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A PERMANENCE THEOREM FOR EXOTIC CLASSES

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The purpose of this paper is to prove a permanence formula for the characteristic classes of an important class of foliations (see §1). This formula relates certain characteristic classes of these foliations to characteristic classes for an associated flat vector bundle.

More precisely, let F be a codimension-n foliation of the complement of the zero section of a flat $(n + 1)$-dimensional vector bundle V which arises from the linear action of a group and an appropriate vector field commuting with the action of this group. Let $h_I c_J$ be a class in $H^*(WO_n)$ (I and J both multi-indices). This formula relates the characteristic class $h_I c_J(F)$ to the characteristic class $h_{i_1} c_J(F)$ (i_1 is the smallest index in I) and the characteristic classes $h_i(V)$ for the flat vector bundle V. Applications follow from the fact that the $h_i(V)$ and the integral over the fiber of $h_{i_1} c_J(F)$ lie in the image of the relative Lie algebra cohomology.

We give two applications. The first is a (weak) independence result for certain classes $h_I c_J$ in $H^*(B\Gamma_n)$ (for n odd). The second is a variation result which shows that most non-rigid classes (in odd codimension) do vary. Slight modification should yield the even codimension case.

The first permanence theorem of this type was proved by Kamber and Tondeur (see [8, 7.59, 7.83] and [9, §7]) and Shulman and Tischler [5, 5.1]. These authors considered the case of locally homogeneous foliations. If we take our vector field to be the radial field, we get a locally homogeneous foliation and in particular the theorem of [5].

Our first application is already proved in [8, 7.93 and 7.95] and a special case in [5, (5.1)]. This method, pushed a bit in an obvious way, shows some further variation and independence results for the higher classes $h_I c_J$.

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1. Introduction

(1.1) Heitsch foliations. See [3] and also [4]. Let X be a vector field in R^{n+1} with the following properties:

1. The one form $w(-) = \langle -, X \rangle$ defines a codimension-one foliation on

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R^{n+1}-(singularities of X) with a compact leaf M ($\langle\ ,\ \rangle$ is the standard inner product).

2. Each integral curve of X intersects M transversally and exactly once and intersects S^n transversally and exactly once.

Now suppose $G \subset SL(n+1)$ commutes with X. Let K be the maximal compact subgroup of G, and Γ a cocompact subgroup. We assume K preserves M. We have the following objects.

(1.2) The flat vector bundle $V = G/K \times_\Gamma R^{n+1}$ with codimension-$(n+1)$ foliation F^{n+1} coming from the flat structure.

(1.3) A codimension-n foliation F^n of $\Gamma \setminus G \times_K S^n$. We obtain this by starting with the codimension-n foliation of $G/K \times$ (R^{n+1}-singularities of X) whose leaves are $G/K \times$ (integral curve of X). G and Γ preserve these leaves giving a codimension-n foliation F^n of $G/K \times_\Gamma$ (R^{n+1}-singularities) $\cong \Gamma \setminus G \times_K$ (R^{n+1}-singularities). (The indicated diffeomorphism is given by $(g, v) \to (g, g^{-1}v)$.) The intersection of this foliation with $\Gamma \setminus G \times_K S^n$ is a codimension-n foliation which we call F^n. The same considerations applied to M instead of S^n yield a codimension-n foliation of $\Gamma \setminus G \times_K M$. We will also call this foliation F^n.

(1.4) Remark. We can describe the foliation F^n as arising directly from a flat Diff(S^n) structure on $\Gamma \setminus G \times_K S^n$. Namely for $g \in G$ and $m \in S^n$ let γ be the integral curve of X through m, and let gm be the intersection of $g\gamma$ with S^n. This action gives us a flat Diff(S^n) bundle $G/K \times_\Gamma S^n \cong \Gamma/G \times_K S^n$, and the foliation coming from the flat structure is F^n. The same considerations apply to $\Gamma \setminus G \times_K M$.

(1.5) Example. We will be concerned with the case where $n+1$ is even, $G = SL(k_1) \times \cdots \times SL(k_p)$, $K = SO(k_1) \times \cdots \times SL(k_p)$ and

$$X = \sum_{j=1}^{p} \lambda_j \left(x_{l_j+1} \frac{\partial}{\partial x_{l_j+1}} + \cdots + x_{l_j+k_j} \frac{\partial}{\partial x_{l_j+k_j}} \right),$$

where $l_j = k_1 + \cdots + k_{j-1}$, and $\lambda_1, \cdots, \lambda_p$ are *positive* numbers.

(1.6) Connections. Let $\tilde{D}$ be the globally flat connection on R^{n+1} relative to $\{\partial/\partial x_1, \cdots, \partial/\partial x_{n+1}\}$. $K \subset O(n+1)$, and so $\tilde{D}$ is K invariant. We can construct a K-invariant connection D on R^{n+1} which, away from the singularities of X, is given by

$$D_Y Z = \frac{\langle Y, X \rangle}{\langle X, X \rangle} [X, Z] + \tilde{D}_{\pi Y} Z,$$

where π is the orthogonal projection perpendicular to X. Let $\tilde{H}$ and H be the horizontal distributions on Frames(R^{n+1}) corresponding to $\tilde{D}$ and D. Both $\tilde{H}$ and H are K-invariant. On $G/K \times_\Gamma$ Frames(R^{n+1}) $\cong$

$\Gamma \setminus G \times_K \text{Frames}(R^{n+1})$ we have the foliation $\tilde{F}$ coming from the flat structure. Then

$$T(\tilde{F}) \oplus (\Gamma \setminus G \times_K \tilde{H}) \text{ and } T(\tilde{F}) \oplus (\Gamma \setminus G \times_K H)$$

are connections on $\Gamma \setminus G \times_K \text{Frames}(R^{n+1})$, and give covariant derivatives $\tilde{\nabla}$ and ∇ on $\Gamma \setminus G \times_K T(R^{n+1})$ which can be identified with the normal bundle to the foliation F^{n+1}, and both $\tilde{\nabla}$ and ∇ are Bott connections. Further $\tilde{\nabla}$ is a flat connection and ∇ is an X basic connection ($\nabla_X s = [X, s]$ appropriately interpreted, see [3] or [4, §2]). The standard innerproduct on R^{n+1} gives rise to an innerproduct on $\Gamma \setminus G \times_K T(R^{n+1})$. Let π_1 and π_2 be the projections perpendicular to and in the direction of X respectively (away from the singularities of X). Let $\nabla^1 = \pi_1 \nabla$ and $\nabla^2 = \pi_2 \nabla$. Then ∇^1 is easily seen to be a Bott connection for F^n.

(1.7) characteristic classes. For a treatment of characteristic classes for flat bundles see [7, §4] and [8, §3].

Briefly, $H^*(sl(n+1), SO(n+1))$ yields characteristic classes for the flat vector bundle $V = G/K \times_\Gamma R^{n+1}$ Namely, for $n+1$ even, $H^*(sl(n+1), SO(n+1)) = \Lambda(h_3, h_5, \cdots, h_n, \chi)$, $\dim h_i = 2i - 1$, $\dim \chi = n + 1$. Then let $h_i(V) = \Delta_{c_i}(\tilde{\nabla}, D^R)$ where D^R is any Riemannian connection. Now the zero section $\Gamma \setminus G/K \to G/K \times_\Gamma R^{n+1}$ pulls the $h_i(V)$ to $\Gamma \setminus G/K$. $\chi(V)$ will be the Euler class of V. For F^n we have the classes $h_I c_J(F^n)$ coming from $H^*(WO(n))$. Finally, for an X basic connection ∇, $\Delta_{c_i}(\nabla, D^R) c_J(K_\nabla)$ is a closed form for $i + |J| = n + 1$ on $\Gamma \setminus G \times_K R^{n+1}$ (see [3] and also [4]).

(1.8) Gauss-Codazzi equation. Let ∇^1 and ∇^0 be two connections on any vector bundle, and let φ be any invariant polynomial. Let ρ^1 and ρ^0 be local connection matrices for ∇^1 and ∇^0, and let $\alpha = \rho^1 - \rho^0$, $K_0 = d\rho^0 + \rho^0 \wedge \rho^0$, and $\Theta = d\alpha + \alpha \wedge \rho^0 + \rho^0 \wedge \alpha$. These quantities are tensorial, i.e., they transform by $\text{Ad}(a^{-1})$. A simple calculation yields

$$\Delta_\varphi(\nabla^1, \nabla^0) = Deg\ \varphi \sum_{\substack{i+j+k+1 \\ = \text{Deg}\ \varphi}} \int_0^1 t^l \varphi(\alpha \wedge \Theta^i \wedge \alpha^{2j} \wedge K_0^k)\, dt,$$

where l is an integer function of i, j, k. α, K_0, Θ are all tensorial.

2. Permanence theorem and applications

We use the notation of the first section. In addition, let D^2 be the connection on $\Gamma \setminus G \times_K (X)$ which is globally flat relative to the framing X, and D^1 any Riemannian connection on $\Gamma \setminus G \times_K (X)^\perp$. Then $D^1 + D^2$ is a Riemannian connection on $\Gamma \setminus G \times_K T(R^{n+1})$.

(2.1) Technical theorem. *If $i + |J| \geqslant n + 1$, then*

$$\Delta_{c_i}(\nabla, \nabla^1 + \nabla^2) c_J(K_{\nabla^1}) = 0,$$

$$\Delta_{c_i c_J}(\nabla, D^1 + D^2) = \Delta_{c_i c_J}(\nabla^1, D^1) + \textit{exact},$$

$$\Delta_{c_i}(\nabla^2, D^2) c_J(K_{\nabla^1}) \equiv 0.$$

If $i + |J| > n + 1$, then $\Delta_{c_i}(\nabla, \tilde{\nabla}) c_J(K_{\nabla^1}) \equiv 0$.

Proof. Γ is a discrete group of diffeomorphisms of $G/K \times R^{n+1}$. Thus for a fixed point in $G/K \times R^{n+1}$ we can find an open set $W = W_1 \times W_2$ such that $W \cap \gamma W = \varphi$ for $\gamma \in \Gamma$, $\gamma \neq 1$. Choose $Y_1, \cdots, Y_N$ to be a framing of $T(W_1)$. In W_2 choose local coorindates $x_1, \cdots, x_{n+1}$ such that $X = \partial/\partial x_1$, and let $X_i = \partial/\partial x_i$ for $i = 1, \cdots, n + 1$. Let Y_j, X_i also denote the images of these vector fields in $G/K \times_\Gamma R^{n+1}$. Then $Y_1, \cdots, Y_N$ locally span $T(F^{n+1})$ and $[Y_j, X_i] = 0$. Let $\eta^1 = \{\pi X_2, \cdots, \pi X_{n+1}\}$ be the local framing of $\Gamma \setminus G \times_K (X)^\perp$ where π is the composite projection

$$T(W) \xrightarrow{\nu} T(W)/T(F^{n+1}) \cong G/K \times_\Gamma T(R^{n+1})$$
$$\cong \Gamma \setminus G \times_K T(R^{n+1}) \to \Gamma \setminus G \times_K (X)^\perp.$$

Similarly let $\eta^2 = \{X\}$ be the framing of $\Gamma \setminus G \times_K (X)$. Then $\Gamma \setminus G \times_K (X)^\perp + \Gamma \setminus G \times_K (X) = \Gamma \setminus G \times_K T(R^{n+1})$, and $\eta^1 + \eta^2$ is a local framing for $\Gamma \setminus G \times_K T(R^{n+1})$. Let θ be the local connection matrix of ∇ relative to $\eta^1 + \eta^2$, θ^1 of ∇^1 relative to η^1, and θ^2 of ∇^2 relative to η^2. Let $\{Y_j^*, X_i^*\}$ be the dual basis to $\{Y_j, X_i\}$. $(X_1^*, \cdots, X_{n+1}^*)$ defines the leaves of F^{n+1} and so is a differential ideal. Away from the singularities of X, $(X_2^*, \cdots, X_{n+1}^*)$ defines F^n and so is a differential ideal. To compute θ, θ^1, θ^2 first notice that $\nabla_{Y_j} \pi X_i = \nu[Y_j, X_i] = 0$ and $\nabla_{Y_j} X_1 = \nu[Y_j, X_1] = 0$, and $\nabla_{X_1} \pi X_i = \nu[X_1, X_i] = 0$ and $\nabla_{X_1} X_1 = \nu[X_1, X_1] = 0$ since ∇ is X basic. Thus θ, θ^1, θ^2 all lie in $(X_2^*, \cdots, X_{n+1}^*)$. Apply (1.4) with $\rho^1 = \theta$, $\rho^0 = \theta^1 + \theta^2$, $\varphi = c_i$ to conclude that α, K_0, and Θ lie in $(X_2^*, \cdots, X_{n+1}^*)$, and so $\Delta_{c_i}(\nabla, \nabla^1 + \nabla^2)$ lies in $(X_2^*, \cdots, X_{n+1}^*)^i$. Also $c_J(K_{\nabla^1})$ is in $(X_2^*, \cdots, X_{n+1}^*)^{|J|}$, and so $\Delta_{c_i}(\nabla, \nabla^1 + \nabla^2) c_J(K_{\nabla^1}) \equiv 0$ for dimension reasons when $i + |J| \geqslant n + 1$. Next $\Delta_{c_i}(\nabla^2, D^2) = \oint_{[0,1]} c_i(dt \wedge \theta^2 + t d\theta^2) = (\text{constant}) \int_0^1 t^{i-1} c_i(\theta^2 \wedge (d\theta^2)^{i-1})\, dt$, and so $\Delta_{c_i}(\nabla^2, D^2)$ lies in $(X_2^*, \cdots, X_{n+1}^*)$, and thus $\Delta_{c_i}(\nabla^2, D^2) c_J(K_{\nabla^1}) \equiv 0$. Now Stokes' theorem says

$$\Delta_{c_i c_J}(\nabla, D^1 + D^2) + \Delta_{c_i c_J}(D^1 + D^2, \nabla^1 + \nabla^2) + \Delta_{c_i c_J}(\nabla^1 + \nabla^2, \nabla) = \text{exact}.$$

The previous argument for $\Delta_{c_i}(\nabla, \nabla^1 + \nabla^2)$ applied to $\Delta_{c_i c_J}(\nabla, \nabla^1 + \nabla^2)$ shows $\Delta_{c_i c_J}(\nabla, \nabla^1 + \nabla^2) \equiv 0$. Now $\Delta_{c_i c_J}(D^1 + D^2, \nabla^1 + \nabla^2) = \Delta_{c_i c_J}(D^1, \nabla^1) + \Delta_{c_i c_J}(D^2, \nabla^2)$ and the argument for $\Delta_{c_i}(\nabla^2, D^2) c_J(K_{\nabla^1})$ applied to $\nabla_{c_i c_J}(D^2, \nabla^2)$ shows $\Delta_{c_i c_J}(D^2, \nabla^2) \equiv 0$. Thus $\Delta_{c_i c_J}(\nabla, D^1 + D^2) = \Delta_{c_i c_J}(\nabla^1, D^1) + \text{exact}$.

Finally let $\tilde{\theta}$ be the local connection matrix of $\tilde{\nabla}$ relative to $\eta^1 + \eta^2$. Then θ and $\tilde{\theta}$ lie in $(X_1^*, \cdots, X_{n+1}^*)$ since $\tilde{\nabla}_{Y_j}\pi X_i = \nu[Y_j, X_i] = 0$ and $\tilde{\nabla}_{Y_j}X_1 = \nu[Y_j, X_1] = 0$. Apply (1.4) with $\rho^1 = \theta$, $\rho^0 = \tilde{\theta}$ to conclude α, K_0, Θ lie in $(X_1^*, \cdots, X_{n+1}^*)$ and so $\Delta_{c_ic_J}(\nabla, \nabla)$ lies in $(X_1^*, \cdots, X_{n+1}^*)^{i+|J|}$. Thus, if $i + |J| > n + 1$, then $\Delta_{c_ic_J}(\nabla, \tilde{\nabla}) \equiv 0$.

(2.2) Theorem (Permanence). *Let $I = (i_1, \cdots, i_r)$, $i_1 + |J| \geqslant n + 1$. Then*

$$h_Ic_J(F^n) = h_{i_2}(V) \cdots h_{i_r}(V)h_{i_1}c_J(F^n),$$

and $h_{i_1}c_J(F^n)$ is presented by $\Delta_{c_ic_J}(\nabla, D^R)$.

Proof.

$$\prod_{j=1}^{r} \Delta_{c_{i_j}}(\nabla, D^1 + D^2)c_J(K_{\nabla^1})$$

$$= \prod_{j=1}^{r} \left\{\Delta_{c_{i_j}}(\nabla^1 + \nabla^2, D^1 + D^2) + \Delta_{c_{i_j}}(\nabla, \nabla^1 + \nabla^2) + \text{exact}\right\}c_J(K_{\nabla^1})$$

$$= \prod_{j=1}^{r} \left\{\Delta_{c_{i_j}}(\nabla^1, D^1) + \Delta_{c_{i_j}}(\nabla^2, D^2) + \Delta_{c_{i_j}}(\nabla, \nabla^1 + \nabla^2) + \text{exact}\right\}c_J(K_{\nabla^1})$$

$$= \prod_{j=1}^{r} \Delta_{c_{i_j}}(\nabla^1, D^1)c_J(K_{\nabla^1}) + \text{exact}, \quad \text{using (2.1).}$$

∇^1 is a Bott connection for F^n, so $\prod_{j=1}^r \Delta_{c_{i_j}}(\nabla^1, D^1)c_J(K_{\nabla^1}) = h_Ic_J(F^n)$. Now

$$\prod_{j=1}^{r} \Delta_{c_{i_j}}(\nabla, D^1 + D^2)c_J(K_{\nabla^1})$$

$$= \Delta_{c_{i_1}}(\nabla, D^1 + D^2)c_J(K_{\nabla^1}) \prod_{j=2}^{r} \left\{\Delta_{c_{i_j}}(\tilde{\nabla}, D^1 + D^2) + \Delta_{c_{i_j}}(\nabla, \tilde{\nabla}) + \text{exact}\right\}.$$

Since $i_j + |J| > n + 1$ for $j \geqslant 2$, $c_J(K_{\nabla^1})\Delta_{c_{i_j}}(\nabla, \tilde{\nabla}) \equiv 0$ for $j \geqslant 2$. So

$$\prod_{j=1}^{r} \Delta_{c_{i_j}}(\nabla, D^1 + D^2)c_J(K_{\nabla^1})$$

$$= \Delta_{c_{i_1}}(\nabla, D^1 + D^2)c_J(K_{\nabla^1}) \prod_{j=2}^{r} \Delta_{c_{i_j}}(\tilde{\nabla}, D^1 + D^2) + \text{exact}.$$

Hence $h_Ic_J(F^n) = h_{i_2}(V) \ldots h_{i_r}(V)h_{i_1}c_J(F^n)$.

Now it is well known that $\Delta_{c_{i_1}}(\nabla^1, D^1)c_J(K_{\nabla^1}) = \Delta_{c_{i_1}c_J}(\nabla^1, D^1) + \text{exact}$, when ∇^1 is a Bott connection for a codimension-n foliation, and $i_1 + |J| \geqslant n + 1$. Thus by (2.1) we have $h_{i_1}c_J(F^n) = \Delta_{c_{i_1}c_J}(\nabla^1, D^1) = \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) +$ exact.

(2.3) Corollary. *If $i_1 + |J| > n + 1$, then $h_Ic_J(F^n) = 0$.*

Proof. $\Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) = \Delta_{c_{i_1}c_J}(\tilde{\nabla}, D^1 + D^2) + \Delta_{c_{i_1}c_J}(\nabla, \tilde{\nabla}) + \text{exact}$. By

(2.1), $\Delta_{c_{i_1}c_J}(\nabla, \nabla) \equiv 0$ if $i_1 + |J| > n + 1$. Now $\Delta_{c_{i_1}c_J}(\tilde{\nabla}, D^1 + D^2) = \Delta_{c_{i_1}}(\tilde{\nabla}, D^1 + D^2)c_J(K_{\tilde{\nabla}})$ + exact since $\tilde{\nabla}$ flat. But $K_{\tilde{\nabla}} \equiv 0$ and so $\Delta_{c_{i_1}c_J}(\tilde{\nabla}, D^1 + D^2) \equiv 0$. For the homogeneous case, this is in [8, (7.95)].

(2.4) Corollary.

$$⨏_{S^n} h_I c_J(F^n) = h_{i_2}(V) \dots h_{i_r}(V) ⨏_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2).$$

Proof. This is the usual permanence type theorem; it follows because the cohomology classes $h_i(V)$ come from $H^*(\Gamma \setminus G/K)$.

Next we prove a theorem which is known. This theorem is one of the applications of the general theory of Kamber-Tondeur; see [8, 7.93 and 7.95], and also [5, 5.1] for a special case.

(2.5) Theorem. *For fixed i_1, J with $i_1 + |J| = n + 1$ and $n + 1$ even, the classes $h_{i_1}h_{i_2} \dots h_{i_r}c_J$ are independent in $H^*(B\Gamma_n)$.*

Proof. Let $G = SL(n + 1)$, $K = SO(n + 1)$ and Γ be a co-compact subgroup. We show that $h_{i_1} \dots h_{i_r}c_J(F^n)$ are independent in $H^*(\Gamma \setminus G \times_K S^n)$. It is enough to show that the integral over the fiber S^n of these classes are independent. We will compute $⨏_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2)$ and use (2.4). Let $J = (j_1, \cdots, j_s)$.

(2.6) Lemma.

$$⨏_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) = a\binom{n+1}{i_1}\binom{n+1}{j_1} \cdots \binom{n+1}{j_s}\chi(V),$$

where a is nonzero number independent of i_1 and J. For the homogeneous case see [8, (7.95)].

The proof will appear at the end of the paper. Now the map $H^*(sl(n + 1), SO(n + 1)) \to H^*(\Gamma \setminus G/K)$ corresponding to the flat vector bundle $V = G/K \times_\Gamma R^{n+1}$ is known to be injective. Injectivity is proven in [7, Theorem 4.19], [8, (4.6)], and also in [6]. $H^*(sl(n + 1), SO(n + 1)) = \Lambda(h_3, h_5, \cdots, h_n, \chi)$ with $h_i \to h_i(V)$, $\chi \to \chi(V)$. The independence of $h_{i_2}(V) \dots h_{i_r}(V)\chi(V)$ in $H^*(\Gamma \setminus G/K)$ follows from the independence of $h_{i_2} \dots h_{i_r}\chi$ in $H^*(sl(n + 1), SO(n + 1))$.

(2.7) Theorem. *Let $n + 1$ be even. If $i_r \leqslant n - 1$ and $i_1 + |J| = n + 1$, then $h_{i_1} \dots h_{i_r}c_J$ in $H^*(B\Gamma_n)$ all vary.*

Remark. With a little thought one should be able to eliminate $i_r \leqslant n - 1$.

Proof. Let us consider $G = SL(n - 1) \times SL(2)$, $K = SO(n - 1) \times SO(2)$, $\underbrace{SL(2) \times \cdots \times SL(2)}_{(n-1)/2} \subset SL(n - 1)$. One can choose a co-compact subgroup Γ_1 of $SL(n - 1)$ whose intersection with $SL(2) \times \cdots \times SL(2)$ is co-compact. Let Γ_2 be a co-compact subgroup of $SL(2)$. Then $\Gamma_1 \times \Gamma_2 = \Gamma$ is a co-compact subgroup of $SL(n - 1) \times SL(2)$ whose intersection Γ' with

$\underbrace{SL(2) \times \cdots \times SL(2)}_{(n+1)/2}$ is co-compact. Let $G' = \underbrace{SL(2) \times \cdots \times SL(2)}_{(n+1)/2}$, $K' = \underbrace{SO(2) \times \cdots \times SO(2)}_{(n+1)/2}$. Then $G' \subset G$, $K' \subset K$. Let $\Gamma' = \Gamma \cap G'$.

Let

$$X = \lambda_1\left(t_1 \frac{\partial}{\partial t_1} + \cdots + t_{n-1}\frac{\partial}{\partial t_{n-1}}\right) + \lambda_2\left(t_n \frac{\partial}{\partial t_n} + t_{n+1}\frac{\partial}{\partial t_{n+1}}\right),$$

with λ_1, λ_2 positive, X commutes with the action of G on R^{n+1}, and so we get codimension-n foliations F^n of $\Gamma \setminus G \times_K S^n$ and F'^n of $\Gamma' \setminus G' \times_{K'} S^n$. The map $\Gamma' \setminus G' \times_{K'} S^n \to \Gamma \setminus G \times_K S^n$ takes leaves of F'^n to leaves of F^n.

We want to show $\fint_{S^n} h_I c_J(F^n)$ is a nonzero varying class in $H^*(\Gamma \setminus G/K)$ which is also in the image of relative Lie algebra cohomology. By (2.4),

$$\fint_{S^n} h_I c_J(F^n) = h_{i_2}(V) \ldots h_{i_r}(V) \fint_{S^n} \Delta_{c_i c_J}(\nabla, D^1 + D^2),$$

in $H^*(\Gamma \setminus G/K)$. First, we have the commutative diagram:

$$\begin{tikzcd}
 & H^*(\Gamma\backslash G/K) \\
H^*(sl(n+1), SO(n+1)) \arrow[ur, "\text{flat bundle}"] \arrow[r] & H^*(sl(n-1) \times sl(2), SO(n-1) \times SO(2)). \arrow[u, "\text{injective}"']
\end{tikzcd}$$

Now $H^*(sl(n-1) \times sl(2), SO(n-1) \times SO(2)) \cong \Lambda(h_3, h_5, \cdots, h_{n-2}, \chi_{n-1}) \otimes \Lambda(\chi_2)$, and under the horizontal map $h_i \to h_i$ for $i \leqslant n-2$ and the map $H^*(sl(n+1), SO(n+1)) \to H^*(\Gamma \setminus G/K)$ sends $h_i \to h_i(V)$. Thus to show that $h_{i_2}(V) \ldots h_{i_r}(V) \fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^2 + D^2)$ is a nonzero class in $H^*(\Gamma \setminus G/K)$, it is enough to show that $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ is in the image of $H^*(sl(n-1) \times sl(2), SO(n-1) \times SO(2))$ and has a component which is a nonzero multiple of the Euler class. To show that this class is in the image we have to show that $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ has a representative which, when pulled up to G/K, is a left invariant form under the action of G. To show that $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ is nonzero and varies, we observe that by naturality $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ pulls back to the $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ for F'^n and $\Gamma' \setminus G' \times_{K'} S^n$, and so it is enough to show that $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ in $H^{n+1}(\Gamma' \setminus G'/K')$ varies with λ_1/λ_2. This has been done in [3] also in [4]. We will prove this here also. If $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2) \neq 0$ in $H^{n+1}(\Gamma' \setminus G'/K')$, it must be a nonzero multiple of the Euler class, and hence $\fint_{S^n} \Delta_{c_{i_1} c_J}(\nabla, D^1 + D^2)$ in $H^{n+1}(\Gamma \setminus G/K)$, by naturality, has a component which is a nonzero multiple of the Euler class. Also, it follows from (2.6) that $\int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2)$ is actually a nonzero multiple of the Euler class. Thus we are done once we show

(2.8) Lemma. *$\int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2)$ + exact pulls up to a G-invariant form on G/K.*

(2.9) Lemma. *Up to a fixed constant depending on the volume of $\Gamma' \setminus G'/K'$,*

$$\int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) = c_{i_1}c_J(\mathrm{Diag}(\lambda_1, \cdots, \lambda_1, \lambda_2))\lambda_1^{n-}\lambda_2$$

$$\cdot \int_M \frac{(t_1^2 - t_2^2)}{\|X\|^2} d\left(\frac{t_1 t_2}{\|X\|^2}\right) \ldots d\left(\frac{t_n^2 - t_{n+1}^2}{\|X\|^2}\right) d\left(\frac{t_n t_{n+1}}{\|X\|^2}\right),$$

where M is the hypersurface $\lambda_1(t_1^2 + \ldots + t_{n-1}^2) + \lambda_2(t_n^2 + t_{n+1}^2) = 1$ in R^{n+1}.

Remark. When $\lambda_1 = \lambda_2$,

$$\int_M = (\text{constant}) \int_{B^{n+1}} (t_1^2 + t_2^2) \ldots (t_n^2 + t_{n+1}^2)\, dt_1 \ldots dt_{n+1} > 0.$$

Hence for λ_1 near λ_2 the $\int_M > 0$ and is independent of $c_{i_1}c_J$. Thus

$$\int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) \text{ varies with } \lambda_1/\lambda_2.$$

3. Proofs of (2.6), (2.8), (2.9)

Let us take the case of a general $G \subset SL(n + 1)$, $n + 1$ even. On $G/K \times_\Gamma R^{n+1}$ we have

$$\Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) - \Delta_{c_{i_1}c_J}(\tilde{\nabla}, D^1 + D^2) + \Delta_{c_{i_1}c_J}(\tilde{\nabla}, \nabla) = \text{exact}.$$

Now $\Delta_{c_{i_1}c_J}(\tilde{\nabla}, D^1 + D^2) + \text{exact} = \Delta_{c_{i_1}}(\tilde{\nabla}, D^1 + D^2)c_J(K_{\tilde{\nabla}}) \equiv 0$ since $\tilde{\nabla}$ is flat. Thus $\int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, D^1 + D^2) = \int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, \tilde{\nabla})$. Now we have a G-equivariant map $M \to S^n$ which takes a point m to the point m' on S^n which is the intersection of the unique integral curve of X through m with S^n. Thus we consider $\Gamma \setminus G \times_K M \to \Gamma \setminus G \times_K S^n$ and we see that

$$\int_M \Delta_{c_{i_1}c_J}(\nabla, \tilde{\nabla}) = \int_{S^n} \Delta_{c_{i_1}c_J}(\nabla, \tilde{\nabla}).$$

Let $x_1, \cdots, x_{n+1}$ be elements in $\mathfrak{g}$, $x_1(t), \cdots, x_{n+1}(t)$ their one-parameter groups in G, and $(X_i)_{gK} = d/dt(gx_i(t)K)_{t=0}$ (a tangent vector at gK in $\Gamma \setminus G/K$). Let $(\bar{X}_i)_{(g,m)} = d/dt(gx_i(t), m)_{t=0}$ which is a vector field in $\Gamma \setminus G \times_K M$ defined along the fiber. Then

$$\left(\int_M \Delta_\varphi(\nabla, \tilde{\nabla})\right)(X_1, \cdots, X_{n+1}) = \int_M \left(\iota(\bar{X}_1) \ldots \iota(\bar{X}_{n+1})\Delta_\varphi(\nabla, \tilde{\nabla})|_{\text{fiber}}\right).$$

Let $j_g\colon M \to \Gamma \setminus G \times_K M$ be given by $j_g(m) = (g, m)$. Then the fiber in question is $j_g(M)$, and the restriction is $j_g^*(\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})\Delta_\varphi(\nabla, \tilde{\nabla}))$. Now let $p\colon \Gamma \setminus G \times M \to \Gamma \setminus G \times_K M$, and let $\overline{X}$ also denote $d/dt(gx(t), m)_{t=0}$ in $\Gamma \setminus G \times M$ and j_g the inclusion of M in $\Gamma \setminus G \times M$. It is immediate that

$$\int_M j_g^*\big(\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})\Delta_\varphi(\nabla, \tilde{\nabla})\big) = \int_M j_g^*\big(\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})p^*\Delta_\varphi(\nabla, \tilde{\nabla})\big).$$

So we need only study $\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})\Delta_\varphi(\nabla, \tilde{\nabla})$ on $\Gamma \setminus G \times M$ (omit the p).

First consider the Gauss-Codazzi equation applied to $\Delta_\varphi(\nabla, \tilde{\nabla})$. Let θ and $\tilde{\theta}$ be local connection matrices for ∇ and $\tilde{\nabla}$. Let $\alpha = \theta - \tilde{\theta}$, $\Theta = d\alpha + \alpha \wedge \tilde{\theta} + \tilde{\theta} \wedge \alpha$, $\tilde{K} = d\tilde{\theta} + \tilde{\theta} \wedge \tilde{\theta}$. Of course $\tilde{K} \equiv 0$. Thus, for φ of degree $n + 1$, we have

$$\begin{aligned}\Delta_\varphi(\nabla, \tilde{\nabla}) &= (n + 1) \sum_{i+j+k=n} \lambda_{(i,j,k)}\varphi(\alpha \wedge \Theta^i \wedge \alpha^{2j} \wedge \tilde{K}^k)\\ &= \sum_{i+j=n} \lambda_{(i,j)}\varphi(\alpha^{2j+1} \wedge \Theta^i).\end{aligned}$$

Now we want to apply $\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})$. First, we show that $j_g^*(\alpha) = j_g^*(\Theta) \equiv 0$ on M. For, $j_g^{-1}(\nabla) = D$ and $j^{-1}(\tilde{\nabla}) = \tilde{D}$ are connections on the vector bundle $T(R^{n+1})$ which agree on vectors tangent to M. Thus $j_g^*(\alpha) \equiv 0$ and hence $j_g^*(\Theta) \equiv 0$.

Thus in expanding $\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})\varphi(\alpha^{2j+1} \wedge \Theta)$, an α or Θ without a $\iota(\overline{X})$ applied to it will restrict to zero on M, and the only terms which can occur are those where $2j + 1 + i \leqslant n + 1$ and $i + j = n$ and so $j = 0$. Therefore the only terms which can be nonzero upon restriction come from $\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})\varphi(\alpha \wedge \Theta^n)$ where each α and Θ has an $\iota(\overline{X})$ applied to it. Thus

$$\begin{aligned}&j_g^*\big(\iota(\overline{X}_1) \ldots \iota(\overline{X}_{n+1})\Delta_\varphi(\nabla, \tilde{\nabla})\big)\\ &\qquad = \sum_\sigma j_g^*\varphi\big(\iota(\overline{X}_1)\alpha \wedge \iota(\overline{X}_2)\Theta \wedge \ldots \wedge \iota(\overline{X}_{n+1})\Theta\big),\end{aligned} \tag{3.1}$$

where Σ_σ denotes the sum with the permutation σ applied to each index and the sign $(-1)^\sigma$.

We have to now evaluate $\alpha(\overline{X}_i)_{(g,m)}$ and $j_g^*(\iota(\overline{X}_i)\Theta_{(g,m)})$. In $\Gamma \setminus G \times R^{n+1}$ let $\tilde{X}_{i,(g,m)} = (d/dt)(gx_i(t), x_i(-t)m)_{t=0}$. Let x_i^* be the vector field in R^{n+1} given by $x_{i,m}^* = (d/dt)(x_i(t)m)_{t=0}$. Then $\tilde{X}_{i,(g,m)} = \overline{X}_{i,(g,m)} - j_g(x_{i,m}^*)$. Now α is tensorial which means that if $\hat{X}_i$ is any tangent vector in frame bundle of $\Gamma \setminus G \times R^{n+1}$ which projects to $\tilde{X}_i$, and if ω and $\tilde{\omega}$ are the connection forms for θ and $\tilde{\theta}$, then $\alpha(\tilde{X}_i) = \omega(\hat{X}_i) - \tilde{\omega}(\hat{X}_i)$. But for $\hat{X}_i$ we can choose $(d/dt)(gx_i(t), x_i(-t)f)_{t=0}$ at a point (g, f) in $\Gamma \setminus G \times$ Frames (TR^{n+1}). Now this $\hat{X}_i$ is horizontal for both ω and $\tilde{\omega}$ by construction. So $\sigma(\tilde{X}_i) = 0$, and

$\alpha(\bar{X}_i)_{(g,m)} = j_g^*(\alpha_{(g,m)})(x_{i,m}^*)$. Let ρ and $\tilde{\rho}$ be local connection matrices for D and $\tilde{D}$ relative to a framing s on $T(R^{n+1})$. Then $\alpha(\bar{X}_i)_{(g,m)} = j_g^*(\alpha_{(g,m)})(x_{i,m}^*) = (\rho - \tilde{\rho})_m(x_{i,m}^*)$, which is independent of g. If we take $s = \{\partial/\partial t_1, \cdots, \partial/\partial t_{n+1}\}$, then $\tilde{\rho} = 0$, $D_Y s = (\langle Y, X\rangle/\langle X, X\rangle)[X, s]$. Thus, if $\omega(Y) = \langle Y, X\rangle$ then $\alpha(\bar{X}_i)_{(g,m)} = \omega(x_{i,m}^*) \otimes L_X(s)_m/s$, where L_X is the Lie derivative and $L_X(s)/s$ is the matrix of $[X, s]$ relative to s. To study $j_g^*(\iota(\bar{X}_i)\Theta_{(g,m)})$ we apply it to a vector field Y tangent to M to get $d\alpha(\bar{X}_i, Y) + [\alpha, \tilde{\theta}](\bar{X}_i, Y)$. Now $\alpha(Y) = j_g^*(\alpha)(Y) = 0$, and relative to the framing s we have $j_g^*(\tilde{\theta}) = \tilde{\rho} = 0$ and therefore $\tilde{\theta}(Y) = 0$. Moreover, $d\alpha(\bar{X}_i, Y)_{(g,m)} = \bar{X}_i \circ \alpha(Y) - Y \circ \alpha(\bar{X}_i) - \alpha([\bar{X}_i, Y])$ and $[\bar{X}_i, Y] = 0$ since they are vector fields on different factors of a product, $\alpha(Y) \equiv 0$ as before, so

$$j_g^*\big(\iota(\bar{X}_i)\Theta_{(g,m)}\big) = -Y_m \circ \alpha\big(\bar{X}_i\big)_{(g,m)} = d\left(\omega(x_{i,m}^*) \otimes \frac{L_X(s)_m}{s}\right)\{Y_m\}.$$

Again this is independent of g.

To summarize, let β be the one-form on R^{n+1} given by $\beta(Y) = \omega(Y) \otimes L_X(s)/s$, and let us consider the function on M given by $m \to \beta(x_i^*)_m$. Then

$$\begin{aligned} &\textstyle\fint_M \Delta_\varphi(\nabla, \tilde{\nabla})_{gK}(X_1, \cdots, X_{n+1}) \\ (3.2)\qquad &= \sum_\sigma \int_M \varphi(\beta(x_1^*)d\beta(x_2^*) \wedge \ldots \wedge d\beta(x_{n+1}^*)). \end{aligned}$$

Now if we consider $\fint \Delta_\varphi(\nabla, \tilde{\nabla})$ pulled up to G/K at the coset gK applied to $X_1, \cdots, X_{n+1}$ where $X_i = d/dt(gx_i(t)K)_{t=0}$, the result is, clearly from (3.2), independent of g. Thus it follows that $\fint_M \Delta_\varphi(\nabla, \tilde{\nabla})$ on G/K is left invariant and hence, on $\Gamma \backslash G/K$, represents an element in the image of $H^*(\mathfrak{g}, K)$. This proves (2.8). Now take the case $G = SL(n+1)$, $K = SO(n+1)$, $n+1$ even, $X = \sum_{j=1}^{n+1} t_j\partial/\partial t_j$ and here $M = S^n$. Then $L_X(s) = -I$, I being the identity matrix. Thus

$$\begin{aligned} &\varphi(\beta(x_1^*)d\beta(x_2^*) \wedge \ldots \wedge d\beta(x_{n+1}^*)) \\ &\qquad = \omega(x_1^*)d(\omega(x_2^*)) \wedge \ldots \wedge d(\omega(x_{n+1}^*))\varphi(I). \end{aligned}$$

Now if $\varphi = c_{i_1}c_{j_1} \ldots c_{j_s}$, then

$$\varphi(I) = c_{i_1}(I)c_{j_1}(I) \ldots c_{j_s}(I) = \binom{n+1}{i_1}\binom{n+1}{j_1} \cdots \binom{n+1}{j_s}.$$

So

$$\begin{aligned} &\textstyle\fint_{S^n} \Delta_{c_{i_1}c_J}(\nabla, \tilde{\nabla})(X_1, \cdots, X_{n+1}) \\ &\qquad = A\binom{n+1}{i_1}\binom{n+1}{j_1} \cdots \binom{n+1}{j_s} \\ &\qquad\qquad \cdot \int_{S^n} \beta(x_1^*)d(\beta(x_2^*)) \wedge \cdots \wedge d(\beta(x_{n+1}^*)), \end{aligned}$$

where A is a constant independent of i_1 and J. In [5] it has been shown that $\int_{S^n} \Delta_{c_1 c_J}(\nabla, \tilde{\nabla})$ is a multiple of $\chi(V)$ by a Lie algebra computation. Since $\int_{S^n} \beta(x_1^*) d(\beta(x_2^*)) \wedge \cdots \wedge d(\beta(x_{n+1}^*))$ is independent of i_1 and J, Lemma (2.6) follows.

Finally, for $G' = SL(2) \times \cdots \times SL(2)$, $K' = SO(2) \times \ldots \times SO(2)$,

$$X = \lambda_1\left(t_1 \frac{\partial}{\partial t_1} + t_2 \frac{\partial}{\partial t_2}\right) + \cdots + \lambda_{(n+1)/2}\left(t_n \frac{\partial}{\partial t_n} + t_{n+1} \frac{\partial}{\partial t_{n+1}}\right),$$

we can choose $x_1, \cdots, x_N$, $N = (n+1)/2$, so that $x_{2i-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $x_{2i} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in the ith $sl(2)$ factor. Then $x_1, \cdots, x_N$ is a complementary basis to k' in $\mathfrak{g}'$ so that $X_1 \wedge \cdots \wedge X_N$ is a well defined N-vector field on $\Gamma' \setminus G'/K'$, and $\int_M \Delta_{c_i c_J}(\nabla, \tilde{\nabla})(X_1, \cdots, X_N)$ is a fixed nonzero constant times $\int_M \Delta_{c_i c_J}(\nabla, \tilde{\nabla})[\Gamma \setminus G/K]$. A simple computation shows

$$x_{2i-1}^* = t_{2i-1} \frac{\partial}{\partial t_{2i-1}} - t_{2i} \frac{\partial}{\partial t_{2i}}, \quad x_{2i}^* = t_{2i} \frac{\partial}{\partial t_{2i-1}}.$$

So

$$\beta(x_{2i-1}^*) = \lambda_i\left(t_{2i-1}^2 + t_{2i}^2\right)\mathrm{Diag}(\lambda_1, \lambda_1, \cdots, \lambda_N, \lambda_N),$$
$$\beta(x_{2i}^*) = \lambda_i t_{2i-1} t_{2i}\, \mathrm{Diag}(\lambda_1, \lambda_1, \cdots, \lambda_N, \lambda_N).$$

Let $\Lambda = \mathrm{Diag}(\lambda_1, \lambda_1, \cdots, \lambda_N, \lambda_N)$. Then from (3.2), $\int_M \Delta_{c_i c_J}(\nabla, \tilde{\nabla})_{gK}(X_1, \cdots, X_N)$ is a constant depending on volume $\Gamma' \setminus G'/K'$ times

$$c_i c_J(\Lambda)\lambda_1^2 \cdots \lambda_N^2 \int_M \frac{(t_1^2 - t_2^2)}{\|X\|^2} d\left(\frac{t_1 t_2}{\|X\|^2}\right) \wedge \cdots \wedge d\left(\frac{t_{n-1}^2 - t_n^2}{\|X\|^2}\right) \wedge d\left(\frac{t_{n-1} t_n}{\|X\|^2}\right),$$

and hence we have (2.9).

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