a parallel vector cross product. Then the minimal model of such manifolds would be formal. We do not know if this is true or not.

The word "calibrated" applies to V^7 since ϕ has comass one. An interesting question remains. What are the associative submanifolds of V^7?

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2. The manifold V^7

Let $K^7 = H(1,2) \times R^2$. Here R^2 is a 2-dimensional vector space and $H(1,2)$ is a generalized Heisenberg group (compare [9]). By definition $H(1,2)$ consists of all matrices of the form

$$\begin{pmatrix} I_2 & X & Z \\ & 1 & y \\ & & 1 \end{pmatrix},$$

where X and Z are 2×1 matrices of real numbers and y is a real number. Let $\{u_1, u_2\}$ be the natural coordinates of R^2 and let $\{X, y, Z\}$ be the coordinates of $H(1,2)$ written as matrices. Put ${}^tX = (x_1, x_2)$ and ${}^tZ = (z_1, z_2)$. Then we have the following left invariant 1-forms on K^7:

$$\begin{aligned} &\alpha_1 = dx_1,\ \alpha_2 = dx_2,\ \beta = dy, \\ &\gamma_1 = dz_1 - x_1 dy,\ \gamma_2 = dz_2 - x_2\, dy, \\ &\eta_1 = du_1,\ \eta_2 = du_2. \end{aligned} \tag{1}$$

Theorem 1. *The 3-form ϕ on K^7 defined by*

$$\begin{aligned} \phi = {} & \gamma_1 \wedge \alpha_1 \wedge \eta_2 + \alpha_2 \wedge \gamma_2 \wedge \eta_2 + \alpha_1 \wedge \gamma_2 \wedge \eta_1 + \alpha_2 \wedge \gamma_1 \wedge \eta_1 \\ & + \beta \wedge \gamma_1 \wedge \gamma_2 + \alpha_1 \wedge \alpha_2 \wedge \beta + \beta \wedge \eta_2 \wedge \eta_1 \end{aligned}$$

is left invariant and closed.

Proof. It is obvious that ϕ is left invariant. On the right-hand side of (1) all of the terms except the third and fourth are closed. But $d(\alpha_1 \wedge \gamma_2 + \alpha_2 \wedge \gamma_1) = 0$, and so the sum of the third and fourth terms is closed.

Let $\Gamma \subset K^7$ be the subgroup whose entries are integers and let $V^7 = \Gamma \setminus K^7$ be the space of right cosets. Denote by μ: $K^7 \to V^7$ the projection.

Theorem 2. *There exists a closed* 3-*form* $\tilde{\phi}$ *on* V^7 *such that* $\mu^*(\tilde{\phi}) = \phi$. *Furthermore* $\tilde{\phi}$ *is the fundamental* 3-*form of a* (*nonparallel*) *vector cross product on* V^7.

Proof. Since the forms given by (1) are left invariant, they descend to K^7. Thus ϕ descends to $\tilde{\phi}$. There is also a metric on K^7 which descends to a metric on V^7. It is given by

$$ds^2 = \alpha_1^2 + \alpha_2^2 + \beta^2 + \gamma_1^2 + \gamma_2^2 + \eta_1^2 + \eta_2^2.$$

Let $\{E_1 \cdots E_7\}$ be the basis dual to $\{\alpha_1, \gamma_2, \eta_2, \eta_1, \alpha_2, \beta, \gamma_1\}$. Then a 2-fold vector cross product P on K^7 is given by $P(E_i, E_j) = -P(E_j, E_i)$, and

$$P(E_i, E_{i+1}) = E_{i+3}, \quad P(E_{i+3}, E_i) = E_{i+1}, \quad P(E_{i+1}, E_{i+3}) = E_i.$$

It is not hard to show that P satisfies the axioms for a 2-fold vector cross product (see [7] and [8]). Since ϕ is the fundamental 3-form of P, and P descends to a vector cross product P on V^7, to show that P is not parallel we prove that $\delta\phi$ is nonzero, where δ denotes the coderivative of K^7 with respect to the metric given above. In fact a calculation shows that $\delta\phi = -(\alpha_2 \wedge \gamma_1 - \alpha_1 \wedge \gamma_2) \neq 0$.

Theorem 3. *The Betti numbers of* V^7 *are as follows*:

$$b_1(V^7) = 5, \quad b_2(V^7) = 13, \quad b_3(V^7) = 21.$$

Proof. Because K^7 is a connected nilpotent Lie group, a theorem of Nomizu [13] implies that the cohomology ring $H^*(V^7, R)$ is isomorphic to $H^*(k)$, where k is the Lie algebra of K^7. The left invariant 1-forms on K^7 are given by (1). From this it is easy to compute the cohomology.

Theorem 4. *The minimal model of* V^7 *is not formal.*

Proof. Since k is a nilpotent Lie algebra, a minimal model for the exterior algebra $\wedge^*(V^7)$ of differential forms on V^7 is the differential algebra (A, d_A) defined as follows. A is the algebra with 7 generators of degree 1, namely those defined by (1). The differential d_A is given by

$$d_A\alpha_i = d_A\beta = d_A\eta_i = 0 \quad (i = 1, 2), \quad d_A\gamma_1 = -\alpha_1 \wedge \beta, \quad d_A\gamma_2 = -\alpha_2 \wedge \beta.$$

To prove that (A, d_A) is not formal it suffices to check that $H^*(A)$ has a nonvanishing Massey product (see for example [6]). Let $[\beta]$ and $[\alpha_1]$ denote the cohomology classes in $H^*(A)$ represented by β and α_1. Then $[\beta] \wedge [\alpha_1] = [d_A\gamma_1] = 0$, so that the Massey product $\langle [\beta], [\alpha_1], [\alpha_1] \rangle$ is well defined. In fact it is nonzero because it is represented by the nonexact form $\gamma_1 \wedge \alpha_1$.

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