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DECOMPOSITION THEOREMS FOR LORENTZIAN MANIFOLDS WITH NONPOSITIVE CURVATURE

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1. Introduction

The Toponogov Splitting Theorem [6] states that a complete Riemannian manifold (H, h) of nonnegative sectional curvatures which contains a line γ: $\mathbf{R} \to H$ (i.e., a complete absolutely minimizing geodesic) must be isometric to a product $\mathbf{R} \times H'$, the first factor being represented by γ. In [6] Cheeger and Gromoll gave a proof of this theorem stemming from their soul construction. Subsequently, Cheeger and Gromoll [5] were able to generalize this Riemannian splitting theorem to the case of nonnegative Ricci curvatures. In [17, p. 696], S. T. Yau raised the question of showing that a geodesically complete Lorentzian 4-manifold of nonnegative timelike Ricci curvature which contains a timelike line (i.e., a complete absolutely maximizing timelike geodesic) is isometrically the Cartesian product of that geodesic and a spacelike hypersurface.

Galloway [9] has recently considered this question for space-times which are spatially closed, i.e., which admit a smooth time function whose level sets are compact (smooth) Cauchy surfaces. Let (M, g) be such a globally hyperbolic space-time which satisfies the strong energy condition $\mathrm{Ric}(v, v) \geqslant 0$ for all timelike vectors v in TM. Suppose further that (M, g) contains a timelike curve which is both future and past complete and that for each $p \in M$, every null geodesic emanating from p contains a past and future null cut point to p.

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Then Galloway shows (M, g) splits isometrically as a Lorentzian product $(\mathbf{R} \times H, -dt^2 \otimes h)$, where (H, h) is a compact Riemannian manifold. The proof employs and extends some results of [1] and [10].

In the present paper, we consider a different class of space-times than those studied in [9] and we use quite different techniques to obtain the following splitting theorem.

Theorem 5.2. *Let (M, g) be a globally hyperbolic space-time of dimension $\geqslant 2$ with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ: $(-\infty, \infty) \to (M, g)$. Then (M, g) is isometric to a product $(\mathbf{R} \times H, -dt^2 \otimes h)$, where (H, h) is a complete Riemannian manifold. The factor $(\mathbf{R}, -dt^2)$ is represented by γ and (H, h) is represented by a level set of a Busemann function associated to γ.*

This theorem provides an affirmative answer to the question raised by Yau for globally hyperbolic space-times with nonpositive timelike sectional curvatures without imposing the assumption of geodesic completeness.

We work directly with the Busemann function in obtaining Theorem 5.2 as in [5] rather than dealing with direct geometric constructions as in the Riemannian proof in [6]. We have also been influenced by a series of papers by R. Greene and H. Wu (cf. [16] for a survey) and by a paper of Eschenburg and Heintze [7].

We would like to thank J.-H. Eschenburg and E. Heintze for providing us with a preprint of [7].

2. Preliminaries

In this paper (M, g) will always be a connected, time oriented Lorentzian manifold which is globally hyperbolic with metric g of signature $(-, +, \cdots, +)$. If A is a subset of M, then $I^+(A) = \{q \in M \mid a \ll q \text{ for some } a \in A\}$ and $I^-(A)$ is defined dually. The sets $I^+(p)$, $I^-(p)$, $I^+(A)$, and $I^-(A)$ are always open. Furthermore, we set $I(A) = I^+(A) \cap I^-(A)$.

Given $p, q \in M$, set $d(p, q) = 0$ if $q \notin J^+(p)$ and let $d(p, q)$ be the supremum of lengths of future directed causal curves from p to q if $q \in J^+(p)$. The *Lorentzian distance function* d satisfies the *reverse triangle inequality* $d(p, q) \geqslant d(p, r) + d(r, q)$ whenever $p \leqslant r \leqslant q$. Since (M, g) is globally hyperbolic, the Lorentzian distance function is both finite valued and continuous, (cf. [13]).

A causal geodesic is *maximal* if the length between any pair of its points is equal to the Lorentzian distance between these points. For a unit speed future directed timelike geodesic γ: $(a, b) \to M$, this means that $d(\gamma(t_1), \gamma(t_2)) = t_2 - t_1$ for all $a < t_1 < t_2 < b$. A maximal timelike geodesic γ is a *line* if it is complete (i.e., $a = -\infty$ and $b = \infty$). A maximal causal geodesic of the form γ: $[a, \infty) \to M$ is called a *causal ray*.

Most of our notational conventions are standard and may be found in [2], [13], and [14].

3. Busemann functions

If γ: $(-\infty, \infty) \to M$ is a future directed timelike line, then for each fixed $r \geqslant 0$ we define b_r^+: $M \to \mathbf{R}$ by $b_r^+(x) = r - d(x, \gamma(r))$. (cf. [3, p. 131], [5, p. 119]). These functions are continuous functions of both x and r because (M, g) is globally hyperbolic.

If $\gamma(r) \notin I^+(x)$, then $d(x, \gamma(r)) = 0$ and $b_r^+(x) = r$. Thus $b_r^+(x)$ is an increasing function of r for fixed x as long as $\gamma(r) \notin I^+(x)$. On the other hand, if $x \ll \gamma(r)$ for some $r \geqslant 0$, then there is a smallest $r_0 \geqslant 0$ such that $x \ll \gamma(r)$ for all $r_0 < r < \infty$. Assuming such an r_0 exists, the reverse triangle inequality implies that $b_r^+(x)$ is a monotone decreasing function of r for all $r > r_0$. If we then allow the possible values of $-\infty$ and $+\infty$, the *Busemann function*

$$b^+(x) = \lim_{r \to \infty} b_r^+(x)$$

exists for all $x \in M$. In the case $x \notin I^-(\gamma)$ we have $b^+(x) = +\infty$. In general, b^+ need not be a continuous function of x for globally hyperbolic space-times. In fact, examples conformal to a subset of the Minkowski plane L^2 may be constructed with b^+ discontinuous.

If $x \ll \gamma(r)$ for all $r_0 < r < \infty$, given a sequence of points $\{x_n\}$ converging to x and a sequence of numbers $\{r_n\}$ diverging to $+\infty$ we will have $x_n \ll \gamma(r_n)$ for all sufficiently large n by the openness of chronological sets. We will implicitly use the properties of limit curves (cf. [2]) to define the notion of co-ray to γ as follows. A (future) *co-ray* to γ from x will be a causal curve starting at x which is future inextendible and is the limit curve of a sequence of maximal length timelike geodesic segments from x_n to $\gamma(r_n)$ for two sequences $\{x_n\}$, $\{r_n\}$ with $x_n \to x$ and $r_n \to \infty$ (cf. [2, pp. 33–45]). Past co-rays are defined dually. This definition of co-ray corresponds to that used by Busemann [3, p. 130]. A (future) co-ray is always a maximal length causal geodesic starting at a point x. However, there may be more than one co-ray to γ from x. Furthermore, a co-ray to γ from x may be a null geodesic. To rule out this possibility, we impose the following condition on (M, g).

Definition 3.1. The globally hyperbolic space-time (M, g) satisfies the *timelike co-ray condition* for the timelike line $\gamma: (-\infty, \infty) \to M$ if for each $x \in I^+(\gamma) \cup I^-(\gamma)$ all future and past co-rays to γ from x are timelike.

The timelike co-ray condition has the following technical consequences which may be obtained using standard arguments.

Lemma 3.2. *Let (M, g) be a globally hyperbolic space-time which satisfies the timelike co-ray condition for the timelike line $\gamma: (-\infty, \infty) \to (M, g)$. Let $x \in I^-(\gamma)$ be arbitrary and $\varepsilon > 0$ be given. Then there exists an integer $N > 0$, a neighborhood $U(x)$ of x with $U(x) \subset I^-(\gamma)$ and an open set V with compact closure $K = \overline{V} \subset I^-(x)$ satisfying the following properties*:

(a) *$K \subset I^+(y)$ for all $y \in U(x)$.*

(b) *Given any $y \in U(x)$ and $q \in K$, we have $d(y, q) < \varepsilon$.*

(c) *If $y \in U(x)$ and $t \geqslant N$, then any maximal timelike geodesic segment from y to $\gamma(t)$ intersects K.*

We now show b^+ is continuous on the set $I^-(\gamma)$.

Lemma 3.3. *Let (M, g) be globally hyperbolic. If (M, g) satisfies the timelike co-ray condition for the timelike line $\gamma: (-\infty, \infty) \to M$, then the Busemann function b^+ is continuous and finite on $I^-(\gamma)$.*

Proof. Let $\varepsilon > 0$ be given and fix $x \in I^-(\gamma)$. Let $U(x)$, N and K be as in Lemma 3.2. Choose any two points y_1, $y_2 \in U(x)$ and any r with $N < r < \infty$. Let G_i be a maximal length timelike geodesic from y_i to $\gamma(r)$ and let $q_i \in G_i \cap K$ for $i = 1, 2$. Then $d(y_1, \gamma(r)) = d(y_1, q_1) + d(q_1, \gamma(r))$ and $d(y_1, q_1) < \varepsilon$ yield $d(q_1, \gamma(r)) > d(y_1, \gamma(r)) - \varepsilon$. Since $y_2 \ll q_1 \ll \gamma(r)$ by Lemma 3.2(a), the reverse triangle inequality yields

$$d(y_2, \gamma(r)) \geqslant d(y_2, q_1) + d(q_1, \gamma(r)).$$

Thus

$$d(y_2, \gamma(r)) > 0 + d(y_1, \gamma(r)) - \varepsilon$$

which implies $\varepsilon > b_r^+(y_2) - b_r^+(y_1)$. Since also $q_2 \in I^+(y_1)$, we may reverse the roles of y_1 and y_2 to obtain

$$|b_r^+(y_1) + b_r^+(y_2)| < \varepsilon.$$

This establishes the equicontinuity of the functions b_r^+ on $U(x)$ for all $N < r < \infty$. Since $b_r^+(x)$ is monotone decreasing for large r, the limit $b^+(x)$ is an element of $\mathbf{R} \cup \{-\infty\}$. If $b^+(x) = -\infty$, then $b_r^+(x) \to -\infty$ as $r \to \infty$ and the last inequality yields $b^+(y) = -\infty$ for all $y \in U(x)$. It follows that b^+ equals

$-\infty$ on an open subset V_1 of $I^-(\gamma)$. On the other hand, if $b^+(x)$ is finite, then the last inequality yields $|b^+(x) - b^+(y)| \leqslant \varepsilon$ for all $y \in U(x)$. In this case the equicontinuous family $\{b_r^+\}$ for $N < r < \infty$ is bounded on $U(x)$ and hence converges to a finite valued continuous function b^+ on $U(x)$. Thus b^+ is a continuous finite valued function on an open subset V_2 of $I^-(\gamma)$ and is equal to $-\infty$ on the open subset V_1 of $I^-(\gamma)$, where $I^-(\gamma) = V_1 \cup V_2$. Using $b^+(\gamma(r_1)) = r_1$ on γ and the connectedness of $I^-(\gamma)$ it follows that $V_2 = I^-(\gamma)$ which establishes the lemma. q.e.d.

Since $\gamma: (-\infty, \infty) \to M$ is a timelike *line* rather than just a ray, one may also define a Busemann function b^- on $I^+(\gamma)$ by

$$b_r^-(x) = r - d(\gamma(-r), x), \qquad r \geqslant 0.$$
$$b^-(x) = \lim_{r \to +\infty} b_r^-(x).$$

Arguments similar to Lemma 3.3 imply that b is continuous on $I^+(\gamma)$. Thus both b^+ and b^- are continuous on $I(\gamma) = I^+(\gamma) \cap I^-(\gamma)$. Given any $x \in I(\gamma)$, choose positive numbers s and t such that $\gamma(-s) \ll x \ll \gamma(t)$. The reverse triangle inequality and $d(\gamma(-s), \gamma(t)) = s + t$ yield $b_t^+(x) + b_s^-(x) \geqslant 0$. Letting $s, t \to \infty$ we obtain the following inequality on $I(\gamma)$:

$$B(x) := b^+(x) + b^-(x) \geqslant 0. \tag{3.1}$$

4. The significance of nonpositive timelike sectional curvature

The two-plane $E = \{u, v\}$ is *timelike* if the metric induced on E is Lorentzian. Thus (M, g) has nonpositive timelike sectional curvatures if for each timelike plane E we have $\langle R(u, v)v, u\rangle \geqslant 0$.

Proposition 4.1. *Let (M, g) be a globally hyperbolic space-time of dimension $\geqslant 2$ with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ. Then the timelike co-ray condition holds on $I(\gamma)$. Thus b^+ and b^- are both continuous on $I(\gamma)$.*

Proof. Assume that $x \in I(\gamma)$ has a future co-ray σ to γ such that σ is null. Then there exist sequences $\{x_n\}$, $\{r_n\}$ and $\{\sigma_n\}$ with $x_n \to x$, $r_n \to \infty$ such that each σ_n is a maximal timelike geodesic segment from x_n to $\gamma(r_n)$ and σ is a limit curve of the sequence $\{\sigma_n\}$. Choose $q \in \gamma \cap I^-(x)$ and let μ_n be a maximal timelike geodesic segment from q to x_n. The segments μ_n are defined for all large n and we may assume that $\{\mu_n\}$ converges to a *timelike* geodesic from q to x. Let γ_n be the segment of γ from q to $\gamma(r_n)$ and set $a_n = L(\mu_n)$, $b_n = L(\sigma_n)$ and $c_n = L(\gamma_n)$, where L denotes arc length. Assuming μ_n, σ_n and

γ_n are parametrized by arclength, define $\beta_n = g(\mu'_n(0), \gamma'_n(0))$ and $\theta_n = g(-\mu'_n(a_n), \sigma'_n(0))$. Then $a_n \to a = d(q, x)$ and $\beta_n \to \beta = g(\mu'(0), \gamma'(0))$. We now apply Harris' triangle comparison theorem (cf. [2, p. 430], [11, p. 303]) to the timelike geodesic triangle $(\mu_n, \sigma_n, \gamma_n)$ using the two-dimensional Minkowski plane as model space. Thus for each n there is a timelike geodesic triangle $(\bar{\mu}_n, \bar{\sigma}_n, \bar{\gamma}_n)$ in L^2 with

$$a_n = L(\bar{\mu}_n), \quad b_n = L(\bar{\sigma}_n), \quad c_n = L(\bar{\gamma}_n), \quad |\bar{\beta}_n| \leqslant |\beta_n|, \quad \bar{\theta}_n \geqslant \theta_n,$$

where $\bar{\beta}_n$ and $\bar{\theta}_n$ are defined analogously to β_n and θ_n. Using $|\bar{\beta}_n| \leqslant |\beta_n|$ and $\beta_n \to \beta$, we find there is some positive C with $|\bar{\beta}_n| \leqslant C$ for all n. The law of cosines for L^2 yields

$$c_n^2 = a_n^2 + b_n^2 + 2a_n b_n \bar{\theta}_n, \tag{4.1}$$

$$b_n^2 = a_n^2 + c_n^2 + 2a_n c_n \bar{\beta}_n. \tag{4.2}$$

Adding (4.2) to (4.1) and solving for $\bar{\theta}_n$ we obtain

$$\bar{\theta}_n = -\frac{c_n \bar{\beta}_n}{b_n} - \frac{a_n}{b_n}.$$

Since $a_n \to a$, $c_n \to \infty$ and $|\bar{\beta}_n| \leqslant C$, equation (4.2) implies $b_n/c_n \to 1$ and $b_n \to \infty$. Consequently, there is some constant C_1 such that $\bar{\theta}_n \leqslant C_1$ for all n. Using $\theta_n \leqslant \bar{\theta}_n$ we find

$$g\big(-\mu'_n(a_n), \sigma'_n(0)\big) \leqslant C_1$$

which implies that the Lorentzian angles between the segments μ_n and σ_n are all bounded by some constant. On the other hand, since σ is assumed to be null, then $\mu_n \to \mu$ and $\sigma_n \to \sigma$ imply these angles must approach ∞. This contradiction establishes the result. q.e.d.

We now show that co-rays to γ are complete given $K \leqslant 0$.

Lemma 4.2. *Let (M, g) be a globally hyperbolic space-time with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ. If η is a future (resp. past) directed timelike co-ray from $x \in I(\gamma)$, then η is future (resp. past) complete.*

Proof. Assume $w \log \eta$ is future directed. Suppose η has finite length L. Let γ: $(-\infty, \infty) \to M$ and η: $[0, L) \to M$ be parametrized with respect to arclength.

By Lemma 3.2 there exists a constant $K > 0$ and "time" T such that if $t > T$ and μ is a maximal segment from x to $\gamma(t)$, then

$$\big|\big\langle \mu'(0), \eta'(0) \big\rangle\big| \leqslant K.$$

Also, by the reverse triangle inequality,

$$d(x, \gamma(t)) \to +\infty \quad \text{as } t \to +\infty.$$

Thus, we can choose $t_0 \geqslant T$ such that

$$d(x, \gamma(t)) \geqslant 3LK \quad \text{for } t \geqslant t_0.$$

Pick a point $y \in \eta$ such that $\gamma(t_0) \notin I^+(y)$. (If no such point existed, then η would be imprisoned in $J^-(\gamma(t_0)) \cap J^+(x)$.) Choose y_1 on η with $y_1 \in I^+(y)$. Since η is a co-ray to γ there exist points $p_n \to y_1$ such that each p_n lies on some maximal timelike geodesic from x to a point q_n of γ. It follows that for sufficiently large n we have $p_n \in I^+(y)$ which implies $q_n \in I^+(y)$. Consequently, there exists a time $t_1 > t_0$ such that $\gamma(t_1) \in \partial I^+(y)$. By the global hyperbolicity of M, $\partial I^+(y) = J^+(y) - I^+(y)$, and hence $d(y, \gamma(t_1)) = 0$. Thus one can choose a time $t_2 > t_1$ such that

$$0 < d(y, \gamma(t_2)) < 3LK/2$$

since the Lorentzian distance function is continuous for globally hyperbolic space-times. Let μ be a maximal segment from x to $\gamma(t_2)$ and let σ be a maximal segment from y to $\gamma(t_2)$. Let ν be the portion of η from x to y. Consider the timelike triangle (ν, μ, σ). Let a, b and c be the lengths of ν, σ and μ, respectively. We have

(1) $c \geqslant 3LK$, $b < c/2$ and $a < L$.

Also, $\beta = \langle \mu'(0), \eta'(0) \rangle$ satisfies

(2) $|\beta| \leqslant K$.

By Harris' triangle comparison theorem there exists a corresponding timelike triangle $(\bar{\mu}, \bar{\mu}, \bar{\sigma})$ in Minkowski space such that $L(\bar{\nu}) = a$, $L(\bar{\mu}) = b$, $L(\bar{\sigma}) = c$ and $|\bar{\beta}| \leqslant |\beta|$. By the law of cosines,

$$b^2 = a^2 + c^2 - 2ac|\bar{\beta}|.$$

Using (1), (2) and $|\bar{\beta}| \leqslant |\beta|$, we obtain

$$b^2 \geqslant c^2 - 2LKc = c^2\left(1 - \frac{2LK}{c}\right) \geqslant \frac{c^2}{3}$$

which contradicts the second inequality in (1). Thus η must be infinite in length. q.e.d.

At this juncture, we do *not* know that the functions $B, b^=$ and b^- defined in §3 are differentiable functions. Thus as in [7], we now need to define smooth local support functions at each $p \in I(\gamma)$ for b^+ and b^-. Fix $p \in I(\gamma)$ and a sequence of real numbers $\{r_n\}$ with $r_n \to +\infty$. Set $\alpha_n = d(p, \gamma(r_n))$. Then α_n is positive for all sufficiently large n, $\alpha_n \to \infty$ by the reverse triangle inequality, and

$$b^+(p) = \lim_{n\to\infty}(r_n - \alpha_n). \tag{4.3}$$

Furthermore, for each sufficiently large n there is some unit future directed timelike vector $v_n^+ \in T_pM$ such that $\exp_p(\alpha_n v_n^+) = \gamma(r_n)$. Using the timelike co-ray condition, we may assume that $v_n^+ \to v^+$, where v^+ is a unit future directed timelike vector at p. The future inextendible timelike geodesic with initial velocity vector v^+ is a co-ray to γ from p and is future complete by Lemma 4.2. Choose a sufficiently small neighborhood $U(p)$ of p, such that $x \ll \gamma(r_n)$ for all n larger than some N and all $x \in U(p)$. Fixing $U(p)$ and N there is some $0 < a_0 < \alpha_n$ such that

$$x \ll \exp_p(a_0 v_n^+) \ll \exp_P(a v_n^+) \ll \exp_p(\alpha_n v_n^+) = \gamma(r_n)$$

for all $n \geqslant N$, each a with $a_0 < a < \alpha_n$, and all $x \in U(p)$. Applying the reverse triangle inequality yields

$$\begin{aligned} d(x, \gamma(r_n)) &\geqslant d\big(x, \exp_p(a v_n^+)\big) + d\big(\exp_p(a v_n^+), \gamma(r_n)\big) \\ &\geqslant d\big(x, \exp_p(a v_n^+)\big) + \alpha_n - a. \end{aligned}$$

We obtain $r_n - \alpha_n + a - d(x, \exp_p(a v_n^+)) \geqslant r_n - d(x, \gamma(r_n))$. In view of equation (4.3), we thus have

$$b^+(p) + a - d\big(x, \exp_p(a v^+)\big) \geqslant b^+(x) \tag{4.4}$$

for all $x \in U(p)$ and $a_0 < a < \infty$. This inequality motivates the following definition of a family of functions $b^+_{p,a}$ (cf. [7]):

$$b^+_{p,a}(x) = b^+(p) + a - d\big(x, \exp_p a v^+\big). \tag{4.5}$$

Equation (4.4) shows that for sufficiently large a these functions are continuous *local super support functions* for the Busemann function b^+; we have $b^+_{p,a}(p) = b^+(p)$ and $b^+_{p,a}(x) \geqslant b^+(x)$ for all x sufficiently close to p.

Now given $p \in I(\gamma)$, construct a unit past directed timelike vector $v^- \in T_pM$ using the same technique as for v^+. Then

$$b^-_{p,a}(x) = b^-(p) + a - d\big(\exp_p(a v^-), x\big) \tag{4.6}$$

provides a family of local super support functions for b^-. That is, $b^-_{p,a}(p) = b^-(p)$ and $b^-_{p,a}(x) \geqslant b^-(x)$ for all x near p and large a for any $p \in I(\gamma)$. For any fixed parameter value a, the nonspacelike cut locus of $\exp_p(a v^+)$ is closed (cf. [2, p. 242]) and since $s \to \exp_p(s v^\pm)$, $s \in [0, a]$ is maximal, we get that there is a neighborhood of p in which $b^+_{p,a}$ (resp., $b^-_{p,a}$) is a *smooth* super support function for the continuous Busemann function b^+ (resp., b^-). Hence $B_{p,a} = b^+_{p,a} + b^-_{p,a}$ is also a smooth super support function for $B = b^+ + b^-$ near p.

In view of the definition of $b^+(x)$ and $b^-(x)$, we now consider functions of the form $f(x) = d(q, x)$ (resp., $d(x, q)$), where q is a given point of M. In general, these functions will fail to be differentiable across null cones as well as on the timelike cut locus of q (cf. [2, p. 105]). Since the null cut locus and nonspacelike cut locus of each $q \in M$ are closed, if $q \ll p$ and p is not in the (future) cut locus of q, then the function $f(x) = d(q, x)$ is smooth on some neighborhood $U(p)$ of p which contains no cut points of q. Furthermore, $\langle \operatorname{grad} f, \operatorname{grad} f \rangle = -1$ on $U(p)$. Conversely, let $f\colon M \to \mathbf{R}$ be a smooth function on an arbitrary Lorentzian manifold (M, g) with $\langle \operatorname{grad} f, \operatorname{grad} f \rangle \equiv -1$. Using the definition of Lorentzian arc length and the reverse Schwartz inequality it is easy to show that any integral curve c of $\operatorname{grad} f$ is a maximal timelike geodesic.

Lemma 4.3. *Let N be an open subset of the Lorentzian manifold (M, g) and assume $f\colon N \to \mathbf{R}$ is a smooth function with $\langle \operatorname{grad} f, \operatorname{grad} f \rangle \equiv -1$ on N. Let $c\colon (a, b) \to N$ be an integral curve of $-\operatorname{grad} f$ and V be any unit parallel vector field which is orthogonal to c. Then*

$$-\langle R(V, c')c', V \rangle \geqslant -(\operatorname{Hess}(f)(V, V) \circ c)' + (\operatorname{Hess}(f)(V, V) \circ c)^2.$$

Let W be an arbitrary parallel field along c. Set

$$\alpha = \alpha(W, c') = \sqrt{\langle W, W \rangle + (\langle W, c' \rangle)^2}.$$

If W is not proportional to c', set

$$V_1 = W + \langle W, c' \rangle c' \quad \text{and} \quad V = V_1 / \| V_1 \|. \tag{4.7}$$

Then $\alpha V = W + \langle W, c' \rangle c'$ and V is a spacelike unit parallel field which is orthogonal to c. Since $\langle \operatorname{grad} f, \operatorname{grad} f \rangle = -1$, one also obtains $\operatorname{Hess}(f)(X, c') = 0$ for any vector field X along c. Thus with α and V as above

$$\operatorname{Hess}(f)(W, W) = \alpha^2 \operatorname{Hess}(f)(V, V) \tag{4.8}$$

and we obtain the following estimate on the Hessian of $d(q, x)$.

Proposition 4.4. *Let (M, g) be a globally hyperbolic space-time of everywhere nonpositive timelike sectional curvature. Fix $p \in M$ and let $q \in I^+(p) \cup I^-(p)$ be any point which is not a cut point of p. Let $c\colon [0, L] \to M$ denote a unit speed maximal timelike geodesic from q to p. If $q \in I^+(p)$, set $f(x) = d(x, q)$*

and if $q \in I^-(p)$, set $f(x) = d(q,x)$. Then

$$-\frac{\alpha^2(w)}{f(p)} \leqslant \mathrm{Hess}(f)(w,w)|_p \tag{4.9}$$

for any $w \in T_pM$, where $\alpha^2(w) = \langle w,w\rangle + (\langle w, c'(L)\rangle)^2$.

Proof. As the proofs for $d(x,q)$ and $d(q,x)$ are dual, we only give the proof for $f(x) = d(q,x)$ and $q \in I^-(p)$. Let N be an open subset of $I^+(q)$ such that $c(s) \in N$ for all $0 < s \leqslant d(q,p)$ and such that N contains no cut points of q. Then c is an integral curve of $-\mathrm{grad}\, f$ for all $0 < s \leqslant d(q,p)$ and f is C^∞ on N. Given a fixed $w \in T_pM$ let V be the unit parallel field along c which satisfies $\alpha V = w + \langle w, c'\rangle c'$ as in equation (4.7). Define $\theta(s) = \mathrm{Hess}(f)(V,V) \circ c(s)$ for all $0 < s \leqslant d(q,p)$. At $x = c(s)$ we have $\mathrm{Hess}(f)(V,V)|_x = -\langle \nabla_V c', V\rangle|_x = S_{c'}(V,V)|_x$, where $S_{c'}$ is the second fundamental form of the distance sphere $\{y \in M \mid d(q,y) = d(q,x)\}$ through x. Thus $\mathrm{Hess}(f)(V,V)|_{c(s)} \to -\infty$ as $s \to 0^+$.

Lemma 4.3 and the curvature assumption yield $\theta^2 - \theta' \leqslant 0$ which implies that θ is nondecreasing and for all $\theta(s) \neq 0$ that

$$\theta^{-1}(s) \leqslant -s \tag{4.10}$$

using $d/ds(\theta^{-1}) = -\theta'/\theta^2$ and $\theta(s) \to -\infty$ as $s \to 0^+$. Thus for all $s > 0$, we have $\theta(s) \geqslant s^{-1}$. Setting $s = d(q,p) = f(q)$ yields

$$\mathrm{Hess}(f)(V,V)|_p \geqslant -\frac{1}{f(p)}. \tag{4.11}$$

The result now follows using equations (4.8) and (4.11).

Corollary 4.5. *Let (M,g) be a globally hyperbolic space-time with $K \leqslant 0$ and suppose that (M,g) contains a timelike line γ: $(-\infty,\infty) \to (M,g)$. Then for any $p \in I(\gamma)$ and $a > 0$, we have*

$$\mathrm{Hess}(b^+_{p,a})(w,w) \leqslant \frac{\alpha^2_+(w)}{a}, \tag{4.12}$$

$$\mathrm{Hess}(b^-_{p,a})(w,w) \leqslant \frac{\alpha^2_-(w)}{a} \tag{4.13}$$

for any $w \in T_pM$, where $\alpha^2_+(w) = \langle w,w\rangle + (\langle w, v^+\rangle)^2$ and $\alpha^2_-(w) = \langle w,w\rangle + (\langle w, v^-\rangle)^2$.

Proof. Since the arguments for (4.12) and (4.13) are similar, it suffices to establish (4.12). Consider the function $f(x) = d(x,q)$ with $q := \exp_p(av^+)$. Since γ satisfies the timelike co-ray condition, $c^+(t) = \exp_p(tv^+)$, $t \geqslant 0$, is a maximal, future directed, future complete timelike geodesic ray. Hence for any $a > 0$, $q = \exp_p(av^+) \in I^+(p)$ is not a cut point to p and $d(p,q) = a$. Thus inequality (4.9) may be applied to $f(x) = d(x,q)$ at $x = p$ to yield inequality (4.12) as $c'(L) = -v^+$. q.e.d.

Using the timelike co-ray condition, Corollary 4.5 and a one-dimensional Calabi-type maximum principle argument, we now show that the function B defined by equation (3.1) vanishes on $I(\gamma)$.

Lemma 4.6. *If $y \in I(\gamma)$ and $B(y) = 0$, then B vanishes on a neighborhood of y. Hence $B \equiv 0$ on $I(\gamma)$.*

Proof. If the conclusion fails, there is some geodesic segment $\sigma: [-1, 1] \to I(\gamma)$ of g with $\sigma(0) = y$ and $B(\sigma(s_1)) > 0$, where $0 < s_1 < 1$. Let $h: [-s_1, s_1] \to \mathbf{R}$ be defined by $h(s) = -\varepsilon(s + s^2)$. If $\varepsilon > 0$ is chosen sufficiently small, the continuous function $B \circ \sigma + h$ is positive at both endpoints of $[-s_1, s_1]$ and zero at $s = 0$. Thus $B \circ \sigma + h$ attains a minimum at some $s_0 \in (-s_1, s_1)$. Let $p = \sigma(s_0)$ and choose a future (resp. past) timelike vector v^+ (resp. v^-) at p tangent to a co-ray to γ from p. Using equations (4.5) and (4.6) define the local support functions $b^+_{p,a}$ (resp., $b^-_{p,a}$) using v^+ (resp., v^-). Then the function $B_{p,a}(x) = b^+_{p,a}(x) + b^-_{p,a}(x)$ is smooth and satisfies $B_{p,a}(p) = B(p)$ and $B_{p,a}(x) \geqslant B(x)$ in some neighborhood $U(p)$ of p. Applying Corollary 4.5 with $w = \sigma'(s_0)$ we obtain

$$\left(B_{p,a} \circ \sigma\right)''\Big|_p \leqslant \left(\alpha_+^2(w) + \alpha_-^2(w)\right)/a.$$

Thus

$$\left(B_{p,a} \circ \sigma + h\right)''\Big|_p \leqslant \left(\alpha_+^2(w) + \alpha_-^2(w)\right)a^{-1} - 2\varepsilon < 0$$

for sufficiently large a. This contradicts

$$B_{p,a} \circ \sigma(s) + h(s) \geqslant B \circ \sigma(s) + h(s) \geqslant B_{p,a}(p) + h(s_0)$$

for all $-s_1 \leqslant s \leqslant s_1$ with $\sigma(s) \in U(p)$. q.e.d.

We have been unable to extend Lemma 4.6 with our proof method to the case of the strong energy condition $\operatorname{Ric}(v, v) \geqslant 0$ for all timelike vectors v because the d'Alembertian operator (which corresponds to the Laplacian in the Riemannian case) is hyperbolic rather than elliptic.

Lemma 4.6 implies $b^+(x) \equiv -b^-(x)$ on $I(\gamma)$.

Lemma 4.7. *The Busemann functions b^+ and b^- are once differentiable on $I(\gamma)$ and the vector field $V = \operatorname{grad} b^+ = -\operatorname{grad} B^-$ is a unit past directed timelike vector field defined on $I(\gamma)$. The vector field V is continuous and at each point $p \in I(\gamma)$ there is a unique future directed co-ray $c^+(t) = \exp_p(-tV)$ and a unique past directed co-ray $c^-(t) = \exp_p(tV)$ to γ. These co-rays to γ at p fit together to form a (distance realizing and complete) timelike line.*

Proof. Choose $p \in I(\gamma)$ and let v^+ and v^- be unit timelike vectors at p which determine future and past directed co-rays at p, respectively. Let $b^+_{p,a}$ and $b^-_{p,a}$ be defined using v^+ and v^-, respectively according to equations (4.5) and (4.6). Now $B_{p,a}(x) = b^+_{p,a}(x) + b^-_{p,a}(x) \geqslant B(x) = 0$ and $B_{p,a}(p) = B(p) = 0$ imply the smooth function $B_{p,a}(x)$ satisfies $\operatorname{grad} B_{p,a}|_p = 0$. Thus $v^+ = -\operatorname{grad} b^+_{p,a}|_p = \operatorname{grad} b^-_{p,a}|_p = -v^-$ so that $v^+ = -v^-$ and the future and past timelike co-rays at p fit together to form a smooth geodesic c.

Also, we have $b^+_{p,a} \geqslant b^+ \geqslant -b^- \geqslant -b^-_{p,a}$ near p with equality at p. Since $\operatorname{grad} b^+_{p,a}(p) = -\operatorname{grad} b^-_{p,a}(p)$, it follows that B^+ and b^- are differentiable at p and $\operatorname{grad} b^+(p) = -\operatorname{grad} b^-(p) = \operatorname{grad} b^+_{p,a}(p) = -\operatorname{grad} b^-_{p,a}(p)$. Now as $v^+ = -\operatorname{grad} b^+_{p,a}(p) = -\operatorname{grad} b^+(p)$ and $v^- = -\operatorname{grad} b^-_{p,a}(p) = -\operatorname{grad} b^-(p)$ the future and past co-rays to γ at p are unique for any $p \in I(\gamma)$. This last fact then implies the continuity of $V = \operatorname{grad} b^+ = -\operatorname{grad} b^-$ since the initial tangent to future co-rays to γ varies continuously with p. Finally, since the restriction to a co-ray is a co-ray, it follows that the geodesic c formed from the union of the co-rays c^+ and c^- to γ at p is distance realizing. Since any timelike co-ray to γ has infinite length, c is also complete.

5. Splitting $I(\gamma)$ and M

We are now ready to show that $I(\gamma)$ is a metric product.

Lemma 5.1. *The set $I(\gamma)$ is isometric to a Lorentzian product $\mathbf{R} \times H$, where (H, h) is a spacelike hypersurface of $I(\gamma)$. Furthermore, each spacelike slice $\{t_0\} \times H$ corresponds to the intersection of $I(\gamma)$ with a level set of b^+ (resp., b^-).*

Proof. Fix $p \in I(\gamma)$ and let c be a geodesic with $c(0) = p$. Since $b^+_{p,a}(x) \geqslant b^+(x) = -b^-(x) \geqslant -b^-_{p,a}(x)$ for all x near p, $b^+ \circ c$ has both super support functions $b^+_{p,a} \circ c$ and subsupport functions $-b^-_{p,a} \circ c$. Corollary 4.5 shows that these support functions have arbitrarily small second derivatives for all t near 0. If $L\colon \mathbf{R} \to \mathbf{R}$ is any affine function, then the same is true of $b^+ \circ c - L$. It follows that $b^+ \circ c$ is an affine function near $t = 0$ and this implies $b^+ \circ c$ is an affine function for any geodesic c with image in $I(\gamma)$. Hence if $H(t_0) = \{q \in I(\gamma) \mid b^+(q) = t_0\}$ is the t_0 level set of b^+ in $I(\gamma)$, and c is any geodesic segment $c\colon [0, a] \to I(\gamma)$ with endpoints in $H(t_0)$, then $b^+ \circ c(0) = b^+ \circ c(a)$ implies $b^+ \circ c(t) = b^+ \circ c(0)$ for all $0 \leqslant t \leqslant a$. Thus $c([0, a]) \subset H(t_0)$, which shows $H(t_0)$ is totally geodesic.

Fixing $p \in I(\gamma)$ we let $e_1 = -\operatorname{grad} b^+(p)$, $e_2, \cdots, e_n$ be an orthonormal basis of T_pM and use this basis to obtain normal coordinates $x_1, x_2, \cdots, x_n$ near p. Any geodesic c with $c(0) = p$ has a representation as $c(t) = (t\alpha_1, \cdots, t\alpha_n)$ near p, where $\alpha_1, \cdots, \alpha_n$ are constants. The affine function $b^+ \circ c$ is given by $b^+ \circ c(t) = At + B_0$, where $B_0 = b^+(p)$. Using $\operatorname{grad} b^+ = -\partial/\partial x_1$ at p we obtain $A = \alpha_1$ and thus $b^+(x) = x_1 + b^+(p)$ in these local coordinates. This shows b^+ is smooth.

The vector field $\operatorname{grad} b^+$ is everywhere orthogonal to the totally geodesic level surfaces $H(t_0)$ and $X = \operatorname{grad} b^+$ is a unit normal field to $H(t_0)$. The second fundamental form S_X must vanish on $H(t_0)$ because this surface is totally geodesic. Thus if v, w are tangent to $H(t_0)$, $S_X(v, w) = \langle -\nabla_v X, w\rangle = 0$. On the other hand, $\langle X, X\rangle \equiv -1$ yields that $\nabla_v X$ is orthogonal to X and hence tangential to $H(t_0)$. Furthermore, X is the unit tangent to the (geodesic) co-ray to γ through each $p \in I(\gamma)$ and hence $\nabla_X X \equiv 0$. Thus $X = \operatorname{grad} b^+$ is a parallel timelike vector field on $I(\gamma)$. Hence $I(\gamma)$ splits locally isometrically by Wu's proof of the local Lorentzian de Rham Theorem (cf. [15, p. 299]).

The vector field $\operatorname{grad} b^+$ is complete since all co-rays to γ are complete geodesics which are contained in $I(\gamma)$. Consequently, the map $I(\gamma) \to I(\gamma)$ given by $p \to \exp_p(t \operatorname{grad} b^+(p))$ is an isometry of $I(\gamma)$ onto $I(\gamma)$ for each fixed $t \in \mathbf{R}$. This isometry takes level sets of b^+ to level sets of b^+. Using the induced metric H on $H(0)$ and the product metric $-dt^2 \otimes h$ on $\mathbf{R} \times H(0)$, we find that F: $\mathbf{R} \times H(0) \to I(\gamma)$ given by $F(t, p_0) = \exp_{p_0}(t \operatorname{grad} b^+(p_0))$ is an isometry onto $I(\gamma)$. This establishes the result. q.e.d.

We are finally ready to prove the main theorem.

Theorem 5.2. *Let (M, g) be a globally hyperbolic space-time of dimension $\geqslant 2$ with everywhere nonpositive timelike sectional curvatures $K \leqslant 0$ which contains a complete timelike line γ: $(-\infty, \infty) \to (M, g)$. Then (M, g) is isometric to a product $(\mathbf{R} \times H, -dt^2 \oplus h)$, where (H, h) is a complete Riemannian manifold. The factor $(\mathbf{R}, -dt^2)$ is represented by γ and (H, h) is represented by a level set of a Busemann function associated to γ.*

Proof. The set $I(\gamma)$ must be strongly causal because it is an open subset of the globally hyperbolic space-time (M, g). Furthermore, $p, q \in I(\gamma)$ implies the compact set $J^+(p) \cap J^-(q)$ also lies in $I(\gamma)$. Thus $I(\gamma)$ is globally hyperbolic. By Lemma 5.1, $I(\gamma)$ is isometric to $\mathbf{R} \times H$. But $\mathbf{R} \times H$ is globally hyperbolic implies H and $\mathbf{R} \times H$ are geodesically complete (cf. [2, p. 65]). Thus $I(\gamma)$ is geodesically complete and consequently, inextendible (cf. [2, p. 160]). Hence $I(\gamma) = M$.

Corollary 5.3. *(M, g) is geodesically complete and the level surfaces of the Busemann functions b^+ and b^- are complete (spacelike) Cauchy hypersurfaces of (M, g).*

By somewhat similar techniques, the following related results may be obtained.

Proposition 5.4. *Let (M, g) be a globally hyperbolic space-time with $\operatorname{Ric}(v, v) \geqslant 0$ on all timelike vectors v. Assume that (M, g) contains a complete timelike line γ such that every co-ray to γ is timelike and without focal points. Then (M, g) is isometric to a product $(\mathbf{R} \times H, -dt^2 \otimes h)$.*

Theorem 5.5. *Let (M, g) be a space-time with a compact Cauchy surface and everywhere nonpositive timelike sectional curvatures $K \leqslant 0$. Then either M is timelike geodesically incomplete or else M splits as in Theorem 5.2 with H compact.*

The proof of Theorem 5.5 uses a result of Harris [12] to show under the given hypotheses that there exists a null cut point along each future or past inextendible null geodesic. This ensures the existence of a timelike line.

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