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A METHOD OF CLASSIFYING EXPANSIVE SINGULARITIES

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Introduction

To study singularities is in a sense to study the classification of germs of varieties. It is therefore important to give a method of classification. The purpose of this paper is to show the classification of a class of germs of varieties, which will be called *expansive singularities* in this paper, is included in that of Lie algebras of formal vector fields. As a matter of course, the classification of the latter does not seem easy. However, note that such a Lie algebra is given by an inverse limit of finite dimensional Lie algebras of polynomial vector fields truncated at the order k, $k \geqslant 0$. Therefore such Lie algebras can be understood by step by step method in the order k.

Let $\mathbf{C}^n$ be the Cartesian product of n copies of complex numbers $\mathbf{C}$ with natural coordinate system $(x_1, \cdots, x_n)$. By $\mathcal{O}$ we mean the ring of all convergent power series in $x_1, \cdots, x_n$ centered at the origin 0. Let V be a germ of variety in $\mathbf{C}^n$ at 0, and $\mathcal{I}(V)$ the ideal of V in $\mathcal{O}$ (cf. [2, pp. 86–87] for the definitions). Two germs V, V' are said to be *bi-holomorphically equivalent* if there is a germ of holomorphic diffeomorphism φ such that $\varphi(0) = 0$ and $\varphi(V) = V'$.

Let $\mathfrak{X}$ be the Lie algebra of all germs of holomorphic vector fields at 0, and $\mathfrak{X}(V)$ the subalgebra defined by

$$\mathfrak{X}(V) = \{u \in \mathfrak{X}; u\mathcal{I}(V) \subset \mathcal{I}(V)\}.$$

$\mathfrak{X}(V)$ is then an $\mathcal{O}$-module. If there are $v_1, \cdots, v_s$ linearly independent at 0, then Corollary 3,4 of [9] shows that V is bi-holomorphically equivalent to the direct product $\mathbf{C}^s \times W$, where $W \subset \mathbf{C}^{n-s}$. Thus for the structure of singularities we have only to consider the germ W. Taking this fact into account, we may restrict our concern to the varieties such that all $u \in \mathfrak{X}(V)$ vanish at 0. Throughout this paper we shall assume this, i.e., $\mathfrak{X}(V)(0) = \{0\}$.

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$u \in \mathfrak{X}(V)(u(0) = 0)$ is called a *semi-simple expansive vector field*, if after a suitable bi-holomorphic change of variables at 0, u can be written in the form

$$u = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i, \tag{1}$$

where $\hat{\mu}_1, \cdots, \hat{\mu}_n$ lie in the same open half-plane in $\mathbf{C}$ about the origin. (See also §2.A for a justification of this definition.) The origin 0 is called an *expansive singularity*, if $\mathfrak{X}(V)$ contains a semi-simple expansive vector field. If V is given by the locus of zeros of a weighted homogeneous polynomial, then V has an expansive singularity at 0. The advantage of existence of such a vector field u is that one can extend through $\exp tu$ a germ V to a subvariety $\tilde{V}$ in $\mathbf{C}^n$. In this paper we restrict our concern to the germs of varieties with expansive singularities at the origin. For such $\mathfrak{X}(V)$, we set $\mathfrak{X}_k(V) = \{u \in \mathfrak{X}(V); j^k u = 0\}$, where $j^k u$ is the k-th jet at 0. Since $\mathfrak{X}(V) = \mathfrak{X}_0(V)$, $\mathfrak{X}_k(V)$ is a finite codimensional ideal of $\mathfrak{X}(V)$ such that $[\mathfrak{X}_k(V), \mathfrak{X}_l(V)] \subset \mathfrak{X}_{k+l}(V)$ and $\bigcap \mathfrak{X}_k(V) = \{0\}$. We denote by $\mathfrak{g}(V)$ the inverse limit of $\{\mathfrak{X}(V)/\mathfrak{X}_k(V)\}_{k \geqslant 0}$ with the inverse limit topology. Since $\mathfrak{X}(V)/\mathfrak{X}_k(V)$ is finite dimensional, $\mathfrak{g}(V)$ is a Frechét space such that the Lie bracket product $[\ ,\ ]$: $\mathfrak{g}(V) \times \mathfrak{g}(V) \mapsto \mathfrak{g}(V)$ is continuous, namely, $\mathfrak{g}(V)$ is a Frechét-Lie algebra. It is obvious that $\mathfrak{g}(V)$ is a Lie algebra of formal vector fields, where a formal vector field u is a vector field $u = \sum_{i=1}^{n} u_i \partial/\partial x_i$ such that each u_i is a formal power series in $x_1, \cdots, x_n$ without constant terms. The statement to be proved in this paper is as follows.

Theorem I. *Let V, V' be germs of varieties with expansive singularities at the origins of $\mathbf{C}^n$, $\mathbf{C}^{n'}$ respectively, and use the same notation and assumptions as above. Then V and V' are bi-holomorphically equivalent if and only if $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ are isomorphic as topological Lie algebras.*

By the above result, we see especially that any isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$ preserves orders, that is, $\Phi\mathfrak{g}_k(V) = \mathfrak{g}_k(V')$ for every k. Hence to classify $\mathfrak{g}(V)$ is to classify the inverse system $\{\mathfrak{X}(V)/\mathfrak{X}_k(V)\}_{k \geqslant 0}$. Note that $\mathfrak{X}(V)/\mathfrak{X}_k(V)$ is an extension of $\mathfrak{X}(V)/\mathfrak{X}_{k-1}(V)$ with an abelian kernel $\mathfrak{X}_{k-1}(V)/\mathfrak{X}_k(V)$. Such extensions can be classified by representations and second cohomologies (cf. [6]).

The proof of the above theorem is divided into several steps as follows.

Step 1. We define the concept of Cartan subalgebras and prove the conjugacy of Cartan subalgebras.

Step 2. Using the assumption that V (resp. V') has an expansive singularity at 0, we prove that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(V)$ such that $\mathfrak{h} \subset \mathfrak{X}(V)$ (resp. $\mathfrak{h}' \subset \mathfrak{X}(V')$). By a suitable bi-holomorphic change of variables, every element of $\mathfrak{h}$ (resp. $\mathfrak{h}'$) can be changed simultaneously into a normal form, which is a polynomial vector field. Moreover, every eigenvector with respect to $\mathrm{ad}(\mathfrak{h})$ is a polynomial vector field.

Step 3. Now suppose there is an isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then by definition $\Phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$. Hence by Steps 1, 2 we may assume that $\Phi(\mathfrak{h}) \subset \mathfrak{X}(V')$. Thus considering the eigenspace decomposition of $\mathfrak{g}(V)$, $\mathfrak{g}(V')$ with respect to $\mathrm{ad}(\mathfrak{h})$ $\mathrm{ad}(\mathfrak{h}')$ respectively we see that Φ induces an isomorphism of $\mathfrak{p}$ onto $\mathfrak{p}'$, where $\mathfrak{p}$ (resp. $\mathfrak{p}'$) is the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) which can be expressed as a polynomial vector field with respect to the local coordinate system normalizing $\mathfrak{h}$ (resp. $\mathfrak{h}'$).

Step 4. From isomorphism $\Phi: \mathfrak{p} \to \mathfrak{p}'$, we conclude by the same procedure as in [5] that there is a bi-holomorphic diffeomorphism φ of $\mathbf{C}^n$ onto $\mathbf{C}^{n'}$ such that $\varphi(0) = 0$ and $d\varphi\mathfrak{p} = \mathfrak{p}'$. The main idea of making such φ is roughly in the fact that every maximal subalgebra of $\mathfrak{p}$ corresponds to a point. However, since $\mathfrak{p}(0) = \{0\}$, the situation is much more difficult than that of [1]. Existence of expansive vector field plays an important role at this step as well as in the above steps.

Step 5. Recapturing V from the Lie algebra $\mathfrak{p}$, we can conclude $\varphi(V) = V'$.

The theorem is proved by this way. Note that the converse is trivial.

1. Conjugacy of Cartan subalgebras

We denote a formal power series f in a form $f = \sum_{|\alpha|\geq 0} a_\alpha x^\alpha$, where $a_\alpha \in \mathbf{C}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. We denote by $\mathfrak{F}$ the Lie algebra of all formal vector fields, and $\mathfrak{F}_k$ the subalgebra

$$\left\{ u \in \mathfrak{F};\ u = \sum_{i=1}^{n} \sum_{|\alpha|>k} a_{i,\alpha} x^\alpha \partial/\partial x_i \right\}.$$

$\mathfrak{F}$ is then regarded as the inverse limit of the system $\{\mathfrak{F}/\mathfrak{F}_k; p_k\}$ where $p_k: \mathfrak{F}/\mathfrak{F}_{k+1} \mapsto \mathfrak{F}/\mathfrak{F}_k$ is the natural projection. We denote by $\tilde{p}_k$ the projection of $\mathfrak{F}$ onto $\mathfrak{F}/\mathfrak{F}_k$. p_k and $\tilde{p}_k$ are sometimes called forgetful mappings. Since $\mathfrak{F}/\mathfrak{F}_k$ is a finite dimensional vector space over $\mathbf{C}$, $\mathfrak{F}$ is a Fréchet space, and the Lie bracket product is continuous.

Let $\mathfrak{g}$ be a closed Lie subalgebra of $\mathfrak{F}$, and $\mathfrak{g}_k = \mathfrak{F}_k \cap \mathfrak{g}$. The closedness of $\mathfrak{g}$ implies that $\mathfrak{g}$ is the inverse limit of the system $\{\mathfrak{g}/\mathfrak{g}_k; p_k\}_{k\geq 0}$. In this paper, we restrict our concern to a closed subalgebra $\mathfrak{g}$ of $\mathfrak{F}_0$. For any subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, we denote by $\mathfrak{n}(\mathfrak{s})$ the normalizer of $\mathfrak{s}$, i.e., $\mathfrak{n}(\mathfrak{s}) = \{u \in \mathfrak{g};\ [u, \mathfrak{s}] \subset \mathfrak{s}\}$, and by $\mathfrak{g}^{(0)}(\mathfrak{s})$ the 0-eigenspace of $\mathrm{ad}(\mathfrak{s})$, i.e., $\mathfrak{g}^{(0)}(\mathfrak{s})$ is the totality of $v \in \mathfrak{g}$ satisfying that there are nonnegative integers m_k, $k \geq 0$, (depending on v)

such that $\mathrm{ad}(s)^{m_k}v \in \mathfrak{g}_k$ for all $s \in \mathfrak{s}$ and all $k \geqslant 0$, where $\mathrm{ad}(u)v = [u, v]$. If $\mathfrak{s}$ is nilpotent, then $\mathfrak{g}^{(0)}(\mathfrak{s}) \supset \mathfrak{n}(\mathfrak{s})$. Therefore, if $\mathfrak{g}^{(0)}(\mathfrak{s}) = \mathfrak{s}$, then $\mathfrak{n}(\mathfrak{s}) = \mathfrak{s}$. The converse is also true if $\dim \mathfrak{g}^{(0)}(\mathfrak{s}) < \infty$ (cf. [6]).

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra of* $\mathfrak{g}$, if the following conditions are satisfied:

($\mathfrak{h}$, 1) $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}$ such that $\tilde{p}_k\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geqslant 0$.

($\mathfrak{h}$, 2) $\mathfrak{h} = \mathfrak{g}^{(0)}(\mathfrak{h})$.

Note that if $\dim \mathfrak{g} < \infty$ above $\mathfrak{h}$ is a usual Cartan subalgebra. The statement to be proved in this section is as follows.

Proposition A. *Let* $\mathfrak{g}$ *be a closed subalgebra of* $\mathfrak{F}_0$. *Then there exists a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$, *and for Cartan subalgebras* $\mathfrak{h}$, $\hat{\mathfrak{h}}$ *of* $\mathfrak{g}$ *there is an inner automorphism A of* $\mathfrak{g}$ *such that* $A\mathfrak{h} = \tilde{\mathfrak{h}}$.

1.A. Automorphisms of $\mathfrak{g}$

Let $\mathfrak{g}$ be a closed Lie subalgebra of $\mathfrak{F}_0$, and $\mathfrak{g}_k = \mathfrak{g} \cap \mathfrak{F}_k$. For every $u \in \mathfrak{g}$ the adjoint action $\mathrm{ad}(u)$ leaves each $\mathfrak{g}_k$ invariant, hence $\mathrm{ad}(u)$ induces a linear mapping $a_k(u)$ of $\mathfrak{g}/\mathfrak{g}_k$ into itself. $\mathrm{ad}(u)$ is then regarded as the inverse limit of the system $\{a_k(u)\}_{k\geqslant 0}$. Define a linear mapping $e^{t\cdot \mathrm{ad}(u)}\colon \mathfrak{g} \mapsto \mathfrak{g}$ by the inverse limit of $\{e^{t\cdot a_k(u)}\}_{k\geqslant 0}$. Since $\mathrm{ad}(u)$ is a derivation of $\mathfrak{g}$, $e^{t\cdot\mathrm{ad}(u)}$ is a one-parameter family of automorphisms of $\mathfrak{g}$. The group $\mathfrak{a}(\mathfrak{g})$ generated by $\{e^{\mathrm{ad}(u)}; u \in \mathfrak{g}\}$ is called the group of *inner* automophisms of $\mathfrak{g}$. The purpose of this section is to investigate the structure of $\mathfrak{a}(\mathfrak{g})$.

Let $\hat{\mathcal{O}}$ be the ring of all formal power series $\Sigma_{|\alpha|\geqslant 0}\, a_\alpha x^\alpha$, and $\hat{\mathcal{O}}_k$ the ideal given by $\hat{\mathcal{O}}_k = \{\Sigma_{|\alpha|\geqslant k+1}\, a_\alpha x^\alpha\}$. $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is then a finite dimensional algebra over $\mathbf{C}$. We denote by $\tilde{\pi}_k$, π_k the projections $\hat{\mathcal{O}} \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, $\hat{\mathcal{O}}/\hat{\mathcal{O}}_{k+1} \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ respectively. Every $u \in \mathfrak{F}_0$ acts naturally on $\hat{\mathcal{O}}$ as a derivation such that $u\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}_k$ for every k. Conversely, $u \in \mathfrak{F}_0$ can be characterized by the above property. Every $u \in \mathfrak{F}_0$ induces therefore a derivation $u^{(k)}$ of the algebra $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, and $u^{(k)}$ is canonically identified with $\tilde{p}_k u$. Conversely, for every derivation δ of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ such that $\delta\hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k$ there is an element $u \in \mathfrak{F}_0$ such that $\delta = \tilde{p}_k u$.

Since a derivation $u\colon \hat{\mathcal{O}} \mapsto \hat{\mathcal{O}}$ can be regarded as an inverse limit of derivations $\{\tilde{p}_k u\colon \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k\}$, we define an automorphism $\exp u$ of $\hat{\mathcal{O}}$ by an inverse limit of $\{e^{\tilde{p}_k u}\}$. We denote by G' the group generated by $\{\exp u;\, u \in \mathfrak{g}\}$.

Define an automorphism $\mathrm{Ad}(\exp u)$ of $\mathfrak{F}$ by

$$(\mathrm{Ad}(\exp u)v)f = (\exp u)v(\exp - u)f, \quad f \in \hat{\mathcal{O}}. \tag{2}$$

Since $(d/dt)_{t=0}(\exp tu)f = uf$, we see easily that

$$\frac{d}{dt}\mathrm{Ad}(\exp tu)v = [u, \mathrm{Ad}(\exp tu)v]. \tag{3}$$

On the other hand $e^{t\cdot \mathrm{ad}(u)}$ satisfies the same differential equation. Thus by uniqueness we obtain

$$\mathrm{Ad}(\exp u) = e^{\mathrm{ad}(u)}. \tag{4}$$

Especially, if $\mathfrak{g}$ is a closed Lie subalgebra of $\mathfrak{F}_0$, then $\mathrm{Ad}(\exp u)\mathfrak{g} = \mathfrak{g}$ for every $u \in \mathfrak{g}$. Since

$$e^{\mathrm{ad}(u)}e^{\mathrm{ad}(v)} = \mathrm{Ad}(\exp u \cdot \exp v),$$

we have that $\mathfrak{a}(\mathfrak{g}) = \{\mathrm{Ad}(g); g \in G'\}$.

Let $G^{(k)}$ be the group generated by $\{e^{\tilde{p}_k u}; u \in \mathfrak{g}\}$. Since $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is finite dimensional, $G^{(k)}$ is a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{g}_k$. For every integer l such that $l \leqslant k$, the group $G^{(k)}$ leaves $\mathfrak{g}_l/\mathfrak{g}_k$ invariant. Hence $\{G^{(k)}\}_{k\geqslant 0}$ forms an inverse system. We denote by G the inverse limit. Obviously, G' is a subgroup of G. However, note that if a sequence $(u_0, u_1, \cdots, u_n, \cdots)$ satisfies $u_l \in \mathfrak{g}_l$ for every $l \geqslant 0$, then $\exp u_0 \cdot \exp u_1 \cdots \exp u_n \cdots$ is an element of G. Since $G^{(k)}$ is a Lie group, G is a topological group under the inverse limit topology. The purpose of the remainder of this section is to show $G = G'$ and that G is a regular Frechét-Lie group with Lie algebra $\mathfrak{g}$, cf. [9].

Let $G_1^{(k)}$, $k \geqslant 1$, be the group generated by $\{e^{\tilde{p}_k u}; u \in \mathfrak{g}_1\}$, and G_1 the inverse limit of $\{G_1^{(k)}\}_{k\geqslant 1}$.

1.1. Lemma. $\exp$ *is a bijective mapping of* $\mathfrak{g}_1$ *onto* G_1.

Proof. Let $\exp_k$ be the exponential mapping of $\mathfrak{g}_1/\mathfrak{g}_k$ into $G_1^{(k)}$, i.e., $\exp_k u = e^{\tilde{p}_k u}$. Since $\exp\colon \mathfrak{g}_1 \mapsto G_1$ is defined by the inverse limit of $\{\exp_k\}$, we have only to show that $\exp_k\colon \mathfrak{g}_1/\mathfrak{g}_k \mapsto G_1^{(k)}$ is bijective. Since $\mathfrak{g}_1/\mathfrak{g}_k = \tilde{p}_1\mathfrak{g}_1$ is a nilpotent Lie algebra, we see that $\exp_k$ is regular and surjective (cf. [3, p. 229]). However, the derivation $\tilde{p}_k u\colon \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is expressed by a triangular matrix with zeros in the diagonal. Therefore one can define $\log(1 + N)$ by $\sum_{n=1}^{\infty}(-1)^{n-1}N^n/n$, which gives the inverse of $\exp_k$. Thus $\exp_k$ is bijective.

1.2. Corollary. $G' = G$.

Proof. We have only to show $G' \supset G$. Since $G^{(1)} = G/G_1$ is generated by $\{\tilde{p}_1 u; u \in \mathfrak{g}\}$, every $g \in G$ can be written in the form $g = \exp u_1 \cdot \exp u_2 \cdots \exp u_m \cdot h$, where $u_1, \cdots, u_m \in \mathfrak{g}$ and $h \in G_1$. Thus the above lemma shows $G \subset G'$.

We next prove that G is a Frechét-Lie group. Although such a structure of G has no direct relevance to our present purpose, there is an advantage of making analogies easy from the theory of finite dimensional Lie groups.

Let $\sigma\colon \tilde{p}_1\mathfrak{g} \mapsto \mathfrak{g}$ be a linear mapping such that $\tilde{p}_1\sigma\tilde{u} = \tilde{u}$ for $\tilde{u} \in \tilde{p}_1\mathfrak{g}$. It is not hard to see that $\xi(u) = \exp \sigma\tilde{p}_1 u \cdot \exp(u - \sigma\tilde{p}_1 u)$ gives a homeomorphism of an open neighborhood U of 0 of $\mathfrak{g}$ onto an open neighborhood $\tilde{U}$ of the identity e of G. Since G is a topological group, there is an open neighborhood V of 0 of $\mathfrak{g}$ such that $\xi(V)^{-1} = \xi(V)$, $\xi(V)^2 \subset \xi(U)$. We set $\eta(u, v) = \xi^{-1}(\xi(u)\xi(v))$ and $i(u) = \xi^{-1}(\xi(u)^{-1})$ for $u, v \in V$. Next we have to prove the differentiability of η and i. However, the differentiability is defined by inverse limits of differentiable mappings, hence that of η and i is trivial in our case. Thus we get the following.

1.3. Lemma. *G is a regular Frechét-Lie group with Lie algebra* $\mathfrak{g}$.

1.B. Simultaneous normalization and eigenspace decomposition

For any $u \in \mathfrak{F}_0$, the linear mapping $u^{(k)}: \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ splits uniquely into a sum of semi-simple part $u_s^{(k)}$ and nilpotent part $u_N^{(k)}$ such that $[u_s^{(k)}, u_N^{(k)}] = 0$. Using eigenspace decomposition of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, we see that $u_s^{(k)}$ is also a derivation of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, and hence so is $u_N^{(k)}$. For $u^{(k+1)}$, we have that $[p_k u_s^{(k+1)}, p_k u_N^{(k+1)}] = 0$, $p_k u_N^{(k+1)}$ is nilpotent, and that $p_k u_s^{(k+1)}$ is semi-simple by considering eigenspace decomposition of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_{k+1}$. Therefore $p_k u_s^{(k+1)} = u_s^{(k)}$ and $p_k u_N^{(k+1)} = u_N^{(k)}$. Hence taking inverse limit we get formal vector fields u_s, u_N which will be called the *semi-simple part* and the *nilpotent part* of u respectively. A formal vector field is said to be *semi-simple* if it has no nilpotent part.

Let $\mathfrak{s}^k$ be a nilpotent subalgebra of $\mathfrak{F}_0/\mathfrak{F}_k$ for an arbitrarily fixed k. Set $\mathfrak{s}_s^k = \{u_s^{(k)}; u^{(k)} \in \mathfrak{s}^k\}$, and denote by p_k^l the forgetful projection of $\mathfrak{F}_0/\mathfrak{F}_k$ onto $\mathfrak{F}_0/\mathfrak{F}_l$, that is, $p_k^l = p_l p_{l+1} \cdots p_{k-1}$. Since $p_k^1 \mathfrak{s}^k$ is a nilpotent subalgebra of $\mathfrak{F}_0/\mathfrak{F}_1$, there is a basis $(f_1^{(1)}, \cdots, f_n^{(1)})$ of $\hat{\mathcal{O}}_0/\hat{\mathcal{O}}_1$ such that every $u^{(1)} \in p_k^1 \mathfrak{s}^k$ is represented by an upper triangular matrix. Let $(\mu_1(u^{(1)}), \cdots, \mu_n(u^{(1)}))$ be the diagonal part. μ_j is then a linear mapping of $p_k^1 \mathfrak{s}^k$ into $\mathbf{C}$ for every j, which one may regard as a linear mapping of $\mathfrak{s}^k$ into $\mathbf{C}$. Since $u_s^{(1)}$ is the semi-simple part of $u^{(1)}$, it must satisfy

$$u_s^{(1)} f_j^{(1)} = \mu_j(u^{(1)}) f_j^{(1)}. \tag{5}$$

By a simple linear algebra, we see that there are $f_1^{(k)}, \cdots, f_n^{(k)} \in \hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k$ such that

$$u_s^{(k)} f_j^{(k)} = \mu_j(u^{(k)}) f_j^{(k)}, \quad \pi_k^l f_j^{(k)} = f_j^{(l)} \quad (1 \leqslant j \leqslant n) \tag{6}$$

for every $u^{(k)} \in \mathfrak{s}^k$, where π_k^l is the forgetful projection of $\hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k$ onto $\hat{\mathcal{O}}_0/\hat{\mathcal{O}}_l$, that is, $\pi_k^l = \pi_l \pi_{l+1} \cdots \pi_{k-1}$.

Since $f_j^{(k)} \in \hat{\mathcal{O}}_0/\hat{\mathcal{O}}_k$, $f_j^{(k)}$ is expressed in the form

$$f_j^{(k)} = \sum_{0 < |\alpha| < k} a_{j,\alpha} x^\alpha. \tag{7}$$

Set $y_j = \Sigma_{0<|\alpha|\leqslant k}\, a_{j,\alpha}x^\alpha$. Since $f_1^{(1)}, \cdots, f_n^{(1)}$ are linearly independent, these give a formal change of variables and every $u_s^{(k)}$ can be written in the form

$$u_s^{(k)} = \sum_{i=1}^{n} \mu_i(u^{(k)})y_i\partial/\partial y_i. \tag{8}$$

Since $[\mathfrak{s}_s^k, \mathfrak{s}^k] = 0$, for $\mathfrak{s}^k$ is nilpotent, every $u^{(k)} \in \mathfrak{s}^k$ should be written in the form

$$u^{(k)} = \sum_{i=1}^{n} \sum_{\substack{\langle\alpha,\mu\rangle=\mu_i\\ 0<|\alpha|<k}} a_{i,\alpha}y^\alpha\partial/\partial y_i, \tag{9}$$

where $\langle\alpha, \mu\rangle = \alpha_1\mu_1 + \cdots + \alpha_n\mu_n$. It should be noted that the semi-simple part $u_s^{(k)}$ of $u^{(k)}$ has been changed into a linear diagonal vector field such as (8).

Let $\mathfrak{s}^{k+1}$ be another nilpotent subalgebra of $\mathfrak{F}_0/\mathfrak{F}_{k+1}$ such that $p_k\mathfrak{s}^{k+1} \subset \mathfrak{s}^k$, and let $\mathfrak{s}_s^{k+1} = \{u_s^{(k+1)}; u^{(k+1)} \in \mathfrak{s}^{k+1}\}$. Since $p_{k+1}^1\mathfrak{s}^{k+1} \subset p_k^1\mathfrak{s}^k$, the equality (5) holds also for every $u^{(1)} \in p_{k+1}^1\mathfrak{s}^{k+1}$, and the equality (6) does for every $p_k\mathfrak{s}^{k+1}$. By a simple linear algebra, we see that there are $f_1^{(k+1)}, \cdots, f_n^{(k+1)} \in \hat{\mathcal{O}}_0/\hat{\mathcal{O}}_{k+1}$ such that

$$u_s^{(k+1)}f_j^{(k+1)} = \mu_j(u^{(k+1)})f_j^{(k+1)}, \quad \pi_k f_j^{(k+1)} = f_j^{(k)}. \tag{10}$$

Note that $f_j^{(k+1)} = f_j^{(k)} + \Sigma_{|\alpha|=k+1}\, a_{j,\alpha}x^\alpha$. Hence by putting

$$y_j = \sum_{0<|\alpha|<k+1} a_{j,\alpha}x^\alpha \tag{11}$$

instead of (7), we get the same equations as (8) and (9) with respect to $\mathfrak{s}^k$. Moreover we have

$$u_s^{(k+1)} = \sum_{i=1}^{n} \mu_i(u^{(k+1)})y_i\partial/\partial y_i, \tag{12}$$

$$u^{(k+1)} = \sum_{i=1}^{n} \sum_{\substack{\langle\alpha,\mu\rangle=\mu_i\\ 0<|\alpha|<k+1}} a_{i,\alpha}y^\alpha\partial/\partial y_i \tag{13}$$

for every $u^{(k+1)} \in \mathfrak{s}^{k+1}$. Especially we obtain the following.

1.4. Lemma. *Notations and assumptions being as above, the forgetful projection p_k: $\mathfrak{s}_s^{k+1} \mapsto \mathfrak{s}_s^k$ is injective.*

Let $\{\mathfrak{s}^k\}_{k\geqslant 1}$ be a series of nilpotent subalgebras $\mathfrak{s}^k$ of $\mathfrak{F}_0/\mathfrak{F}_k$ such that $p_k\mathfrak{s}^{k+1} \subset \mathfrak{s}^k$ for every $k \geqslant 1$. We denote by $\mathfrak{s}$ the inverse limit, and set $\mathfrak{s}_s = \{u_s; u \in \mathfrak{S}\}$. Note that $\dim \mathfrak{s}_s^k \leqslant n$ for every $k \geqslant 1$. Thus there is an integer k_0 such that p_k: $\mathfrak{s}_s^{k+1} \mapsto \mathfrak{s}_s^k$ is bijective for every $k \geqslant k_0$. By a method of inverse limit, we see that there is a formal change of variables

$$y_j = f_j(x_1, \cdots, x_n),\ 1 \leqslant j \leqslant n, f_j \in \hat{\mathcal{O}}_0 \tag{14}$$

such that (8) and (9) hold for every $u^{(k)} \in \mathfrak{s}^k (k \geqslant 1)$, and

$$u_s = \sum_{i=1}^{n} \mu_i(u) y_i \partial/\partial y_i, \tag{15}$$

$$u = \sum_{i=1}^{n} \sum_{\langle \alpha, \mu \rangle = \mu_i} a_{i,\alpha} y^\alpha \partial/\partial y_i \tag{16}$$

for every $u \in \mathfrak{s}$.

Now let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_0$, and suppose the above $\mathfrak{s}^k$'s are subalgebras of $\mathfrak{g}/\mathfrak{g}_k$ respectively. Hence the inverse limit $\mathfrak{s}$ is a closed subalgebra of $\mathfrak{g}$. We next consider the eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathrm{ad}(\mathfrak{s})$. Since $\mathrm{ad}(u)$: $\mathfrak{F}_0 \mapsto \mathfrak{F}_0$ leaves $\mathfrak{g}$ invariant for every $u \in \mathfrak{s}$, and $[\mathrm{ad}(u), \mathrm{ad}(u_s)] = 0$, we see that $\mathrm{ad}(u_s)$: $\mathfrak{F}_0 \mapsto \mathfrak{F}_0$ is the semi-simple part of $\mathrm{ad}(u)$ and hence $\mathrm{ad}(u_s)\mathfrak{g} \subset \mathfrak{g}$. Therefore we have only to consider the eigenspace decomposition with respect to $\mathrm{ad}(\mathfrak{s}_s)$.

For a linear mapping λ of $\tilde{p}_1 \mathfrak{s}_s$ into $\mathbf{C}$, i.e., $\lambda \in (\tilde{p}_1 \mathfrak{s}_s)^*$, we denote by $\mathfrak{F}_\lambda$ the subspace

$$\left\{ u \in \mathfrak{F}_0;\ u = \sum_{i=1}^{n} \sum_{\langle \alpha, \mu \rangle - \mu_i = \lambda} a_{i,\alpha} y^\alpha \partial/\partial y_i \right\}.$$

Note that $\mathfrak{F}_\lambda = \{0\}$ for almost all $\lambda \in (\tilde{p}_1 \mathfrak{s}_s)^*$ except countably many λ's. By $\pi(\mathfrak{s})$ we denote the set of all $\lambda \in (\tilde{p}_1 \mathfrak{s}_s)^*$ such that $\mathfrak{F}_\lambda \neq \{0\}$. If $\tilde{p}_1 \mathfrak{s}_s = \{0\}$, then we set $\pi(\mathfrak{s}) = 0$, because all μ_j's are zeros.

1.5. Lemma. *If $\tilde{p}_1 \mathfrak{s}_s = 0$, then $\mathfrak{g}^{(0)}(\mathfrak{s}) = \mathfrak{g}$.*

Proof. By (16), every $u \in \mathfrak{s}$ can be written in the form $u = u_1 + u_2$ such that

$$u_1 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_j^i y_j \partial/\partial y_i, \quad u_2 = \sum_{i=1}^{n} \sum_{|\alpha| \geqslant 2} a_{i,\alpha} y^\alpha \partial/\partial y_i.$$

The reason for the shape of u_1 is that the linear part of u is an upper triangular matrix. Therefore for every $k \geqslant 1$ there is an integer m_k such that $\mathrm{ad}(u)^{m_k} \mathfrak{F}_0 \subset \mathfrak{F}_k$ for every $u \in \mathfrak{s}$. This means $\mathfrak{g} = \mathfrak{g}^{(0)}(\mathfrak{s})$ by definition.

Now we set $\mathfrak{g}^{(\lambda)}(\mathfrak{s}) = \mathfrak{g} \cap \mathfrak{F}_\lambda$ for every $\lambda \in \pi(\mathfrak{s})$.

1.6. Lemma. *Every $u \in \mathfrak{g}$ can be rearranged in the form*

$$u = \sum_{\lambda \in \pi(\mathfrak{s})} u_\lambda, \quad u_\lambda \in \mathfrak{F}_\lambda.$$

Moreover, every u_λ is contained in $\mathfrak{g}^{(\lambda)}(\mathfrak{s})$.

Proof. Since the first assertion is trivial, we have only to show the second one. Since $\pi(\mathfrak{s})$ is a countable set, there is $v_0 \in \mathfrak{s}_s$ such that $\lambda(v_0^{(1)}) \neq \lambda'(v_0^{(1)})$ for any $\lambda, \lambda' \in \pi(\mathfrak{s})$ satisfying $\lambda \neq \lambda'$. For every k, let $u^{(k)}$ be the truncation of $u \in \mathfrak{g}$ at the order k. $u^{(k)}$ is canonically identified with $\tilde{p}_k u$, and can be rearranged in the form $u^{(k)} = \sum_{\lambda \in \pi(\mathfrak{s})} u_\lambda^{(k)}$, where each $u_\lambda^{(k)}$ is the truncation of u_λ at the order k. Since $\mathfrak{g}/\mathfrak{g}_k$ is finite dimensional, only finite number of $u_\lambda^{(k)}$'s do not vanish. Applying $\mathrm{ad}(v_0^{(k)})^l$ to $u^{(k)}$ we have, since $\mathrm{ad}(\mathfrak{s}_s)\mathfrak{g} \subset \mathfrak{g}$,

$$\mathrm{ad}(v^{(k)})^l u^{(k)} = \sum_{\lambda \in \pi(\mathfrak{s})} \lambda(v_0)^l u_\lambda^{(k)} \in \mathfrak{g}/\mathfrak{g}_k.$$

Hence considering Vandermonde's matrix we get $u_\lambda^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$. Thus taking inverse limit we get $u_\lambda \in \mathfrak{g}$; hence the desired result.

1.7. Corollary. *$\tilde{p}_k\mathfrak{g}^{(0)}(\mathfrak{s})$ is the zero-eigenspace of* $\mathrm{ad}(\tilde{p}_k\mathfrak{s})$: $\mathfrak{g}/\mathfrak{g}_k \mapsto \mathfrak{g}/\mathfrak{g}_k$.

Proof. It is trivial that $\tilde{p}_k\mathfrak{g}^{(0)}(\mathfrak{s})$ is contained in the zero-eigenspace of $\mathrm{ad}(\tilde{p}_k\mathfrak{s})$, for $[\mathfrak{s}_s, \mathfrak{g}^{(0)}(\mathfrak{s})] = \{0\}$. Thus we have only to show the converse. The zero-eigenspace of $\mathrm{ad}(\tilde{p}_k\mathfrak{s})$ is equal to that of $\mathrm{ad}(\tilde{p}_k\mathfrak{s}_s)$, that is, the space of all $v^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$ such that $[\tilde{p}_k\mathfrak{s}_s, v^{(k)}] = \{0\}$. Thus $v^{(k)}$ should be written in the form (9). Let $v \in \mathfrak{g}$ be an element such that $\tilde{p}_k v = v^{(k)}$, and let $v = \sum_{\lambda \in \pi(\mathfrak{s})} v_\lambda$ be the decomposition in accordance with the above lemma. Then it is clear that $\tilde{p}_k v_0 = v^{(k)}$. Since $v_0 \in \mathfrak{g}^{(0)}(\mathfrak{s})$, we get the desired result.

1.C. Existence and conjugacy of Cartan subalgebras

Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_0$. If $\mathfrak{g}/\mathfrak{g}_1 = \{0\}$, then $\mathfrak{g}/\mathfrak{g}_k$ is nilpotent for every $k \geqslant 1$, for $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$. Therefore by Lemma 1.5 we see that $\mathfrak{g}$ itself is the only Cartan subalgebra of $\mathfrak{g}$, and the conjugacy is hence trivial in this case.

Now suppose $\mathfrak{g}/\mathfrak{g}_1 \neq \{0\}$, and let $\mathfrak{h}^1$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_1$.

1.8. Lemma. *Let $\mathfrak{h}^1, \cdots, \mathfrak{h}^k$ be a series of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_1, \cdots, \mathfrak{g}/\mathfrak{g}_k$ respectively such that $p_{l-1}\mathfrak{h}^l = \mathfrak{h}^{l-1}$ for $2 \leqslant l \leqslant k$. Then there is a Cartan subalgebra $\mathfrak{h}^{k+1}$ of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k\mathfrak{h}^{k+1} = \mathfrak{h}^k$.*

Proof. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$. We prove at first that $p_k\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$. Since $\mathfrak{h}'$ is nilpotent, so is $p_k\mathfrak{h}'$. Let $\mathfrak{h}'_s = \{u_s^{(k+1)}; u^{(k+1)} \in \mathfrak{h}'\}$, and let $v^{(k)}$ be an element of the zero-eigenspace of $p_k\mathfrak{h}'$. Then $[v^{(k)}, p_k\mathfrak{h}'_s] = \{0\}$, and hence $v^{(k)}$ can be written in the form (9). Let $v^{(k+1)}$ be an element of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k v^{(k+1)} = v^{(k)}$. Using the eigenspace decomposition of $\mathfrak{g}/\mathfrak{g}_{k+1}$ with respect to $\mathrm{ad}(\mathfrak{h}'_s)$, we see that $v^{(k+1)} = \sum_{\lambda \in \pi(\mathfrak{h}')} v_\lambda^{(k+1)}$. Note that this decomposition is given by only rearranging of the terms of $v^{(k+1)}$ (cf. Lemma 1.6). Hence it is clear that $p_k v_0^{(k+1)} = v^{(k)}$, where $v_0^{(k+1)}$ is an element of the zero-eigenspace of $\mathfrak{h}'_s$. However, since $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$, we get $v_0^{(k+1)} \in \mathfrak{h}'$. Thus $v^{(k)} \in p_k\mathfrak{h}'$, and $p_k\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

By the well-known conjugacy of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$, there is an inner automorphism A such that $A(p_k\mathfrak{h}') = \mathfrak{h}^k$. Since there is a natural projectioin of $G^{(k+1)}$ onto $G^{(k)}$ (cf. §1.A), there is an inner automorphism A' of $\mathfrak{g}/\mathfrak{g}_{k+1}$ which induces naturally A. Thus, by setting $A'\mathfrak{h}' = \mathfrak{h}^{k+1}$, $\mathfrak{h}^{k+1}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k\mathfrak{h}^{k+1} = \mathfrak{h}^k$.

By the above lemma, we have a series $\{\mathfrak{h}^k\}_{k\geqslant 1}$ of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_k\mathfrak{h}^{k+1} = \mathfrak{h}^k$. Let $\mathfrak{h}$ be the inverse limit of $\mathfrak{h}^k$.

1.9. Lemma. *With the same notations and assumptions as above, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. Since $\tilde{p}_k\mathfrak{h} = \mathfrak{h}^k$, $\tilde{p}_k\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geqslant 1$. By Corollary 1.7, $\tilde{p}_k\mathfrak{g}^{(0)}(\mathfrak{h})$ is the zero-eigenspace of $\operatorname{ad}(p_k\mathfrak{h})$. Since $\tilde{p}_k\mathfrak{h} = \mathfrak{h}^k$ is a Cartan subalgebra, we have $\tilde{p}_k\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathfrak{h}^k$ and hence $\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathfrak{h}$. Thus $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

We next consider the converse of the above lemma.

1.10. Lemma. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, $\tilde{p}_k\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geqslant 1$.*

Proof. By Corollary 1.7, the zero-eigenspace of $\operatorname{ad}(\tilde{p}_k\mathfrak{h})$ is equal to $\tilde{p}_k\mathfrak{g}^{(0)}(\mathfrak{h})$. Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we see $\tilde{p}_k\mathfrak{g}^{(0)}(\mathfrak{h}) = \tilde{p}_k\mathfrak{h}$. Thus $\tilde{p}_k\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

As in §1.A, we denote by $G^{(k)}$ the Lie group generated by $\{e^{\tilde{p}_k u}; u \in \mathfrak{g}\}$. Let π_k: $G^{(k+1)} \mapsto G^{(k)}$ be the natural projection. We shall next prove the conjugacy of Cartan subalgebras, which completes the proof of Proposition A. Let $\mathfrak{h}$, $\hat{\mathfrak{h}}$ be Cartan subalgebras of $\mathfrak{g}$. By the argument in the first part of this section, we may assume $\mathfrak{g}/\mathfrak{g}_1 \neq \{0\}$. Since $\tilde{p}_1\mathfrak{h}$, $\tilde{p}_1\hat{\mathfrak{h}}$ are Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_1$, there is $g_1 \in G^{(1)}$ such that $\operatorname{Ad}(g_1)(\tilde{p}_1\hat{\mathfrak{h}}) = \tilde{p}_1\mathfrak{h}$. Therefore one may assume without loss of generality that $\tilde{p}_1\mathfrak{h} = \tilde{p}_1\hat{\mathfrak{h}}$. Let $G_l^{(k)}$ be the Lie group generated by $\{e^{\tilde{p}_k u}; u \in \mathfrak{g}_l\}$ for any $l, l \leqslant k$.

1.11. Lemma. *Let $\mathfrak{h}$, $\hat{\mathfrak{h}}$ be Cartan subalgebras of $\mathfrak{g}$ such that $\tilde{p}_k\mathfrak{h} = \tilde{p}_k\hat{\mathfrak{h}}$. Then there is $g_{k+1} \in G_k^{(k+1)}$ such that $\operatorname{Ad}(g_{k+1})(\tilde{p}_{k+1}\mathfrak{h}) = \tilde{p}_{k+1}\hat{\mathfrak{h}}$.*

Proof. Since $\tilde{p}_k\mathfrak{h} = \tilde{p}_k\hat{\mathfrak{h}}$, $\tilde{p}_{k+1}\mathfrak{h}$ and $\tilde{p}_{k+1}\hat{\mathfrak{h}}$ are Cartan subalgebras of $p_k^{-1}\tilde{p}_k\mathfrak{h} = p_k^{-1}\tilde{p}_k\hat{\mathfrak{h}}$. Let $p_k^{-1}\tilde{p}_k\mathfrak{h} = \tilde{p}_{k+1}\mathfrak{h} \oplus \Sigma_{\lambda\neq 0}\,\mathfrak{g}'_\lambda$, $p_k^{-1}p_k\hat{\mathfrak{h}} = \tilde{p}_{k+1}\hat{\mathfrak{h}} \oplus \Sigma_{\lambda\neq 0}\,\mathfrak{g}''_\lambda$ be the eigenspace decompositions with respect to $\operatorname{ad}(\tilde{p}_{k+1}\mathfrak{h})$ and $\operatorname{ad}(\tilde{p}_{k+1}\hat{\mathfrak{h}})$ respectively. Since $p_k\tilde{p}_{k+1}\mathfrak{h} = p_k\tilde{p}_{k+1}\hat{\mathfrak{h}} = p_k\mathfrak{h}$, we see that $\Sigma\,\mathfrak{g}'_\lambda \subset \mathfrak{g}_k/\mathfrak{g}_{k+1}$ and $\Sigma\,\mathfrak{g}''_\lambda \subset \mathfrak{g}_k/\mathfrak{g}_{k+1}$. It is well-known (cf. [6, pp. 59–66]) that there are $v_1, \cdots, v_m \in \Sigma_{\lambda\neq 0}\,\mathfrak{g}'_\lambda$, $w_1, \cdots, w_l \in \Sigma_{\lambda\neq 0}\,\mathfrak{g}''_\lambda$ such that

$$\operatorname{Ad}(\exp v_1) \cdots \operatorname{Ad}(\exp v_m)\operatorname{Ad}(\exp w_1) \cdots \operatorname{Ad}(\exp w_l)\tilde{p}_{k+1}\mathfrak{h} = \tilde{p}_{k+1}\hat{\mathfrak{h}}.$$

Since $\exp v_i$, $\exp w_j \in G_k^{(k+1)}$, there is $g_{k+1} \in G_k^{(k+1)}$ such that $\operatorname{Ad}(g_{k+1})(\tilde{p}_{k+1}\mathfrak{h}) = \tilde{p}_{k+1}\hat{\mathfrak{h}}$.

Let G_k be the subgroup of G generated by $\{e^u; u \in \mathfrak{g}_k\}$. For Cartan subalgebras $\mathfrak{h}$, $\hat{\mathfrak{h}}$ of $\mathfrak{g}$, the above lemma shows that there are elements $g_1, \cdots, g_k, \cdots$ such that $g_k \in G_k$ and

$$\operatorname{Ad}(g_1) \cdots \operatorname{Ad}(g_k)\hat{\mathfrak{h}} = \mathfrak{h}, \quad \operatorname{mod} \mathfrak{g}_{k+1}.$$

Noting that $g_1 \cdots g_k \cdots \in G$ and putting $g = g_1 \cdots g_k \cdots$, we see $\mathrm{Ad}(g)\mathfrak{h} = \hat{\mathfrak{h}}$. This shows the conjugacy of Cartan subalgebras, and Proposition A is thereby proved.

2. Cartan subalgebras at expansive singularities

2.A. Semi-simple expansive vector fields

In this section, the notation is the same as in the introduction. A germ of holomorphic vector field $u \in \mathfrak{X}(V)$ is said to be *expansive*, if the eigenvalues of the linear term of u at 0 lie in the same open half-plane in **C** about the origin. u is said to be *semi-simple expansive* if u is expansive and semi-simple as a formal vector field. The purpose of this section is to show the following.

2.1. Lemma. *Let $u \in \mathfrak{X}(V)$ be a semi-simple expansive vector field. Then there is a germ $y_j = f_j(x_1, \cdots, x_n)$, $1 \leqslant j \leqslant n$, of biholomorphic change of variables such that u can be written in the form*

$$u = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i.$$

Proof. By a suitable change of variables $y_j = \sum_{0<|\alpha|\leqslant k} a_{j,\alpha} x^\alpha$ such as in (7), u can be written in the form

$$u = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i + w, \quad w \in \mathfrak{X}_k(V)$$

for sufficiently large k. For the proof that u is linearizable, it is enough to show that there are holomorphic functions $f_1, \cdots, f_n$ in $y_1, \cdots, y_n$ such that $uf_j = \hat{\mu}_j f_j$ $(1 \leqslant j \leqslant n)$ and $f_j = y_j +$ higher order terms. Set $f_j = y_j + g_j$, and consider the equation $u(y_j + g_j) = \hat{\mu}_j(y_j + g_j)$. Then we get

$$(u - \hat{\mu}_j)g_j = -wy_j. \tag{17}$$

Since k is sufficiently large, we have

$$\lim_{t\to\infty} e^{-t(u-\hat{\mu}_j)} w\, y_j = 0, \tag{18}$$

and

$$-\int_0^\infty e^{-t(u-\hat{\mu}_j)} w\, y_j\, dt \tag{19}$$

exists as a germ of holomorphic functions (cf. [5]). Set $g_j = -\int_0^\infty e^{-t(u-\hat{\mu}_j)} w\, y_j\, dt$. Then

$$(u - \hat{\mu}_j)g_j = \int_0^\infty \frac{d}{dt} e^{-t(u-\hat{\mu}_j)} w\, y_j\, dt$$

$$= \Big[e^{-t(u-\hat{\mu}_j)} w\, y_j \Big]_0^\infty = -w\, y_j.$$

2.B. Lie algebras containing semi-simple expansive vector fields

Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_0$ such that $\mathfrak{g}$ contains a semi-simple expansive vector field X.

2.2. Lemma. *Let X be a semi-simple expansive vector field in $\mathfrak{g}$. Then there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing X.*

Proof. By the same proof as in the above lemma, we see that X can be linearizable by a suitable formal change of variables, and hence we may assume that X can be written in the form $X = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i$, $\operatorname{Re} \hat{\mu}_i > 0$. Let $\mathfrak{g}^{(0)}(X) = \{u \in \mathfrak{g};\, [X, u] = 0\}$. Since every $u \in \mathfrak{g}^{(0)}(X)$ can be written in the form

$$u = \sum_{i=1}^{n} \sum_{\langle \alpha, \hat{\mu} \rangle = \hat{\mu}_i} a_{i,\alpha} y^{\alpha} \partial/\partial y_i, \tag{20}$$

we see that $\mathfrak{g}^{(0)}(X)$ is a finite dimensional Lie subalgebra of $\mathfrak{g}$. Since $\operatorname{ad}(X)$: $\mathfrak{g}^{(0)}(X) \mapsto \mathfrak{g}^{(0)}(X)$ is of diagonal type, there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{(0)}(X)$ containing X. We shall show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. For that purpose we have only to show $\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathfrak{h}$. Since $X \in \mathfrak{h}$, we see $\mathfrak{g}^{(0)}(\mathfrak{h}) \subset \mathfrak{g}^{(0)}(X)$, and hence $\mathfrak{g}^{(0)}(\mathfrak{h})$ is the zero-eigenspace of $\operatorname{ad}(\mathfrak{h})$ in $\mathfrak{g}^{(0)}(X)$. However, since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{(0)}(X)$, we have $\mathfrak{h} = \mathfrak{g}^{(0)}(\mathfrak{h})$.

2.3. Corollary. *If $\mathfrak{g}$ has a semi-simple expansive vector field, then every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is finite dimensional, and $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is finite dimensional for every $\lambda \in \pi(\mathfrak{h})$.*

Proof. By the above lemma, there is a finite dimensional Cartan subalgebra of $\mathfrak{g}$. However from Proposition A it follows that all Cartan subalgebras are finite dimensional, and every Cartan subalgebra contains a semi-simple expansive vector field. Note that

$$\mathfrak{F}_\lambda = \left\{ u \in \mathfrak{F}_0;\, u = \sum_{i=1}^{n} \sum_{\langle \alpha, \mu \rangle - \mu_i = \lambda} a_{i,\alpha} y^{\alpha} \partial/\partial y_i \right\}.$$

Since $\mathfrak{h}$ contains an expansive vector field, we see that $\dim \mathfrak{F}_\lambda < \infty$ and hence $\dim \mathfrak{g}^{(\lambda)}(\mathfrak{h}) < \infty$.

2.4. Corollary. *With the same notation as in the introduction, if $\mathfrak{X}(V)$ contains a semi-simple expansive vector field X, then there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(V)$ such that $\mathfrak{h} \subset \mathfrak{X}(V)$. Moreover, for that $\mathfrak{h}$, $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is contained in $\mathfrak{X}(V)$ for every $\lambda \in \pi(\mathfrak{h})$.*

Proof. Since $X \in \mathfrak{X}(V)$, Lemma 2.1 shows that X can be written in the form $X = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i$ by a suitable biholomorphic change of variables. Therefore every $u \in \mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is contained in $\mathfrak{X}(V)$, as u is a polynomial vector field in $y_1, \cdots, y_n$.

2.C. Isomorphisms of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$

Let V, V' be germs of varieties in $\mathbf{C}^n$, $C^{n'}$ respectively. Supose there is a bicontinuous isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

2.5. Lemma. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(V)$. Then so is $\Phi(\mathfrak{h})$ of $\mathfrak{g}(V')$.*

Proof. Set $\mathfrak{h}' = \Phi(\mathfrak{h})$. Since $\Phi\colon \mathfrak{g}(V) \mapsto \mathfrak{g}(V')$ is continuous, for every k' there is an integer $k = k(k')$ such that $\Phi(\mathfrak{g}(V)) \subset \mathfrak{g}_{k'}(V')$. Thus $\tilde{p}_{k'}\mathfrak{h}'$ is a nilpotent subalgebra of $\mathfrak{g}(V')/\mathfrak{g}_{k'}(V')$, and $\mathfrak{g}^{(0)}(\mathfrak{h}') \supset \Phi(\mathfrak{g}^{(0)}(\mathfrak{h}))$. Replacing Φ by Φ^{-1} we hence get the desired result.

Now suppose that V and V' have expansive singularities at the origins respectively. By Corollary 2.4, $\mathfrak{X}(V)$ and $\mathfrak{X}(V')$ contain Cartan subalgebras of $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ respectively.

2.6. Corollary. *Under the same assumptions as above, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(V)$ contained in $\mathfrak{X}(V)$, and suppose there is a bicontinuous isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then there is a bicontinuous isomorphism Ψ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$ such that $\Psi(\mathfrak{h}) \subset \mathfrak{X}(V')$, that is, $\Psi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$ contained in $\mathfrak{X}(V')$.*

Proof. By the above lemma, $\Phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$. By Corollary 2.4, there is a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}(V')$ contained in $\mathfrak{X}(V')$. By Proposition A, there is $g \in G$ such that $\mathrm{Ad}(g)\Phi(\mathfrak{h}) = \mathfrak{h}'$. Noting that $\mathrm{Ad}(g)\colon \mathfrak{g}(V') \mapsto \mathfrak{g}(V')$ is a bicontinuous isomorphism, we thus obtain the desired $\Psi = \mathrm{Ad}(g)\Phi$.

In the remainder of this section we assume that there is a bicontinuous isomorphism $\Phi\colon \mathfrak{g}(V) \mapsto \mathfrak{g}(V')$ such that $\Phi(\mathfrak{h}) = \mathfrak{h}'$ where $\mathfrak{h}$, $\mathfrak{h}'$ are Cartan subalgebras of $\mathfrak{g}(V)$, $\mathfrak{g}(V')$ respectively such that $\mathfrak{h} \in \mathfrak{X}(V)$ and $\mathfrak{h}' \subset \mathfrak{X}(V')$. By Corollaries 2.3 and 2.4, there is a local coordinate system $(y_1, \cdots, y_n)$ related biholomorphically to the original one such that every $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is a finite dimensional space of polynomial vector fields in $y_1, \cdots, y_n$. We choose such a local coordinate system $(z_1, \cdots, z_{n'})$ for $\mathfrak{g}(V')$. Let $\mathfrak{p}(V; y_1, \cdots, y_n)$ (resp. $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$) be the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) such that u can be expressed as a polynomial vector field in $y_1, \cdots, y_n$ (resp. $z_1, \cdots, z_{n'}$). $\mathfrak{p}(V; y_1, \cdots, y_n)$ and $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$ are Lie subalgebras of $\mathfrak{X}(V)$, $\mathfrak{X}(V')$ respectively. Since $\mathfrak{g}^{(\lambda)}(\mathfrak{h}) \subset \mathfrak{p}(V; y_1, \cdots, y_n)$ for every $\lambda \in \pi(\mathfrak{h})$, we get the following.

2.7. Corollary. *With the same notation and assumptions as above, the above isomorphism $\Phi\colon \mathfrak{g}(V) \mapsto \mathfrak{g}(V')$ induces an isomorphism of $\mathfrak{p}(V; y_1, \cdots, y_n)$ onto $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$.*

Proof. Note that $\Phi(\mathfrak{g}^{(\lambda)}(\mathfrak{h})) = \mathfrak{g}^{(\lambda)}(\mathfrak{h}')$, because $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is an eigenspace of $\mathrm{ad}(\mathfrak{h})$. Every $u \in \mathfrak{p}(V; y_1, \cdots, y_n)$ can be written in the form $u = \sum_{\lambda \in \pi(\mathfrak{h})} u_\lambda$, but the summation in this case is a finite sum. Since $\Phi(u) = \sum_{\lambda \in \pi(\mathfrak{h})} \Phi(u_\lambda)$ and $\Phi(u_\lambda) \in \mathfrak{g}^{(\lambda)}(\mathfrak{h}')$, we see that $\Phi(u) \in \mathfrak{p}(V'; z_1, \cdots, z_{n'})$. Replacing Φ by Φ^{-1} gives the desired result.

Let $\mathbf{C}[y_1, \cdots, y_n]$ be the ring of all polynomials in $y_1, \cdots, y_n$. Then, since $\mathfrak{g}(V)$ is an $\hat{\mathcal{O}}$-module, $\mathfrak{p}(V; y_1, \cdots, y_n)$ is a $\mathbf{C}[y_1, \cdots, y_n]$-module.

3. Theorem of Pursell-Shanks' type

In this section we consider two Lie algebras $\mathfrak{p}(V; y_1, \cdots, y_n)$ and $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$ of polynomial vector fields such that they are $\mathbf{C}[y_1, \cdots, y_n]$ and $\mathbf{C}[z_1, \cdots, z_{n'}]$-module respectively and that there is an isomorphism Φ of $\mathfrak{p}(V; y_1, \cdots, y_n)$ onto $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$. The goal is as follows.

Theorem II. *With the same notation and assumptions as above, there is a biholomorphic mapping φ of $\mathbf{C}^n$ onto $\mathbf{C}^{n'}$ such that $d\varphi\mathfrak{p}(V; y_1, \cdots, y_n) = \mathfrak{p}(V'; z_1, \cdots, z_{n'})$. Moreover, $\varphi(V) = V'$ as germs of varieties.*

Note at first that Theorem II implies Theorem I in the introduction, for Corollaries 2.6 and 2.7 show that an isomorphism between $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ induces an isomorphism between $\mathfrak{p}(V; y_1, \cdots, y_n)$ and $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$.

3.A. Characterization of maximal subalgebras

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$, and denote by $\mathfrak{h}^{(\infty)}$ the ideal consisting of all $u \in \mathfrak{h}$ such that $\mathrm{ad}(v_1) \cdots \mathrm{ad}(v_k)u \in \mathfrak{h}$ for every $k \geqslant 0$ and any $v_1, \cdots, v_k \in \mathfrak{p}(V; y_1, \cdots, y_n)$. Let $V_\mathfrak{p}$ be the set of all points $q \in \mathbf{C}^n$ such that $\mathfrak{p}(V; y_1, \cdots, y_n)$ does not span n-dimensional vector space at q, that is, $\dim \mathfrak{p}(V; y_1, \cdots, y_n)(q) < n$. For a point $p \in \mathbf{C}^n$, let $\mathfrak{p}_p$ be the isotropy subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$ at p, i.e., $\mathfrak{p}_p = \{u \in \mathfrak{p}(V; y_1, \cdots, y_n); u(p) = 0\}$.

3.1. Lemma. *For a point $p \in \mathbf{C}^n - V_\mathfrak{p}$, $\mathfrak{p}_p$ is a maximal finite codimensional subalgebra such that $\mathfrak{p}_p^{(\infty)} = \{0\}$.*

Proof. Since $p \in \mathbf{C}^n - V_\mathfrak{p}$, there are $u_1, \cdots, u_n \in \mathfrak{p}(V; y_1, \cdots, y_n)$ such that $u_j(p) = \partial/\partial y_j|_p$ for $1 \leqslant j \leqslant n$. Consider

$$\left(\mathrm{ad}(u_1)^{l_1} \cdots \mathrm{ad}(u_n)^{l_n} v\right)(p) = 0$$

for any $l_1, \cdots, l_n$, and we get easily that $\mathfrak{p}_p^{(\infty)} = \{0\}$.

We next prove the maximality of $\mathfrak{p}_p$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$ such that $\mathfrak{h} \supsetneq \mathfrak{p}_p$. There is then an element $v \in \mathfrak{h}$ such that

$v(p) \neq 0$. By a suitable linear change of variables, we may assume that v is written in the form

$$(21) \qquad v = g\frac{\partial}{\partial y_1} + \sum_{i=2}^{n} h_j \frac{\partial}{\partial y_j}, \quad g(p) \neq 0, h_j(p) = 0.$$

Let $(p_1, \cdots, p_n)$ be the coordinate of p. Then $(y_1 - p_1)u_j \in \mathfrak{p}_p$ for $1 \leqslant j \leqslant n$, and therefore $[v, (y_1 - p_1)u_j] = v(y_1)u_j + (y_1 - p_1)[v, u_j] \in \mathfrak{h}$. Since $v(y_1)(p) = g(p) \neq 0$, we have $\mathfrak{h}(p) = \mathfrak{p}(V; y_1, \cdots, y_n)(p)$ and hence $\mathfrak{h} = \mathfrak{p}(V; y_1, \cdots, y_n)$.

Let $\mathfrak{W}_\mathfrak{p}$ be the set of all points q such that $\mathfrak{p}_q$ is a maximal subalgebra and $\mathfrak{p}_q^{(\infty)} = \{0\}$. By the above lemma, $\mathfrak{W}_\mathfrak{p}$ contains $\mathbf{C}^n - V_\mathfrak{p}$. The goal of this section is as follows.

3.2. Proposition. *Let $\mathfrak{g}$ be a maximal, finite codimensional subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$. Then there is a unique point $p \in \mathfrak{W}_\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{p}_p$.*

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$, and let $J = \{f \in \mathbf{C}[y_1, \cdots, y_n]; f\mathfrak{p}(V; y_1, \cdots, y_n) \subset \mathfrak{g}\}$. Obviously, J is an ideal of $\mathbf{C}[y_1, \cdots, y_n]$, for $\mathfrak{p}(V; y_1, \cdots, y_n)$ is a $\mathbf{C}[y_1, \cdots, y_n]$-module.

3.3. Lemma. *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$ such that $\mathbf{C}[y_1, \cdots, y_n]\mathfrak{g} = \mathfrak{p}(V; y_1, \cdots, y_n)$. Then $J\mathfrak{p}(V; y_1, \cdots, y_n)$ is an ideal of $\mathfrak{p}(V; y_1, \cdots, y_n)$ contained in $\mathfrak{g}$.*

Proof. By definition $J\mathfrak{p}(V; y_1, \cdots, y_n) \subset \mathfrak{g}$. Since $(uf)v = [u, fv] - f[u, v]$, we have $\mathfrak{g}J \subset J$, and hence $(\mathbf{C}[y_1, \cdots, y_n]\mathfrak{g})J \subset J$. By the assumption, we get $\mathfrak{p}(V; y_1, \cdots, y_n)J \subset J$, so that $J\mathfrak{p}(V; y_1, \cdots, y_n)$ is an ideal of $\mathfrak{p}(V; y_1, \cdots, y_n)$.

By the above lemma, we see also that $J\mathfrak{p}(V; y_1, \cdots, y_n) \subset \mathfrak{g}^{(\infty)}$. The next lemma is due to Amamiya [1]. Although the proof can be seen in [5], we repeat it here for the sake of selfcontainedness.

3.4. Lemma. *Let $\mathfrak{g}$ be a finite codimensional subalgebra of $\mathfrak{p}(V; y_1, \cdots, y_n)$. Then $J \neq \{0\}$.*

Proof. Set $\mathfrak{g}^{(1)} = \{u \in \mathfrak{g}; [u, \mathfrak{p}(V; y_1, \cdots, y_n)] \subset \mathfrak{g}\}$. Since codim $\mathfrak{g} < \infty$ and ad(u) for every $u \in \mathfrak{g}$ induces a linear mapping of $\mathfrak{p}(V; y_1, \cdots, y_n)/\mathfrak{g}$ into itself, we see that codim $\mathfrak{g}^{(1)} < \infty$ and hence in particular $\mathfrak{g}^{(1)} \neq \{0\}$.

Let v be a nontrivial element in $\mathfrak{g}^{(1)}$, and f a polynomial such that $vf \neq 0$. Consider a sequence $fv, f^2v, f^3v, \cdots$. Since codim $\mathfrak{g}^{(1)} < \infty$, there is a polynomial $P(t)$ in t such that $P(f)v \in \mathfrak{g}^{(1)}$.

We next prove that if v and gv are contained in $\mathfrak{g}^{(1)}$, then $(vg)^2 \in J$. For this purpose, let w be an arbitrary element of $\mathfrak{p}(V'; y_1, \cdots, y_{n'})$. Then we have

$$[v, gw] = (vg)w + g[w, v] \in \mathfrak{g},$$
$$[gv, w] = -(wg)v + g[w, v] \in \mathfrak{g}.$$

Hence

$$(vg)w + (wg)v \in \mathfrak{g} \tag{22}$$

for every $w \in \mathfrak{p}(V; y_1, \cdots, y_n)$. In (22) replacing w by $(wg)v$ and $(vg)w$ in turn, we have $(vg)(wg)v \in \mathfrak{g}$ and

$$(vg)^2 w + (vg)(wg)v \in \mathfrak{g}.$$

Hence $(vg)^2 w \in \mathfrak{g}$, which implies that $(vg)^2 \in J$. Set $g = P(f)$. Then $v, gv \in \mathfrak{g}^{(1)}$ and $vg \neq 0$ because of $vf \neq 0$. Thus we get $J \neq \{0\}$.

3.5. Corollary. *Let* $\mathfrak{g}$ *be a maximal finite codimensional subalgebra of* $\mathfrak{p}(V; y_1, \cdots, y_n)$ *such that* $\mathfrak{g}^{(\infty)} = \{0\}$. *Then* $\mathfrak{g}$ *is a* $\mathbf{C}[y_1, \cdots, y_n]$-*module.*

Proof. We have only to show that $\mathbf{C}[y_1, \cdots, y_n]\mathfrak{g} \subsetneqq \mathfrak{p}(V; y_1, \cdots, y_n)$, because if so, the maximality of $\mathfrak{g}$ shows that $\mathbf{C}[y_1, \cdots, y_n]\mathfrak{g} = \mathfrak{g}$. Thus we can assume that $\mathbf{C}[y_1, \cdots, y_n]\mathfrak{g} = \mathfrak{p}(V; y_1, \cdots, y_n)$. Then by the above lemma, we get that $\mathfrak{g}^{(\infty)} \supset J\mathfrak{p}(V; y_1, \cdots, y_n) \neq 0$, contradicting the assumption.

Now we have only to consider a maximal finite codimensional subalgebra $\mathfrak{g}$ of $\mathfrak{p}(V; y_1, \cdots, y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$ and $\mathfrak{g}$ is a $\mathbf{C}[y_1, \cdots, y_n]$-module. Let $M_p = \{f \in \mathbf{C}[y_1, \cdots, y_n]; f(p) = 0\}$.

3.6. Lemma. *For a* $\mathbf{C}[y_1, \cdots, y_n]$-*submodule* $\mathfrak{g}$ *of* $\mathfrak{p}(V; y_1, \cdots, y_n)$, *if*

$$\mathfrak{g} + M_p\mathfrak{p}(V; y_1, \cdots, y_n) = \mathfrak{p}(V; y_1, \cdots, y_n)$$

for every $p \in \mathbf{C}^n$, *then* $\mathfrak{g} = \mathfrak{p}(V; y_1, \cdots, y_n)$.

Proof. By Nakayama's lemma, we see that for each $p \in \mathbf{C}^n$ there is $f_p \in \mathbf{C}[y_1, \cdots, y_n]$ such that $f_p(p) \neq 0$ and $f_p\mathfrak{p}(V; y_1, \cdots, y_n) \subset \mathfrak{g}$. Since the ideal $\mathcal{I}$ generated by $\{f_p; p \in \mathbf{C}^n\}$ has no common zero, we see that $\mathcal{I} = \mathbf{C}[y_1, \cdots, y_n]$ and hence there are $f_{p_1}, f_{p_2}, \cdots, f_p, g_1, g_2, \cdots, g_l \in \mathbf{C}[y_1, \cdots, y_n]$ such that $1 = \sum_{j=1}^{l} g_j f_{p_j}$. Therefore

$$\mathfrak{p}(V; y_1, \cdots, y_n) = \left(\sum_{j=1}^{l} g_j\right)\mathfrak{g} \subset \mathfrak{g}.$$

3.7. Corollary. *Let* $\mathfrak{g}$ *be a maximal finite codimensional subalgebra of* $\mathfrak{p}(V; y_1, \cdots, y_n)$ *such that* $\mathfrak{g}^{(\infty)} = \{0\}$. *Then there exists uniquely a point* $p \in \mathfrak{W}_{\mathfrak{p}}$ *such that* $\mathfrak{g} = \mathfrak{p}_p$.

Proof. By Corollary 3.5, $\mathfrak{g}$ is a $\mathbf{C}[y_1, \cdots, y_n]$-module, and hence there is a point $p \in \mathbf{C}^n$ such that $\mathfrak{g} + M_p\mathfrak{p}(V; y_1, \cdots, y_n) \subsetneqq \mathfrak{p}(V; y_1, \cdots, y_n)$. Thus $\mathfrak{g} \supset M_p\mathfrak{p}(V; y_1, \cdots, y_n)$ by the maximality of $\mathfrak{g}$. It is easy to see that such a point is unique, because $M_p + M_q = \mathbf{C}[y_1, \cdots, y_n]$ if $p \neq q$.

If $\mathfrak{p}(V; y_1, \cdots, y_n)(p) = \{0\}$, then $M_p\mathfrak{p}(V; y_1, \cdots, y_n)$ is an ideal of $\mathfrak{p}(V; y_1, \cdots, y_n)$. Thus it must be contained in $\mathfrak{g}^{(\infty)}$, and hence must be $\{0\}$ by the assumption, contradicting the assumption. Therefore

$\mathfrak{p}(V; y_1, \cdots, y_n)(p) \neq \{0\}$. Now there are $u \in \mathfrak{p}(V; y_1, \cdots, y_n)$ and $f \in \mathbf{C}[y_1, \cdots, y_n]$ such that $u(p) \neq 0$, $f(p) = 0$ and $(uf)(p) \neq 0$. For every $v \in \mathfrak{p}(V; y_1, \cdots, y_n)$, fv is an element of $\mathfrak{g}$. Therefore if u were contained in $\mathfrak{g}$, then $[u, fv] \in \mathfrak{g}$. Thus $(uf)v \in \mathfrak{g}$. It follows that $(uf)(p)v \in (uf - (uf)(p))v + \mathfrak{g} \subset \mathfrak{g}$. Since $(uf)(p) \neq 0$, we get $v \in \mathfrak{g}$, hence $\mathfrak{g} = \mathfrak{p}(V; y_1, \cdots, y_n)$, contradicting the assumption.

From the above argument it follows that $\mathfrak{g} \subset \mathfrak{p}_p$, so that $\mathfrak{g} = \mathfrak{p}_p$ by the maximality of $\mathfrak{g}$. Since $\mathfrak{g}^{(\infty)} = \{0\}$, we see $p \in \mathfrak{W}_\mathfrak{p}$ by definition.

This completes the proof of Proposition 3.2. (See also added in proof.)

3.B. A diffeomorphism induced from Φ

Let $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$ be another Lie algebra of polynomial vector fields on $\mathbf{C}^{n'}$. Subsets $V_{\mathfrak{p}'}$, $\mathfrak{W}_{\mathfrak{p}'}$ are defined in the same way as in $\mathfrak{p}(V; y_1, \cdots, y_n)$. Suppose there is an isomorphism Φ of $\mathfrak{p}(V; y_1, \cdots, y_n)$ onto $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$. For a point $p \in \mathfrak{W}_{\mathfrak{p}'}$, $\mathfrak{p}_p$ is a maximal finite codimensional subalgebra such that $\mathfrak{p}_p^{(\infty)} = 0$. Then $\Phi(\mathfrak{p}_p)$ has the same property. Hence there is a point $\varphi(p) \in \mathfrak{W}_{\mathfrak{p}'}$ such that $\Phi(\mathfrak{p}_p) = \mathfrak{p}'_{\varphi(p)}$, where $\mathfrak{p}'_{\varphi(p)}$ is defined in the same manner as in $\mathfrak{p}(V; y_1, \cdots, y_n)$. φ: $\mathfrak{W}_\mathfrak{p} \mapsto \mathfrak{W}_{\mathfrak{p}'}$ is a bijective mapping. The goal of this section is as follows.

3.8. Proposition. *With the same notation and assumptions as above, assume further that $\mathfrak{p}(V; y_1, \cdots, y_n)$ (resp. $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$) contains a vector field X (resp. X') such that $X = \sum_{j=1}^{n} \hat{\mu}_j y_j \partial/\partial y_j$ (resp. $X' = \sum_{j=1}^{n'} \hat{\mu}'_j z_j \partial/\partial z_j$). Then φ can be extended to a holomorphic diffeomorphism of $\mathbf{C}^n$ onto $\mathbf{C}^{n'}$ such that $\varphi(V_\mathfrak{p}) = V_{\mathfrak{p}'}$.*

Note that the existence of X and X' are obtained by Lemma 2.1.

Let $\Psi_\mathfrak{p}$ be the totality of $\mathbf{C}$-valued functions f on $\mathfrak{W}_\mathfrak{p}$ such that fu can be extended to an element of $\mathfrak{p}(V; y_1, \cdots, y_n)$ for every $u \in \mathfrak{p}(V; y_1, \cdots, y_n)$. Remark that the existence of fu is unique, because $\mathfrak{W}_\mathfrak{p}$ is dense in $\mathbf{C}^n$. $\Psi_\mathfrak{p}$ is a ring, and $\mathfrak{p}(V; y_1, \cdots, y_n)$ is an $\Psi_\mathfrak{p}$-module. For $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$, we define $\Psi_{\mathfrak{p}'}$ in the same manner as above.

3.9. Lemma. *With the same notation and assumptions as above, φ induces an isomorphism of $\Psi_{\mathfrak{p}'}$ onto $\Psi_\mathfrak{p}$.*

Proof. Let $f \in \Psi_{\mathfrak{p}'}$, and p be an arbitrary point in $\mathfrak{W}_\mathfrak{p}$. By definition, $f\Phi(u)$ can be extended to an element of $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$, which will be denoted by the same notation. $f\Phi(u) - f(\varphi(p))\Phi(u) \in \mathfrak{p}'_{\varphi(p)}$ hence $\Phi^{-1}(f\Phi(u) - f(\varphi(p))\Phi(u)) \in \mathfrak{p}_p$, that is, $\Phi^{-1}(f\Phi(u) - f(\varphi(p))\Phi(u))(p) = 0$. Therefore $\Phi^{-1}(f\Phi(u))(p) = f(\varphi(p))u$, that is, $\Phi^{-1}(f\Phi(u)) = (\varphi^*f)u$. Since the left-hand member is contained in $\mathfrak{p}(V; y_1, \cdots, y_n)$, we have $\varphi^*f \in \Psi_\mathfrak{p}$. It is easy to see that φ^*: $\Psi_{\mathfrak{p}'} \mapsto \Psi_\mathfrak{p}$ is an isomorphism.

3.10. Lemma. *Under the same assumption as in the statement of Proposition* 3.8, *we have* $\Psi_{\mathfrak{p}} = \mathbf{C}[y_1, \cdots, y_n]$, *and hence* φ *is a bi-holomorphic diffeomorphism of* $\mathbf{C}^n$ *onto* $\mathbf{C}^{n'}$.

Proof. Obviously $\Psi_{\mathfrak{p}} \supset \mathbf{C}[y_1, \cdots, y_n]$. For any $f \in \Psi_{\mathfrak{p}}$, fX is an element of $\mathfrak{p}(V; y_1, \cdots, y_n)$. Thus $fy_1, \cdots, fy_n \in \mathbf{C}[y_1, \cdots, y_n]$, and it is not hard to see $f \in \mathbf{C}[y_1, \cdots, y_n]$.

3.11. Lemma. $\varphi(C^n - V_{\mathfrak{p}}) = C^{n'} - V_{\mathfrak{p}'}$.

Proof. By the above lemma we have $n = n'$. Let p be a point of $C^n - V_{\mathfrak{p}}$. Then $\operatorname{codim} \mathfrak{p}_p = n$, and therefore $\operatorname{codim} \mathfrak{p}'_{\varphi(p)} = n$, because $\mathfrak{p}'_{\varphi(p)} = \Phi(\mathfrak{p}_p)$. Hence we see $\varphi(\mathbf{C}^n - V_{\mathfrak{p}}) = \mathbf{C}^{n'} - V_{\mathfrak{p}'}$.

This completes the proof of Proposition 3.8.

3.C. Recapture of the germ

Recall that V is a germ of variety with 0 as an expansive singularity. Hence there is $X = \sum_{i=1}^n \hat{\mu}_i y_i \partial/\partial y_i \in \mathfrak{X}(V)$ such that $\operatorname{Re} \hat{\mu}_i > 0$ for $1 \leqslant i \leqslant n$. Since X is a linear vector field, $\exp tX$ is a bi-holomorphic diffeomorphism of $\mathbf{C}^n$ onto itself. Remark that $(\exp tX)V = V$ as germs of varieties for $X\mathfrak{I}(V) \subset \mathfrak{I}(V)$ where $\mathfrak{I}(V)$ is the ideal of V in $\mathcal{O}$. Let $\tilde{V} = \bigcup_{t \in \mathbf{R}} (\exp tX)V$. Though V is a germ of variety at 0, the expansive property of X yields that $\tilde{V}$ is a closed subset of $\mathbf{C}^n$ such that $(\exp tX)\tilde{V} = \tilde{V}$. Obviously, $\tilde{V} = V$ as germs of varieties.

In this section we shall prove that $V_{\mathfrak{p}} = \tilde{V}$, so that $V_{\mathfrak{p}} = V$ as germs of varieties. Let $\hat{\mathfrak{I}}(V)$ be the closure of $\mathfrak{I}(V)$ in $\hat{\mathcal{O}}$. Note that $\mathfrak{g}(V)$ is also the closure of $\mathfrak{X}(V)$ in $\mathfrak{F}_0$. Hence $\mathfrak{g}(V)\hat{\mathfrak{I}}(V) \subset \hat{\mathfrak{I}}(V)$. Recall that $\mathfrak{p}(V; y_1, \cdots, y_n)$ is given by using the eigenspace decomposition of $\mathfrak{g}(V)$ with respect to $\operatorname{ad}(X)$, that is, every $u \in \mathfrak{g}(V)$ can be rearranged in the form $u = \sum u_\lambda$ as in Lemma 1.6, and $\mathfrak{p}(V; y_1, \cdots, y_n)$ is generated by the u_λ's. Similarly, we decompose $\hat{\mathfrak{I}}(V)$ into eigenspaces of X. Let f be an element of $\hat{\mathfrak{I}}(V)$. Then f can be rearranged in the form

$$f = \sum_{\nu} f_\nu, \quad f_\nu = \sum_{\langle \alpha, \hat{\mu} \rangle = \nu} a_\alpha y^\alpha. \tag{23}$$

Thus f_ν is a polynomial such that $Xf_\nu = \nu f_\nu$. By the same proof as in Lemma 1.6 we see that $f_\nu \in \hat{\mathfrak{I}}(V)$. We denote by $I_{\mathfrak{p}}$ the ideal of $\mathbf{C}[y_1, \cdots, y_n]$ generated by all f_νs with $f \in \hat{\mathfrak{I}}(V)$.

3.11. Lemma. $I_{\mathfrak{p}} \subset \mathfrak{I}(V)$.

Proof. Let $f \in \mathfrak{I}(V)$. f can be rearranged in the form $f = \sum_{i=1}^\infty f_{\nu_i}$, $f_{\nu_i} = \sum_{\langle \alpha, \hat{\mu} \rangle = \nu_i} a_\alpha y^\alpha$. We may assume $0 < \nu_1 < \cdots \nu_k < \cdots$. First of all, we shall show $f_{\nu_i} \in \mathfrak{I}(V)$. Note that $e^{\nu_1 t}(\exp - tX)f = \sum e^{-(\nu_j - \nu_1)t} f_{\nu_j} \in \mathfrak{I}(V)$ for $t > 0$. Suppose f is defined on a neighborhood N of 0 in $\mathbf{C}^n$. Then $(\exp - tX)f$ is defined on $(\exp tX)N$. Note that $\bigcup_{t > 0} (\exp tX)N = \mathbf{C}^n$ and

$\bigcup_{t>0}(\exp tX)(N \cap V) = \tilde{V}$. Since $e^{\nu_1 t}(\exp - tX)f = 0$ on $(\exp tX)(N \cap V)$, taking $\lim_{t\mapsto\infty}$ we see that $f_{\nu_1} = 0$ on $\tilde{V}$. Since $\tilde{V} = V$ as germs of varieties, we have $f_{\nu_1} \in \mathcal{I}(V)$. Applying the same procedure to $f - f_{\nu_1}$, we get $f_{\nu_2} \in \mathcal{I}(V)$, and so on. Hence $f_{\nu_j} \in \mathcal{I}(V)$.

Let $f \in \hat{\mathcal{I}}(V)$. Then there is a sequence $\{f^{(m)}\}$ in $\mathcal{I}(V)$ such that $\lim f^{(m)} = f$ in the topology of formal power series. For any eigenvalue ν of X: $\hat{\mathcal{O}} \mapsto \hat{\mathcal{O}}$, we see $f_\nu^{(m)} \in \mathcal{I}(V)$, and $\lim_{m\to\infty} f_\nu^{(m)} = f_\nu$ as polynomials, because the degrees of $f_\nu^{(m)}$, f_ν are bounded from above by a number related only to $\hat{\mu}_1, \cdots, \hat{\mu}_n$ and ν. Since $f_\nu^{(m)}|V \equiv 0$, we have $f_\nu|V \equiv 0$, hence $f_\nu \in \mathcal{I}(V)$. Recalling that the f_ν's generate $I_\mathfrak{p}$, we thus see $I_\mathfrak{p} \subset \mathcal{I}(V)$.

3.12. Lemma. *With the same notations and assumptions as above, a polynomial vector field u with $u(0) = 0$ is contained in $\mathfrak{p}(V; y_1, \cdots, y_n)$ if and only if $uI_\mathfrak{p} \subset I_\mathfrak{p}$.*

Proof. For $u \in \mathfrak{g}(V)$ and $f \in \hat{\mathcal{I}}(V)$, let $u = \Sigma_\lambda u_\lambda$ and $f = \Sigma_\nu f_\nu$ be the decompositions of eigenvectors with respect to $\mathrm{ad}(X)$, X respectively. Then $u_\lambda \in \mathfrak{p}(V; y_1, \cdots, y_n)$, $f_\nu \in I_\mathfrak{p}$. Since $Xu_\lambda f_\nu = [X, u_\lambda]f_\nu + u_\lambda X f_\nu = (\lambda + \nu)u_\lambda f_\nu$, $u_\lambda f_\nu$ is also an eigenvector of X. Since $uf \in \hat{\mathcal{I}}(V)$, the $u_\lambda f_\nu$'s appear in the eigenspace decomposition of uf, and hence $u_\lambda f_\nu \in I_\mathfrak{p}$. Thus we have $\mathfrak{p}(V; y_1, \cdots, y_n)I_\mathfrak{p} \subset I_\mathfrak{p}$.

Conversely, if $uI_\mathfrak{p} \subset I_\mathfrak{p}$ for a polynomial vector field u with $u(0) = 0$, then $u\hat{\mathcal{I}}(V) \subset \hat{\mathcal{I}}(V)$ by taking the closure in the formal power series. Note that $u\mathcal{I}(V) \subset \mathcal{O} \cap \hat{\mathcal{I}}(V)$. We next prove that $\mathcal{I}(V) = \mathcal{O} \cap \hat{\mathcal{I}}(V)$. For that purpose, we have only to show $\mathcal{I}(V) \supset \mathcal{O} \cap \hat{\mathcal{I}}(V)$, because the converse is trivial. Let $f \in \mathcal{O} \cap \hat{\mathcal{I}}(V)$, and let $f = \Sigma_\nu f_\nu$ be the eigenvector decomposition of f with respect to X. Then by Lemma 3.11 we have $(f_\nu \in I_\mathfrak{p} \subset \mathcal{I}(V))$. Thus $f_\nu = 0$ on V, hence $f = 0$ on V. This means $f \in \mathcal{I}(V)$. Thus $uI_\mathfrak{p} \subset I_\mathfrak{p}$ yields $u \in \mathfrak{X}(V) \subset \mathfrak{g}(V)$. However u is a polynomial vector field in $y_1, \cdots, y_n$, hence $u \in \mathfrak{p}(V; y_1, \cdots, y_n)$.

3.13. Lemma. $V_\mathfrak{p} = V_{I_\mathfrak{p}}$, *the locus of zeros of* $I_\mathfrak{p}$.

Proof. Let p be a point in $\mathbf{C}^n - V_\mathfrak{p}$. By definition there are $u_1, \cdots, u_n \in \mathfrak{p}(V; y_1, \cdots, y_n)$ such that $u_1(p), \cdots, u_n(p)$ are linearly independent. Assume for a while that $p \in V_{I_\mathfrak{p}}$. Since $u_i I_\mathfrak{p} \subset I_\mathfrak{p}$, we have

$$(u_1^{l_1} \cdots u_n^{l_n} f)(p) = 0$$

for every $f \in I_\mathfrak{p}$ and any $l_1, \cdots, l_n$. Thus $f = 0$, contradicting the fact $I_\mathfrak{p} \neq \{0\}$. Therefore $V_\mathfrak{p} \supset V_{I_\mathfrak{p}}$.

Conversely, let $p \in \mathbf{C}^n - V_{I_\mathfrak{p}}$. There is then $g \in I_\mathfrak{p}$ such that $g(p) \neq 0$. By Lemma 3.12, $g\partial/\partial y_1, \cdots, g\partial/\partial y_n \in \mathfrak{p}(V; y_1, \cdots, y_n)$, which are linearly independent at p. Hence $p \in \mathbf{C}^n - V_\mathfrak{p}$. Thus $V_{I_\mathfrak{p}} \supset V_\mathfrak{p}$.

3.14. Lemma. $V_{I_{\mathfrak{p}}} = V$ *as germs of varieties.*

Proof. By Lemma 3.11 we have $\mathcal{O}I_{\mathfrak{p}} \subset \mathcal{I}(V)$, which implies that $V_{I_{\mathfrak{p}}} \supset V$. Assume for a while that $V_{I_{\mathfrak{p}}} \supsetneqq V$. Then there is $f \in \mathcal{I}(V)$ such that $f \not\equiv 0$ on V. Let $f = \sum_\nu f_\nu$ be the eigenvector decomposition of f. Then $f_\nu \in I_{\mathfrak{p}}$, so that $f_\nu = 0$ on V. Hence $f = 0$ on V contradicting the assumption. Thus we get $V_{I_{\mathfrak{p}}} = V$ as germs of varieties, and hence $V_{I_{\mathfrak{p}}} = \tilde{V}$.

From the above result it follows that $\varphi\colon \mathbf{C}^n \to \mathbf{C}^{n'}$ maps $\tilde{V}$ onto $\tilde{V}'$ and $\varphi(V) = V'$ as germs. This implies that $\varphi^*\mathcal{I}(V') = \mathcal{I}(V)$ so that $d\varphi\mathfrak{X}(V) = \mathfrak{X}(V')$. Hence the proof of Theorem I in the introduction is complete.

Added in proof. The proof of Corollary 3.7 and also Lemma 2.15 in [5] contains a slight gap, for it is not trivial that $\mathfrak{g} + M_p\mathfrak{p}$ is a subalgebra, where $\mathfrak{p} = \mathfrak{p}(V; y_1, \cdots, y_n)$. This is proved as follows.

Since $\mathfrak{p}/\mathfrak{g}$ is a finite dimensional $\mathbf{C}[y_1, \cdots, y_n]$-module, there is l such that $M_p^{l+1}(\mathfrak{p}/\mathfrak{g}) = M_p^l(\mathfrak{p}/\mathfrak{g})$. By Nakayama's lemma, there is a polynomial Q such that $Q(p) = 1$ and $QM_p^l(\mathfrak{p}/\mathfrak{g}) = \{0\}$. As $QM_p^l = M_p^l$, we have $M_p^l\mathfrak{p} \subset \mathfrak{g}$, hence $J \supset M_p^l$. Therefore codim $J < \infty$. Now assume for a while that $\mathfrak{g}(p) \neq \{0\}$. By a suitable linear change of coordinate $\mathfrak{g}$ contains $u = g_1\partial/\partial x_1 + \sum_{j=2}^n h_j\partial/\partial x_j$ such that $g_1(p) = 1$, $h_j \in M_p$. As codim $J < \infty$, there is a polynomial $P(x_1)$ of one variable such that $P \in J$. We get easily $(u^kP)(p) \neq 0$ for some k. Since $\mathfrak{g}J \subset J$, it follows $J(p) \neq \{0\}$. This implies $J \ni 1$ for $J \supset M_p^l$. Hence $\mathfrak{g} = \mathfrak{p}$, contradicting the assumption.

Therefore $\mathfrak{g} \subset \mathfrak{p}_p$. If $\mathfrak{p} = \mathfrak{p}_p$, then $\mathfrak{g} + M_p\mathfrak{p}$ is a subalgebra of $\mathfrak{p}$. Thus $\mathfrak{g} \supset M_p\mathfrak{p}$ because of the maximality. But since $M_p\mathfrak{p}$ is an ideal of $\mathfrak{p}$, we get $M_p\mathfrak{p} \subset \mathfrak{g}^{(\infty)} = \{0\}$. Therefore we have $\mathfrak{p} \neq \mathfrak{p}_p$. Since $\mathfrak{g} \subset \mathfrak{p}_p$, we get $\mathfrak{p}_p = \mathfrak{g} \supset M_p\mathfrak{p}$.

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