J. DIFFERENTIAL GEOMETRY
19 (1984) 125–171

KOSZUL COHOMOLOGY AND THE GEOMETRY OF PROJECTIVE VARIETIES

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0. Introduction

There are a number of interesting problems and results which involve being able to compute Koszul cohomology groups; for example, the local Torelli problem, understanding the canonical ring of a variety of general type, Petri's work on the ideal of a special curve, Mumford's projective normality theorem, and Donagi's work on the global Torelli theorem for projective hypersurfaces.

Received by the editors June 6, 1983. The research was partially supported by N.S.F. Grant MCS 82-00924.

Unfortunately, there seem to be fewer ways to compute Koszul cohomology groups than reasons to compute them. It seemed fruitful to try to find a few techniques which would make it easier to approach these problems.

For V a finite dimensional complex vector space, $S(V)$ the symmetric algebra over V, and $B = \bigoplus_{q\in\mathbf{Z}} B_q$ a graded $S(V)$-module, we have the *Koszul complex*

$$(0.1)\qquad \cdots \to \wedge^{p+1}V\otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \wedge^p V\otimes B_q \xrightarrow{d_{p,q}} \wedge^{p-1}V\otimes B_{q+1}\to\cdots.$$

The *Koszul cohomology groups* are defined by

$$(0.2)\qquad \mathcal{K}_{p,q}(B,V) = \frac{\ker d_{p,q}}{\operatorname{im} d_{p+1,q-1}}.$$

If we have a *minimal free resolution*

$$(0.3)\qquad \cdots\to\bigoplus_{q\geq q_1} M_{1,q}\otimes S(V)(-q)\to\bigoplus_{q\geq q_0} M_{0,q}\otimes S(V)(-q)\to B\to 0,$$

then a well-known result is the *Syzygy Theorem* (1.b.4)

$$(0.4)\qquad \mathcal{K}_{p,q}(B,V)\simeq M_{p,p+q}(B,V).$$

The situation we will study in this paper is

$$(0.5)\qquad \begin{cases} X & \text{a compact complex manifold,}\\ L\to X & \text{an analytic line bundle,}\\ \mathcal{F}\to X & \text{a coherent analytic sheaf,}\\ W\subseteq H^0(X,L) & \text{a linear subspace.}\end{cases}$$

We then take

$$(0.6)\qquad B=\bigoplus_{q\in\mathbf{Z}} H^i(X,\mathcal{F}\otimes qL),\qquad V=W$$

and denote

$$(0.7)\qquad \mathcal{K}^i_{p,q}(X,\mathcal{F},L,W)=\mathcal{K}_{p,q}(B,V)$$

with the conventions

(1) If $\mathcal{F}=\mathcal{O}_X(E)$, we write $\mathcal{K}^i_{p,q}(X,E,L,W)$.

(2) If $\mathcal{F}=\mathcal{O}_X$, we suppress it and write $\mathcal{K}^i_{p,q}(X,L,W)$.

(3) If $W=H^0(X,L)$, we suppress it and write $\mathcal{K}^i_{p,q}(X,\mathcal{F},L)$.

(4) If $i=0$, we may suppress it and write $\mathcal{K}_{p,q}(X,\mathcal{F},L,W)$.

By Serre Duality, we have

$$(0.8)\qquad \mathcal{K}^i_{p,q}(X,E,L,W)^*\simeq\mathcal{K}^{n-i}_{\dim W-p,-q}(X,K_X\otimes E^*,L,W),$$

where dim $X = n$. There is the spectral sequence for *Koszul cohomology* which abuts to zero and has

(0.9) $$E_2^{p,q} = \mathcal{K}^q_{-p,k+p}(X, \mathcal{F}, L, W),$$

where k is a constant. A consequence of this is the Duality Theorem (2.c.6)

(0.10) $$\mathcal{K}_{p,q}(X, E, L, W)^* \simeq \mathcal{K}_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W),$$

where dim $X = n$ and dim $W = r + 1$, provided that

(0.11) $$W \subseteq H^0(X, L) \quad \text{is base-point free}$$

and

(0.12) $$\begin{aligned} &H^i(X, E \otimes (q - i)L) = 0, \\ &H^i(X, E \otimes (q - i - 1)L) = 0, \quad \text{for } i = 1, 2, \cdots, n - 1. \end{aligned}$$

Note that the hypothesis (0.12) is vacuous when X is a curve. When X is Kahler, W is base-point free, and dim $\varphi_{mL}(X) = n$ for some $m > 0$, then we have (2.c.10)

(0.13) $$\mathcal{K}_{p,q}(X, K_X, L, W)^* \simeq \mathcal{K}_{r-n-p,n+1-q}(X, L, W)$$

for $q \geq n + 1$, and, if either $n = 1$ or $h^{0,n-1}(X) = 0$, for $q = n$, as the hypothesis (0.12) of the Duality Theorem holds by Mumford's variant of the Kodaira Vanishing Theorem. In particular, when the hypotheses of (0.13) hold,

(0.14) $$\mathcal{K}_{r-n,n+1}(X, K_X, L, W) \simeq \mathbf{C}.$$

The Theorem of the Gaussian class (2.b.9) shows that the geometrically defined Koszul class, the *Gaussian class* or *extrinsic fundamental class*

$$\gamma \in \mathcal{K}_{r-n,n+1}(X, K_X, L, W)$$

defined using the tangent planes to $\varphi_L(X)$, is a generator provided dim $\varphi_L(X) = n$.

In §3, there are three computational results. The Vanishing Theorem (3.a.1) says

(0.15) $$\mathcal{K}_{p,q}(X, E, L, W) = 0 \quad \text{if } H^0(X, E \otimes qL) \leq p.$$

Although this is an elementary result, it has turned out to be quite useful, especially in tandem with the Duality Theorem. The "Lefschetz Theorems" relate the Koszul cohomology of a variety X and a smooth hyperplane section $X \cap H$; the main result (3.b.7) is that

(0.16) $$\mathcal{K}_{p,q}(X, L) \simeq \mathcal{K}_{p,q}(X \cap H, L)$$

if $X \cap H$ is connected and

(0.17) $$H^1(X, qL) = 0 \quad \text{for all } q \geq 0.$$

The hypothesis (0.17) is true for ample bundles on varieties X of dimension $\geqslant 2$ with $K_X \leqslant 0$, so (0.16) holds for $K-3$ surfaces, Fano 3-folds, etc.

In many ways, the most delicate result we prove here is the $\mathcal{K}_{p,1}$ Theorem (3.c.1), which says that if $m = \dim \varphi_L(X)$ and $h^0(X, L) = r+1$, then

$$(0.18) \qquad \mathcal{K}_{p,1}(X, L) = 0 \quad \text{for } p > r - m,$$

$$(0.19) \qquad \mathcal{K}_{r-m,1}(X, L) = 0 \quad \text{unless } \varphi_L(X) \text{ is an } m\text{-fold of minimal degree},$$

$$(0.20) \qquad \mathcal{K}_{r-m-1,1}(X, L) = 0 \begin{cases} \text{unless either } \deg \varphi_L(X) \leqslant r+2-m \text{ or } \varphi_L(X) \\ \text{lies on an } (m+1)\text{-fold of minimal degree.} \end{cases}$$

In order to prove (0.20), we need the *Strong Castelnuovo Lemma* (3.c.6) that if $P_1, \cdots, P_d$ are points in general position in $\mathbf{P}_n$, then

$$(0.21) \qquad \begin{matrix} P_1, \cdots, P_n \text{ lie on a} \\ \text{rational normal curve} \end{matrix} \Leftrightarrow \mathcal{K}_{r-1,1}(P_1, P_2, \cdots, P_d) \neq 0.$$

When $r+4 \leqslant d \leqslant 2r+2$, this is stronger than Castelnuovo's Lemma.

One application of the Vanishing Theorem and the Duality Theorem is (4.a.1), which says that for a smooth curve C of genus g and an analytic line bundle $L \to C$ of degree d,

$$(0.22) \qquad \varphi_L(C) \text{ is projectively normal if } d \geqslant 2g+1,$$

$$(0.23) \qquad I_*(\varphi_L(C)) \text{ is generated by quadrics if } d \geqslant 2g+2,$$

$$(0.24) \qquad \begin{matrix} \text{the syzygies in } I_*(\varphi_L(C)) \text{ are} \\ \text{generated by those of the form} \\ \sum_i L_i Q_i = 0, \ \deg L_i = 1 \end{matrix} \qquad \text{if } d \geqslant 2g+3,$$

etc. Here (0.22) is Mumford's projective normality theorem, (0.23) was proved by Saint-Donat and Fujita, and the statements about syzygies are new. Actually, (4.a.1) says more, and in conjunction with an existence result (4.a.2) of F. Schreyer gives a fairly good picture of what a minimal free resolution of the ideal sheaf of $\varphi_L(C)$ looks like for d large relative to g. For varieties of higher dimension and sufficiently ample line bundles, there is a similar result, the *Theorem of the Top Row* (4.a.4).

The Arbarello-Sernesi module of X, L is the $S(H^0(X, L))$-module $\bigoplus_{q \in \mathbf{Z}} H^0(X, K_X \otimes qL)$. If $|L|$ is base-point free and $\dim \varphi_L(X) = \dim X = n$, we show in Theorem (4.b.2) that, with certain exceptional cases, the Arbarello-Sernesi module is generated in degree $\leqslant n-1$ and its relations are generated in degrees $\leqslant n$. Petri obtained this result for curves in the case when L is special; however, this hypothesis is unnecessary. If $L = K_X$, we obtain Theorem (4.c.1) about generators and relations of the canonical ring of a variety of general type.

The H^1 Lemma (4.d.1) and its improvement (4.d.7) deal with the question of when the map

$$H^1(X, E) \to \operatorname{Hom}(W, H^1(X, E \otimes L)) \tag{0.25}$$

induced by cup product is injective, where $E \to X$ is an analytic vector bundle and $W \subseteq H^0(X, L)$ is a linear subspace. One version is that if $L \simeq L_1 \otimes L_2 \otimes \cdots \otimes L_k$, $W = H^0(X, L)$, and

$$\text{the base locus of each } |L_i| \text{ has codimension} \geqslant 2, \tag{0.26}$$

$$h^0(W, E \otimes L \otimes L_i) \leqslant h^0(X, L_i) - 2, \tag{0.27}$$

then (0.25) is injective. From this, one obtains a Local Torelli Theorem of Kii (4.d.9) and splitting lemmas (4.d.11), (4.d.12).

For $L \to C$ and $M \to C$ analytic line bundles over a smooth curve and $W \subseteq H^0(C, L)$ a base-point free linear system, the H^0 Lemma (4.e.1) states that the multiplication map

$$W \otimes H^0(C, M) \to H^0(C, L \otimes M) \tag{0.28}$$

is surjective if

$$H^1(C, M \otimes L^{-1}) \leqslant \dim W - 2. \tag{0.29}$$

When $\dim W = 2$, this is the base-point free pencil trick. The Explicit H^0 Lemma (4.e.4) states that

$$H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$$

is surjective if $|L|$ is base-point free, $\deg L \leqslant \deg M$, and either $\deg L + \deg M \geqslant 4g + 2$ or $\deg M = 2g + 1$, $\deg L = 2g$, extending a result of Mumford.

For $L \to X$ a sufficiently positive line bundle, we obtain from the spectral sequence for Koszul cohomology the interesting representation (4.f.1)

$$H^q(X, \Omega_X^p) \simeq \mathcal{K}_{h^0(X, L)-q-1, q+1}(X, \Omega_X^p, L) \tag{0.30}$$

for the Hodge groups of a projective variety, involving only holomorphic sections of analytic line bundles.

Finally, there is a section on open problems and an Appendix. In the Appendix, which is joint work with R. Lazarsfeld, it is shown that on a compact complex manifold X with analytic line bundles $M_i \to X$, $i = 1, 2$, and $h^0(X, M_i) = r_i + 1$, $r_i \geqslant 1$, then

$$\mathcal{K}_{r_1+r_2-1,1}(X, L) \neq 0 \quad \text{for } L \simeq M_1 \otimes M_2. \tag{0.31}$$

In particular, this proves one direction of the Noether-Enriques-Petri conjecture.

The results of §§1 and 2, with the possible exception of the Theorem of the Gaussian Class, are classical–so classical, indeed, that I could not find out to whom they should be attributed.

The author is grateful to Ron Donagi, David Eisenbud, Phillip Griffiths, and Robert Lazarsfeld for some extremely helpful discussions. Especially useful was a small seminar in which this work was presented for the first time, which was of tremendous help to me; I wish to thank all the participants, Koji Cho, Ron Donagi, Phillip Griffiths, Stefanos Pantazis, Igor Reider, and Wu Xian.

1. Algebraic preliminaries

(a) The Koszul cohomology groups. We will consider

$$\text{(1.a.1)}\qquad \begin{cases} k & \text{a field,}\\ V & \text{a finite-dimensional vector space over } k,\\ S(V) & \text{the symmetric algebra on } V,\\ B=\bigoplus_{q\in\mathbf{Z}} B_q & \text{a graded } S(V)\text{-module.}\end{cases}$$

In this situation, there is a Koszul complex

$$\text{(1.a.2)}\qquad \begin{aligned}\cdots\to\wedge^{p+1}V\otimes B_{q-1}&\xrightarrow{d_{p+1,q-1}}\wedge^{p}V\otimes B_q\xrightarrow{d_{p,q}}\wedge^{p-1}V\otimes B_{q+1}\\ &\xrightarrow{d_{p-1,q+1}}\wedge^{p-2}V\otimes B_{q+2}\to\cdots\end{aligned}$$

constructed as follows. Let $\iota\in V^*\otimes V$ be the identity element. There is a natural contraction map

$$\text{(1.a.3)}\qquad \wedge^p V\xrightarrow{\lrcorner\iota}\wedge^{p-1}V\otimes V$$

dual to the exterior product map $\wedge^{p-1}V^*\otimes V^*\to\wedge^p V^*$. Given the multiplication $V\otimes B_q\xrightarrow{m_q}B_{q+1}$, we define $d_{p,q}$ by the commutative diagram:

$$\text{(1.a.4)}\qquad \begin{tikzcd}\wedge^p V\otimes B_q \arrow[r, "(\lrcorner\iota)\otimes\mathrm{Id}"] \arrow[dr, "d_{p,q}"'] & \wedge^{p-1}V\otimes V\otimes B_q \arrow[d, "\mathrm{Id}\otimes m_q"]\\ & \wedge^{p-1}V\otimes B_{q+1}\end{tikzcd}$$

Since

$$\wedge^p V\xrightarrow{(\lrcorner\iota)\circ(\lrcorner\iota)}\wedge^{p-2}V\otimes S^2V$$

is the zero map,

(1.a.5) $$d_{p-1,q+1} \circ d_{p,q} = 0.$$

So (1.a.2) is a complex. We note that in defining $d_{p,q}$, we are using the convention

(1.a.6) $$\wedge^p V = 0 \quad \text{if } p < 0 \text{ or } p > \dim V$$

and that (1.a.5) continues to be true.

Definition. *The Koszul cohomology* groups of B are

(1.a.7) $$\mathcal{K}_{p,q}(B, V) = \frac{\ker d_{p,q}}{\operatorname{im} d_{p+1,q-1}}.$$

By the convention (1.a.6), we have automatically

(1.a.8) $$\mathcal{K}_{p,q}(B, V) = 0 \quad \text{if } p < 0 \text{ or } p > \dim V.$$

A standard fact about the cohomology of complexes implies that for any m,

(1.a.9) $$\sum_{p+q=m} (-1)^p \dim_k(\mathcal{K}_{p,q}(B, V)) = \sum_{p+q=m} (-1)^p \binom{\dim V}{p} \dim_k(B_q).$$

Consider

(1.a.10) $$\begin{cases} V^1, V^2 & \text{finite dimensional vector spaces over } k, \\ B^1, B^2 & \text{graded } S(V_1)\text{-, } S(V_2)\text{-modules respectively,} \\ V^1 \xrightarrow{L} V^2 & \text{a linear transformation,} \\ S(V^1) \xrightarrow{\tilde{L}} S(V^2) & \text{the map induced by } L \text{ on symmetric algebras,} \\ B^1 \xrightarrow{\hat{L}} B^2 & \text{a linear transformation preserving the gradings.} \end{cases}$$

We will say $(\hat{L}, L)$ is a morphism of graded modules if

(1.a.11) $$\tilde{L}(h) \cdot \hat{L}(b) = h \cdot b \quad \text{for all } h \in S(V^1), b \in B^1.$$

For such a morphism of graded modules, the map

$$\wedge^p V^1 \otimes B^1_q \xrightarrow{\wedge^p L \otimes \hat{L}_q} \wedge^p V^2 \otimes B^2_q$$

descends to Koszul cohomology to give a map

(1.a.12) $$\mathcal{K}_{p,q}(B^1, V^1) \xrightarrow{\hat{L}_*} \mathcal{K}_{p,q}(B^2, V^2)$$

which is the induced map on Koszul cohomology and has the functorial property $(\widehat{L \circ M})_* = \hat{L}_* \circ \hat{M}_*$. If $V^1 = V^2 = V$ and $L = \mathrm{Id}$ we have

$$\mathcal{K}_{p,q}(B^1, V) \xrightarrow{\hat{L}_*} \mathcal{K}_{p,q}(B^2, V) \tag{1.a.13}$$

and this notation will always assume $L = \mathrm{Id}$ unless it is indicated otherwise. In this case, the condition (1.a.11) is just that $\hat{L}$ is a grading-preserving graded $S(V)$-module morphism.

(b) Syzygies. Returning to the general situation (1.a.1), assume that B has a *minimal free resolution* of the form

$$\cdots \to \bigoplus_{q \geqslant q_1} S(V)(-q) \otimes M_{1,q} \to \bigoplus_{q > q_1} S(V)(-q) \otimes M_{0,q} \to B \to 0, \tag{1.b.1}$$

where the $M_{p,q}(B, V)$ are finite dimensional vector spaces over k. Such a resolution exists provided:

(1) $\dim_k(B_q) < \infty$ for all q.

(2) $q \in \mathbf{Z} \mid B_q \neq 0\}$ is bounded from below.

Definition (1.b.2). The *syzygies of order p and weight q* for the $S(V)$-module B are $M_{p,q}(B, V)$.

Alternatively, these are defined inductively as follows:

$$\begin{aligned} M_{0,q}(B,V) &= \text{generators of degree } q \text{ for } B \text{ as an} \\ & \ S(V)\text{-module}, \\ M_{1,q}(B,V) &= \text{primitive relations of weight } q \\ & \ \text{for } B \text{ as an } S(V)\text{-module} \\ M_{2,q}(B,V) &= \text{primitive syzygies of weight } q \text{ among} \\ & \ \text{the relations for } B, \\ &\cdots \end{aligned} \tag{1.b.3}$$

These are to be interpreted as follows. If $x_1, x_2, \ldots$ are generators for B with $\deg x_i = e_i$, then a relation of weight q among the generators is one of the form

$$\sum_i u_i x_i = 0, \qquad u_i \in S^{q-e_i}(V).$$

A primitive relation of weight q is one that is not an $S(V)$-linear combination of relations of lower weight. If $\sum_i u_i^\nu x_i = 0$ are a basis for the primitive relations of weights e^ν respectively, a syzygy of weight q is a relation of the form

$$\sum_\nu w_\nu u_i^\nu = 0 \quad \text{for all } i, \text{ with } w_\nu \in S^{q-e_\nu}(V)$$

and so on inductively.

Theorem (1.b.4). *As vector spaces over* k, $\mathcal{K}_{p,q}(B,V) \simeq M_{p,p+q}(B,V)$.

Proof. We need

$$\mathcal{K}_{p,q}(S(V),V) \simeq \begin{cases} \mathbf{C} & \text{if } p=q=0, \\ 0 & \text{otherwise,} \end{cases} \tag{1.b.5}$$

or equivalently that the complex

$$\cdots \to \wedge^2 V \otimes S^{l-2}V \to V \otimes S^{l-1}V \to S^l V \to 0 \tag{1.b.6}$$

is exact unless $l=0$ when the complex reduces to $0 \to S^0V \to 0$. This is well known (see [3]); it follows from the same proof as the usual Poincaré Lemma when one dualizes the complex.

Consider the bigraded complex

$$A^{-p,-q} = \begin{cases} \wedge^p V \otimes \bigoplus_{k\geqslant 0} \left(S^kV \otimes M_{q,d-p-k}\right), & q \geqslant 0, \\ \wedge^p V \otimes B_{d-p}, & q=-1, \\ 0, & q<-1, \end{cases} \tag{1.b.7}$$

where $d \in \mathbf{Z}$ is fixed. As maps, we take

$$A^{p,q} \xrightarrow{d} A^{p+1,q}, \quad A^{p,q} \xrightarrow{\delta} A^{p,q+1}, \tag{1.b.8}$$

where for $q \geqslant 0$, d comes from the complex (1.b.6) and for $q=-1$, d is the map from (1.a.2), while δ is $\wedge^{-p}V$ tensored with $(-1)^p$ times the degree $(d-p)$ terms of the minimal free resolution (1.b.1). Note $d^2=0$, $\delta^2=0$, and $d\delta + \delta d = 0$. There are thus (see [8]) two spectral sequences $'E$, $''E$ abutting to the cohomology of the total complex with

$$'E_1^{p,q} = H_\delta^q(A^{p,\cdot}) = 0 \quad \text{for all } p,q, \tag{1.b.9}$$

$$''E_1^{p,q} = H_d^p(A^{0,q}) = \begin{cases} \mathcal{K}_{-p,d-p}(B,V), & q=-1, \text{ any } p, \\ M_{-q,d}(B,V), & q \geqslant 0, p=0, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$''E_r^{p,q} \xrightarrow{d_r} {''E_r^{p-(r-1),q+r}}.$$

The maps

$$M_{-q,d}(B,V) \xrightarrow{d_1} M_{-q+1,d}(B,V)$$

are zero by minimality of the resolution (1.b.1), and thus the only nonzero d_r's are

$$''E_{q+1}^{0,-q} \xrightarrow{d_{q+1}} ''E_{q+1}^{-q,-1}.$$

Since $''E_\infty^{p,q} = 0$ for all p, q as $'E_1 = 0$ and $'E, E''$ have the same abutment, we conclude d_{q+1} is an isomorphism, so

$$M_{q,d}(B, V) \xrightarrow{d_{q+1}} \mathcal{K}_{q,d-q}(B, V) \quad \text{for all } q \geq 0 \tag{1.b.10}$$

which completes the proof.

Remark. It is possible to make the isomorphism of Theorem (1.b.4) more explicit by expressing the intrinsic part of the maps in the minimal resolution (1.b.1) in terms of the $\mathcal{K}_{p,q}(B, V)$. One small fact along these lines we will want later is that *the multiplication map*

$$\mathcal{K}_{p,q}(B, V) \otimes S^k V \to \mathcal{K}_{p,q+k}(B, V), \qquad k > 0, \tag{1.b.11}$$

is zero. Since it is clear from the definition of the $M_{p,q}$'s that the multiplication map

$$M_{p,d}(B, V) \otimes S^k V \to M_{p,d+k}(B, V), \qquad k > 0,$$

is zero, we notice from the proof that if we tensor the bigraded complex (1.b.7) by $S^k V$, we get a commutative diagram

$$\begin{array}{ccc} M_{p,d}(B, V) \otimes S^k V & \xrightarrow{\cong} & \mathcal{K}_{p,d-p}(B, V) \otimes S^k V \\ \downarrow & & \downarrow \\ M_{p,d+k}(B, V) & \xrightarrow{\cong} & \mathcal{K}_{p,d+k-p}(B, V) \end{array}$$

and so conclude (1.b.11).

(c) Cohomology operations. If B, C are graded $S(V)$-modules, there is a natural map

$$\left(\wedge^{p_1} V \otimes B_{q_1}\right) \otimes \left(\wedge^{p_2} V \otimes C_{q_2}\right) \to \wedge^{p_1+p_2} V \otimes (B \otimes C)_{q_1+q_2} \tag{1.c.1}$$

by wedging on the first factors and tensoring on the second. This descends to Koszul cohomology to give the *cup product map*

$$\mathcal{K}_{p_1,q_1}(B, V) \otimes \mathcal{K}_{p_2,q_2}(C, V) \xrightarrow{\cup} \mathcal{K}_{p_1+p_2,q_1+q_2}(B \otimes C, V). \tag{1.c.2}$$

If B is a graded $S(V)$-algebra, we have $S(V)$-module map $B \otimes B \to B$ from multiplication. By (1.a.13), this induces a map on Koszul cohomology

$$\mathcal{K}_{p,q}(B \otimes B, V) \to \mathcal{K}_{p,q}(B, V)$$

which composed with (1.c.2) yields the *cup product map for a graded $S(V)$-algebra*

$$\mathcal{K}_{p_1,q_1}(B,V)\otimes\mathcal{K}_{p_2,q_2}(B,V)\overset{\cup}{\to}\mathcal{K}_{p_1+p_2,q_1+q_2}(B,V). \tag{1.c.3}$$

If B is a commutative algebra, we have

$$\alpha_1\cup\alpha_2=(-1)^{p_1p_2}\alpha_2\cup\alpha_1, \tag{1.c.4}$$

where $\alpha_i\in\mathcal{K}_{p_i,q_i}(B,V)$.

It is also possible to define *Massey products*. If $\alpha_i\in\mathcal{K}_{p_i,q_i}(B,V)$, $i=1,2,3$, and

$$\alpha_1\cup\alpha_2=0,\quad \alpha_2\cup\alpha_3=0, \tag{1.c.5}$$

then we can write

$$\begin{aligned}\alpha_1\wedge\alpha_2&=d_{p_1+p_2+1,q_1+q_2-1}(\rho),\quad \rho\in\textstyle\bigwedge^{p_1+p_2+1}V\otimes B_{q_1+q_2-1},\\ \alpha_2\wedge\alpha_3&=d_{p_2+p_3+1,q_2+q_3-1}(\tau),\quad \tau\in\textstyle\bigwedge^{p_2+p_3+1}V\otimes B_{q_2+q_3-1},\end{aligned} \tag{1.c.6}$$

and

$$d_{p_1+p_2+p_3+1,q_1+q_2+q_3-1}\left(\alpha_1\wedge\tau+(-1)^{p_1+1}\rho\wedge\alpha_3\right)=0, \tag{1.c.7}$$

so we get an element

$$\alpha_1\wedge\tau+(-1)^{p_1+1}\rho\wedge\alpha_3\in\mathcal{K}_{p_1+p_2+p_3+1,q_1+q_2+q_3-1}(B,V). \tag{1.c.8}$$

Choosing different ρ and τ changes this element by something in

$$\begin{aligned}&\mathcal{K}_{p_1,q_1}(B,V)\cup\mathcal{K}_{p_2+p_3+1,q_2+q_3-1}(B,V)\\ &\qquad+\mathcal{K}_{p_1+p_2+1,q_1+q_2-1}(B,V)\cup\mathcal{K}_{p_3,q_3}(B,V),\end{aligned}$$

so we obtain a well-defined element

$$M(\alpha_1,\alpha_2,\alpha_3)\in\mathcal{K}_{p_1+p_2+p_3+1,q_1+q_2+q_3-1}(B,V)/\mathfrak{D}(B,V), \tag{1.c.9}$$

where

$$\begin{aligned}\mathfrak{D}(B,V)=\Big(&\big(\mathcal{K}_{p_1,q_1}(B,V)\cup\mathcal{K}_{p_2+p_3+1,q_2+q_3-1}(B,V)\big)\\ &+\big(\mathcal{K}_{p_1+p_2+1,q_1+q_2-1}(B,V)\cup\mathcal{K}_{p_3,q_3}(B,V)\big)\Big).\end{aligned}$$

(d). The spectral sequence relating Koszul cohomology groups of an exact complex. Let $B^{\cdot}$ be a complex of graded $S(V)$-modules with maps preserving the gradings

$$0\to B^1\to B^2\to B^3\to\cdots\to B^{n-1}\to B^n\to 0. \tag{1.d.1}$$

Consider the bigraded complex of

$$C^{p,q} = \wedge^{l-q} V \otimes B_q^p, \qquad l \text{ fixed},$$

with maps

$$C^{p,q} \xrightarrow{d} C^{p+1,q}, \quad C^{p,q} \xrightarrow{\delta} C^{p,q+1},$$

where d comes from the complex $B^\cdot$ tensored with $\wedge^{l-q}V$ and δ comes from $(-1)^p$ times the map for the complex (1.d.1) for each fixed p. Thus $d^2 = 0$, $\delta^2 = 0$ and $d\delta + \delta d = 0$. So we obtain two spectral sequences $'E$, $''E$ abutting to the cohomology of the total complex with

$$\text{(1.d.2)} \qquad \begin{aligned} 'E_1^{p,q} &= H_\delta^q(A^{p,\cdot}) = \mathcal{K}_{l-q,q}(B^p, V), \\ ''E_1^{p,q} &= H_d^p(A^{\cdot,q}) = \wedge^{l-q} V \otimes H^p(B_q^\cdot). \end{aligned}$$

If the complex $B^\cdot$ is exact, then $''E_1^{p,q} = 0$ and thus $'E_\infty^{p,q} = 0$. Thus we have

Proposition (1.d.3). *Let $B^\cdot$ be an exact complex of graded $S(V)$-modules where the maps preserve the gradings. Then there is a spectral sequence with*

$$E_1^{p,q} = \mathcal{K}_{l-q,q}(B^p, V)$$

that abuts to zero.

Corollary (1.d.4) (*Long Exact Sequence for Koszul Cohomology*). *If $0 \to A \to B \to C \to 0$ is a short exact sequence of graded $S(V)$-modules with maps preserving the gradings, there is a long exact sequence*

$$\text{(1.d.5)} \qquad \begin{aligned} \cdots &\to \mathcal{K}_{1,q-1}(A, V) \to \mathcal{K}_{1,q-1}(B, V) \to \mathcal{K}_{1,q-1}(C, V) \\ &\to \mathcal{K}_{0,q}(A, V) \to \mathcal{K}_{0,q}(B, V) \to \mathcal{K}_{0,q}(C, V) \to 0. \end{aligned}$$

Proof. The only nonzero d_r's are the

$$\mathcal{K}_{p,q}(A, V) \xrightarrow{d_1} \mathcal{K}_{p,q}(B, V), \quad \mathcal{K}_{p,q}(B, V) \xrightarrow{d_1} \mathcal{K}_{p,q}(C, V),$$

$$\big(\ker d_1 : \mathcal{K}_{p,q}(A, V) \to \mathcal{K}_{p,q}(B, V)\big)$$

$$\xrightarrow{d_2} \big(\operatorname{coker} d_1 : \mathcal{K}_{p+1,q-1}(B, V) \to \mathcal{K}_{p+1,q-1}(C, V)\big)$$

from which the long exact sequence follows.

2. The Duality Theorem

(a) Transition to the setting of complex manifolds. The constructions of §1 will be of interest to us primarily in the case

$$(2.a.1)\qquad \begin{cases} X & \text{a compact complex manifold,} \\ L \to X & \text{an analytic line bundle,} \\ \mathfrak{F} & \text{a coherent analytic sheaf of } \mathcal{O}_X\text{-modules,} \\ W \subseteq H^0(X, L) & \text{a linear subspace,} \end{cases}$$

where we take

$$(2.a.2)\qquad V = W, \quad B = \bigoplus_{q \in \mathbf{Z}} H^0(X, \mathfrak{F} \otimes \mathcal{O}_X(qL)).$$

Our basic notation will be

$$(2.a.3)\qquad \mathcal{K}^i_{p,q}(X, \mathfrak{F}, L, W) = \mathcal{K}_{p,q}(B, V)$$

with the further notational conventions:

(1) If $W = H^0(X, L)$, we will drop the W and write $\mathcal{K}^i_{p,q}(X, \mathfrak{F}, L)$.

(2) If $\mathfrak{F} = \mathcal{O}_X(E)$, where $E \to X$ is an analytic vector bundle, we will write $\mathcal{K}^i_{p,q}(X, E, L, W)$.

(3) If $\mathfrak{F} = \mathcal{O}_X$, we will drop the $\mathfrak{F}$ and write $\mathcal{K}^i_{p,q}(X, L, W)$.

(4) If $i = 0$, we will drop the i and write $\mathcal{K}_{p,q}(X, \mathfrak{F}, L, W)$.

If $X \xrightarrow{f} Y$ is an analytic map, and

$$L_X = f^*L_Y, \qquad \mathfrak{F}_X = f^*(\mathfrak{F}_Y)$$

we have the pullback maps

$$H^0(Y, L_Y) \xrightarrow{f^*} H^0(X, L_X),$$

$$H^i(Y, \mathfrak{F}_Y \otimes \mathcal{O}_Y(qL_Y)) \xrightarrow{f^*} H^i(X, \mathfrak{F}_X \otimes \mathcal{O}_X(qL_X)).$$

If $W_X = f^*W_Y$ then by (1.a.12) there is an induced pullback map on Koszul cohomology

$$(2.a.4)\qquad \mathcal{K}^i_{p,q}(Y, \mathfrak{F}_Y, L_Y, W_Y) \xrightarrow{f^*} \mathcal{K}^i_{p,q}(X, \mathfrak{F}_X, L_X, W_X)$$

and also

$$\mathcal{K}^i_{p,q}(Y, \mathfrak{F}_Y, L_Y) \xrightarrow{f^*} \mathcal{K}^i_{p,q}(X, \mathfrak{F}_X, L_X),$$

$$\mathcal{K}^i_{p,q}(Y, L_Y) \xrightarrow{f^*} \mathcal{K}^i_{p,q}(X, L_X),$$

where in all cases

(2.a.5) $$(g \circ f)^* = f^* \circ g^*.$$

We can apply Theorem (1.b.4) to the situation (1.b.1) provided that

(2.a.6) $$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(qL)) = 0 \quad \text{for } q \text{ sufficiently negative.}$$

Condition (2.a.6) holds if L is ample, and we will assume henceforth that (2.a.6) holds. If so, then

(2.a.7)
$$\begin{aligned}
&H^i_{0,d}(X, \mathcal{F}, L, W) \\
&\quad = \textit{generators of the } S(W)\textit{-module } \bigoplus_{q \in \mathbf{Z}} H^i(X, \mathcal{F} \otimes \mathcal{O}_X(qL)) \\
&\qquad \textit{of degree } d, \\
&H^i_{1,d}(X, \mathcal{F}, L, W) \\
&\quad \simeq \textit{primitive relations of weight } d+1 \textit{ among the generators} \\
&\qquad \textit{of the } S(W)\textit{-module } \bigoplus_{q \in \mathbf{Z}} H^i(X, \mathcal{F} \otimes \mathcal{O}_X(qL)), \\
&H^i_{2,d}(X, \mathcal{F}, L, W) \\
&\quad \simeq \textit{primitive syzygies of weight } d+2 \textit{ of the } S(W)\textit{-module} \\
&\quad \bigoplus_{q \in \mathbf{Z}} H^i(X, \mathcal{F} \otimes \mathcal{O}_X(qL)),
\end{aligned}$$

or in general

(2.a.8)
$$\begin{aligned}
\mathcal{K}^i_{p,q}(X, \mathcal{F}, L, W) &\simeq \textit{primitive } p\textit{'th syzygies of weight } d+p \\
&\quad \textit{of the } S(W)\textit{-module } \bigoplus_{q \in \mathbf{Z}} H^i(X, \mathcal{F} \otimes \mathcal{O}_X(qL)).
\end{aligned}$$

We denote by $X \overset{\varphi_W}{\to} \mathbf{P}(W^*)$ the rational map defined by the linear system W when the base locus of W has codimension $\geqslant 2$, and

$$X \overset{\varphi_L}{\to} \mathbf{P}(H^0(X, L)^*)$$

the map φ_W when $W = H^0(X, L)$. In view of the equivalence

$$\left\{ \begin{array}{l} S^q H^0(X, L) \to H^0(X, qL) \\ \text{is onto for all } q \geqslant 2 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} H^0(X, L) \otimes H^0(X, (q-1)L) \\ \to H^0(X, qL) \text{ is onto for all } q \geqslant 2 \end{array} \right\}$$

we see that

(2.a.9) $$|L| \textit{ is projectively normal} \leftrightarrow \mathcal{K}_{0,q}(X, L) = 0 \quad \forall q \geqslant 1.$$

We denote

$$I(\varphi_L(X)) = \bigoplus_{q \geq 2} I_q(\varphi_L(X)) \tag{2.a.10}$$

the ideal of $\varphi_L(X)$ in $\mathbf{P}(H^0(X, L)^*)$. If $|L|$ is projectively normal, we have the short exact sequence of graded modules

$$0 \to I(\varphi_L(X)) \to S(H^0(X, L)) \to \bigoplus_{q \geq 0} H^0(X, qL) \to 0 \tag{2.a.11}$$

and thus by comparing minimal free resolutions of $I(\varphi_L(X))$ and $\bigoplus_{q \geq 0} H^0(X, qL)$, we have

$$\begin{aligned} &M_{p-1,q}\big(I(\varphi_L(X)), H^0(X, L)\big) \\ &\qquad \simeq M_{p,q}\Big(\bigoplus_{q \geq 0} H^0(X, qL), H^0(X, L)\Big) \end{aligned} \tag{2.a.12}$$

for $|L|$ projectively normal. Thus

$$\begin{cases} \mathcal{K}_{1,q}(X, L) & \simeq \quad I_{q+1}(\varphi_L(X))/H^0(X, L)I_q(\varphi_L(X)), \\ \mathcal{K}_{2,q}(X, L) & = \quad \text{primitive relations of weight } q + 2 \\ & \qquad \text{among the generators of the } S(H^0(X, L))\text{-} \\ & \qquad \text{module } I(\varphi_L(X)), \\ \quad \vdots & \\ \text{for } |L| & \qquad \text{projectively normal.} \end{cases} \tag{2.a.13}$$

In general, if $\mathcal{F}$ is a coherent analytic sheaf of $\mathcal{O}_P$-modules on a projective space $P = \mathbf{P}(V^*)$ and $B = \bigoplus_{q \in \mathbf{Z}} H^0(P, \mathcal{F}(q))$ then

$$\begin{aligned} \cdots &\to \bigoplus_{q \geq q_1} M_{1,q}(B, V) \otimes \mathcal{O}_P(-q) \to \bigoplus_{q \geq q_0} M_{0,q}(B, V) \otimes \mathcal{O}_P(-q) \\ &\to \mathcal{F} \to 0 \end{aligned} \tag{2.a.14}$$

is called a minimal resolution of $\mathcal{F}$ by free $\mathcal{O}_P$-modules; the fact it is a resolution is a consequence of Theorems A and B. Thus in particular

Theorem (2.a.15). *Let X be a compact complex manifold, $L \to X$ an analytic line bundle, and $\mathcal{I}_{\varphi_L(X)}$ the ideal sheaf of $\varphi_L(X)$ in $P = \mathbf{P}(H^0(X, L)^*)$. If $|L|$ is projectively normal, then*

$$\begin{aligned} \cdots &\to \bigoplus_{q \geq 0} \mathcal{K}_{1,q}(X, L) \otimes \mathcal{O}_P(-1 - q) \\ &\to \bigoplus_{q \geq 0} \mathcal{K}_{0,q}(X, L) \otimes \mathcal{O}_P(-q) \to \mathcal{I}_{\varphi_L(X)} \to 0 \end{aligned} \tag{2.a.16}$$

is a minimal resolution of $\mathcal{I}_{\varphi_L(X)}$ by free $\mathcal{O}_P$-modules.

A few elementary properties of the Koszul cohomology groups in this setting are

(2.a.17) $$\mathcal{K}^i_{p,q}(X, \mathcal{F} \otimes kL, L, W) \simeq \mathcal{K}^i_{p,q+k}(X, \mathcal{F}, L, W),$$

(2.a.18) $\mathcal{K}_{p,q}(X, L, W) = 0$ if $q < 0$ for L not the trivial bundle,

(2.a.19) $$\mathcal{K}_{p,0}(X, L, W) = \begin{cases} \mathbf{C} & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

(2.a.20) $$\mathcal{K}^i_{p,q}(X, E, L, W)^* \simeq \mathcal{K}^{n-i}_{\dim W-p,-q}(X, K_X \otimes E^*, L, W),$$

where $\dim X = n$.

Property (2.a.17) is clear from the definition. Property (2.a.18) follows because we cannot have both $H^0(X, L) \neq 0$ and $H^0(X, qL) = 0$ for some $q < 0$ unless L is the trivial bundle. Property (2.a.19) follows because by definition

$$\mathcal{K}_{p,0}(X, L, W) = \ker(\wedge^p W \to \wedge^{p-1} W \otimes H^0(X, L))$$

which is 0 if $p \neq 0$ and $\mathbf{C}$ if $p = 0$. Finally, (2.a.20) follows from Serre duality and the fact that

$$\int_X (s\alpha) \wedge \beta = \int_X \alpha \wedge (s\beta)$$

for $s \in H^0(X, L)$, $\alpha \in \mathcal{A}^{0,i}(X, E \otimes qL)$ and

$$\beta \in \mathcal{A}^{0,n-i}(X, K_X \otimes E^* \otimes (-1-q)L),$$

for then under Serre duality the Koszul complex

$$\cdots \to \wedge^{p+1} W \otimes H^i(X, E \otimes (q-1)L) \to \wedge^p W \otimes H^i(X, E \otimes qL)$$
$$\to \wedge^{p-1} W \otimes H^i(X, E \otimes (q+1)L) \to \cdots$$

goes to

$$\cdots \to \wedge^{p-1} W^* \otimes H^{n-i}(X, K_X \otimes E^* \otimes (-1-q)L)$$
$$\to \wedge^p W^* \otimes H^{n-i}(X, K_X \otimes E^* \otimes (-q)L)$$
$$\to \wedge^{p+1} W^* \otimes H^{n-i}(X, K_X \otimes E^* \otimes (1-q)L) \to \cdots$$

which, tensored with $\wedge^{\dim W} W$ and contracting, gives the Koszul complex

$$\cdots \to \wedge^{\dim W-p+1} W \otimes H^{n-i}(X, K_X \otimes E^* \otimes (-1-q)L)$$
$$\to \wedge^{\dim W-p} W \otimes H^{n-i}(X, K_X \otimes E^* \otimes (-q)L)$$
$$\to \wedge^{\dim W-p-1} W \otimes H^{n-i}(X, K_X \otimes E^* \otimes (1-q)L) \to \cdots.$$

(b) The Gaussian class. Let $s_0, s_1, \cdots, s_r$ be a basis for $W \subseteq H^0(X, L)$ and $e_0, \cdots, e_r$ the dual basis for W^*. There is a natural element

$$\gamma_p(X, L, W) \in \wedge^{p+1} W^* \otimes H^0(X, \Omega_X^p \otimes (p+1)L)$$

defined by

$$\gamma_p(X, L, W) = \sum_{j_1 \cdots j_p = 1}^{\dim X} \sum_{i_1, \cdots, i_{p+1}=0}^{r} e_{i_1} \wedge \cdots \wedge e_{i_{p+1}} \cdot s_{i_1} \frac{\partial s_{i_2}}{\partial z_{j_1}} \frac{\partial s_{i_3}}{\partial z_{j_2}} \cdots \frac{\partial s_{i_{p+1}}}{\partial z_{j_p}} dz_{j_1} \wedge \cdots \wedge dz_{j_p}, \tag{2.b.1}$$

where $z_1, \cdots, z_n$ are local coordinates on X. If we regard

$$s = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_r \end{pmatrix}$$

as a section of $W^* \otimes L$, then

$$\gamma_p(X, L, W) = \sum_{j_1, \cdots, j_p = 1}^{\dim X} s \wedge \frac{\partial s}{\partial z_{j_1}} \wedge \cdots \wedge \frac{\partial s}{\partial z_{j_p}} dz_{j_1} \wedge \cdots \wedge dz_{j_p}. \tag{2.b.2}$$

A more intrinsic representation of $\gamma_p(X, L, W)$ is to let

$$\partial s = \begin{pmatrix} \partial s_0 \\ \vdots \\ \partial s_n \end{pmatrix}, \qquad \partial s \in W^* \otimes L \otimes \Omega_X^1 \mod s,$$

where ∂s is defined mod s because, if in local coordinates, $s^\alpha = \xi_{\alpha\beta} s^\beta$ then

$$\partial s^\alpha = (\partial \xi_{\alpha\beta}) s^\beta + \xi_{\alpha\beta} \partial s^\beta$$

and thus ∂s transforms as a W^*-valued section of $\Omega_X^1 \otimes L$ modulo s. Then

$$\gamma_p(X, L, W) = s \wedge \underbrace{\partial s \wedge \cdots \wedge \partial s}_{p \text{ times}}. \tag{2.b.3}$$

Under the (noncanonical) identification

$$\wedge^{r+1} W \simeq \mathbf{C} \tag{2.b.4}$$

there is an isomorphism

$$\wedge^k W^* \simeq \wedge^{r+1-k} W, \quad \forall k \tag{2.b.5}$$

so that we may consider

(2.b.6) $$\gamma_p(X, L) \in \wedge^{r-p} W \otimes H^0(X, \Omega^p \otimes (p+1)L).$$

Under (2.b.5), the map

$$\wedge^{p-1} W^* \xrightarrow{\wedge s} \wedge^p W^* \otimes L$$

is dual to

$$\wedge^p W \xrightarrow{\lrcorner \iota} \wedge^{p-1} W \otimes L$$

so that

$$\iota \lrcorner \gamma_p(X, L, W) = 0$$

and thus we obtain an element

(2.b.7) $$\bar{\gamma}_p(X, L, W) \in \mathcal{K}_{r-p,p+1}(X, \Omega^p_X, L, W)$$

called the *Gaussian class of order p of X, L*. If $p = n$, we obtain

(2.b.8) $$\bar{\gamma} = \bar{\gamma}_n \in \mathcal{K}_{r-n,n+1}(X, K_X, L, W)$$

which we call simply the *Gaussian class of X, L*, or the *extrinsic fundamental class*.

Theorem (2.b.9) (*Theorem of the Gaussian Class*). *If W is base-point free, X is Kähler and*

(2.b.10) $$\dim \varphi_{mL}(X) \geq p \quad \textit{for some } m > 0,$$

then

(2.b.11) $$\bar{\gamma}_p(X, L, W) \neq 0$$

as an element of $\mathcal{K}_{r-p,p+1}(X, \Omega^p_X, L, W)$.

Proof. If $\bar{\gamma}_p(X, L, W) = 0$, then

$$s \wedge \partial s \wedge \cdots \wedge \partial s = s \wedge \alpha$$

for some $\alpha \in \wedge^p W^* \otimes H^0(X, \Omega^p_X \otimes pL)$. If we choose a lifting

(2.b.12) $$g \in W^* \otimes \mathcal{Q}^0(x, \Omega^1_X \otimes L)$$

so $g = \partial s \bmod s$, then

(2.b.13) $$\bar{\partial} g = As,$$

where

(2.b.14) $$A \in \mathcal{Q}^{0,1}(X, \Omega^1_X)$$

is a representative of the extension class of the first prolongation bundle of L, and hence of the first Chern class $c_1(L)$. We have $s \wedge g \wedge \cdots \wedge g = s \wedge \alpha$ so

$$g \wedge g \wedge \cdots \wedge g - \alpha = s \wedge E_1, \qquad E_1 \in \wedge^{p-1} W^* \otimes \mathcal{Q}^0(X, \Omega_X^p \otimes pL).$$

Then taking $\bar{\partial}$ of both sides,

$$p(\bar{\partial} g) \wedge g \wedge \cdots \wedge g = s \wedge \bar{\partial} E_1$$

or

$$pAs \wedge g \wedge \cdots \wedge g = s \wedge \bar{\partial} E_1.$$

Thus

$$pAg \wedge \cdots \wedge g - \bar{\partial} E_1 = s \wedge E_2,$$
$$E_2 \in \wedge^{p-2} W^* \otimes \mathcal{Q}^{0,1}(X, \Omega_X^p \otimes (p-1)L).$$

Taking $\bar{\partial}$ of both sides and rearranging as before, we get

$$p(p-1)s \wedge A \wedge A \wedge g \wedge \cdots \wedge g = s \wedge \bar{\partial} E_2.$$

Continuing inductively, we obtain

$$\text{(2.b.15)} \qquad p!\underbrace{A \wedge \cdots \wedge A}_{p \text{ times}} = \bar{\partial} E_{p+1}, \qquad E_{p+1} \in \mathcal{Q}^{0,p-1}(\Omega_X^p),$$

from which we conclude

$$\text{(2.b.16)} \qquad c_1(L)^p = 0 \quad \text{in } H^p(X, \Omega_X^p).$$

However, as a (1, 1)-form on X, $c_1(L)$ is proportional to the class represented by φ_{mL}^* of the Fubini-Study form on $\mathbf{P}(H^0(X, mL)^*)$ for any $m > 0$, and thus $\wedge^p c_1(L)$ is $\geqslant 0$ pointwise and positive somewhere if $\dim \varphi_L(X) \geqslant p$. This contradicts the assumption $\bar{\gamma}_p(X, L, W) = 0$. q.e.d.

Corollary (2.b.17). *If $W \subseteq H^0(X, L)$ is base-point free, and X is Kähler then* $\dim \varphi_W(X) = \dim \varphi_{mL}(X)$ *for any* $m > 0$.

Proof. If $p = \dim \varphi_W(X)$, then

$$\gamma_{p+1}(X, L, W) = s \wedge \overbrace{\partial s \wedge \cdots \wedge \partial s}^{p+1 \text{ times}} = 0.$$

Thus

$$c_1(L)^{p+1} = 0 \quad \text{in } H^{p+1}(X, \Omega^{p+1})$$

by the proof of (2.b.9). Hence

$$\dim \varphi_{mL}(X) \leqslant p \quad \text{for all } m > 0.$$

So

$$\dim \varphi_{mL}(X) \leqslant \dim \varphi_W(X) \quad \text{for all } m > 0$$

while the opposite inequality is automatic.

(c) The Duality Theorem. There are two main results:

Theorem (2.c.1). *Let X be a compact complex manifold of dimension n and $L \to X$ an analytic line bundle. Assume*

$$W \subseteq H^0(X, L) \quad \textit{is base-point free} \tag{2.c.2}$$

with $\dim W = r + 1$ *and*

$$\begin{aligned} &H^i(X, (i - (n+1))L) = 0, && i = 1, 2, \cdots, n-1, \\ &H^i(X, (i-n)L) = 0, && i = 1, 2, \cdots, n-1. \end{aligned} \tag{2.c.3}$$

Then

$$\mathcal{K}_{r-n,n+1}(X, K_X, L, W) \simeq \mathbf{C} \tag{2.c.4}$$

and furthermore, if

$$\dim \varphi_{mL}(X) = n \quad \textit{for some } m > 0, \tag{2.c.5}$$

then the Gaussian class is a generator.

Remark. By a generalization of Mumford's variant of the Kodaira Vanishing Theorem (see [5]), the hypothesis (2.c.3) is implied by (2.c.2) and (2.c.5) if X is Kähler.

Theorem (2.c.6) (*Duality Theorem*). *Let X be a compact complex manifold of dimension n, $L \to X$ an analytic line bundle and $E \to X$ an analytic vector bundle. Assume*

$$W \subseteq H^0(X, L) \quad \textit{is base-point free}$$

with $\dim W = r + 1$ *and*

$$\begin{aligned} &H^i(X, E \otimes (q-i)L) = 0, && i = 1, 2, \cdots, n-1, \\ &H^i(X, E \otimes (q-i-1)L) = 0, && i = 1, 2, \cdots, n-1. \end{aligned} \tag{2.c.7}$$

Then

$$\mathcal{K}_{p,q}(X, E, L, W)^* \simeq \mathcal{K}_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W). \tag{2.c.8}$$

Under the further assumptions (2.c.3), *the duality is given by the cup product map*

$$\begin{aligned} &\mathcal{K}_{p,q}(X, E, L, W) \otimes \mathcal{K}_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W) \\ &\qquad \xrightarrow{\cup} \mathcal{K}_{r-n,n+1}(X, K_X, L, W) \simeq \mathbf{C} \end{aligned} \tag{2.c.9}$$

which is then a perfect pairing.

Corollary (2.c.10). *Let X be a compact Kähler manifold of dimension n and $L \to X$ an analytic line bundle. Assume*

(1) *$W \subseteq H^0(X, L)$ is base-point free and let* $\dim W = r + 1$,

(2) $\dim \varphi_{mL}(X) = n$ *for some* $m > 0$.

Then

(2.c.11) $$\mathcal{K}_{p,q}(X, K_X, L, W)^* \simeq \mathcal{K}_{r-n-p,n+1-q}(X, L, W)$$

if $q \geqslant n + 1$. *If either* $n = 1$ *or* $H^{n-1}(X, \mathcal{O}_X) = 0$, *then* (2.c.11) *also holds for* $q = n$.

Remark. If (1) and (2) hold, we thus conclude the cup product (2.c.9) is a perfect pairing.

Proof of Corollary (2.c.10). Under our hypotheses, by Mumford's variant of the Kodaira Vanishing Theorem (see [5]),

$$H^i(X, kL) = 0 \quad \text{for } k < 0 \text{ and } i \leqslant n - 1,$$

and thus by Serre duality

$$H^i(X, K_X \otimes kL) = 0 \quad \text{for } k > 0 \text{ and } i > 0.$$

Thus if $q \geqslant n + 1$ or, if $H^{n-1}(X, \mathcal{O}_X) = 0$ for $q = n$, the cohomology hypotheses (2.c.7) hold, so the Duality Theorem applies. q.e.d.

The main element of the proof of the two theorems is the following result.

Theorem (2.c.12) (*The Spectral Sequence for Koszul Cohomology*). *Let X be a compact complex manifold of dimension n, $L \to X$ an analytic line bundle and $E \to X$ an analytic vector bundle. Assume*

$$W \subseteq H^0(X, L) \quad \textit{is base-point free.}$$

Then for any $k \in \mathbf{Z}$, there is a spectral sequence $E_r^{p,q}$ abutting to zero with

(2.c.13) $$E_2^{p,q} = \mathcal{K}^q_{-p,k+p}(X, E, L, W)$$

and with maps

(2.c.14) $$E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}.$$

Proof of Theorem (2.c.12). Consider the bigraded complex:

(2.c.15) $$C^{p,q} = \wedge^p W \otimes \mathcal{A}^{0,q}(X, E \otimes (k-p)L).$$

The rows of (2.c.15) are obtained by taking global $\mathcal{C}^\infty(0, q)$ forms of the sheaf Koszul complex

(2.c.16) $$\begin{aligned}\cdots &\to \wedge^{p+1} W \otimes E \otimes (k-1-p)L \to \wedge^p W \otimes E \otimes (k-p)L \\ &\to \wedge^{p-1} W \otimes E \otimes (k+1-p)L \to \cdots\end{aligned}$$

which is exact if W is a base-point free linear system. Since $\mathcal{C}^\infty$ forms are fine sheaves, the rows of (2.c.15) are exact.

The columns of (2.c.15) are $\wedge^p W$ tensored with the Dolbeault complex. Associated to this bigraded complex (see [8]) are two spectral sequences with the same abutment. One of these spectral sequences has as E_1 term the cohomology of the rows, hence is zero. The other spectral sequence has E_1 term the cohomology of the columns, hence has

$$E_1^{p,q} = \wedge^{-p} W \otimes H^q(X, E \otimes (k+p)L).$$

The d_1's are just the maps of the Koszul complex, so

$$E_2^{p,q} = \mathcal{K}^q_{-p,k+p}(X, E, L, W)$$

and the d_r's go as indicated. This spectral sequence abuts to zero because the first one does.

Proof of Theorems (2.c.1) *and* (2.c.6). As a corollary of the theorem just proved, we see that

$$\mathcal{K}^n_{p+n+1,q-n-1}(X, E, L, W) \xrightarrow{d_{n+1}} \mathcal{K}^0_{p,q}(X, E, L, W) \tag{2.c.17}$$

is defined and an isomorphism provided that

$$\begin{aligned} &\mathcal{K}^i_{p+i+1,q-i-1}(X, E, L, W) = 0, \quad i = 1, 2, \cdots, n-1, \\ &\mathcal{K}^i_{p+i,q-i}(X, E, L, W) = 0, \qquad i = 1, 2, \cdots, n-1. \end{aligned} \tag{2.c.18}$$

In the situation of Theorems (2.c.1) or (2.c.6), the hypotheses (2.c.3) or respectively (2.c.7) imply (2.c.18). Now by (2.a.17) and (2.c.17), we have

$$\mathcal{K}^0_{p,q}(X, E, L, W)^* \simeq \mathcal{K}^0_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W)$$

for Theorem (2.c.6) and, specializing,

$$\mathcal{K}^0_{0,0}(X, L, W)^* \simeq \mathcal{K}_{r-n,n+1}(X, K_X, L, W)$$

for Theorem (2.c.1), and by (2.a.16),

$$\mathcal{K}^0_{0,0}(X, L, W) \simeq \mathbf{C}.$$

To see that the cup product map (2.c.9) gives the duality in case both (2.c.3) and (2.c.9) are true, let

$$\alpha \in \mathcal{K}_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W)$$

and let

$$\tilde{\alpha} \in \wedge^{r-n-p} W \otimes H^0(X, K_X \otimes E^* \otimes (n+1-q)L, W)$$

represent α. Because $\tilde{\alpha}$ is holomorphic and $\iota \lrcorner \tilde{\alpha} = 0$, we have $d\tilde{\alpha} = 0$, $\delta\tilde{\alpha} = 0$ and thus tracing through the spectral sequence

$$d_{n+1}(\alpha \wedge \beta) = \alpha \wedge d_{n+1}(\beta) \quad \text{for} \quad \beta \in \mathcal{K}_{p,q}(X, E, L, W).$$

Thus we have the commutative diagram

(2.c.9)

$$\begin{array}{ccc}
\mathcal{K}_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W) \otimes \mathcal{K}^*_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W) & \xrightarrow{\text{contraction}} & H^n(X, K_X) \\
\downarrow 1 \otimes d_{n+1} & & \downarrow d_{n+1} \\
\mathcal{K}_{r-n-p,n+1-q}(X, K_X \otimes E^*, L, W) \otimes \mathcal{K}_{p,q}(X, E, L, W) & \xrightarrow{\cup'} & \mathcal{K}_{r-n-p,n+1}(X, K_X, L, W)
\end{array}$$

which shows the cup product gives the duality.

Finally, the statement that the Gaussian class is a generator of $\mathcal{K}_{r-n-p,n+1}(X, K_X, L, W)$ is a consequence of (2.b.9).

3. Computational techniques for Koszul cohomology

(a) A vanishing theorem. We want to prove

Theorem (3.a.1) (*Vanishing Theorem*). *Let X be a compact complex manifold, $L \to X$ an analtyic line bundle, $W \subseteq H^0(X, L)$ a linear subspace, and $E \to X$ an analytic vector bundle. Then*

(3.a.2) $$\mathcal{K}_{p,q}(X, E, L, W) = 0 \quad \textit{if}\ h^0(E \otimes qL) \leqslant p.$$

Proof. Let $P_1, P_2, \cdots, P_{r+1}$ be generic points of X, and choose $s_1, s_2, \cdots, s_{r+1}$ a basis for W so that

(3.a.3) $$s_i(P_j) = \delta_{ij}.$$

If $\alpha \in \wedge^p W \otimes H^0(E \otimes qL)$ we may consider, if $\dim W = r+1$, that

$$\alpha \in \wedge^{r+1-p} W^* \otimes H^0(E \otimes qL)$$

and then the condition $\iota \lrcorner \alpha = 0$ becomes

(3.a.4) $$s_{i_1}\alpha_{i_2,\cdots,i_{r+2-p}} + s_{i_2}\alpha_{i_3,\cdots,i_{r+2-p}i_1} + \cdots + s_{i_{r+2-p}}\alpha_{i_1,\cdots,i_{r+1-p}} = 0.$$

Evaluating at $P_{i_{r+2-p}}$, we obtain

(3.a.5) $$\alpha_{i_1,\cdots,i_{r+1-p}}(P_j) = 0 \quad \text{if}\ j \neq i_1, i_2, \cdots, i_{r+1-p}.$$

If $P_1, \cdots, P_{r+1}$ are generic, and if $h^0(X, E \otimes qL) \leqslant p$, then any $\alpha \in H^0(X, E \otimes qL)$ vanishing at p of the points $P_1, \cdots, P_{r+1}$ is zero. Thus, by (3.a.5),

$$\alpha_{i_1,\cdots,i_{r+1-p}} = 0 \quad \text{for all}\ i_1, \cdots, i_{r+1-p}$$

and thus $\alpha = 0$. q.e.d.

An application of the Vanishing Theorem which we will need in §3c is

Corollary (3.a.6). *Let $H \to \mathbf{P}_1$ be the hyperplane bundle. Then for $a, d \in \mathbf{Z}$, $d > 0$,*

$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0, \tag{3.a.7}$$

unless $0 \leqslant a \leqslant 2d - 2$ *and* $a - d + 1 \leqslant p \leqslant a$.

Proof. $h^0(\mathbf{P}_1, aH) = a + 1$ so (3.a.2) becomes

$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0 \quad \text{if } a + 1 \leqslant p.$$

By the Duality Theorem

$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH)^* \simeq \mathcal{K}_{d-1-p,2}(\mathbf{P}_1, (-2 - a)H, dH) \tag{3.a.8}$$

and by (3.a.2),

$$\mathcal{K}_{d-1-p,2}(\mathbf{P}_1, (-2 - a)H, dH) = 0 \quad \text{if } 2d - 1 - a \leqslant d - 1 - p$$

so

$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0 \quad \text{if } p \leqslant a - d.$$

Combining these,

$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0 \quad \text{unless } a - d + 1 \leqslant p \leqslant a \tag{3.a.9}$$

From (3.a.8) and the definition, we have

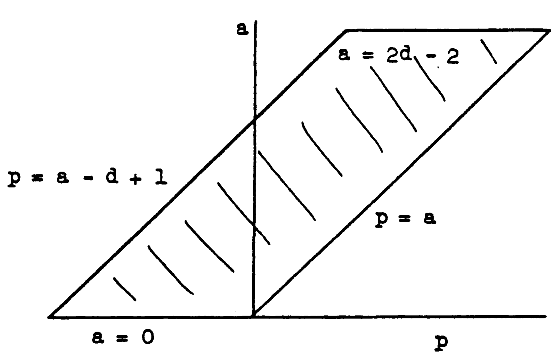
$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0 \quad \text{unless } a \geqslant 0 \text{ and } 2d - 2 - a \geqslant 0$$

so

$$\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0 \quad \text{unless } 0 \leqslant a \leqslant 2d - 2. \tag{3.a.10}$$

Now (3.a.9) and (3.a.10) together are (3.a.7).

Remark. Corollary (3.a.6) can be rephrased as $\mathcal{K}_{p,0}(\mathbf{P}_1, aH, dH) = 0$ *outside the closed parallelogram*:



(b) The "Lefschetz Theorem". We wish to consider several variants of the situation where $Y \subset X$ has codimension 1, and relate the Koszul cohomology groups of X and Y.

Theorem (3.b.1). *Let X be a compact complex manifold. $L \to X$ an analytic line bundle, $Y \subset X$ a smooth hypersurface with $[Y] = M$ the analytic line bundle associated to Y and L_Y the restriction of L to Y. Assume*

$$\begin{aligned} &H^0(X, L - M) = 0, \\ &H^1(X, qL - M) = 0 \quad \text{for all } q \geqslant 0. \end{aligned} \tag{3.b.2}$$

Then there is a long exact sequence

$$\begin{aligned} \cdots &\to \mathcal{K}_{1,q-1}(X, L) \to \mathcal{K}_{1,q-1}(Y, L_Y) \to \mathcal{K}_{0,q}(X, M^*, L) \\ &\to \mathcal{K}_{0,q}(X, L) \to \mathcal{K}_{0,q}(Y, L_Y) \to 0. \end{aligned} \tag{3.b.3}$$

Proof. Consider the graded $S(H^0(X, L))$-modules

$$B^1 = \bigoplus_{q \geqslant 0} H^0(X, M^* \otimes qL), \quad B^2 = \bigoplus_{q \geqslant 0} H^0(X, qL), \quad B^3 = \bigoplus_{q \geqslant 0} H^0(Y, qL_Y).$$

The hypotheses (3.b.2) insure that we have a short exact sequence of graded modules $0 \to B^1 \to B^2 \to B^3 \to 0$ and an isomorphism $H^0(X, L) = H^0(Y, L_Y)$ from the short exact sequence

$$0 \to \mathcal{O}_X((q-p)L \otimes M^*) \to \mathcal{O}_X((q-p)L) \to \mathcal{O}_Y((q-p)L_Y) \to 0$$

using the hypotheses (3.b.2). By the long exact sequence for Koszul cohomology (1.d.4), we obtain the long exact sequence (3.b.3).

Corollary (3.b.4). *If $Y = \operatorname{div} u$, where $u \in \operatorname{im} S^k H^0(X, L) \subset H^0(X, kL)$ for some $k \geqslant 2$, and*

$$H^1(X, qL) = 0 \quad \text{for all } q \geqslant -k, \tag{3.b.5}$$

then

$$\mathcal{K}_{p,q}(Y, L_Y) \simeq \mathcal{K}_{p,q}(X, L) \oplus \mathcal{K}_{p-1,q+1-k}(X, L). \tag{3.b.6}$$

Proof. We need only see that the map

$$\mathcal{K}_{p,q-k}(X, L) \xrightarrow{\text{mult } u} \mathcal{K}_{p,q}(X, L)$$

is the zero map, which follows from (1.b.11). q.e.d.

For a hyperplane section, we have

Theorem (3.b.7). *Let X be a compact complex manifold, $L \to X$ an analytic line bundle, $Y \subset X$ a connected hypersurface in the linear system $|L|$ and let L_Y denote the restriction of L to Y. Assume*

$$H^1(X, qL) = 0 \quad \text{for all } q \geqslant 0. \tag{3.b.8}$$

Then $\mathcal{K}_{p,q}(X, L) \simeq \mathcal{K}_{p,q}(Y, L_Y)$ for all p, q.

Proof. We have an exact sequence

$$0 \to H^0(X, \mathcal{O}) \to H^0(X, L) \to H^0(Y, L_Y) \to 0$$

and thus

$$\wedge^p H^0(X, L) \simeq \begin{cases} \wedge^p H^0(Y, L_Y) \oplus \wedge^{p-1} H^0(Y, L_Y), & p \geq 1, \\ \mathbf{C}, & p = 0. \end{cases}$$

We thus have a short exact sequence of graded $S(H^0(X, L))$-modules

$$0 \to B^1 \to B^2 \to B^3 \to 0,$$

where

$$B^1 = \bigoplus_{q \geq 0} H^0(X, (q-1)L), \quad B^2 = \bigoplus_{q \geq 0} H^0(X, qL), \quad B^3 = \bigoplus_{q \geq 0} H^0(Y, qL_Y).$$

We thus obtain a long exact sequence

$$\begin{aligned} \cdots &\to \mathcal{K}_{p,q-p-1}(X, L) \to \mathcal{K}_{p,q-p}(X, L) \\ &\to \mathcal{K}_{p,q-p}(Y, L_Y) \oplus \mathcal{K}_{p-1,q-p}(Y, L_Y) \to \mathcal{K}_{p-1,q-p}(X, L) \to \cdots \end{aligned} \tag{3.b.9}$$

by (1.d.4). Now by (1.b.11), the maps

$$\mathcal{K}_{p,q-p-1}(X, L) \to \mathcal{K}_{p,q-p}(X, L)$$

are zero, so we obtain

$$\begin{aligned} &\mathcal{K}_{p,q-p}(Y, L_Y) \oplus \mathcal{K}_{p-1,q-p}(Y, L_Y) \\ &\qquad \simeq \mathcal{K}_{p,q-p}(X, L) \oplus \mathcal{K}_{p-1,q-p}(X, L) \end{aligned} \tag{3.b.10}$$

for all p, q. For $p = 0$, we obtain

$$\mathcal{K}_{0,q}(Y, L_Y) \simeq \mathcal{K}_{0,q}(X, L)$$

and for $p = 1$,

$$\mathcal{K}_{1,q}(Y, L_Y) \oplus \mathcal{K}_{0,q}(Y, L_Y) \simeq \mathcal{K}_{1,q}(X, L) \oplus \mathcal{K}_{0,q}(X, L)$$

and thus

$$\mathcal{K}_{1,q}(Y, L_Y) \simeq \mathcal{K}_{1,q}(X, L)$$

and, continuing inductively, obtain the theorem.

(c) The $\mathcal{K}_{p,1}$ Theorem. This result has the most delicate proof of any in this paper. Once it is established, a variety of interesting geometric results–e.g. the Enriques-Petri-Babbage Theorem on the ideal of a canonical curve–follow from it and the Duality Theorem.

Theorem (3.c.1) (*The $\mathcal{K}_{p,1}$ Theorem*). *Let X be a compact complex manifold, $L \to X$ an analytic line bundle with $h^0(X, L) = r + 1$ and let $m = \dim \varphi_L(X)$. Then*

(1) $\mathcal{K}_{p,1}(X, L) = 0$ *for* $p > r - m$.

(2) $\mathcal{K}_{r-m,1}(X, L) = 0$ *unless* $\varphi_L(X)$ *is an m-fold of minimal degree.*

(3) $\mathcal{K}_{r-m-1,1}(X, L) = 0$ *unless either* $\deg \varphi_L(X) \leqslant r + 2 - m$ *or* $\varphi_L(X)$ *lies on an* $(m+1)$*-fold of minimal degree.*

Remark. In [5], we proved a preliminary version of this result, obtaining (1) and (2) as above, but getting (3) with a much worse bound for $\deg \varphi_L(X)$. A del Pezzo surface $X \subseteq \mathbf{P}_9$ has degree 9, does not lie on a threefold of minimal degree, and $\mathcal{K}_{6,1}(X, L) \neq 0$ by the theorem in the appendix. Note that $\deg \varphi_L(X) = r + 2 - m$ in this case. Thus the bound in (2) cannot be improved.

Proof. Let

$$\alpha \in \wedge^p H^0(X, L) \otimes H^0(X, L)$$

represent a nonzero class in $\mathcal{K}_{p,1}(X, L)$. Then

$$\iota \lrcorner \alpha \in \wedge^{p-1} H^0(X, L) \otimes I_2(\varphi_L(X)).$$

Regarding

$$\iota \lrcorner \alpha \in \mathrm{Hom}\big(\wedge^{p-1} H^0(X, L)^*, I_2(\varphi_L(X))\big) \tag{3.c.2}$$

it is proved in [5] that

$$\dim(\mathrm{im}(\iota \lrcorner \alpha)) \geqslant \binom{p+1}{2}. \tag{3.c.3}$$

From this and Castelnuovo's Lemma, one can prove (1), (2), and also (3) with a weaker conclusion about $\deg \varphi_L(X)$. Our strategy here is to use a strengthened version of the Castelnuovo's Lemma. We need some further notation to state it.

Let V be a vector space of dimension $r + 1$ and

$$P_1, \cdots, P_d \in \mathbf{P}(V^*).$$

We set

$$B_q = \mathrm{im}\left(H^0(\mathbf{P}(V^*), qH) \to \bigoplus_{i=1}^{d} H^0(P_i, qH) \right), \tag{3.c.4}$$

where $H \to \mathbf{P}(V^*)$ is the hyperplane bundle. Then $B = \oplus_{q \geqslant 0} B_q$ is an $S(V)$-module, since $V \simeq H^0(\mathbf{P}(V^*), H)$. We will then denote

$$\mathcal{K}_{p,q}(P_1, \cdots, P_d) = \mathcal{K}_{p,q}(B, V) \tag{3.c.5}$$

using the definition of §1a. We then have

Theorem (3.c.6) (*Strong Castelnuovo Lemma*). *Let* $P_1, P_2, \cdots, P_d \in \mathbf{P}_r$ *be points in general position. Then*

$$\left\{\begin{matrix} P_1, P_2, \cdots, P_d \text{ lie on a} \\ \text{rational normal curve} \end{matrix}\right\} \leftrightarrow \mathcal{K}_{r-1,1}(P_1, P_2, \cdots, P_d) \neq 0.$$

Remark. By (3.c.3), one has that

$$\mathcal{K}_{r-1,1}(P_1, \cdots, P_d) \neq 0 \to P_1, \cdots, P_d$$

lie on at least $\binom{r}{2}$ linearly independent quadrics. Thus when $d \geqslant 2r+3$, the Strong Castelnuovo Lemma does not say any more than the usual Castelnuovo Lemma. If $d \leqslant r+3$, any d points in general position lie on a rational normal curve. However, when d lies in the range $r+4 \leqslant d \leqslant 2r+2$, then (3.c.6) does say something new, and this is what allows us to obtain the bound on $\deg \varphi_L(X)$ in (3) of the $\mathcal{K}_{p,1}$ Theorem.

Proof of (3.c.6). The ideal $\mathcal{I}_C$ of a rational normal curve C in $\mathbf{P}_r$ has minimal resolution by an Eagon-Northcott complex

$$(3.c.7) \quad 0 \to \bigoplus_{(r-1)\binom{r}{r}} \mathcal{O}_{\mathbf{P}_r}(-r) \to \cdots \to \bigoplus_{2\binom{r}{3}} \mathcal{O}_{\mathbf{P}_r}(-3) \to \bigoplus_{\binom{r}{2}} \mathcal{O}_{\mathbf{P}_r}(-2) \to \mathcal{I}_C \to 0.$$

By Theorem (2.a.15), we conclude

$$\mathcal{K}_{r-1,1}(C, H) \simeq \mathbf{C}^{r-1}.$$

If $P_1, \cdots, P_d$ lie on C, then $I_2(C) \subseteq I_2(P_1, \cdots, P_d)$ and so all syzygies, syzygies among syzygies, etc. of C map to syzygies, syzygies among syzygies, etc. of $P_1, \cdots, P_d$. By degree, any syzygy of depth p and weight $p+1$ is primitive, so the map

$$\mathcal{K}_{r-1,1}(C, H) \to \mathcal{K}_{r-1,1}(P_1, \cdots, P_d)$$

is injective. Thus

$$\mathcal{K}_{r-1,1}(P_1, \cdots, P_d) \neq 0$$

if $P_1, \cdots, P_d$ lie on a rational normal curve C, which proves one direction of (3.c.6).

Conversely, assume we are given $P_1, \cdots, P_d$ with $\mathcal{K}_{r-1,1}(P_1, \cdots, P_d) \neq 0$. If $d \leqslant r+3$, we are done, as any $r+3$ points in general position lie on a rational normal curve. If $d \geqslant r+3$, let C be the unique rational normal curve containing $P_1, \cdots, P_{r+3}$. As before, let

$$B_q = \operatorname{im}\big(H^0(\mathbf{P}_r, qH) \to H^0(P_1 + \cdots + P_{r+3}, qH)\big)$$

and set

$$B = \bigoplus_{q \in \mathbf{Z}} B_q, \qquad A = \bigoplus_{q \in \mathbf{Z}} H^0(C, qH),$$

$$R = \bigoplus_{q \in \mathbf{Z}} \left(\ker\left(H^0(C, qH) \to H^0(P_1 + \cdots + P_{r+3}, qH)\right)\right).$$

Note

$$R \simeq \bigoplus_{q \in \mathbf{Z}} H^0(C, \mathcal{O}_C(qH - P_1 - \cdots - P_{r+3})).$$

Since rational normal curves are projectively normal, the restriction map $A \to B$ is surjective, and thus we have a short exact sequence of graded $S(V)$-modules $0 \to R \to A \to B \to 0$. Thus by (1.d.4) there is a long exact sequence

$$\text{(3.c.8)} \qquad \cdots \to \mathcal{K}_{r-1,1}(A, V) \to \mathcal{K}_{r-1,1}(B, V) \to \mathcal{K}_{r-1,1}(R, V) \to \cdots.$$

Now

$$\mathcal{K}_{r-2,2}(R, V) \simeq \mathcal{K}_{r-2,2}(C, \mathcal{O}_C(-P_1 - \cdots - P_{r+3}), H).$$

If $\mathbf{P}_1$ is the underlying projective line of the rational curve C, and $L \to \mathbf{P}_1$ the hyperplane bundle for $\mathbf{P}_1$, then $H \simeq rL$ and $\mathcal{O}_C(-P_1 - \cdots - P_{r+3}) \simeq -(r+3)L$. Thus

$$\mathcal{K}_{r-2,2}(R, V) \simeq \mathcal{K}_{r-2,2}(\mathbf{P}_1, -(r+3)L, rL) \simeq \mathcal{K}_{r-2,0}(\mathbf{P}_1, (r-3)L, rL).$$

We can now invoke (3.a.7) to conclude that $\mathcal{K}_{r-2,2}(R, V) = 0$. Thus

$$\begin{array}{ccc} \mathcal{K}_{r-1,1}(C, H) & \twoheadrightarrow & \mathcal{K}_{r-1,1}(P_1, \cdots, P_{d+3}) \\ & & \uparrow \\ & & \mathcal{K}_{r-1,1}(P_1, \cdots, P_d). \end{array}$$

Now if $\alpha \in \mathcal{K}_{r-1,1}(P_1, \cdots, P_d)$ then it is the image of $\tilde{\alpha} \in \mathcal{K}_{r-1,1}(C, H)$. By (3.c.3), $\dim \operatorname{im}(\iota \lrcorner \tilde{\alpha}) \geq \binom{r}{2}$ and thus $\operatorname{im}(\iota \lrcorner \tilde{\alpha}) = I_2(C)$. However, the quadrics in $\operatorname{im}(\iota \lrcorner \tilde{\alpha})$ all contain $P_1, \cdots, P_d$, so

$$I_2(C) \subseteq I_2(P_1, \cdots, P_d).$$

Since a rational normal curve is cut out by quadrics, we conclude $P_1, \cdots, P_d \in C$ which proves the lemma. q.e.d.

We now return to the proof of Theorem (3.c.1). If $\mathcal{K}_{p,1}(X, L) \neq 0$ let π be a generic $(p+1)$-plane in $\mathbf{P}_r$. Then

$$\mathcal{K}_{p,1}(\pi \cap \varphi_L(X), H) \neq 0$$

since a syzygy restricted to a generic linear space does not vanish. Now by the Strong Castelnuovo Lemma,

$$\pi \cap \varphi_L(X) \subseteq \text{a rational normal curve.}$$

In particular,

$$\dim(\pi \cap \varphi_L(X)) = m - r + p + 1 \leqslant 1$$

so $p \leqslant r - m$ proving (1). If $p = r - m$, then

$$\deg(\pi \cap \varphi_L(X)) \leqslant r + 1 - m$$

and thus

$$\deg \varphi_L(X) \leqslant r + 1 - m$$

which proves (2).

If

$$\alpha \in \wedge^{r-m-1} H^0(X, L) \otimes H^0(X, L)$$

represents a nonzero class in $\mathcal{K}_{r-m-1,1}(X, L)$, let $Y = \mathrm{Var}(\mathrm{im}(\iota \lrcorner \alpha))$. If π is a $(r - m)$-plane in $\mathbf{P}_r$ corresponding to an $(r - m + 1)$-dimensional subspace $W \subset H^0(X, L)^*$, then we define

$$\alpha_\pi \in \wedge^{r-m-1} W^* \otimes W^*$$

to be the image of α under the maps dual to $W \to H^0(X, L)^*$. Then

$$\mathrm{im}(\iota \lrcorner \alpha)|_\pi = \mathrm{im}(\iota \lrcorner \alpha_\pi)$$

and thus

$$\pi \cap Y = \mathrm{Var}(\mathrm{im}(\iota \lrcorner \alpha_\pi)).$$

By the proof of the Strong Castelnuovo Lemma, for π generic, and $d \geqslant r - m + 3$,

$$\mathrm{Var}(\mathrm{im}(\iota \lrcorner \alpha)) = C_\pi,$$

where C_π is the rational normal curve whose existence is guaranteed by the lemma. Thus either $\deg \varphi_L(X) \leqslant r - m + 2$ or $Y \cap \pi$ is a rational normal curve for a generic π. In the second case, Y is a variety of minimal degree. This proves the $\mathcal{K}_{p,1}$ Theorem. q.e.d.

4. Applications

(a). The Theorem of the Top Row. In [11], Mumford proved that for a smooth curve of genus g and a holomorphic line bundle $L \to C$ of degree d, that $\varphi_L(C)$ is projectively normal if $d \geqslant 2g + 1$. He also proved the ideal of

$\varphi_L(C)$ is generated by quadrics if $d \geqslant 3g + 1$; Saint-Donat and Fujita showed $\varphi_L(C)$ is cut out by quadrics if $d \geqslant 2g + 2$.

Theorem (4.a.1). *Let C be a smooth curve of genus g and $L \to C$ an analytic line bundle of degree d. Then*:

(1) $\mathcal{K}_{p,q}(C, L) = 0$ *for* $q \geqslant 3$ *if* $h^1(L) = 0$.

(2) $\mathcal{K}_{p,2}(C, L) = 0$ *if* $d \geqslant 2g + 1 + p$.

Thus:

(1) $\varphi_L(C)$ *is projectively normal if* $d \geqslant 2g + 1$.

(2) *The ideal of* $\varphi_L(C)$ *is generated by quadrics if* $d \geqslant 2g + 2$.

(3) *The syzygies among the quadrics in the ideal of* $\varphi_L(C)$ *are generated in weight* 3 *if* $d \geqslant 2g + 3$, *etc.*

Proof. By the Duality Theorem,

$$\mathcal{K}_{p,q}(C, L)^* \simeq \mathcal{K}_{r-1-p,2-q}(C, K, L).$$

Now

$$h^1(L) = 0 \to h^0(K - (q-2)L) = 0 \quad \text{for } q \geqslant 3$$

which proves (1). If $q = 2$, by the Vanishing Theorem,

$$\mathcal{K}_{r-1-p,0}(C, K, L) = 0 \quad \text{if } h^0(K) \leqslant r - 1 - p.$$

By Riemann-Roch, $r = d - g$. So

$$\mathcal{K}_{p,2}(C, L) = 0 \quad \text{if } g \leqslant d - g - 1 - p$$

or

$$\mathcal{K}_{p,2}(C, L) = 0 \quad \text{if } d \geqslant 2g + 1 + p.$$

The remaining results are just reinterpretations of the first two. q.e.d.

Theorem (4.a.1) is precise, due to the following result of F. Schreyer [12].

(4.a.2) For each genus g, there exists a number $d_0(g)$ so that if $d \geqslant d_0(g)$,

$$\mathcal{K}_{p,2}(C, L) \neq 0 \quad \text{if } r - 1 \geqslant p \geqslant r - g.$$

Thus, when d is large, we have the following picture of a minimal free resolution for the ideal sheaf of $\varphi_L(C)$:

$$(4.a.3) \qquad \begin{array}{ccccccccc} 0 & 0 & \mathcal{K}_{r-1,2} & \cdots & \mathcal{K}_{r-g,2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathcal{K}_{r-1,1} & \mathcal{K}_{r-2,1} & \cdots & & & \mathcal{K}_{2,1} & \mathcal{K}_{1,1} \end{array}$$

where the entries marked by dots in the top row are nonzero.

Theorem (4.a.4) (*Theorem of the Top Row*). *Let X be a compact Kähler manifold of dimension n and $L \to X$ an analytic line bundle with $h^0(X, L) = r+1$. Then for L sufficiently positive,*

(1) $\mathcal{K}_{p,q}(X, L) = 0$ *for* $q \geq n+2$, *and*

(2) $\mathcal{K}_{p,n+1}(X, L) = 0$ *for* $p \leq r - n - h^{0,n}(X)$.

Note. By L sufficiently positive, we mean that there exists a bundle L_0- so that the theorem is true if $L \otimes L_0^* \geq 0$.

Remark. Once again, Schreyer's result gives that

$$\text{(4.a.5)} \quad \mathcal{K}_{p,n+1}(X, L) \neq 0 \quad \text{for } r-n \geq p \geq r+1-n-h^{0,n}(X)$$

if L is sufficiently positive. Thus the resolution of the module $\oplus_{q \geq 0} H^0(X, qL)$ has the picture

$$\text{(4.a.6)} \quad \begin{array}{cccccccccc} 0 & 0 & \mathcal{K}_{r-n,n+1} & \cdots & \mathcal{K}_{r+1-h^{n,0}(X),n+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathcal{K}_{r-n,n} & & & & & & \mathcal{K}_{0,n} \\ \vdots & \vdots & \vdots & & & & & & \vdots \\ 0 & 0 & \mathcal{K}_{r-n,1} & \cdots & \cdots & \cdot & \cdot & \cdots & \mathcal{K}_{0,1} \end{array}$$

where by Schreyer's result the indicated $h^{n,0}(X)$ entries in the top row are nonzero.

Proof of (4.a.4). By taking L sufficiently positive, we can arrange that $|L|$ is base-point free and

$$H^i(X, qL) = 0 \quad \text{if} \begin{cases} i = 0 \text{ and } q < 0, \\ 1 \leq i \leq n-1, \quad q \neq 0, \\ i = n, \quad q > 0. \end{cases}$$

For $q \geq n+1$, the cohomology hypotheses of the Duality Theorem are satisfied, so

$$\mathcal{K}_{p,q}(X, L)^* \simeq \mathcal{K}_{r-n-p,n+1-q}(X, K_X, L).$$

For $q \geq n+2$, we get zero since

$$h^0(X, K_X - (q-(n+1))L) = 0.$$

For $q = n+1$, we get

$$\mathcal{K}_{r-n-p,0}(X, K_X, L) = 0 \quad \text{if } h^0(K_X) \leq r-n-p$$

by the Vanishing Theorem.

(b) The Arbarello-Sernesi module and Petri's analysis of the ideal of a special curve. Petri and later Arbarello and Sernesi [2] studied the ideal of a special curve by looking at generators and relations of

$$\bigoplus_{q \in \mathbf{Z}} H^0(C, K_C \otimes qL)$$

as an $S(H^0(C, L))$-module. We will generalize their results both by dropping the requirement that L be special and by extending their results to higher dimensions.

Definition (4.b.1). For X a compact complex manifold and $L \to X$ an analytic line bundle, the *Arbarello-Sernesi module* of X, L is

$$\bigoplus_{q \in \mathbf{Z}} H^0(X, K_X \otimes qL)$$

viewed as an $S(H^0(X, L))$-module.

Theorem (4.b.2). *Let X be a compact Kähler manifold of dimension n and $L \to X$ an analytic line bundle with $h^0(X, L) = r + 1$. Assume $|L|$ is base-point free and that* $\dim \varphi_{kL}(X) = n$ *for some $k > 0$. Then*

(1) *The Arbarello-Sernesi module of X, L is generated in degree $\leqslant q$ if*

(a) $q \geqslant n + 1$;

(b) $q = n$ *and* $r \neq n$;

(c) $q = n - 1$; *either* $\dim \varphi_L(X) = n - 1$ *and* $\deg \varphi_L(X) \geqslant r + 4 - n$ *or* $\dim \varphi_L(X) = n$; *and $\varphi_L(X)$ does not lie on an n-fold of minimal degree; and either $n = 1$ or $h^{n-1}(X, \mathcal{O}_X) = 0$.*

(2) *The relations among generators of the Arbarello-Sernesi module are generated in weight $\leqslant q$ if*

(a) $q \geqslant n + 2$;

(b) $q = n + 1$ *and* $r \neq n + 1$;

(c) $q = n$; $\dim \varphi_L(X) = n$; *either $n = 1$ or $h^{n-1}(X, \mathcal{O}_X) = 0$;* $\deg \varphi_L(X) \geqslant r + 3 - n$; *and $\varphi_L(X)$ does not lie on an $(n + 1)$-fold of minimal degree.*

Proof. By the Duality Theorem's Corollary (2.c.10),

$$\mathcal{K}_{p,q}(X, K_X, L)^* \simeq \mathcal{K}_{r-n-p,n-1-q}(X, L)$$

if $q \geqslant n + 1$, and also for $q = n$ if either $n = 1$ or $h^{n-1}(X, \mathcal{O}_X) = 0$. Since

$$\mathcal{K}_{r-n-p,n+1-q}(X, L) = 0 \quad \text{for } q \geqslant n + 2$$

and

$$\mathcal{K}_{r-n-p,n+1-q}(X, L) = 0 \quad \text{for } q = n + 1 \text{ unless } p = r - n,$$

we obtain (1a), (1b), (2a), and (2b). Since

$$\mathcal{K}_{0,n}(X, K_X, L)^* \simeq \mathcal{K}_{r-n,1}(X, L)$$
$$\mathcal{K}_{1,n}(X, K_X, L)^* \simeq \mathcal{K}_{r-n-1,1}(X, L),$$

we obtain (1c) and (2c) from the $\mathcal{K}_{p,1}$ Theorem (3.c.1).

Remark. We also obtain results about all the syzygies of the Arbarello-Sernesi module from the fact that, under the hypotheses of the theorem,

(4.b.3) $$\mathcal{K}_{p,q}(X, K_X, L) = 0 \quad \text{for } q \geq n+2, \text{ any } p,$$

(4.b.4) $$\mathcal{K}_{p,n+1}(X, K_X, L) = 0 \quad \text{for } p \neq r - n$$

as stated above.

(c) The canonical ring of a variety of general type. Since writing [5], our point of view has evolved somewhat. It is now easier to obtain those results, and they may be extended to syzygies.

Theorem (4.c.1). *Let X be a smooth n-fold of general type. Assume that $|K_X|$ is base-point free. Then*

(1) $\mathcal{K}_{p,q}(X, K_X)^* \simeq \mathcal{K}_{h^0(X, K_X)-(n+1+p),n+2-q}(X, K_X)$ *if* $q \geq n+2$ *or if* $q = n+1$ *and* $H^{n-1}(X, \mathcal{O}_X) = 0$.

(2) $\mathcal{K}_{p,q}(X, K_X) = 0$ *if* $q \geq n+3$.

(3)
$$\mathcal{K}_{p,n+2}(X, K_X) = \begin{cases} \mathbf{C} & \text{if } p = h^0(X, K_X) - (n+1), \\ 0 & \text{otherwise}. \end{cases}$$

(4)
$$\mathcal{K}_{0,n+1}(X, K_X) = \begin{cases} \mathbf{C}^{h^0(X, K_X)-n-1} & \text{if } \varphi_K(X) \text{ is an } n\text{-fold of minimal degree}, \\ 0 & \text{if } \varphi_K(X) \text{ is not on an } n\text{-fold of minimal degree} \end{cases}$$

provided in both cases that $H^{n-1}(X, \mathcal{O}_X) = 0$ *and* $\dim \varphi_K(X) \geq n-1$.

(5) $\mathcal{K}_{1,n+1}(X, K_X) = 0$ *if* $\dim \varphi_K(X) = n$, $H^{n-1}(X, \mathcal{O}_X) = 0$, *and* $\varphi_K(X)$ *does not lie on an* $(n+1)$*-fold of minimal degree and* $\deg \varphi_K(X) \neq h^0(X, K_X) - n$ *or* $h^0(X, K_X) - n + 1$.

Proof. (1) is a consequence of (2.c.11), and (1) implies (2). Since

$$\mathcal{K}_{p,0}(X, K_X) = \begin{cases} \mathbf{C} & \text{if } p = 0, \\ 0 & \text{otherwise}, \end{cases}$$

we obtain (3). The $\mathcal{K}_{p,1}$ theorem implies (4) and (5). q.e.d.

Note. In case $n = 1$, the cohomological hypotheses of the Duality Theorem are vacuous, so in that case the hypothesis $H^{n-1}(X, \mathcal{O}_X)$ can be eliminated wherever it occurs; thus, one recovers the Enriques-Babbage-Petri theorem. Our theorem is an extension of a theorem of Arbarello and Sernesi (see [2]). We also note that if φ_K is birational to its image, then in (5) the possibilities $\deg \varphi_K(X) = h^0(X, K_X) - n$ or $h^0(X, K_X) - n + 1$ can be eliminated, as for $n > 1$ these are rational varieties, and for $n = 1$ they are rational or elliptic.

(d) The H^1 Lemma, a theorem of Kii, and a splitting lemma.

Theorem (4.d.1) (*The H^1 Lemma*). *Let X be a compact complex manifold, $L \to X$ an analytic line bundle, $W \subseteq H^0(X, L)$ a linear subspace and $E \to X$ an analytic vector bundle. Assume*:

(1) *The base locus of W has codimension* ≥ 2.

(2) $h^0(X, E \otimes 2L) \leq \dim W - 2$.

Then the map

$$H^1(X, E) \to W^* \otimes H^1(X, E \otimes L) \tag{4.d.2}$$

induced by the cup product map is injective.

Proof. The kernel of the map (4.d.2) is just $\mathcal{K}^1_{\dim W,0}(X, E, L, W)$. Let $\mathcal{U}$ be a sufficiently fine open cover of X and let

$$\mathcal{C}^q(\mathcal{U}, E \otimes kL) = q\text{th Čech cochains of } \mathcal{U} \text{ for } E \otimes kL.$$

If we take the bigraded complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \to & \mathcal{C}^1(E) & \to & W^* \otimes \mathcal{C}^1(E \otimes L) & \to & \wedge^2 W^* \otimes \mathcal{C}^1(E \otimes 2L) & \to & \cdots \\
 & & \uparrow\delta & & \uparrow\delta & & \uparrow\delta & & \\
0 & \to & \mathcal{C}^0(E) & \to & W^* \otimes \mathcal{C}^0(E \otimes L) & \to & \wedge^2 W^* \otimes \mathcal{C}^0(E \otimes 2L) & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

then we obtain two spectral sequences $'E_r^{p,q}$, $''E_r^{p,q}$ with the same abutment (see [8]). The rows are exact at the first term automatically and at the second term because the base locus of W has codimension ≥ 2. Thus

$$''E_1^{0,q} = 0 \quad \text{and} \quad ''E_1^{1,q} = 0 \quad \text{for all } q.$$

Hence

$$''E_\infty^{0,q} = 0 \quad \text{and} \quad ''E_\infty^{1,q} = 0 \quad \text{for all } q.$$

Thus, since $'E$, $''E$ have the same abutment, $'E_\infty^{0,1} = 0$. On the other hand,

$$'E_2^{p,q} = \mathcal{K}^q_{\dim W-p,p}(X, E, L, W) \quad \text{for all } p, q.$$

We conclude that

$$\begin{array}{ccc}
'E_2^{0,1} & \xrightarrow{\quad d_2 \quad} & 'E_2^{2,0} \\
\wr\| & & \wr\| \\
\mathcal{K}^1_{\dim W,0}(X, E, L, W) & & \mathcal{K}^0_{\dim W-2,2}(X, E, L, W)
\end{array}$$

is injective. Now

$$\mathcal{K}^0_{\dim W-2,2}(X, E, L, W) = 0$$

by the Vanishing Theorem (3.a.1) and the hypothesis $h^0(X, E \otimes 2L) \leqslant \dim W - 2$. So

$$\mathcal{K}^1_{\dim W,0}(X, E, L, W) = 0$$

and (4.d.2) is injective. q.e.d.

Remark. In Theorem (4.d.1), it is clear from the proof that we may replace hypothesis (2) by

(2)′ $\mathcal{K}_{\dim W-2,2}(X, E, L, W) = 0$.

Furthermore, if W is base-point free, (4.d.2) is injective if and only if (2)′ holds.

Corollary (4.d.3). *Let X be a compact Kähler manifold of dimension n. If $|K_X|$ is base-point free, then the derivative of the period map in* $\mathrm{Hom}(H^{n,0}(X), H^{n-1,1}(X))$,

$$H^1(X, \Theta_X) \xrightarrow{P_*} H^0(X, K_X)^* \otimes H^1(X, \Omega_X^{n-1}) \tag{4.d.4}$$

is injective if and only if

$$\mathcal{K}_{h^0(X, K_X)-2,1}(X, \Omega_X^{n-1}, K_X) = 0. \tag{4.d.5}$$

Proof. This follows from the remark.

Corollary (4.d.6). *Let X be a compact Kähler manifold of dimension n. If the base locus of $|K_X|$ has codimension $\geqslant 2$, then* (4.d.4) *is injective, and hence the Local Torelli Theorem holds for X, provided*

$$\mathcal{K}_{\dim W-2,1}(X, \Omega_X^{n-1}, K_X) = 0.$$

Proof. This follows from the remark following the proof of Theorem (4.d.1). q.e.d.

Theorem (4.d.7) (*Improvement of the H^1 Lemma*). *Let X be a compact complex manifold, $L_i \to X$ analytic line bundles, $i = 1,\cdots,k$, $L = L_1 \otimes L_2 \otimes \cdots \otimes L_k$ and $E \to X$ an analytic vector bundle. Assume that for all $i = 1,\cdots,k$,*

(1) *the base locus of $|L_i|$ has codimension $\geqslant 2$, and*

(2) $h^0(X, E \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes 2L_i) \leqslant h^0(X, L_i) - 2$. *Then the map*

$$H^1(X, E) \to H^0(X, L)^* \otimes H^1(X, E \otimes L) \tag{4.d.8}$$

induced by cup product is injective.

Proof. By the H^1 Lemma, the maps

$$H^1(X, E \otimes L_1 \otimes \cdots \otimes L_{i-1}) \to H^0(X, L_i)^* \otimes H^1(X, E \otimes L_1 \otimes \cdots \otimes L_i)$$

are injective. Thus, if $\eta \in H^1(X, E)$ there exists $s_1 \in H^0(X, L_1)$ so

$$\eta s_1 \neq 0 \quad \text{in } H^1(X, E \otimes L_1)$$

and $s_2 \in H^0(X, L_2)$ so

$$\eta s_1 s_2 \neq 0 \quad \text{in } H^1(X, E \otimes L_1 \otimes L_2)$$

and so on inductively, until

$$\eta s_1 s_2 \cdots s_k \neq 0 \quad \text{in } H^1(X, E \otimes L).$$

Since

$$s_1 s_2 \cdots s_k \in H^0(X, L_1 \otimes \cdots \otimes L_k) \simeq H^0(X, L)$$

we are done.

Remark. We can replace (2) by the hypothesis

(2)′ $h^0(X, E \otimes L \otimes L_i) \leqslant h^0(X, L_i) - 2$, $i = 1, \cdots, k$,

since if $s_j \in H^0(X, L_j)$, $s_j \neq 0$, for each $j > i$, then multiplication by $s_{i+1}, \cdots, s_k$ gives an injection

$$H^0(X, E \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes 2L_i) \hookrightarrow H^0(X, E \otimes L \otimes L_i)$$

so

$$h^0(X, E \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes 2L_i) \leqslant h^0(X, E \otimes L \otimes L_i).$$

Corollary (4.d.9). *Let X be a compact Kähler manifold of dimension n, and assume $K_X \simeq L_1 \otimes \cdots \otimes L_k$, where $L_i \to X$ is an analytic line bundle. Assume for all $i = 1, 2, \cdots, k$ that*:

(1) *The base locus of $|L_i|$ has codimension $\geqslant 2$.*

(2) $h^0(X, \Theta_X \otimes L_1 \otimes \cdots \otimes L_{i-1} \otimes 2L_i) \leqslant h^0(X, L_i) - 2$.

Then the map

$$H^1(X, \Theta_X) \xrightarrow{P_*} H^0(X, K_X)^* \otimes H^1(X, \Omega_X^{n-1})$$

is injective and thus the local Torelli Theorem is true for X.

Note. By the remark above, hypothesis (2) can be replaced by

(2)′ $h^0(X, \Omega_X^{n-1} \otimes L_i) \leqslant h^0(X, L_i) - 2$.

In this form, the result is due to Kii [10], who derives from it the Local Torelli Theorem in a number of cases. This approach to Local Torelli was also used by Lieberman, Peters, and Wilsker.

Definition. An exact sequence $0 \to E \to F \to G \to 0$ of analytic vector bundles on X *splits on sections* if the map $H^0(X, F) \to H^0(X, G)$ is surjective.

Corollary (4.d.10). *Let X be a compact complex manifold, $E \to X$ an analytic vector bundle and $L \to X$ an analytic vector bundle. Assume*:

(1) *The base locus of $|L|$ has codimension $\geqslant 2$.*

(2) $h^0(X, E \otimes L) \leqslant h^0(X, L) - 2$.

Then any analytic extension $0 \to E \to F \to L \to 0$ of E by L splits on sections if and only if it splits analytically, i.e. $F \simeq E \oplus L$.

Proof. If F splits, then automatically it splits on sections. Let us show the converse. Let $e \in H^1(X, E \otimes L^*)$ be the extension class of F. If F splits on sections, then

$$e \in \ker H^1(X, E \otimes L^*) \to H^0(X, L)^* \otimes H^1(X, E)$$

so

$$e \in \mathcal{K}^1_{h^0(X,L),-1}(X, E, L).$$

By the H^1 Lemma,

$$e = 0 \quad \text{if } h^0(X, E \otimes L) \leqslant h^0(X, L) - 2.$$

But

$$e = 0 \leftrightarrow F \simeq E \oplus L \quad \text{analytically.}$$

Remark. Corollary (4.d.10) remains true if we replace the hypothesis (2) by (2)′ $\mathcal{K}_{h^0(X,L)-2,1}(X, E, L) = 0$.

There are two refinements of (4.d.10).

Corollary (4.d.11). *Let X be a compact complex manifold,*

$$0 \to E \to F \to L \to 0$$

an exact sequence of analytic vector bundles on X with $L \to X$ a line bundle and let

$$W = \operatorname{im}\big(H^0(X, F) \to H^0(X, L)\big).$$

If

(1) *the base locus of W has codimension* $\geqslant 2$,

(2) $h^0(X, E \otimes L) \leqslant \dim W - 2$,

then $F \simeq E \oplus L$ analytically.

Corollary (4.d.12). *Let X be a compact complex manifold, and*

$$0 \to E \to F \to mL \to 0, \qquad m > 0, \tag{4.d.13}$$

an exact sequence of analytic vector bundles, with L a line bundle. Assume:

(1) *The base locus of $|L|$ has codimension* $\geqslant 2$.

(2) $h^0(X, E \otimes L) \leqslant h^0(X, L) - 2$.

Then the sequence (4.d.13) *splits analytically if and only if it is split on sections.*

(e) The H^0 Lemma.

Theorem (4.e.1) (*The H^0 Lemma*). *Let C be a smooth curve with $L \to C$, $M \to C$ analytic line bundles. Let $W \subseteq H^0(C, L)$ be a base-point free linear subsystem. Then the multiplication map*

$$W \otimes H^0(C, M) \to H^0(C, L \otimes M) \tag{4.e.2}$$

is onto if

$$h^1(C, M \otimes L^{-1}) \leqslant \dim W - 2. \tag{4.e.3}$$

Remark. For $\dim W = 2$, this is just the base-point free pencil trick.

Corollary (4.e.4) (*The Explicit H^0 Lemma*). *Let C be a smooth curve of genus g, and $L \to C$ and $M \to C$ analytic line bundles. Assume that* $\deg L \leqslant \deg M$ *and that $|L|$ is base-point free. If either*

(4.e.5) $$\deg L + \deg M \geqslant 4g + 2$$

or

(4.e.6) $$\deg M = 2g + 1, \quad \deg L = 2g$$

then the multiplication map

(4.e.7) $$H^0(C, L) \otimes H^0(C, M) \to H^0(C, L \otimes M)$$

is surjective.

Remark. This improves a result of Mumford [11] that $\deg M \geqslant 2g + 1$ and $\deg L \geqslant 2g$ imply (4.e.7) is surjective.

Proof of (4.e.1). We want to show $\mathcal{K}_{0,1}(C, M, L, W) = 0$. By the Duality Theorem,

$$\mathcal{K}_{0,1}(C, M, L, W)^* \simeq \mathcal{K}_{r-1,1}\left(C, K_C \otimes M^{-1}, L, W\right),$$

where $\dim W = r + 1$. Now by the Vanishing Theorem, we are done if

$$h^0\left(C, K_C \otimes M^{-1} \otimes L\right) \leqslant r - 1$$

or, equivalently,

$$h^1\left(C, M \otimes L^{-1}\right) \leqslant \dim W - 2. \quad \text{q.e.d.}$$

Proof of (4.e.4). If $h^1(C, M \otimes L^{-1}) = 0$ we are done by (4.e.1), so assume $M \otimes L^{-1}$ is special. Now

$$h^1\left(C, M \otimes L^{-1}\right) = g - 1 + \deg L - \deg M + h^0\left(C, M \otimes L^{-1}\right).$$

So we are done if

$$g - 1 + \deg L - \deg M + h^0\left(C, M \otimes L^{-1}\right) \leqslant h^0(C, L) - 2$$

or, equivalently, if

$$g - 1 + \deg L - \deg M + h^0\left(C, M \otimes L^{-1}\right) \leqslant -1 - g + \deg L + h^1(C, L)$$

which simplifies to

(4.e.8) $$2g + h^0\left(C, M \otimes L^{-1}\right) \leqslant \deg M + h^1(C, L).$$

If $h^0(C, M \otimes L^{-1}) = 0$ we are done, as in this case we need only show $2g \leqslant \deg M + h^1(C, L)$ and we are given

$$2 \deg M \geqslant \deg M + \deg L \geqslant 4g + 2$$

so $\deg M \geqslant 2g + 1$. Thus, we are reduced to considering the situation

$$h^0(C, M \otimes L^{-1}) \neq 0 \quad \text{and} \quad h^1(C, M \otimes L^{-1}) \neq 0.$$

Now, by Clifford's Theorem,

$$h^0(C, M \otimes L^{-1}) - 1 \leqslant \tfrac{1}{2}(\deg M - \deg L).$$

Now (4.e.8) would follow if we knew

$$2g + 1 + \tfrac{1}{2}(\deg M - \deg L) \leqslant \deg M + h^1(C, L)$$

which is equivalent to

$$2g + 1 \leqslant \tfrac{1}{2}(\deg M + \deg L) + h^1(C, L)$$

which follows from (4.e.5). In the case of (4.e.6), $\deg(M \otimes L^{-1}) = 1$ so $h^0(C, M \otimes L^{-1}) \leqslant 1$ and so (4.e.8) becomes

$$2g + 1 \leqslant \deg M + h^1(C, L).$$

This follows from (4.e.6) and completes the proof of (4.e.4). q.e.d.

(f) A holomorphic representation of the $H^{p,q}$ groups of a smooth variety.

Theorem (4.f.1). *Let X be a smooth projective variety and $L \to X$ an analytic line bundle. If L is sufficiently positive, then*

$$H^q(X, \Omega_X^p) \simeq \mathcal{K}_{h^0(X, L)-q-1, q+1}(X, \Omega_X^p, L).$$

Remark. This expression for the $H^{p,q}$ groups has some affinity with the Poincaré Residue. We say the representation is "holomorphic" because it is entirely in terms of H^0's of analytic bundles. The term "L sufficiently positive" has the same meaning as in the note to Theorem (4.a.4).

Proof. For L sufficiently positive, $|L|$ is base-point free and

$$H^i(X, \Omega_X^p \otimes qL) = 0 \quad \text{for } 1 \leqslant i \leqslant \dim X, q > 0.$$

Thus

$$H^q(X, \Omega_X^p) \simeq \mathcal{K}^q_{h^0(X, L), 0}(X, \Omega_X^p, L).$$

We argue inductively that for $2 \leqslant r \leqslant q$,

$$d_r = 0 \quad \text{on } H^q(X, \Omega_X^p),$$

$$\operatorname{im} d_r = 0 \quad \text{in } \mathcal{K}^0_{h^0(X, L)-1-1, q+1}(X, \Omega_X^p, L)$$

because $H^{q-r+1}(X, \Omega_X^p \otimes rL) = 0$ and $H^{r-1}(X, \Omega_X^p \otimes (q + 1 - r)L) = 0$. Further,

$$H^q(X, \Omega_X^p) \xrightarrow{d_{q+1}} \mathcal{K}^0_{h^0(X, L)-1-1, q+1}(X, \Omega_X^p, L)$$

must be an isomorphism, as the spectral sequence abuts to zero and all further d_r's are zero because they run out of room.

5. Open problems and conjectures

The first conjecture we would like to formulate is

Conjecture (5.1) (*Noether-Enriques-Petri Conjecture*). *Let C be a smooth curve of genus g. Then*

$$\mathcal{K}_{p,1}(C, K_C) \neq 0 \leftrightarrow$$

has a g_d^r with $d \leqslant g-1$, $r \geqslant 1$, and $d - 2r \leqslant g - 2 - p$.

Remarks. (1) The direction $\leftarrow$ of the conjecture is proved in a joint appendix with R. Lazarsfeld that follows this paper.

(2) When $p = g-2$, this conjecture is Noether's Theorem; when $p = g-3$, it is the Enriques-Petri-Babbage Theorem.

(3) By the Duality Theorem, a minimal free resolution of $\mathcal{I}_{\varphi_K(C)}$ for C nonhyperelliptic has the picture

$$(5.2)\qquad \begin{array}{ccccccccccc} \mathcal{K}_{g-2,3} & 0 & 0 & & \cdots & 0 & 0 & \cdots & 0 & 0 & \to \\ & \mathcal{K}_{g-3,2} & \cdot & \cdot & \cdots & \mathcal{K}_{g-2-p_{\max},2} & 0 & \cdots & 0 & 0 & \\ & & & \cdot & & \cdots & & & & & \\ 0 & & 0 & \mathcal{K}_{p_{\mathrm{mam}},1} & & & & & \mathcal{K}_{2,1} & \mathcal{K}_{1,1} & \end{array}$$

where

$$(5.3)\qquad p_{\max} = \max\{p \mid \mathcal{K}_{p,1}(C, K_C) \neq 0\}.$$

Note

$$\mathcal{K}_{p,q}(C, K_C)^* \simeq \mathcal{K}_{g-2-p,3-q}(C, K_C).$$

The conjecture states that $p_{\max} = g - 2 - \nu_{\min}$, where

$$(5.4)\qquad \nu_{\min} = \min\{d - 2r \mid C \text{ has a } g_d^r \text{ with } r \geqslant 1 \text{ and } d \leqslant g-1\}.$$

(4) From (5.2), it is clear that we must have $p_{\max} \geqslant g - 3 - p_{\max}$ so $p_{\max} \geqslant (g-3)/2$. So

$$(5.5)\qquad p_{\max} \geqslant \begin{cases} (g-3)/2, & g \text{ odd}, \\ (g-2)/2, & g \text{ even}. \end{cases}$$

Since every curve of genus g has a $g^1_{[(g+3)/2]}$,

$$\nu_{\min} \leqslant [(g+3)/2] - 2 = [(g-1)/2].$$

Then

$$g - 2 - \nu_{\min} \geqslant g - 2 - \left[\frac{g-1}{2}\right] = \begin{cases} \dfrac{g-3}{2} & \text{if } g \text{ is odd}, \\ \dfrac{g-2}{2} & \text{if } g \text{ is even}. \end{cases}$$

This is consistent with the Noether-Enriques-Petri Conjecture. It is natural to conjecture:

Conjecture (5.6). *For a generic curve of genus g,*

$$(5.7) \qquad p_{\max} = \begin{cases} (g-3)/2, & g \text{ odd}, \\ (g-2)/2, & g \text{ even}. \end{cases}$$

A problem related to the Noether-Enriques-Petri Conjecture is the following question, which is a slight modification of a conjecture of J. Harris and D. Mumford.

Conjecture (5.8) (*Harris-Mumford Conjecture*). *Let S be a smooth $K-3$ surface, $L \to S$ an ample line bundle. Then $\nu_{\min}(C)$ is constant for all smooth $C \in |L|$.*

Remarks. (1) Donagi has constructed an example S, L where there exist $C_1, C_2 \in |L|$ so C_1 has a g_4^1, but C_2 does not. However, all $C \in |L|$ have either a g_4^1 or a g_6^2. Donagi has some partial results on this conjecture.

(2) The Harris-Mumford Conjecture would be a corollary of the Noether-Enriques-Petri Conjecture. For if $C \in |L|$ is smooth, then by the Lefschetz Theorem (3.b.7) and the adjunction formula

$$K_C \simeq K_S \otimes L|_C \simeq L|_C$$

we have

$$\mathcal{K}_{p,q}(C, K_C) \simeq \mathcal{K}_{p,q}(S, L).$$

Thus

$$(5.9) \qquad p_{\max}(C) \quad \text{is constant for smooth } C \in |L|.$$

If the Noether-Enriques-Petri Conjecture is true, then

$$\nu_{\min}(C) = g - 2 - p_{\max}(C)$$

so this is also constant.

A problem intimately related to the Noether-Enriques-Petri Conjecture is

Problem (5.10). *Generalize the $\mathcal{K}_{p,1}$ Theorem.*

Remarks. We would like to be able to say

$$(5.11) \qquad \mathcal{K}_{p,1}(X, L) \neq 0 \leftrightarrow \varphi_L(X) \quad \text{lies on a member of some class of varieties of low degree.}$$

For $p \geqslant h^0(X, L) - \dim \varphi_L(X) - 2$, this is covered by the $\mathcal{K}_{p,1}$ Theorem. For $p = 1$,

$$\mathcal{K}_{1,1}(X, L) \neq 0 \leftrightarrow \varphi_L(X) \quad \text{lies on a quadric.}$$

A natural starting place might be to try to generalize the Strong Castelnuovo Lemma (3.c.6). For example:

Problem (5.12). *For $P_1, P_2, \cdots, P_d \in \mathbf{P}_r$ points in general position, is it true that*

$$\mathcal{K}_{r-2,1}(P_1, \cdots, P_d) \neq 0 \leftrightarrow P_1, \cdots, P_d \quad \textit{lie on a surface of minimal degree.}$$

The direction $\leftarrow$ is known, and the case $r = 4$ would appear to be the first unknown case for the direction $\rightarrow$.

The Theorem of the Top Row (4.a.4) gives a description of the top row of the $\mathcal{K}_{p,q}(X, L)$ for L sufficiently ample. This might generalize

Problem (5.13). *On a smooth n-fold X, if $L \to X$ is a sufficiently ample analytic line bundle, which $\mathcal{K}_{p,q}(X, L)$ must be zero?*

A variant of Problem (5.13) would be to take $L \to X$ an ample bundle and ask which $\mathcal{K}_{p,q}(X, kL)$ must be zero when k is sufficiently large.

A potentially rich area of study is

Problem (5.14). *What is the variational theory of the $\mathcal{K}_{p,q}(X, L)$? What do they look like for X generic or for X and L generic?*

Here is a special case of (5.14). If a general curve C of genus g has a g_d^r which is special, then if L is the g_d^r,

$$H^0(C, K_C - 2L) = 0 \quad \text{(see [1])}, \tag{5.15}$$

$$\ker\Big(H^0(C, L) \otimes H^0(C, K_C - L) \overset{\mu_0}{\to} H^0(C, K_C)\Big) = 0 \quad \text{(see [9])}, \tag{5.16}$$

where (5.15) follows from the study of the Gaussian system of C, L and (5.16) is Petri's Conjecture. By the Duality Theorem,

$$\mathcal{K}_{p,q}(C, L) \simeq \mathcal{K}_{r-1-p,2-q}(C, K_C, L)$$

so (5.15) and (5.16) are equivalent to

$$\mathcal{K}_{p,q}(C, L) = 0 \quad \text{for } q \geqslant 4, \tag{5.17}$$

$$\mathcal{K}_{r-2,3}(C, L) = 0. \tag{5.18}$$

A consequence of these is that

$$\mathcal{K}_{p,3}(C, L) = 0 \quad \text{for } p \leqslant r - 2. \tag{5.19}$$

Combining this with the Duality Theorem, we obtain

$$\mathcal{K}_{p,3}(C, L) \simeq \begin{cases} H^1(C, L) & \text{if } p = r - 1, \\ 0 & \text{if } p \neq r - 1. \end{cases} \tag{5.20}$$

This gives a complete description of the top row of the $\mathcal{K}_{p,q}$'s in this case. It would be interesting to know what the other $\mathcal{K}_{p,q}$'s look like in this situation.

Problem (5.21). *When X, L is projectively normal, how do the Koszul cohomology groups $\mathcal{K}_{p,q}(X, L)$ relate to the stability of X, L? Can one apply them to moduli question?*

An example of what might happen in the second half of (5.21) is Sernesi's work on moduli of curves of genus [13].

The Gaussian class or extrinsic fundamental class of §2b and the representation of the Hodge groups in §4b seem quite hopeful.

Problem (5.22) (*suggested by P. Griffiths*). *Work out the relative theory of the extrinsic fundamental class for a pair of varieties $X \subset Y$.*

Problem (5.23). *Can the representation of the Hodge groups in* §4d *be used to compute the derivatives of normal functions* (*see* [7])?

A final question is

Problem (5.24). *Can Koszul cohomology be used to make further progress on the Local and Degree One Torelli Problems?*

There are some hopeful signs in this direction–the work of Kii described in §4c, Donagi's work on Degree One Torelli for smooth hypersurfaces in $\mathbf{P}_N$ (see [4]), and a recent paper of the author's [6].

Appendix: The nonvanishing of certain Koszul cohomology groups

MARK GREEN & ROBERT LAZARSFELD

Theorem. *Let X be a compact complex manifold, and L, M_1, M_2 analytic line bundles on X with $L \simeq M_1 \otimes M_2$. Assume*

$$h^0(X, M_i) = r_i + 1, \qquad r_i \geqslant 1, \quad i = 1, 2.$$

Then $\mathcal{K}_{r_1+r_2-1,1}(X, L) \neq 0$.

Corollary. *If a smooth curve C of genus g has a g_d^r with $r \geqslant 1$ and $d \leqslant g-1$, then*

$$\mathcal{K}_{g-(d-2r+2),1}(C, K_C) \neq 0.$$

Remark. In terms of the language of §5, the corollary is equivalent to

$$p_{\max} \geqslant g - 2 - \nu_{\min}.$$

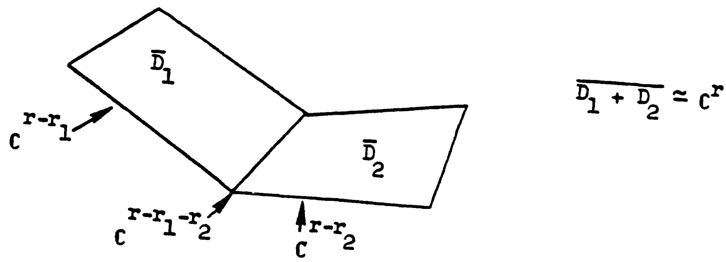
Equality would be the Noether-Enriques-Petri Conjecture.

The Corollary follows from the Theorem by letting M_1 be the g_d^r, M_2 be the residual $g_{2g-2-d}^{r+g-d-1}$, and $L = K_C$.

Proof of Theorem. Let

$$D_1 \in |L - M_1|, \qquad D_2 \in |L - M_2|$$

so $D_1 + D_2 \in |L|$. Letting $\overline{D}_1$ denote the linear span of D_1 pulled up to $\mathbf{C}^{r+1}$, etc., we have the following picture in $\mathbf{C}^{r+1} \simeq H^0(X, L)^*$:



Choose a basis $s_0,\cdots,s_{r+1}$ for $H^0(X, L)$ with dual basis $e_0,\cdots,e_{r+1}$ for $H^0(X, L)^*$ so that

$$\begin{aligned} e_1,\cdots,e_{r-r_1} &\quad \text{is a basis for } \overline{D}_1, \\ e_{r_2+1},\cdots,e_r &\quad \text{is a basis for } \overline{D}_2, \\ e_{r_2+1},\cdots,e_{r-r_1} &\quad \text{is a basis for } \overline{D}_1 \cap \overline{D}_2. \end{aligned}$$

Note

$$\overline{D}_1 = \{s_0 = s_{r-r_1+1} = \cdots = s_r = 0\},$$
$$\overline{D}_2 = \{s_0 = s_1 = \cdots = s_{r_2} = 0\}.$$

Now let

$$\iota = \sum_{i=1}^{r-r_1} e_i \otimes s_i, \qquad s = \sum_{i=0}^{r} e_i \otimes s_i.$$

Consider

$$\alpha = \iota \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} = \iota \wedge \textstyle\bigwedge^{r-r_1-r_2}(\overline{D}_1 \cap \overline{D}_2).$$

While $\iota \in \overline{D}_1 \otimes H^0(X, L)$ we see that α involves only $s_1,\cdots,s_{r_2}$ and these are all zero on $\overline{D}_2$. Thus

$$\alpha \in \textstyle\bigwedge^{r-r_1-r_2+1}\overline{D}_1 \otimes H^0(X, L - [D_2]).$$

Furthermore, $\iota = s$ on D_1 because $s_{r-r_1+1} = \cdots = s_r = s_0 = 0$ on D_1. Thus

$$s \wedge \alpha \in \textstyle\bigwedge^{r-r_1-r_2+2} H^0(X, L)^* \otimes H^0(X, L \otimes (L - [D_1] - [D_2])).$$

Since $L \simeq [D_1] \otimes [D_2]$ we have

$$s \wedge \alpha \in \textstyle\bigwedge^{r-r_1-r_2+2} H^0(X, L)^* \otimes H^0(X, L).$$

Further, as $s \wedge (s \wedge \alpha) = 0$ we have

$$s \wedge \alpha \in \mathcal{K}_{r_1+r_2-1,1}(X, L)$$

using the isomorphism

$$\wedge^{r-r_1-r_2+2} H^0(X, L)^* \simeq \wedge^{r_1+r_2-1} H^0(X, L).$$

It remains to show that $s \wedge \alpha$ is nontrivial in Koszul cohomology. Assume on the contrary that

$$s \wedge \alpha = s \wedge \beta, \qquad \beta \in \wedge^{r-r_1-r_2+1} H^0(X, L)^*.$$

Then

$$\beta = \alpha = \iota \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \quad \text{mod } s.$$

Thus

$$s \wedge e_j \wedge \beta = 0 \quad \text{for } r_2 + 1 \leqslant j \leqslant r - r_1.$$

So

$$\sum_{i=0}^{r} s_i e_i \wedge e_j \wedge \beta = 0 \quad \text{for } r_2 + 1 \leqslant j \leqslant r - r_1$$

and thus

$$e_i \wedge e_j \wedge \beta = 0 \quad \text{for all } 0 \leqslant i \leqslant r, r_2 + 1 \leqslant j \leqslant r - r_1.$$

If $r - r_1 - r_2 + 2 < r + 1$, that is, if $r_1 + r_2 \geqslant 2$ which is true by hypothesis, we may conclude

$$e_j \wedge \beta = 0 \quad \text{for } r_2 + 1 \leqslant j \leqslant r - r_1.$$

Thus

$$\beta = c \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \quad \text{for some } c \in H^0(X, L)^*.$$

Now returning to the equation $s \wedge \alpha = s \wedge \beta$ we get

$$s \wedge (\iota - c) \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} = 0.$$

Since

$$(\iota - c) \wedge e_{r_2+1} \wedge \cdots \wedge e_{r-r_1} \in \wedge^{r-r_1-r_2+1} \overline{(D_1 \cup c)}$$

we conclude $s \in \overline{D_1 \cup c}$. Thus $\sum_{i=0}^{r} s_i e_i \in \overline{D_1 \cup c}$ and hence $e_i \in \overline{D_1 \cup c}$ for all i. So $\overline{D_1 \cup c} = H^0(X, L)^* = \mathbf{C}^{r+1}$. However,

$$\dim \overline{D_1 \cup c} \leqslant \dim \overline{D}_1 + 1 = r - r_1 + 1 < r + 1$$

which is a contradiction. So

$$s \wedge \alpha \neq 0 \quad \text{in } \mathcal{K}_{r_1+r_2-1,1}(X, L).$$

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