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PSEUDO-EINSTEIN REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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Introduction

The purpose of the present paper is to study real hypersurfaces in complex space forms with certain condition on the Ricci tensor. Cartan and Thomas [18], have shown that an Einstein hypersurface of Euclidean space is a hypersphere if its scalar curvature is positive, and Fialkow [2] classified Einstein hypersurfaces in spaces of constant curvature (see also [5] and [11]). We shall show that any real hypersurface of a complex projective space is not Einsteinian (Theorem 4.3). So we introduce the notion of pseudo-Einstein real hypersurfaces in a Kaehlerian manifold.

Let M be a real hypersurface of a Kaehlerian manifold $\bar{M}$. Denote by J the almost complex structure of $\bar{M}$, and by C a unit normal of M in $\bar{M}$. Put $JC = -U$. Then U is a unit vector field tangent to M. Let g be the Riemannian metric tensor field of $\bar{M}$ as well as the one induced on M. Now we put $f(X) = g(X, U)$ for any vector field X tangent to M. If the Ricci tensor S of M is of the form $S(X, Y) = ag(X, Y) + bf(X)f(Y)$ for some constants a and b, then M is called a *pseudo-Einstein* real hypersurface of $\bar{M}$. If $b = 0$, then M is *Einsteinian*. Pseudo-Einstein real hypersurfaces of a complex projective space $P^n(C)$ are studied also by Maeda [7]. Our aim is to determine all connected complete pseudo-Einstein real hypersurfaces in a complex projective space $P^n(C)$ $(n \geqslant 3)$ and a complex number space C^n $(n \geqslant 3)$.

In §1 we state basic formulas for real hypersurfaces in a complex space form. In §2 we prove some lemmas for real hypersurfaces in a complex space form. §3 is devoted to a study of examples of pseudo-Einstein real hypersurfaces in a complex projective space $P^n(C)$, and in §4 we determine connected complete pseudo-Einstein real hypersurfaces in $P^n(C)$. First of all, we prove that any connected pseudo-Einstein real hypersurfaces M of $P^n(C)$ has at most three constant prinipal curvatures (Proposition 4.1). On the other hand, Takagi [13], [14] classified connected complete real hypersurfaces in

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$P^n(C)$ with two or three constant principal curvatures. Combining these results, we have our theorem (Theorem 4.1). In the last §5 we give some examples of pseudo-Einstein real hypersurfaces in a complex number space C^n, and determine all connected complete pseudo-Einstein real hypersurfaces in C^n.

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1. Preliminaries

Let $\bar{M}$ be a Kaehlerian manifold of complex dimension n (real dimension $2n$) with almost complex structure J, and M a connected Riemannian real hypersurface of $\bar{M}$ with the induced metric. The Riemannian metric tensor field of $\bar{M}$ will be denoted by g, that induced on M is also denoted by the same g, and all metric properties of M refer to this metric. We denote by C a unit normal of M in $\bar{M}$. For any vector field X tangent to M we put

$$JX = \phi X + f(X)C, \quad JC = -U, \tag{1.1}$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1,1), f is a 1-form, and U is a unit vector field on M. Then they satisfy

$$\phi^2 X = -X + f(X)U, \quad \phi U = 0, \quad f(\phi X) = 0 \tag{1.2}$$

for any vector field X tangent to M. Thus (ϕ, f) defines an almost contact structure on M. Moreover we have

$$\begin{aligned} g(\phi X, Y) + g(X, \phi Y) = 0, \quad f(X) = g(X, U), \\ g(\phi X, \phi Y) = g(X, Y) - f(X)f(Y). \end{aligned} \tag{1.3}$$

By $\bar{\nabla}$ we denote the operator of covariant differentiation in $\bar{M}$, and by ∇ the one in M determined by the induced metric. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)C, \quad \bar{\nabla}_X C = -AX$$

for any vector fields X and Y tangent to M. We call A the *second fundamental form* of M, which can be considered as a symmetric $(2n-1, 2n-)$-matrix. We recall that the rank of A at a point x of M is called the *type number* at x and is denoted by $t(x)$.

Now we assume that $\bar{M}$ is of constant holomorphic sectional curvature $4c$. Then $\bar{M}$ is called a *complex space form* and is denoted by $\bar{M}^n(c)$. Let R

denote the Riemannian curvature tensor of M. Then we obtain

(1.4)
$$\begin{aligned} R(X, Y)Z = c(&g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ &- g(JX, Z)JY + 2g(X, JY)JZ) + g(AY, Z)AX \\ &- g(AX, Z)AY - g((\nabla_X A)Y, Z)C + g((\nabla_Y A)X, Z)C. \end{aligned}$$

Comparing the tangential and normal parts in (1.4), we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z = c(&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &+ 2g(X, \phi Y)\phi Z) + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{1.5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c(f(X)\phi Y - f(Y)\phi X + 2g(X, \phi Y)U). \tag{1.6}$$

In particular, we have

$$g((\nabla_X A)U, U) = g((\nabla_U A)X, U). \tag{1.7}$$

From (1.5) the Ricci tensor S of M is given by

$$\begin{aligned} S(X, Y) = {} & (2n + 1)cg(X, Y) - 3cf(X)f(Y) \\ & + Hg(AX, Y) - g(AX, AY), \end{aligned} \tag{1.8}$$

where we have put $H = \operatorname{trace} A$. Therefore the scalar curvature k of M is given by

$$k = 4(n^2 - 1)c + H^2 - \operatorname{trace} A^2. \tag{1.9}$$

If H vanishes identically, then M is said to be *minimal*. If the Ricci tensor S of M is of the form $S(X, Y) = ag(X, Y) + bf(X)f(Y)$ for some constants a and b, then M is said to be *pseudo-Einstein*. When $b = 0$, M is an Einstein manifold. If the second fundamental form A of M is of the form $AX = \alpha X + \beta f(X)U$, where α and β are functions on M, then M is said to be *totally η-umbilical*. When α and β are constant, totally η-umbilical real hypersurfaces of a complex space form are necessarily pseudo-Einstein. If $\beta = 0$, then M is *totally umbilical*. But, if $c \neq 0$, by (1.6) we see that there exists no totally umbilical real hypersurfaces of $\overline{M}^n(c)$ (see Tashiro-Tachibana [16]).

2. Basic formulas and lemmas

In this section we prepare some basic formulas and lemmas for real hypersurfaces of a complex space form. Let M be a connected real hypersurface of a complex space form $\overline{M}^n(c)$ with constant holomorphic sectional curvature $4c$. First of all, from (1.1) and Gauss and Weingarten formulas we have

$$(2.1) \qquad \nabla_X U = \phi AX,$$

$$(2.2) \qquad (\nabla_X \phi) Y = f(Y)AX - g(AX, Y)U$$

for any vector fields X and Y tangent to M.

Now we assume that the vector U is an eigenvector of A, that is, $AU = \alpha U$. Then (2.1) implies that

$$(\nabla_X A)U = (X\alpha)U + \alpha\phi AX - A\phi AX,$$

from which it follows that

$$(2.3) \qquad g((\nabla_X A)Y, U) = (X\alpha)f(Y) + \alpha g(Y, \phi AX) - g(Y, A\phi AX).$$

By Codazzi equation (1.6) and (2.3) we have

$$(2.4) \qquad \begin{aligned} 2cg(X, \phi Y) = {} & (X\alpha)f(Y) - (Y\alpha)f(X) + \alpha g((\phi A + A\phi)X, Y) \\ & -2g(A\phi AX, Y). \end{aligned}$$

Putting $X = U$ or $Y = U$ in (2.4), we see that $X\alpha = (U\alpha)f(X)$ and $Y\alpha = (U\alpha)f(Y)$, and hence (2.4) reduces to

$$(2.5) \qquad 2cg(X, \phi Y) = \alpha g((\phi A + A\phi)X, Y) - 2g(A\phi AX, Y).$$

In the following we suppose that $\dim M = 2n - 1 \geqslant 3$, i.e., $n \geqslant 2$.

Lemma 2.1. *Let M be a real hypersurface of a complex space form $\overline{M}^n(c)$. If $\phi A + A\phi = 0$, then $c \leqslant 0$. Moreover if $c = 0$, then $t(x) \leqslant 1$ at all x.*

Proof. Since $\phi A + A\phi = 0$, we have $\phi AU = 0$ and hence $AU = f(AU)U$. This means that the vector U is an eigenvector of A. We now put $\alpha = f(AU)$. Then (2.5) implies that

$$cg(X, \phi Y) = -g(\phi AX, AY) = g(A\phi X, AY).$$

From this we see that $cg(\phi X, \phi X) = -g(A\phi X, A\phi X) \leqslant 0$. Since the rank of ϕ is $2n - 2$ and $n \geqslant 2$, we must have $c \leqslant 0$. Furthermore, if $c = 0$ we have $g(A\phi X, A\phi X) = 0$ and hence $A\phi X = -\phi AX = 0$. Therefore we obtain $AX = \alpha f(X)U$ for any vector field X tangent to M. Thus we have $t(x) \leqslant 1$ at each point x of M. This completes our assertion.

Lemma 2.2. *Let M be a real hypersurface of a complex space form $\overline{M}^n(c)$ $(c > 0)$. If U is an eigenvector of A, then $\alpha = f(AU)$ is constant.*

Proof. Since we have $X\alpha = (U\alpha)f(X)$, we see that $\nabla_X \operatorname{grad} \alpha = (X\beta)U + \beta\phi AX$, where we have put $\beta = U\alpha$. From this we have

$$(2.6) \qquad (Y\beta)f(X) - (X\beta)f(Y) = \beta g(\phi AX, Y) - \beta g(\phi AY, X),$$

because of the fact that $g(\nabla_X \operatorname{grad} \alpha, Y) = g(\nabla_Y \operatorname{grad} \alpha, X)$. Putting $X = U$ or $Y = U$ in (2.6), we obtain $X\beta = (U\beta)f(X)$ and $Y\beta = (U\beta)f(Y)$. Therefore we have $\beta g((\phi A + A\phi)X, Y) = 0$. From this and Lemma 2.1, we have $\beta = 0$ and hence α is constant.

Next we consider the type number of a real hypersurface of a complex space form, and have

Lemma 2.3. *Let M be a real hypersurface of a complex space form $\overline{M}^n(c)$ $(c \neq 0)$. Then $t(x) > 1$ at some point x of M.*

Proof. Let us assume that the type number of M is $t(x) \leqslant 1$ at any point x of M. We can choose an orthonormal frame field of M for which the second fundamental form of M can be diagonal, that is, $Ae_i = 0$, $i = 1, \cdots, 2n-2$ and $Ae_{2n-1} = \lambda e_{2n-1}$. Let $M' = \{x \in M: \lambda_x \neq 0\}$. Then M' is an open set of M. In the following our calculation is considered on M'. Then we obtain

$$g((\nabla_{e_i}A)e_j, e_k) = 0 \quad \text{for } i, j, k = 1, \cdots, 2n-2.$$

From this and (1.6) we have

$$f(e_i)g(\phi e_j, e_k) - f(e_j)g(\phi e_i, e_k) + 2f(e_k)g(e_i, \phi e_j) = 0.$$

Putting $j = k$ in this equation, we see that

$$f(e_j)g(e_i, \phi e_j) = 0, \tag{2.7}$$

which implies that

$$\sum_{i=1}^{2n-2} f(e_j)g(e_i, \phi e_j)g(e_i, \phi e_{2n-1})$$
$$= f(e_j)g(\phi e_j, \phi e_{2n-1}) = -f(e_j)f(e_j)f(e_{2n-1}) = 0.$$

Consequently we see that $f(e_j) = 0$ for $j = 1, \cdots, 2n-2$ or $f(e_{2n-1}) = 0$. If $f(e_j) = 0$ for $j = 1, \cdots, 2n-2$, then $f(e_{2n-1}) = 1$ and hence $e_{2n-1} = U$. Since we have $g((\nabla_{e_i}A)e_j, U) = 0$ for $i, j = 1, \cdots, 2n-2$, (1.6) implies $g(e_i, \phi e_j) = 0$. Thus we have that

$$\sum_{i,j=1}^{2n-2} g(e_i, \phi e_j)g(e_i, \phi e_j) = 2n-2 = 0,$$

or $n = 1$. This is a contradiction. Next we suppose that $f(e_{2n-1}) = 0$. Then we have $AU = 0$ and hence $(\nabla_X A)U + A\phi AX = 0$. If $AX \neq 0$, we have $A\phi X = 0$. Thus we have $(\nabla_X A) = 0$ for any vector field X tangent to M. From this and (1.6) we obtain $g(X, \phi Y) = 0$ for any vectors X and Y. This is a contradiction. Therefore we see that M' is empty, that is, M is totally geodesic. But this contradicts that M is not totally umbilical. Therefore we must have $t(x) > 1$ at some point x of M.

Lemma 2.4. *Let M be a real hypersurface of a complex space form $\overline{M}^n(c)$ $(c \neq 0)$. If $\phi A = A\phi$, then M has at most three constant principal curvatures.*

Proof. From the assumption, we see that U is an eigenvector of A. From this and (2.6) we obtain $\beta g(\phi AX, Y) = 0$. If $\beta \neq 0$ at some point x of M, then $\phi AX = 0$ and hence (2.5) implies that $cg(X, \phi Y) = 0$. From this we get $c = 0$.

This is a contradiction. Thus we have $\beta = 0$ and hence β is constant. On the other hand, from (2.5) it follows that

$$\phi A^2X - \alpha\phi AX - c\phi X = 0. \tag{2.8}$$

Using (1.2) and (2.8) we obtain

$$A^2X - \alpha AX - cX + cf(X)U = 0. \tag{2.9}$$

Furthermore, we may assume that $Ae_i = \lambda_i e_i$, $i = 1, \cdots, 2n - 2$ and $Ae_{2n-1} = \alpha e_{2n-1}$, $e_{2n-1} = U$. Then (2.9) implies that at most two λ_i are distinct, which will be denoted by λ and μ. Then $\lambda + \mu = \alpha$ and $\lambda\mu = -c$. Therefore λ and μ are constant. This proves our assertion.

If M is totally η-umbilical, that is, if the second fundamental form A of M is of the form $AX = aX + bf(X)U$ for some scalar functions a and b on M, then we have $\phi A = A\phi$. Therefore Lemma 2.4 implies that

Lemma 2.5. *Let M be a totally η-umbilical real hypersurface of a complex space form $\overline{M}^n(c)$ $(c \neq 0)$. Then M has two constant principal curvatures.*

Proof. From the assumption on the second fundamental form, we see that M has two principal curvatures. From Lemma 2.4 these two principal curvatures are constant.

In the sequel, we study a real hypersurface M of a complex space form $\overline{M}^n(c)$ under the assumption that $A\phi + \phi A = k\phi$ for some constant $k \neq 0$. Then the vector U is an eigenvector of A. Therefore (2.5) implies

$$2cg(X, \phi Y) = \alpha kg(\phi X, Y) - 2g(A\phi AX, Y). \tag{2.10}$$

On the other hand, in the proof of Lemma 2.2 we have already shown that $\beta g((\phi A + A\phi)X, Y) = 0$ where $\beta = U\alpha$. Thus $\beta kg(\phi X, Y) = 0$. Since $k \neq 0$, we obtain $\beta = 0$ and hence α is constant. From the assumption and (2.10) we also have

$$2\phi A^2X - 2k\phi AX + \alpha k\phi X + 2c\phi X = 0,$$

which implies that

$$2A^2X - 2kAX + (\alpha k + 2c)X - 2(\alpha^2 + c)f(X)U + k\alpha f(X)U = 0. \tag{2.11}$$

From this the eigenvalues of A, which will be denoted by λ_i $(i = 1, \cdots, 2n - 2)$, α satisfies the following quadratic equation

$$2t^2 - 2kt + (\alpha k + 2c) = 0.$$

Therefore at most two λ_i are distinct, and hence M has at most three principal curvatures λ, μ and α. Since α, k and c are constant, λ and μ are also constant. If $AX = \lambda X$, then $A\phi X = (k - \lambda)\phi X = \mu\phi X$. Therefore the multiplicities of λ and μ are equal to $n - 1$. If $\lambda = \mu$, then $A\phi = \phi A$, and therefore $2A\phi = 2\phi A = k\phi$ which implies that $-2AX + 2\alpha f(X)U = -kX + kf(X)U$, that is, we have $AX = \frac{1}{2}kX + \frac{1}{2}(k - 2\alpha)f(X)U$. Consequently M is totally η-umbilical.

Lemma 2.6. *Let M be a real hypersurface of a complex space form $\overline{M}^n(c)$. If $\phi A + A\phi = k\phi$ for some constant $k \neq 0$, then M has at most three constant principal curvatures λ, μ and α. If $\lambda \neq \mu$, then the multiplicities of λ and μ are equal.*

3. Examples

In this section we give examples of pseudo-Einstein real hypersurfaces in a complex projective space $P^n(C)$ with constant holomorphic sectional curvature 4. First of all, we describe real hypersurfaces in $P^n(C)$ with two or three constant principal curvatures (see Takagi [13], [14]).

Let C^{n+1} be the space of $(n+1)$-tuples of complex numbers $(z_1, \cdots, z_{n+1})$. Put $S^{2n+1} = \{(z_1, \cdots, z_{n+1}) \in C^{n+1}\colon \sum_{j=1}^{n+1}|z_j|^2 = 1\}$. For a positive number r we denote by $M_0'(2n, r)$ a hypersurface of S^{2n+1} defined by

$$\sum_{j=1}^{n} |z_j|^2 = r|z_{n+1}|^2, \quad \sum_{j=1}^{n+1} |z_j|^2 = 1. \tag{3.1}$$

For an integer m $(2 \leqslant m \leqslant n-1)$ and a positive number s, a hypersurface $M'(2n, m, s)$ of S^{2n+1} is defined by

$$\sum_{j=1}^{m} |z_j|^2 = s \sum_{j=m+1}^{n+1} |z_j|^2, \quad \sum_{j=1}^{n+1} |z_j|^2 = 1. \tag{3.2}$$

For a number t $(0 < t < 1)$ we denote by $M'(2n, t)$ a hypersurface of S^{2n+1} defined by

$$\left|\sum_{j=1}^{n+1} z_j^2\right|^2 = t, \quad \sum_{j=1}^{n+1} |z_j|^2 = 1. \tag{3.3}$$

Let π be the natural projection of S^{2n+1} onto $P^n(C)$. Then $M_0(2n-1, r) = \pi(M_0'(2n, r))$ is a connected compact real hypersurface of $P^n(C)$ with two constant principal curvatures. We call $M_0(2n-1, r)$ a *geodesic hypersphere* of $P^n(C)$. Moreover $M(2n-1, m, s) = \pi(M'(2n, m, s))$ $(n \geqslant 3)$ and $M(2n-1, t) = \pi(M'(2n, t))$ $(n \geqslant 2)$ are connected compact real hypersurfaces in $P^n(C)$ with three constant principal curvatures. Then Takagi [13], [14] proved the following theorems.

Theorem A (*Takagi* [13]). *If M is a connected complete real hypersurface in $P^n(C)$ $(n \geqslant 2)$ with two constant principal curvatures, then M is a geodesic hypersphere.*

Theorem B (*Takagi* [14]). *If M is a connected complete real hypersurface in*

$P^n(C)$ $(n \geqslant 3)$ with three constant principal curvatures, then M is congruent to some $M(2n-1, m, s)$ or $M(2n-1, t)$.

Real hypersurfaces $M_0(2n-1, r)$, $M(2n-1, m, s)$ and $M(2n-1, t)$ are said to be of types A_1, A_2 and B respectively in Takagi [13]. We denote by $\xi_1, \cdots, \xi_j$ the principal curvatures of M in $P^n(C)$, and by $m(\xi_1), \cdots, m(\xi_j)$ their multiplicities. Then Takagi [13] gave the following table:

TABLE

	dim M	j	ξ_i	$m(\xi_i)$
A_1	$2n-1$ $(n \geqslant 2)$	2	$\xi_1 = \cot\theta$ $\xi_2 = 2\cot 2\theta$	$m(\xi_1) = 2(n-1)$ $m(\xi_2) = 1$
A_2	$2(p+q)-3$ $(p \geqslant q \geqslant 2)$	3	$\xi_1 = \cot\theta$ $\xi_2 = -\tan\theta$ $\xi_3 = 2\cot 2\theta$	$m(\xi_1) = 2(p-1)$ $m(\xi_2) = 2(q-1)$ $m(\xi_3) = 1$
B	$2p-3$ $(p \geqslant 3)$	$3\xi_2 = -\tan(\theta - /4)$	$\xi_1 = \cot(\theta - /4)$ $m(\xi_2) = p-2$ $\xi_3 = 2\cot 2\theta$	$m(\xi_1) = p-2$ $m(\xi_3) = 1$

Here we notice that the vector U is an eigenvector of A with respect to ξ_3. Any geodesic hypersphere $M_0(2n-1, r)$ is pseudo-Einsteinian. In the next place we show that $M(2n-1, m, (m-1)/(n-m))$ and $M(2n-1, 1/(n-1))$ are pseudo-Einsteinian. From (1.8) and Table we see that $M(2n-1, m, s)$ is pseudo-Einsteinian if and only if

$$H\cot\theta - \cot^2\theta = -H\tan\theta - \tan^2\theta. \tag{3.4}$$

Since $H = p\cot\theta - (2n-2-p)\tan\theta + 2\cot 2\theta$, where p denotes the multiplicity of $\cot\theta$, (3.4) implies that $\sin^2\theta = p/(2n-2)$. On the other hand, a hypersurface $M'(2n, m, s)$ of S^{2n+1} has two principal curvatures $\cot\theta$ and $-\tan\theta$ with multiplicities $p+1$ and $2n-1-p$ respectively (see Takagi [14, p. 515]). Thus $p = 2m-2$ and

$$M' = S^{2m-1}\left(\frac{n-1}{m-1}\right) \times S^{2(n-m)+1}\left(\frac{n-1}{n-m}\right),$$

where $(n-1)/(m-1) = \xi_1^2 + 1$ and $(n-1)/(n-m) = \xi_2^2 + 1$. From this and (3.2) we obtain $s = \dfrac{m-1}{n-m}$. Thus $M(2n-1, m, \dfrac{m-1}{n-m})$ is pseudo-Einsteinian, and the Ricci tensor S of $M(2n-1, m, \dfrac{m-1}{n-m})$ is of the form $S(X, Y) = ag(X, Y) + bf(X)f(Y)$ for some constants a and b. Next we determine a and b. The constant a is given by $a = (2n+1) + H\cot\theta - \cot^2\theta$ by (1.8). Since $\sin^2\theta = p/(2n-2)$, $H\cot\theta - \cot^2\theta = -1$ and hence

$a = 2n$. Moreover, from (1.8) it follows that b is given by $b = -2 + 2H \cot 2\theta - 4 \cot^2 2\theta$. By this we obtain $b = -2$. Thus the Ricci tensor S of $M(2n - 1, m, (m - 1)/(n - m))$ is of the form $S(X, Y) = 2ng(X, Y) - 2f(X)f(Y)$.

Furthermore, from (1.8) and Table we see that $M(2n - 1, t)$ is pseudo-Einsteinian if and only if

$$(3.5) \quad H \cot\left(\theta - \frac{\pi}{4}\right) - \cot^2\left(\theta - \frac{\pi}{4}\right) = -H \tan\left(\theta - \frac{\pi}{4}\right) - \tan^2\left(\theta - \frac{\pi}{4}\right),$$

which together with

$$H = (n - 1)\left[\cot\left(\theta - \frac{\pi}{4}\right) - \tan\left(\theta - \frac{\pi}{4}\right)\right] + 2 \cot 2\theta$$

gives that $\sin^2 2\theta = 1/(n - 1)$. On the other hand, from the results of Nomizu [9, Theorem 1, p. 1186] and Takagi [14, p. 515] it follows that a hypersurface $M'(2n, t)$ of S^{2n+1} has four constant principal curvatures $\cot(\theta - \pi/4)$, $\cot \theta$, $\cot(\theta + \pi/4) = -\tan(\theta - \pi/4)$ and $\cot(\theta + \pi/2)$ with multiplicities $n - 1$, 1, $n - 1$ and 1 respectively, and that t is given by $t = \sin^2 2\theta$ (see also Takagi [15]). Consequently we obtain $t = 1/(n - 1)$. Thus $M(2n - 1, 1/(n - 1))$ is pseudo-Einsteinian. Moreover we have a $a = 2n$ and $b = 2 - 4n$, and hence the Ricci tensor S of $M(2n - 1, 1/(n - 1))$ is given by $S(X, Y) = 2ng(X, Y) + (2 - 4n)f(X)f(Y)$.

Next, in consequence of (3.4), $M(2n - 1, m, (m - 1)/(n - m))$ is minimal if and only if $\sin^2 \theta = \cos^2 \theta$, $\sin^2 \theta = \frac{1}{2}$. Since $\sin^2 \theta = (m - 1)/(n - 1)$, we have $m = (n + 1)/2$. Thus $M(2n - 1, (n + 1)/2, 1)$ is a pseudo-Einstein real minimal hypersurface in $P^n(C)$. In this case, n must be odd.

If we suppose that $M(2n - 1, 1/(n - 1))$ is minimal, (3.5) implies that $\cot^2(\theta - \pi/4) = \tan^2(\theta - \pi/4)$. From this we have $\sin 2\theta = 0$. This is a contradiction to the fact that $\sin^2 2\theta = 1/(n - 1)$. Therefore $M(2n - 1, 1(n - 1))$ is not minimal.

A geodesic hypersphere $M_0(2n - 1, r)$ is minimal if and only if $H = (2n - 2) \cot \theta + 2 \cot 2\theta = 0$, i.e., $\cos^2 \theta = 1/2n$. Then we have (see Takagi [13, p. 51])

$$M_0' = S^{2n-1}\left(\frac{2n}{2n - 1}\right) \times S^1(2n),$$

where $2n/(2n - 1) = \xi_1^2 + 1$ and $2n = 1/\xi_1^2 + 1$. Thus from (3.1) we have $r = 2n - 1$. Therefore a geodesic hypersphere $M_0(2n - 1, 2n - 1)$ is minimal. For a constant a of $M_0(2n - 1, r)$ we obtain $a = 2n + (2n - 2) \cot^2 \theta$ by using (1.8). Thus we have $a > 2n$, and also $b = -2n$.

From these considerations we see that $M_0(2n-1, r)$, $M(2n-1, m, (m-1)/(n-m))$ and $M(2n-1, 1/(n-1))$ are not Einsteinian.

Now we summarize some results from the previous sections. First of all, we notice the following fact. Let λ, μ and α be principal curvatures of $M(2n-1, m, s)$ or $M(2n-1, t)$, and let $T_\lambda = \{X : AX = \lambda X\}$, $T_\mu = \{X : AX = \mu X\}$. Then $\phi T_\lambda \subset T_\lambda$ and $\phi T_\mu \subset T_\mu$ on $M(2n-1, m, s)$, and $\phi T_\lambda \subset T_\mu$ and $\phi T_\mu \subset T_\lambda$ on $M(2n-1, t)$ (see Takagi [14, Lemma 3.4, p. 513]). If $A\phi = \phi A$, then $\phi T_\lambda \subset T_\lambda$ and $\phi T_\mu \subset T_\mu$. Thus by Lemma 2.4 and Theorems A, B we obtain

Theorem 3.1 (*Okumura* [10]). *Let M be a connected complete real hypersurface in $P^n(C)$ $(n \geqslant 3)$. If $A\phi = \phi A$, then M is congruent to some $M_0(2n-1, r)$ or $M(2n-1, m, s)$.*

From Lemma 2.5 and Theorem A we have

Theorem 3.2 (*Takagi* [13]). *If M is a connected complete totally η-umbilical real hypersurface in $P^n(C)$ $(n \geqslant 2)$, then M is a geodesic hypersphere $M_0(2n-1, r)$.*

Furthermore, by Lemma 2.6 and Theorems A, B we obtain

Theorem 3.3. *Let M be a connected complete real hypersurface in $P^n(C)$ $(n \geqslant 3)$. If $\phi A + A\phi = k\phi$ for some constant $k \neq 0$, then M is congruent to some $M_0(2n-1, r)$ or $M(2n-1, t)$.*

Remark. In Theorem 3.3 if $k = 0$, then by Lemma 2.1 there is no real hypersurface in $P^n(C)$.

4. Pseudo-Einstein real hypersurface in $P^n(C)$

Let M be a connected real hypersurface of a complex space form $\overline{M}^n(c)$ $(n \geqslant 3)$. We can choose a local field of orthonormal frames $e_1, \cdots, e_{2n-1}$, e_{2n} in $\overline{M}^n(c)$ in such a way that, restricted to M, $e_1, \cdots, e_{2n-1}$ are tangent to M, and $e_{2n-1} = U$, $e_{2n} = Je_{2n-1} = C$. Then for a suitable choice of $e_1, \cdots, e_{2n-2}$, the second fundamental form A is represented by a matrix form

$$(4.1) \qquad A = \left(\begin{array}{ccc|c} \lambda_1 & & 0 & h_1 \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ 0 & & \lambda_{2n-2} & h_{2n-2} \\ \hline h_1 & \cdots\ \cdots & h_{2n-2} & \alpha \end{array}\right),$$

where we have put $h_i = g(Ae_i, U)$, $i = 1, \cdots, 2n-2$, and $\alpha = g(AU, U)$.

In the following we assume that M is a pseudo-Einstein real hypersurface in $\overline{M}^n(c)$. Then (1.8) reduces to

$$\begin{aligned} &ag(X, Y) + bf(X)f(Y) \\ &\quad = (2n+1)cg(X, Y) - 3cf(X)f(Y) + Hg(AX, Y) - g(AX, AY) \end{aligned} \tag{4.2}$$

for any vector fields X and Y tangent to M, where a and b are constants. From (4.1) and (4.2) we have the following equations:

$$g(Ae_i, Ae_j) = 0 \quad \text{for } i \neq j, \quad i, j = 1, \cdots, 2n-2,$$

$$Hg(Ae_i, U) - g(Ae_i, AU) = 0 \quad \text{for } i = 1, \cdots, 2n-2.$$

By these equations we obtain

$$h_i h_j = 0, \; i \neq j, \; i, j = 1, \cdots, 2n-2, \tag{4.3}$$

$$h_i(H - \lambda_i - \alpha) = 0, \; i = 1, \cdots, 2n-2. \tag{4.4}$$

Equations (4.3) show that at most one h_i does not vanish. Thus we can assume $h_i = 0$ for $i = 2,, \cdots, 2n-2$. Then (4.4) implies

Lemma 4.1. *Let M be a connected real hypersurface of a complex space form $\overline{M}^n(c)$. If M is pseudo-Einsteinian, then $H = \lambda_1 + \alpha$ or $h_1 = 0$.*

On the other hand, by (4.2) we obtain the following equations:

$$a = (2n+1)c + H\lambda_i - \lambda_i^2, \quad i = , \cdots, 2n-2, \tag{4.5}$$

$$a = (2n+1)c + H\lambda_1 - \lambda_1^2 - h_1^2, \tag{4.6}$$

$$a = (2n-2)c - b + H\alpha - \alpha^2 - h_1^2. \tag{4.7}$$

In the sequel, we take $P^n(C)$ as an ambient manifold. Then we can have

Lemma 4.2. *Let M be a connected pseudo-Einstein real hypersurface in $P^n(C)$. Then $h_1 = 0$.*

Proof. Suppose that $H = \lambda_1 + \alpha$. Then (4.6) and (4.7) imply $b = -3$. Therefore (4.2) can be rewritten as

$$ag(X, Y) = (2n+1)g(X, Y) + Hg(AX, Y) - g(AX, AY). \tag{4.8}$$

Here we take a new local field of orthonormal frames $e_1, \cdots, e_{2n-1}$ of M for which the second fundamental form A can be represented by a diagonal matrix form, i.e., $Ae_i = \beta_i e_i$ $(i = 1, \cdots, 2n-1)$. Then (4.8) implies

$$\beta_i^2 - H\beta_i + a - (2n+1) = 0. \tag{4.9}$$

Therefore each principal curvatures β_i satisfies the quadratic equation

$$t^2 - Ht + a - (2n+1) = 0. \tag{4.10}$$

Thus at most two principal curvatures can be distinct at each point. Let us denote them by λ and μ with $\lambda \geqslant \mu$. Since M is not totally umbilical, we may suppose $\lambda \neq \mu$ at some point. Then from (4.10) we see

$$H = \lambda + \mu, \qquad \lambda\mu = a - (2n + 1). \tag{4.11}$$

Let p be the multiplicity of λ. Then we have $H = p\lambda + (2n - 1 - p)\mu$. Combining this with (4.11) gives

$$(p - 1)\lambda + (2n - 2 - p)\mu = 0. \tag{4.12}$$

Suppose $a > (2n + 1)$. Then the second equation of (4.11) shows that λ and μ have the same sign at some point. Therefore (4.12) implies that $p = 1$ and $n = 3/2$, which is a contradiction. If $a < (2n + 1)$ and $\lambda = \mu$ at some point, then we have $(2n - 2)\lambda^2 = a - (2n + 1) < 0$ by (4.10). This is also a contradiction. Hence M has exactly two distinct principal curvatures $\lambda > \mu$ at each point. Then we see $1 < p < 2n - 2$ from (4.12), and

$$\lambda_2 = -\frac{(2n - 2 - p)(a - 2n - 1)}{(p - 1)}, \qquad \mu^2 = -\frac{(p - 1)(a - 2n - 1)}{(2n - 2 - p)},$$

from (4.11) and (4.12). Therefore the two principal curvatures λ and μ are constant. Thus applying Lemma 3.3 of Takagi [13] we must have $p = 1$ or $p = 2n - 2$. This is also a contradiction. Next we assume that $a = (2n + 1)$. Then the product of two principal curvatures is zero, and (4.10) shows that $\lambda^2 - H\lambda = 0$, from which $(p - 1)\lambda^2 = 0$. This gives $t(x) \leqslant 1$ at each point. This contradicts Lemma 2.3.

From Lemma 4.2 we see that the vector U is an eigenvector of A, i.e., $AU = \alpha U$. Therefore from (4.2) the principal curvatures λ_i satisfy

$$\lambda_i^2 - H\lambda_i + a - (2n + 1) = 0, \quad i = 1, \cdots, 2n - 2. \tag{4.13}$$

Thus each λ_i satisfies the quadratic equation (4.10). Therefore at most two λ_i can be distinct. Let us denote them by λ and μ with $\lambda \geqslant \mu$. Consequently M has at most three principal curvatures λ, μ and α.

Next we prove that λ, μ and α are constant. From Lemma 2.2 we have already seen that α is constant.

Proposition 4.1. *Let M be a connected pseudo-Einstein real hypersurface in $P^n(C)$ $(n \geqslant 3)$. Then M has at most three constant principal curvatures.*

Proof. First of all, (4.2) gives

$$a = (2n - 2) - b + H\alpha - \alpha^2. \tag{4.14}$$

If $\alpha \neq 0$, then H is constant by (1.14), and (4.13) implies that λ and μ are constant. Next we suppose that $\alpha = 0$. Then we have $H = p\lambda + (2n - 2 - p)\mu$, where p denotes the multiplicity of λ.

Let $a > (2n + 1)$. If $\lambda \neq \mu$ at some point x of M, then from $H = \lambda + \mu$, we get $(p - 1)\lambda + (2n - 3 - p)\mu = 0$. Since $\lambda\mu = a - (2n + 1) > 0$, we conclude that $p = 1$ and $2n - 3 = p$ and hence $n = 2$. This is a contradiction to the assumption $n \geqslant 3$. Thus we must have $\lambda = \mu$ at each point. Then (4.13) implies that $(2n - 3)\lambda^2 = a - (2n + 1)$ showing that λ is a constant.

Suppose $a < (2n + 1)$. If $\lambda = \mu$ at some point, then we have $(2n - 3)\lambda^2 = a - (2n + 1) < 0$ by (4.13). This is a contradiction. Therefore $\lambda \neq \mu$ at each point, and using (4.10) we obtain $H = p\lambda + (2n - 2 - p)\mu = \lambda + \mu$ and $\lambda\mu = a - (2n + 1)$ giving

$$\lambda^2 = -\frac{(2n - 3 - p)(a - 2n - 1)}{(p - 1)}, \qquad \mu^2 = -\frac{(p - 1)(a - 2n - 1)}{(2n - 3 - p)}.$$

Consequently the principal curvatures λ and μ are constant.

Next we assume that $a = (2n + 1)$. In this case the product of two principal curvatures is zero. Thus if $\lambda \neq 0$, then $H = P\lambda$, and (4.13) implies $(p - 1)\lambda^2 = 0$. Hence $p = 1$, and $t(x) \leqslant 1$ at each point. This is a contradiction by Lemma 2.3. Consequently M has at most three constant principal curvatures.

From Theorems A, B of Takagi [13], [14] and Proposition 4.1 we have

Theorem 4.1. *If M is a connected complete pseudo-Einstein real hypersurface in $P^n(C)$ $(n \geqslant 3)$, then M is congruent to some geodesic hypersphere $M_0(2n - 1, r)$ or $M(2n - 1, m, (m - 1)/(n - m))$ or $M(2n - 1, 1/(n - 1))$.*

From Theorem 4.1 and the argument in §3 we have

Theorem 4.2. *If M is a connected complete pseudo-Einstein real minimal hypersurface in $P^n(C)$ $(n \geqslant 3)$, then M is congruent to $M_0(2n - 1, 2n - 1)$ or $M(2n - 1, (n + 1)/2, 1)$. In the later case, n is odd.*

If a real hypersurface M of $P^n(C)$ is Einsteinian, then it is obviously pseudo-Einsteinian and has at most three constant principal curvatures. From this and Theorem 4.1, the argument in §3 gives

Theorem 4.3. *Let M be a connected complete real hypersurface in $P^n(C)$ $(n \geqslant 3)$. Then M is not Einstein.*

Corollary 4.1. *Let M be a connected complete pseudo-Einstein real hypersurface in $P^n(C)$ $(n \geqslant 3)$. Then we have $a \geqslant 2n$. If $a \neq 2n$, then M is congruent to some geodesic hypersphere $M_0(2n - 1, r)$. If $a = 2n$ and $b = -2$, then M is congruent to some $M(2n - 1, m, (m - 1)/(n - m))$. If $a = 2n$ and $b = 2 - 4n$, then M is congruent to $M(2n - 1, 1/(n - 1))$.*

5. Pseudo-Einstein real hypersurfaces in C^n

In this section we study a connected complete pseudo-Einstein real hypersurface M in a complex number space C^n $(n \geqslant 3)$. First of all, we give some examples of connected complete pseudo-Einstein real hypersurfaces in C^n $(= R^{2n})$.

(1) Hyperplanes: $M = R^{2n-1}, A = 0$.

(2) Spheres: $M = S^{2n-1}(c) = \{(z_1, \cdots, z_n) \in C^n: \Sigma_{j=1}^n |z_j|^2 = 1/c\}$, $A = \sqrt{c}\, I$.

(3) Cylinders over $(2n-2)$-spheres: $M = S^{2n-2}(c) \times R$, $A = \sqrt{c}\, I_{2n-2} \oplus 0$.

(4) Cylinders over complete plane curves: $M = \gamma \times R^{2n-2}$, where γ is a curve:$-\infty < s < \infty \to \gamma(s)$ in a plane R^2 perpendicular to R^{2n-2}, $A = \lambda I_1 \oplus 0$ for some scalar function λ on γ.

If M is an Einstein real hypersurface in C^n, then M is a sphere, a hyperplane, or a cylinder over a complete plane curve (cf. Ryan [11, Theorem 3.3, p. 376]).

From Lemma 4.1 we can consider two cases: (I) $H = \lambda_1 + \alpha$, (II) $h_1 = 0$, and hence U is an eigenvector of A.

If $H = \lambda_1 + \alpha$, then (4.6) and (4.7) imply $b = 0$, and hence M is an Einstein manifold. Thus we have

Lemma 5.1. *Let M be a connected pseudo-Einstein real hypersurface of C^n. If $H = \lambda_1 + \alpha$, then M is an Einstein manifold.*

Next we assume that $h_1 = 0$. Then we see that $Ae_i = \lambda_i e_i$ ($i = 1, \cdots, 2n-2$), and $AU = \alpha U$. Moreover (4.5), (4.6) and (4.7) reduce to

$$a = H\lambda_i - \lambda_i^2, \quad i = 1, \cdots, 2n-2, \tag{5.1}$$

$$a + b = H\alpha - \alpha^2. \tag{5.2}$$

Thus each λ_i satisfies the quadratic equation

$$t^2 - Ht + a = 0,$$

and hence we can have at most two distinct λ_i, which are denoted by λ and μ with $\lambda \geqslant \mu$. Consequently M has at most three principal curvatures λ, μ and α. Since U is an eigenvector of A, by the similar method like that in the proof of Lemma 2.2, we have $\beta g(\phi AX + A\phi X, Y) = 0$. Therefore from Lemma 2.1 we have

Lemma 5.2. *Let M be a connected pseudo-Einstein real hypersurface of C^n. If $h_1 = 0$, then $\phi A + A\phi = 0$ or $\beta = 0$. Moreover if $\phi A + A\phi = 0$, then $t(x) \leqslant 1$ at any point x of M.*

If $t(x) \leqslant 1$ at any point x of M, then M is locally isometric to R^{2n-1}. Furthermore, if M is complete, by a theorem of Hartman-Nirenberg [4], M is a cylinder over a complete plane curve (for the proof of the theorem of Hartman-Nirenberg see also Nomizu [8]). If $t(x) = 0$ for all x, then M is totally geodesic and is a hyperplane.

In the following we assume that $\beta = 0$, that is, α is constant. Here we need the following theorem due to Cartan [1] (see also Gray [3]).

Theorem C (*Cartan* [1]). *Let M be a hypersurface in C^n whose principal curvatures are constant. Then at most two of them are distinct.*

Suppose $\alpha \neq 0$. Then (5.2) shows that H is also constant, and hence λ and μ are constant by (5.1). Therefore, from Theorem C, M has at most two distinct principal curvatures. If $\alpha = \lambda$ or $\alpha = \mu$, then (5.1) and (5.2) imply that $b = 0$. Thus M is an Einstein manifold. Next we assume that $\lambda = \mu$ and $\lambda \neq \alpha$. Then the equation (1.5) of Gauss implies

$$g(X, R(X, Y)Y) = \lambda\alpha \quad \text{for } X \in T_\lambda,\ Y \in T_\alpha, \tag{5.3}$$

where we have put $T_\lambda = \{X\colon AX = \lambda X\}$ and $T_\alpha = \{X\colon AX = \alpha X\}$. Since λ and α are constant, both distributions T_λ and T_α are parallel (see Ryan [11, pp. 372–374]). Therefore $g(X, R(X, Y)Y) = 0$ for $X \in T_\lambda$, $Y \in T_\alpha$, and hence $\lambda\alpha = 0$. By the assumption, $\alpha \neq 0$ and hence $\lambda = 0$. Consequently $t(x) = 1$ on M.

Next suppose $\alpha = 0$. Then (5.2) implies

$$a + b = 0. \tag{5.4}$$

Let $a > 0$. If $\lambda \neq \mu$ at some point x of M, then $\lambda\mu = a > 0$ and λ, μ have the same sign. On the other hand, $\lambda + \mu = H = p\lambda + q\mu$, where p and q denote the multiplicities of λ and μ respectively, from which $p = 1$ and $q = 1$. Since this contradicts the assumption $n \geqslant 3$, we have $\lambda = \mu$ at any point of M. Hence $a = (2n - 3)\lambda^2$, and λ is constant with multiplicity $p = 2n - 2$.

Let $a < 0$. Then $\lambda\mu < 0$. If $\lambda = \mu$ at some point x of M, then we get a contradiction. Thus $\lambda \neq \mu$ at any point on M, and $H = \lambda + \mu = p\lambda + q\mu$, $\lambda\mu = a$, from which it follows that

$$\lambda^2 = \frac{-a(2n - 2 - p)}{p}, \quad \mu^2 = \frac{-ap}{(2n - 2 - p)}.$$

Therefore λ, μ and α are constant. This contradicts to Theorem C.

Suppose $a = 0$. Then (5.1) implies $(p - 1)\lambda^2 = 0$. If $\lambda \neq 0$, then $p = 1$. Consequently $t(x) \leqslant 1$ on M. On the other hand, if $a = 0$, then by (5.4) we have $b = 0$, and M is Einsteinian.

When $a > 0$, M has two constant principal curvatures λ and $\alpha = 0$ with multiplicities $2n - 2$ and 1 respectively. Then, if M is complete, M is congruent to a cylinder over $(2n - 2)$-sphere $S^{2n-2}(c) \times R$. Indeed, the Riemannian curvature tensor R of M satisfies $R(X, Y) \cdot R = 0$, and hence a theorem of Nomizu [8] implies our assertion. From these we get

Theorem 5.1. *Let M be a connected complete pseudo-Einstein real hypersurface in C^n $(n \geqslant 3)$. Then M is congruent to a hyperplane R^{2n-1}, a sphere $s^{2n-1}(c)$, a cylinder over a $(2n - 2)$-sphere $S^{2n-2}(c) \times R$, or a cylinder over a complete plane curve $\gamma \times R^{2n-2}$.*

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