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CIRCLES INVARIANT UNDER DIFFEOMORPHISMS OF FINITE ORDER

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1. Introduction

If a compact surface M in $\mathbf{R}^3$ is rotated into itself, then we prove that there is always an embedded circle on the surface which is invariant under the rotation. A natural question to ask is whether this result can be generalized to prove that every isometry of a compact surface has an invariant circle.

In this paper we shall prove a general structure theorem on the existence and topological properties of invariant circles. Except for a recent result on representing one-dimensional homology classes by embedded circles, our techniques of proof will be classical. These techniques are based on the theory of branched covering spaces.

We follow the proof of our structure theorem with some special results on the existence of invariant circles. This latter work leads us to make the following conjecture: for an infinite number of compact surfaces, every diffeomorphism of finite order has an invariant circle.

Throughout the paper, M will be a compact orientable surface, and f: $M \to M$ will denote an orientation-preserving diffeomorphism of finite order. If the order of f is n, and M_f denotes th orbit space of f with the quotient topology, then the natural projection P: $M \to M_f$ is a cyclic branched covering space of the orientable surface M_f. The structure of the branched covering space P: $M \to M_f$ is the key to proving our results. The structure of P: $M \to M_f$ can be analyzed by various representation theorems which we shall now recall for the reader.

Given a finite subset $B \subset M$ and an onto representation ρ: $\pi_1(M - B) \to Z_n$, there is an associated cyclic n-sheeted branched covering space P: $\overline{M} \to M$, where cyclic means that the group of covering transformations of the unbranched covering space $\overline{P}$: $\overline{M} - P^{-1}(B) \to M - B$ is generated by a diffeomorphism f: $\overline{M} - P^{-1}(B) \to \overline{M} - P^{-1}(B)$. By the Riemann extension theorem, $\bar{f}$ extends to a diffeomorphism f: $\overline{M} \to \overline{M}$ of order n. With respect to f: $\overline{M} \to \overline{M}$, P: $\overline{M} \to M$ is the same branched covering space as the natural

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projection P: $M \to M_f$. Also recall that when f: $M \to M$ has order n, the branched covering space P: $M \to M_f$ can be constructed abstractly from a canonical representation ρ: $\pi_1(M - B) \to Z_n$ where B is the branch locus of P.

1. Since local perturbations of one invariant circle gives rise to new invariant circles, we shall consider two invariant circles to be equivalent if they are isotopic through invariant circles. When we talk about an invariant circle, we shall frequently be referring to the equivalence class of an embedded circle invariant under some diffeomorphism.

Structure theorem. *Suppose M has positive genus and f: $M \to M$.*

1. *There exists an infinite number of distinct homology classes representable by invariant circles if and only if $f^2 = \mathrm{id}$ or $M_f \neq S^2$.*

2. *If $f^2 \neq \mathrm{id}$ and $M_f = S^2$, then each invariant circle disconnects M.*

We will prove the structure theorem by a series of lemmas. The proof of the first lemma is an elementary proof in covering space theory and we leave it to the reader to verify. We shall use the phrase "γ' lifts to γ" to mean that γ is the inverse image of γ' under the projection P: $M \to M_f$.

Lemma 1. *Suppose f: $M \to M$, and ρ: $\pi_1(M_f - B) \to Z_n$ is the representation for the branched cover P: $M \to M_f$. Then $\gamma \subset M - P^{-1}(B)$ is an invariant circle if and only if γ is the lift of an embedded circle $\gamma' \subset M_f - B$ with $\rho(\gamma')$ a generator of Z_n.*

Lemma 2. *If f: $M \to M$ and $M_f \neq S^2$, then there exists an infinite number of distinct homology classes which can be represented by invariant circles not passing through branch points of P: $M \to M_f$.*

Proof. Since Z_n is abelian, the representation for P: $M \to M_f$ factors through the first homology group by

$$\begin{tikzcd}
\pi_1(M_f - B) \arrow[rr, "\sim"] \arrow[dr, "\rho"'] & & H_1(M_f - B) \arrow[dl, "\tilde{\rho}"] \\
& Z_n &
\end{tikzcd}$$

Let B denote the branch locus for P: $M \to M_f$, i: $M_f - B \to M_f$ be the inclusion map, and i_*: $H_1(M_f - B) \to H_1(M_f)$ be the induced map on homology.

We will call a class $\omega \in H_1(M)$ primitive if $\omega = 0$ or if ω is not a nontrivial multiple of any other integer homology class. Rank arguments from linear algebra imply the existence of an infinite number of homology classes $\{\omega_J\}_{J=1}^{\infty} \subset H_1(M_f - B)$ such that

(1) $\tilde{\rho}(\omega_J) = 1$ for all J,

(2) the elements of $S = \{i_*(\omega_J)\}_{J=1}^{\infty} \subset H_1(M_f)$ are distinct primitive homology classes.

By results in [4], each ω_J can be represented by an embedded circle γ_J'. By Lemma 1, the circles γ_J' have lifts γ_J which are circles invariant under f: $M \to M$. Since $P_*([\gamma_J]) = n\omega_J \in H_1(M_f)$ and the classes ω_J are distinct on M_f, the homology classes represented by the invariant circles γ_J must represent distinct homology classes on M. This completes the proof of Lemma 2. q.e.d.

The following lemma together with Lemma 2 completes the forward implication in part 1 of Theorem 1.

Lemma 3. *If f: $M \to M$ has order 2 and $M_f = S^2$, then there exists an infinite number of distinct homology classes represented by invariant circles passing through the branch points of P: $M \to M_f$.*

Proof. There is a "canonical" automorphism H: $M \to M$ of a surface of genus g having order two and with $M_H = S^2$. This automorphism is accomplished geometrically by rotating a standard g holed surface along its central axis by $180°$. H is classically known as the hyperelliptic automorphism.

From this geometric representation of H: $M \to M$, it is easy to see that there is an infinite number of invariant circles representing distinct homology classes. Also it is elementary to show that any other diffeomorphism f: $M \to M$ of order 2 with $M_f = S^2$ is conjugate to H: $M \to M$. Hence such an f: $M \to M$ has an infinite number of homology classes representable by invariant circles.

Lemma 4. *If f: $M \to M$, and γ is an invariant circle passing through a branch point of P: $M \to M_f$, then the order of f is 2.*

Proof. If $q \in \gamma$ is a branch point of P: $M \to M_f$, then for some K less than the order of f we have $f^K(q) = q$. Since K is less than the order of f, and an isometry of M is determined by its values on an infinite subset, we have $f^K|\gamma \neq \mathrm{id}_\gamma$. Thus f^K restricts to an isometry of a circle with a fixed point, and f^K is not the identity map on the circle. The Lefschetz theorem implies that $f^K|\gamma$ is orientation-reversing on γ, and hence $f|\gamma$ is orientation-reversing on γ. Another application of the Lefschetz theorem shows that $f|\gamma$: $\gamma \to \gamma$ has order 2. This in turn implies that f: $M \to M$ has order 2. q.e.d.

The next lemma completes the proof of the structure theorem for invariant circles.

Lemma 5. *If f: $M \to M$ with $M_f = S^2$ and $f^2 \neq \mathrm{id}_M$, then each invariant circle disconnects M.*

Proof. Let γ be an invariant circle, and suppose B is the branch locus of the natural projection P: $M \to M_f$. Lemma 4 implies that $\gamma \subset M - P^{-1}(B)$. If the homology class $[\gamma]$ is not the zero homology class, then the arguments in the proof of Lemma 4 imply $f_*[\gamma] = [\gamma]$. But a nontrivial invariant homology class on M descends to a nontrivial homology class on $M_f = S^2$ which is absurd. Therefore $[\gamma] = 0$ which implies γ disconnects M.

2. While the structure theorem gives a fairly complete description of the number and topological structure of invariant circles, it does not prove that every f: $M \to M$ has an invariant circle. In fact, there are f: $M \to M$ with no invariant circles.

Proposition 1. *There exists an f: $M \to M$ of order 30 on a surface of genus 11 with the followng properties*:

(1) *f has no invariant circle.*

(2) *If g: $M \to M$ has no invariant circles, then g is conjugate to f^r for some r relatively prime to* 30.

Proof. Let γ_1, γ_2 and γ_3 be small circles centered at some theree points p_1, p_2, and p_3 on S^2. Orient these circles counter-clockwise with respect to the usual orientation on S^2. Let P: $M \to S^2$ be the cyclic branched covering space associated to the representation ρ: $\pi_1(S^2 - \{p_1, p_2, p_3\}) \to Z_{30}$ defined on the generators by $\rho(\gamma_1) = 2$ and $\rho(\gamma_2) = 3$. Let f: $M \to M$ be the diffeomorphism of order 30 which is associated to P: $M \to S^2$. Note that none of the three embedded circles in $S^2 - \{p_1, p_2, p_3\}$ lift to circles invariant under f. By Lemmas 1 and 4, f has no invariant circle.

Suppose g: $M \to M$, and g has no invariant circle. Then by the argument found in the proof of part 2 of the next theorem, g has order 30 and the branch locus of the corresponding P: $M \to S^2$ has 3 points q_1, q_2, q_3. Let γ_1, γ_2 and γ_3 be circles centered at q_1, q_2, and q_3 respectively, and oriented counter-clockwise with respect to the orientation on S^2. Let ρ': $\pi_1(S^2 - \{q_1, q_2, q_3\}) \to Z_{30}$ be the representation for P'. We may assume that the orders of $\rho'(\gamma_1)$, $\rho'(\gamma_2)$ and $\rho'(\gamma_3)$ are 15, 10 and 6 because other possible orders give rise to invariant circles.

If L: $S^2 \to S^2$ is a diffeomorphism with $L(p_1) = q_1$, $L(p_2) = q_2$ and $L(p_3) = q_3$, then L lifts to $\tilde{L}$: $M \to M$ in the following diagram:

$$\begin{tikzcd} M \arrow[r, "\tilde{L}"] \arrow[d, "P"'] & M \arrow[d, "P'"] \\ S^2 \arrow[r, "L"'] & S^2 \end{tikzcd}$$

It is straightforward to verify that $g = \tilde{L} f^r \tilde{L}^{-1}$ for some r relatively prime to 30. q.e.d.

Although a general f: $M \to M$ may not have an invariant circle, there are interesting conditions one can place on M or f to assure that invariant circles do exist. Some of these conditions are outlined in the next theorem.

Theorem 2. (1) *If f: $M \to M$ has order $n = p^i q^j$ with p and q primes, then f has an invariant circle.*

(2) *If the genus of M is less than* 11, *then every f: $M \to M$ has an invariant circle.*

(3) *If $M \neq S^2$, $M \subset \mathbf{R}^3$ and f: $M \to M$ is induced by an isometry of M in $\mathbf{R}^3$, then f has at least* 4 *invariant circles.*

Proof. (1) Suppose $B = \{p_1, p_2, \cdots, p_K\}$ is the branch locus for P: $M \to M_f$, and $\gamma_1, \gamma_2, \cdots, \gamma_K$ are circles on M_f centered at the respective points and oriented counter-clockwise with respect to the point and the orientation of M_f. If $M_f \neq S^2$, then the structure theorem implies the existence of many invariant circles. Therefore we shall assume $M_f = S^2$.

Now consider the integers $\{m_i = \rho(\gamma_i) | i = 1, 2, \cdots, K\}$ where ρ: $\pi_1(M_f - B) \to Z_n$ is the representation for P: $M \to M_f$. If there is an integer m_t not divisible by p or q, then Lemma 1 implies that the loop $\gamma_i \gamma_t$ will lift to an invariant circle. Suppose that no such m_t exists. Since ρ is onto, there do exist integers m_r and m_s which are respectively, relatively prime to p and q but not to both. Thus $\rho(\gamma_r) + \rho(\gamma_s)$ is a generator for Z_n. Hence by Lemma 1, the connected sum of γ_r and γ_s lifts to an invariant circle for f: $M \to M$.

(2) Suppose that M is a surface of least genus which admits an f: $M \to M$ with no invariant circle. By the structure theorem we may assume that $M_f = S^2$. Let $\gamma_1, \gamma_2, \cdots, \gamma_K$ be circles centered at the respective branch points $B = \{q_1, q_2, \cdots, q_K\}$ for P: $M \to M_f$. Let ρ: $\pi_1(M_f - B) \to Z_N$ be the representation giving rise to P.

The total branching order for $P^{-1}(q_i)$ can be calculated to be $N(O_i - 1)/O_i$ where O_i is order of $\rho(\gamma_i)$ in Z_N. For convenience, we reorder the branch points so that the O_i appear in nonincreasing order. By the Riemann Hurwitz formula, the genus of M can be calculated to be

$$g = 1 - N + \frac{N}{2} \sum_{i=1}^{K} \frac{O_i - 1}{O_i}.$$

If more than two of the orders O_i equal 2, then there is a natural onto representation ρ': $\pi_1(S^2 - \{q_1, q_2, \cdots, q_{k-2}\}) \to Z_N$ induced by the original representation ρ. It is easy to check that ρ' gives rise to a g: $M' \to M'$ with no invariant circle, and the genus of M' is less than the genus of M. Since this contradicts the minimality assumption on M, O_i equals 2 for at most two points. Hence $(O_i - 1)/O_i \geqslant 2/3$ except at possibly two points. Plugging this information into the above formula yields

$$g \geqslant 1 + \frac{(2k - 7)}{6} N.$$

Since $N \geqslant 30$, by part 1 of Theorem 2, $g \geqslant 11$ when $k \geqslant 5$. When $k = 3$ and $N = 30$ or $N = 42$, a case by case argument shows $g \geqslant 11$. If $O_3 = 2$, then O_1 and O_2 must be divisible by two distinct primes. This is because N is divisible by three distinct primes and because any two of $\{\rho(\gamma_1), \rho(\gamma_2), \rho(\gamma_3)\}$ generate Z_N. Hence $O_1 \geqslant O_2 \geqslant 15$. This result gives the estimate $g \geqslant 1 + 11/60N$. Part 1 of Theorem 2 implies that $N = 30$ or $N = 42$. By a similar argument, we get $N = 30$ or $N = 42$ when $O_3 > 2$. Hence $g \geqslant 11$ when $k = 3$. The proof of the case $k = 4$ is similar to the above argument for $k = 3$, and we leave it to the reader to verify the details of this case.

(3) Suppose M is contained in $\mathbf{R}^3$, and F: $\mathbf{R}^3 \to \mathbf{R}^3$ is an isometry of $\mathbf{R}^3$ inducing f: $M \to M$. Since M is compact, F is a rotation about some line L. If $L \cap M$ is empty, then every orbit of f contains the same number of points. This implies that P: $M \to M_f$ is a covering space, and hence $M_f \neq S^2$. By the structure theorem, f would have an infinite number of invariant circles.

Suppose now that $M_f = S^2$. Since every branched cover of S^2 with two branch points is again S^2, P: $M \to M_f$ must have more than 2 branch points. Note that in our geometric situation, the branch points of P: $M \to M_f$ are precisely the fixed points of f: $M \to M$, or equivalently, they are the points in $L \cap M$. Hence the size of $L \cap M$ is at least 3. If we consider L as a circle in S^3 and M as a surface in S^3, then intersection theory implies that $L \cap M$ is even and hence at least 4. Therefore P: $M \to S^2$ has at least four branches. Clearly, small circles around these branch points yield distinct circles invariant under f: $M \to M$.

Corollary. *Suppose M is a compact embedded surface in $\mathbf{R}^3$ and f: $M \to M$ is induced by an isometry I: $\mathbf{R}^3 \to \mathbf{R}^3$ with possibly infinite order. Then f: $M \to M$ has an invariant circle.*

Proof. If f: $M \to M$ has finite order and M is not S^2, the result follows from part 3 of Theorem 2. If M is S^2, then f has two fixed points by Lefschetz theorem for isometries of a compact surface. Hence f: $S^2 \to S^2$ has an invariant circle.

If f: $M \to M$ has infinite order, then I: $\mathbf{R}^3 \to \mathbf{R}^3$ is given by rotation by an irrational angle around some line in $\mathbf{R}^3$. This observation shows that closure of the orbit of any nonfixed point of f: $M \to M$ is an invariant circle.

Remark. There is an isometry of a surface of genus 3 in $\mathbf{R}^3$ with precisely 4 invariant circles. However, every isometry of a surface of genus 2 in $\mathbf{R}^3$ has an infinite number of invariant circles.

In conclusion we make the following conjecture: For an infinite number of compact surfaces every diffeomorphism of finite order has an invariant circle. We believe that a proof or a counterexample to this conjecture could be obtained by the methods of proof in Theorem 2.

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