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CONSERVATION LAWS AND DIFFERENTIAL CONCOMITANTS

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1. Introduction

The notion of a conservation law on a manifold has appeared previously in papers by H. Osborn, P. R. Eiseman, D. E. Blair, and the author (see [1], [3], and [7]). In the first three sections of this paper the origin of the conservation law problem for manifolds and some earlier results are reviewed. The last section extends results obtained in [3].

In order to formulate a definition of a conservation law on a manifold let us recall the notion of a conservation law in physics. A conservation law is an equation in the form

$$\frac{\partial u}{\partial r} + \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j} = 0,$$

which simply expresses the fact that the quantity of u contained in a domain D of (x_1, x_2, x_3) space changes at a rate equal to the flux of the vector (v_1, v_2, v_3) into D; i.e.,

$$\frac{d}{dt}\int\int\int_D u\, dx_1 dx_2 dx_3 = \int_\partial\int_D \vec{v} \cdot \vec{n}\, dS.$$

Conservation law form may also sometimes be obtained from a system

$$V_t + AV_x = 0, \tag{1.1}$$

where V is a column vector of n unknown functions, A is a square matrix depending on V, t, and x, and the subscripts t and x denote partial differentiations. If we can write $AV_x = W_x$, then

$$V_t + W_x = 0 \tag{1.2}$$

is a system of conservation laws. These examples then lead (see [8]) to the definition of a conservation law on a manifold. A differential 1-form φ on a manifold M is a conservation law for a linear operator, on 1-forms, if both φ and $\underline{h}\varphi$ are exact. A general problem is then to determine all conservation

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laws for a given endomorphism $\underline{h}$ of differential 1-forms on a given manifold M. Various generalizations of this problem are possible. For example, a conservation law problem could be posed for a family $\{\underline{h}_\alpha\}$ of vector 1-forms. One may restrict the problem by imposing algebraic and analytic conditions on $\underline{h}$. A conservation law problem may also be formulated in terms of differential forms of degree $p, p > 1$. In this paper the problem is restricted to the study of conservation laws for a single vector 1-form, $\underline{h}$. The analysis divides itself into two cases according to whether or not the trace of $\underline{h}$ is constant. The main result, in Section 4, which concerns the case in which the trace of $\underline{h}$ is constant, is that there do exist conservation laws for $\underline{h}$. The case of non-constant trace is reviewed in Section 3.

2. Notation and remarks

Let M denote a compact, orientable, n-dimensional Riemannian manifold without boundary, and suppose E denotes the module of C^∞ differential 1-forms on M. An endomorphism $\underline{h}$ of 1-forms induces mappings

$$h^{(q)}: \Lambda^p E \to \Lambda^p E, \; q \leqslant p \text{ and } 1 \leqslant p \leqslant n,$$

of p-forms which are given by

$$h^{(q)}(\theta_1 \cdots \cdots \wedge \theta_p)$$
$$= \frac{1}{q!\,(p-q)!} \sum_\pi |\pi| \{\underline{h}\theta_{\pi(1)} \wedge \cdots \wedge \underline{h}\theta_{\pi(q)}\} \wedge \theta_{\pi(q+1)} \wedge \cdots \wedge \theta_{\pi(p)},$$

where π runs through all permutations of $(1, \cdots, p)$, and $|\pi|$ denotes the sign of the permutation. In particular, if α and β are 1-forms, then

$$h^{(1)}(\alpha \wedge \beta) = \underline{h}\alpha \wedge \beta + \alpha \wedge \underline{h}\beta,$$
$$h^{(2)}(\alpha \wedge \beta) = \underline{h}\alpha \wedge \underline{h}\beta.$$

Also if $p = n$, then $h^{(q)}(\theta_1 \wedge \cdots \wedge \theta_n) = (-1)^{q-1}\mathfrak{C}_{n-q}(\theta_1 \wedge \cdots \wedge \theta_n)$ where $\mathfrak{C}_{n-q}$ is the $(n-q)$-th coefficient in the characteristic equation $\underline{h}^n = \sum_{j=0}^{n-1} \mathfrak{C}_j \underline{h}^j$ of $\underline{h}$ with $\underline{h}^j = h \circ h \circ \cdots \circ h$, j-times. That is, except for sign, $\mathfrak{C}_j$ is the j-th invariant of $\underline{h}$. In particular $\mathfrak{C}_{n-1}$ is the trace of $\underline{h}$. The induced mappings satisfy an identity

$$h^{(q)} = \frac{1}{q!} \sum_{j=0}^{q-1} (-1)^{q+j+1} (h^{q-j})^{(1)} h^{(j)},$$

where both sides of the above expression are assumed to operate on p-forms, with $q \leqslant p$. This identity is established in [9] and is closely related to a similar identity involving the Newton transformations. It should also be noted that $\underline{h}$

is invertible if and only if $h^{(q)}$ is invertible as an operator on q-forms. In the sequel it is assumed that $\underline{h}$ is invertible.

If the commutator of $h^{(1)}$ and exterior differentiation d is denoted by d_h, so that

$$d_h = h^{(1)}d - dh^{(1)},$$

then d_h reduces to d when $\underline{h}$ is equal to the identity. Moreover under the assumptions that the space is contractible and $\underline{h}$ is nonsingular and has an identically vanishing Nijenhuis tensor, d_h will satisfy a Poincaré lemma [10]. The same cannot be said for the commutators of $h^{(q)}$ and d when $q > 1$, though that is not of interest here. For these commutators one has

$$h^{(q+1)}d - dh^{(q+1)} = d_h h^{(q)},$$

where the above expression is assumed to operate on p-forms. For $q \geqslant 1$, it must be assumed that the Nijenhuis tensor $[\underline{h}, \underline{h}]$ of $\underline{h}$ vanishes identically. The Nijenhuis tensor of $\underline{h}$ is defined, for $\theta \in E$, by the formula

$$[\underline{h}, \underline{h}]\theta = -h^{(2)}d\theta + h^{(1)}d\underline{h}\theta - d\underline{h}^2\theta. \tag{2.1}$$

It is easily checked that $[\underline{h}, \underline{h}] \in \mathrm{End}_A(E, E \wedge E)$, where A denotes the C^∞ functions on M. This formula may be rewritten as

$$[\underline{h}, \underline{h}] = -h^{(2)}d + d_h\underline{h}. \tag{2.2}$$

The definition of the concomitant $[\underline{h}, \underline{h}]$ may be extended so that $[\underline{h}, \underline{h}]$ can be regarded as an element of $\mathrm{End}_A(\Lambda^p E, \Lambda^{p+1}E)$. The formula then becomes

$$[\underline{h}, \underline{h}] = -h^{(2)}d + d_h h^{(1)} + dh^{(2)}, \tag{2.3}$$

where both sides of (2.3) act on p-forms. If $\underline{h}$, $\underline{k}$ are vector 1-forms, the concomitant $[\underline{h}, \underline{k}]$ can be formed and is defined by setting

$$[\underline{h}, \underline{k}] = \tfrac{1}{2}\{d_{hk} - (d_h)_k\}. \tag{2.4}$$

The formulas (2.1)–(2.3) are then recovered by setting $\underline{h} = \underline{k}$ and using the relation $(h^2)^{(1)} - h^{(1)}h^{(1)} = -h^{(2)}$.

In view of the preceding discussion it is obvious that if $[\underline{h}, \underline{h}] = 0$, then θ is a conservation law for $\underline{h}$ if and only if $d_h\theta = d_h\underline{h}\theta = 0$, by virtue of the Poincaré lemma. Moreover if θ is a conservation law for $\underline{h}$, then so are the forms $\underline{h}^i\theta$, $i \geqslant 1$, where $\underline{h}^i$ denotes $\underline{h} \circ \underline{h} \circ \cdots \circ \underline{h}$, i-times. Thus it seems natural to impose the condition that the Nijenhuis tensor $[\underline{h}, \underline{h}]$ vanish identically in studying the conservation law problem. This view is somewhat reinforced by the following additional consideration. If one considers the problem of finding differentiable functions g^i, $i = 1, 2, \cdots$, such that

$\underline{h}df^i = dg^i$, for a given $\underline{h}$ and given functions f^i, then locally that problem is the same as investigating the system

$$\frac{\partial f^i}{\partial x^\alpha} h_j^\alpha = \frac{\partial g^i}{\partial x^j}, \quad i, j = 1, \cdots, n,$$

which arises if $\underline{h}dx^i = h_\alpha^i dx^\alpha$ is given in terms of local coordinates $\{x^1, \cdots, x^n\}$. The obvious integrability condition is therefore

$$\frac{\partial^2 f^i}{\partial x^k \partial x^\alpha} h_j^\alpha - \frac{\partial^2 f^i}{\partial x^j \partial x^\alpha} h_k^\alpha = \frac{\partial f^i}{\partial x^\alpha}\left(\frac{\partial h_k^\alpha}{\partial x^j} - \frac{\partial h_j^\alpha}{\partial x^k}\right),$$

which follows from $\partial^2 g^i / \partial x^j \partial x^k = \partial^2 g^i / \partial x^k \partial x^j$. Now if it happens that

$$h_j^i = \frac{\partial f^i}{\partial x^j},$$

we would then obtain the integrability condition,

$$\text{(2.5)} \qquad \frac{\partial h_k^i}{\partial x^\alpha} h_j^\alpha - \frac{\partial h_j^i}{\partial x^\alpha} h_k^\alpha = h_\alpha^i\left(\frac{\partial h_k^\alpha}{\partial x^j} - \frac{\partial h_j^\alpha}{\partial x^k}\right),$$

which is just the vanishing of the Nijenhuis tensor H_{jk}^i. (2.5) is obtainable from (2.1) if one sets $\theta = dx^i$, and $\underline{h}\theta = h_\alpha^i dx^\alpha$ in (2.1), and assumes that $[\underline{h}, \underline{h}]dx^i = 0$.

3. Preliminary results

The notation and results in this section appear in [2] and [3] and are included in this paper for the convenience of the reader. First note that an inner product on $\Lambda^p E$ is defined by setting

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta,$$

where α and β are p-forms, and $*$ is the Hodge operator. Then if $\underline{h}_t$ and tr $\underline{h}$ denote the transpose and trace of $\underline{h}$, it follows from a direct calculation that

$$\text{(3.1)} \qquad h^{(1)} * + * h_t^{(1)} = (\operatorname{tr} \underline{h}) *.$$

Since

$$(\alpha, h^{(1)}\beta) = \int_M \alpha \wedge * h^{(1)}\beta = \int_M \alpha \wedge \{-h_t^{(1)} * + (\operatorname{tr} \underline{h}) *\}\beta$$

$$= \int_M h_t^{(1)}\alpha \wedge * \beta = (h_t^{(1)}\alpha, \beta),$$

$h_t^{(1)}$ is the adjoint of $h^{(1)}$ relative to the inner product (,). It then follows by an induction argument that $h_t^{(p)}$ is the adjoint of $h^{(p)}$ when $p > 1$. The adjoints of d_h and $[\underline{h}, \underline{k}]$ are denoted respectively by δ_h and $\operatorname{ad} j[\underline{h}, \underline{k}]$, and these are also found by direct calculations. The results are as given in the following formulas, where δ is the adjoint of d, and the operators δ_h and $\operatorname{ad} j[\underline{h}, \underline{k}]$ map p-forms to $(p-1)$-forms. We have

$$\delta_h = \delta h_t^{(1)} - h_t^{(1)}\delta, \tag{3.2}$$

$$\delta_h = (-1)^{np+n+1} * \{d_h + d(\operatorname{tr} \underline{h}) \wedge \cdot\} *, \tag{3.3}$$

$$\begin{aligned} &\operatorname{ad} j[\underline{h}, \underline{k}] \\ &= (-1)^{np+1} * \left\{[\underline{h}, \underline{k}] + \tfrac{1}{2}\{d(\operatorname{tr} \underline{hk}) - \underline{h}d(\operatorname{tr} \underline{k}) - \underline{k}d(\operatorname{tr} \underline{h})\} \wedge \cdot\right\} *. \end{aligned} \tag{3.4}$$

It is then possible to define a generalized Laplacian operator Δ_h by setting $\Delta_h = d_h\delta_h + \delta_h d_h$ which appears in [2]. However, our study of conservation laws does not require use of this operator.

The vanishing of the concomitant $[\underline{h}, \underline{h}]$ implies the vanishing of $[\underline{h}^i, \underline{h}^j]$ for any pair of nonnegative integers i and j. Hence if it is assumed that $[\underline{h}, \underline{h}] = 0$, then formula (3.4) yields the following results:

$$d(\operatorname{tr} \underline{h}^{r+1}) = \underline{h}d(\operatorname{tr} \underline{h}^r) + \underline{h}^r d(\operatorname{tr} \underline{h}), \tag{3.5}$$

$$\underline{h}^r d(\operatorname{tr} \underline{h}) = \frac{1}{(r+1)} d(\operatorname{tr} \underline{h}^{r+1}), \tag{3.6}$$

$$\underline{h}d(\operatorname{tr} \underline{h}^r) = \left(\frac{r}{r+1}\right) d(\operatorname{tr} \underline{h}^{r+1}), \quad \text{for } r = 1, 2, 3, \cdots. \tag{3.7}$$

Formula (3.5) is a consequence of formula (3.4) coupled with the fact that $[\underline{h}, \underline{h}] = 0$ implies $[\underline{h}, \underline{h}^r] = 0$, for integers $r \geqslant 1$. The proof of formula (3.6) proceeds by an induction argument. If $r = 1$, then the vanishing of $[h, h]$ in formula (3.4) yields the formula (3.6). If $r \geqslant 1$, then the assumption that $\underline{h}^r d(\operatorname{tr} \underline{h}) = (1/r+1) d(\operatorname{tr} \underline{h}^{r+1})$ yields $\underline{h}^{r+1} d(\operatorname{tr} \underline{h}) = (1/r+2) d(\operatorname{tr} \underline{h}^{r+2})$ since $[\underline{h}^{r+1}, \underline{h}] = 0$. Formula (3.7) is a straightforward consequence of (3.5) and (3.6). We are then led to the following proposition.

Proposition 3.1. *If* $[\underline{h}, \underline{h}] = 0$ *and* $\operatorname{tr} \underline{h}$ *is not a constant, then*

(a) $d(\operatorname{tr} \underline{h}^r)$ *is a conservation law for* $\underline{h}$, *for all positive integers* r,

(b) $d(\operatorname{tr} \underline{h})$ *is a conservation law for* $\underline{h}^r$, *for all positive integers* r.

The preceding result cannot be a global one in the event $\underline{h}$ is cyclic with generator $d(\operatorname{tr} \underline{h})$. The condition that $\underline{h}$ is cyclic simply means that the characteristic and minimal polynomials of $\underline{h}$ are the same, and thus if $\underline{h}$ were cyclic, then

$$\begin{aligned} &d(\operatorname{tr} \underline{h}) \wedge \underline{h}d(\operatorname{tr} \underline{h}) \wedge \cdots \wedge \underline{h}^{n-1} d(\operatorname{tr} \underline{h}) \\ &\qquad = \frac{1}{n!} d(\operatorname{tr} \underline{h}) \wedge d(\operatorname{tr} \underline{h}^2) \wedge \cdots \wedge d(\operatorname{tr} \underline{h}^n), \end{aligned}$$

and hence $\{\operatorname{tr} \underline{h}^j\}_{j=1}^n$ would serve as coordinates, which is impossible if M is compact. In fact it will never be possible to find n independent global conservation laws even when $\underline{h}$ is globally defined. In any case the existence of conservation laws for $\underline{h}$, or $\underline{h}^i$, is guaranteed if tr $\underline{h}$ is not a constant. When tr $\underline{h}$ is constant then no existence statement concerning conservation laws can be made, at least on the basis of the formulas (3.5), (3.6), and (3.7). In the next section the constant trace case is examined.

4. Constant trace case

If we suppose that tr $\underline{h}$ is a constant with $[\underline{h}, \underline{h}] = 0$, then a solution to the problem of finding conservation laws for $\underline{h}$ is obtained if it is possible to produce a vector 1-form $\underline{k}$ with tr $\underline{k}$ not constant and such that the mixed concomitant $[\underline{h}, \underline{k}]$ vanishes identically. If such an operator $\underline{k}$ exists, then $d(\operatorname{tr} \underline{k})$ is a conservation law for $\underline{h}$. The existence of such a vector 1-form thus hinges on obtaining a solution to a system of $n\binom{n}{2}$ differential equations in n^2 unknowns which arise from the condition $[\underline{h}, \underline{k}] = 0$. This system is overdetermined when $n > 3$. In particular if $\underline{k}$ could be chosen by setting $\underline{k} = \alpha I$, where α is a scalar function, then $d\alpha$ is a conservation law for $\underline{h}$. This simple choice of $\underline{k}$ is possible provided there exists a solution to $[\underline{h}, \alpha I] = 0$, which again involves an overdetermined system but with fewer unknowns. A remark concerning this possibility is made at the conclusion of this section. However rather than constructing any solutions explicitly the approach in this section will be to exhibit a candidate $\underline{k}$ with nonconstant trace and thus verifying that $[\underline{h}, \underline{k}] = 0$ under the added hypothesis that $\underline{h}$ be cyclic. The key to producing the result is the following proposition which is a consequence of a theorem due to E. T. Kobayashi. We suppose that the characteristic equation of $\underline{h}$ has the form $\underline{h}^n = a_0 I + a_1 \underline{h} + \cdots + a_{n-1}\underline{h}^{n-1}$.

Proposition 4.1. *If $\underline{h}$ is cyclic, has an identically vanishing Nijenhuis tensor, and has constant trace, then there exist coordinates $\{x^i\}_{i=1}^n$ such that $\underline{h}dx^i = dx^{i+1}$ when $i = 1, 2, \cdots, n-1$ and*

$$hdx^n = a_0 dx^1 + a_1 dx^2 + \cdots + a_{n-1} dx^n.$$

Proof. The proposition is established by first noting that the conditions of constant trace and vanishing Nijenhuis tensor imply that the coefficients $\{a_i\}$, $i = 0, 1, \cdots, n-1$, of the characteristic equation are constant, as noted in Corollary 3.7 [3], or as a consequence of Proposition 3.9 in the same paper. The existence of the coordinates $\{x^i\}$ then follows from E. T. Kobayashi's theorem [5].

The above proposition then implies that with respect to the basis $\{dx^i\}$, $i = 1, 2, \cdots, n$, of 1-forms, $\underline{h}$ is represented by a companion matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \cdot & & 1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{bmatrix}.$$

This basis is in fact a local basis of conservation laws for $\underline{h}$, and thus one possible solution of the conservation law problem has been obtained. However since the goal was to produce a vector 1-form $\underline{k}$ such that $d(\operatorname{tr} \underline{k})$ is a conservation law for $\underline{h}$, we continue.

If an operator $\underline{k}$ is defined by setting

$$\underline{k}dx^i = dx^{i-1} + x^i dx^n$$

for $i = 1, 2, \cdots, n$ (with $dx^0 \equiv 0$), then $\underline{k}$ is also represented by a companion matrix with respect to the basis $\{dx^i\}$, and $d(\operatorname{tr} \underline{k})$ will be a conservation law for $\underline{h}$. This last statement follows by simply noting the following results:

$$\operatorname{tr} \underline{k} = x^n, \tag{4.1}$$

$$\operatorname{tr} \underline{\underline{hk}} = (n-1) + a_0 x^1 + a_1 x^2 + \cdots + a_{n-1} x^n, \tag{4.2}$$

$$\underline{h} d(\operatorname{tr} \underline{k}) = d(\operatorname{tr} \underline{\underline{hk}}). \tag{4.3}$$

(4.3) would also be an immediate consequence of the vanishing of the concomitant $[\underline{h}, \underline{k}]$ which can be verified directly from formula (2.4), which we rewrite in the form

$$\begin{aligned}[\underline{h}, \underline{k}] = \tfrac{1}{2}\{-[h^{(1)}k^{(1)} - (hk)^{(1)}]d + [h^{(1)}d\underline{k} + k^{(1)}d\underline{h}] \\ -[d\underline{\underline{hk}} + d\underline{\underline{kh}}]\}.\end{aligned} \tag{4.4}$$

It is then routine to check that $[h, k]dx^i \equiv 0$, $i = 1, \cdots, n$, and hence formula (4.3) is a result of formula (3.4).

The preceding discussion serves to establish the following result.

Proposition 4.2. *If $\underline{h}$ is cyclic, has identically vanishing Nijenhuis tensor, and has constant trace, then there exists a vector 1-form $\underline{k}$ with nonconstant trace and identically vanishing concomitant $[\underline{h}, \underline{k}]$.*

Proposition 3.1 may now be combined with the preceding results to obtain the following result.

Proposition 4.3. *Suppose that $\underline{h}$ is cyclic and has identically vanishing Nijenhuis tensor. Then*

(a) *if* tr $\underline{h}$ *is not constant,* $d(\text{tr } \underline{h})$ *is a conservation law for* $\underline{h}$, *and*

(b) *if* tr $\underline{h}$ *is constant, there exists a vector* 1*-form* $\underline{k}$ *with nonconstant trace and identically vanishing concomitant* $[\underline{h}, \underline{k}]$ *such that* $d(\text{tr } \underline{k})$ *is a conservation law for* $\underline{h}$.

It should be noted that the operator $\underline{k}$ as defined above is certainly not unique. One could also, for example, define $\underline{k}$ by setting $\underline{k}dx^i = dx^{i-1} + f_i(x^i)dx^n$, $i = 1, \cdots, n$, provided that $f_n(x^n)$ is not constant and that the functions f_i are differentiable.

Next, it should be noted that for an arbitrary vector 1-form $\underline{k}$ the hypotheses that $[\underline{h}, \underline{h}] = [\underline{h}, \underline{k}] = 0$ are not enough to guarantee the vanishing of either $[\underline{h}, \underline{k}^r]$ or $[\underline{h}^r, \underline{k}]$ when r is an integer larger than 1. This statement is clear, for example, in the case $r = 2$ if one considers the identities

$$[\underline{h}, \underline{hk}] + [\underline{k}, \underline{h}^2] = h^{(1)}[\underline{h}, \underline{k}] + [\underline{k}, \underline{h}]\underline{h} + [\underline{h}, \underline{h}]\underline{k},$$

$$[\underline{h}, \underline{k}^2] + [\underline{k}, \underline{kh}] = k^{(1)}[\underline{h}, \underline{k}] + [\underline{k}, \underline{k}]\underline{h} + [\underline{h}, \underline{k}]\underline{k}.$$

However, for the choice of $\underline{k}$ made earlier in this section, i.e., $\underline{k}dx^i = dx^{i-1} + x^i dx^n$, direct calculations do yield the results that $[\underline{h}^r, \underline{k}] = [\underline{h}, \underline{k}^r] = 0$ for $r \geqslant 2$, and consequently it is possible to obtain the following analog of statements (a) and (b) of Proposition 3.1.

Proposition 4.4. *If $\underline{h}$ is cyclic, has identically vanishing concomitant* $[\underline{h}, \underline{h}]$, *and has constant trace, then there exists a vector* 1*-form $\underline{k}$ such that* $[\underline{h}, \underline{k}] = 0$ *and*

(a) $d(\text{tr } \underline{k})$ *is a conservation law for* $\underline{h}^r$, $r \geqslant 1$,

(b) $d(\text{tr } \underline{k}^r)$ *is a conservation law for* $\underline{h}$, $r \geqslant 1$.

Remark. In the first paragraph of this section it was noted that if one could find a differentiable function α such that $[\underline{h}, \alpha I] = 0$, then $d\alpha$ is a conservation law for $\underline{h}$. This is a simple consequence of formula (3.4) which in this event would yield

$$\underline{h}\, d\alpha = \frac{(\text{tr } \underline{h})}{n} d\alpha,$$

since tr $\underline{h}$ is constant. Thus $d\alpha$ is an eigenvector of $\underline{h}$ with constant eigenvalue $(\text{tr } \underline{h})/n$. For example, if $n = 2$, (x, y) are local coordinates and $\underline{h}$ is given by

$$\underline{h}\, dx = (1 - x)dx - x\, dy,$$

$$\underline{h}\, dy = x\, dx + (1 + x)\, dy,$$

then $[\underline{h}, \underline{h}] = 0$. The choice $\alpha = (x + y)$ thus yields the results $[\underline{h}, \alpha I] = 0$ and $\underline{h}\, d\alpha = d\alpha$, since tr $\underline{h} = 2$. One might then be led to conjecture that under the hypotheses of constant trace for $\underline{h}$ and vanishing Nijenhuis tensor that a differentiable function α exists such that $[\underline{h}, \alpha I] = 0$ if and only if $d\alpha$ is an eigenvector of $\underline{h}$ with constant eigenvalue λ. However a straightforward calculation, for $\theta \in E$, shows that

$$[\underline{h}, \alpha I]\theta = \tfrac{1}{2}\{\underline{h}\, d\alpha \wedge \theta - d\alpha \wedge \underline{h}\theta\},$$

and hence $[\underline{h}, \alpha I]$ need not vanish when $\underline{h}\, d\alpha = \lambda\, d\alpha$.

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