J. DIFFERENTIAL GEOMETRY
14(1979), 317–337

DEFORMATION THEORY FOR HOLOMORPHIC FOLIATIONS

T. DUCHAMP & M. KALKA

Introduction

In this paper we consider deformations of holomorphic foliations on compact manifolds. By a holomorphic foliation we mean a foliation given by local submersions f_α: $U_\alpha \to R^{2q}$ which patch together via maps $\varphi_{\alpha\beta}$: $R^{2q} \to R^{2q}$, which are local biholomorphisms when R^{2q} is identified with $\mathbf{C}^q$.

For $\mathfrak{F}$ a holomorphic foliation on a manifold M, we show that the infinitesimal deformations of $\mathfrak{F}$ correspond to elements of $H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}})$, where $\underset{\sim}{\theta}_{\mathfrak{F}}$ is the sheaf of germs of holomorphic vector fields on the normal bundle of $\mathfrak{F}$ which are constant on the leaves of $\mathfrak{F}$. For example, if $\mathfrak{F}$ is given by the fibers of a submersion onto a complex manifold, then $\underset{\sim}{\theta}_{\mathfrak{F}}$ is the pull-back of the sheaf $\underset{\sim}{\theta}_N$ of germs of holomorphic vector fields on the image. By constructing explicitly a resolution of $\underset{\sim}{\theta}_{\mathfrak{F}}$ by an elliptic complex $(E_Q^{*\bullet}, d_Q)$ we show that $H^\bullet(M, \underset{\sim}{\theta}_{\mathfrak{F}})$ is finite dimensional.

Resolutions of the sheaf of sections of the normal bundle of a C^∞-foliation which are constant on leaves have appeared in the works of Hamilton [4], Heitsch [5], Kamber-Tondeur [6], Mostow [9] and Vaisman [15]. Also in the case where M is a complex manifold and the submersions f_α are holomorphic, Heitsch has constructed a resolution of the sheaf $\underset{\sim}{\theta}_{\mathfrak{F}}$ and shown that its cohomology groups are finite dimensional. Our resolution is different from his and applies to the case where M is only a smooth manifold. Of course the general theory of pseudogroup structures on manifolds developed by Spencer [13] applies to the case of holomorphic foliations on smooth manifolds. However, the relevant pseudogroup is neither elliptic nor complex; hence the Spencer complex associated to such a foliation does not directly lead to finite dimensionality results and the theory of elliptic complexes does not apply to it.

Having constructed a resolution of $\underset{\sim}{\theta}_{\mathfrak{F}}$, we then show how to extend Kuranishi's theorem on the existence of a locally complete finite dimensional holomorphic family for complex structures close to a given complex structure,

Communicated by L. Nirenberg, August 25, 1977, and, in revised form, June 2, 1978.

to holomorphic foliations. In order to do this, it is necessary to define a bracket operation $[\ ,\ ]_Q$: $\underset{\sim}{E}_Q^{*r} \times \underset{\sim}{E}_Q^{*s} \to \underset{\sim}{E}_Q^{*r+s}$ with certain nice properties (2.11–2.14). In general we cannot do this. However, if we assume that there is a C^∞-foliation $\mathfrak{F}^\perp$ transverse to the foliation $\mathfrak{F}$, then such a bracket can be defined. Distributions near the tangent bundle of $\mathfrak{F}$ are given by elements of $\Gamma(E_Q^{*1})$, and the integrability condition in the complex Frobenius theorem takes the form $d_Q - [\ ,\ ]_Q = 0$. The operator d_Q is just the sum of the Dolbeaut operator in the holomorphic directions normal to $\mathfrak{F}$ and the de Rham operator in directions parallel to $\mathfrak{F}$. Note that the transverse foliation $\mathfrak{F}^\perp$ allows us to consider all bundles as sub-bundles of the complexified tangent bundle of M or its dual. The proof proceeds exactly as in Kuranishi [8], only the bundles and the operators have been changed. In fact Kuranishi's theorem is a special case of our theorem, where the leaves of the foliation are the points of M.

We then consider the problem of computing $H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}})$. In particular, we consider the case where $\mathfrak{F}$ is given by a fibration $M \xrightarrow{f} N$ with N a complex manifold and with fiber S. We show that if $H^1_{DR}(S) = 0$ and $H^1(N, \underset{\sim}{\theta}_N) = 0$, then $H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}}) = 0$, where $\underset{\sim}{\theta}_N$ is the sheaf of germs of holomorphic vector fields on N. If the structure group of the fibration is discrete, this implies that there are no small deformations of $\mathfrak{F}$, up to equivalence. This should be compared with Hamilton's result [4] that if $\mathfrak{F}$ is a C^∞-Hausdorff foliation with $H^1_{DR}(L) = 0$, where L is the generic leaf of $\mathfrak{F}$, then $\mathfrak{F}$ is structurally stable.

The paper is organized as follows: In §1 we describe the relevant elliptic complexes and define the operator d_Q. In §2 we define the bracket operator $[\ ,\ ]_Q$ and derive the partial differential equation which is the integrability condition in the complex Frobenius theorem. In §3 we solve this equation and prove Kuranishi's theorem. In §4 we compute $H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}})$ in certain cases. The techniques of §4 are similar to those of Mostow [9]. The main results of this paper are Theorems 1.27, 2.4 and 3.1.

We will use the following notational conventions: Latin subscripts (or superscripts) will run from 1 to p, whereas Greek subscripts (or superscripts) will run from 1 to q where $n = p + 2q$. Also if B is a vector bundle over M, then we will denote by $\underset{\sim}{B}$ the sheaf of germs of sections of B. If $\underset{\sim}{S}$ is a sheaf, the space of global sections of $\underset{\sim}{S}$ will be denoted by $\Gamma(\underset{\sim}{S})$. If B is a bundle, the space of its global sections will be denoted by $\Gamma(B)$. We will use the Einstein summation conventions.

1. Elliptic complexes associated to a holomorphic foliaton

Let M be an n-dimensional C^∞-manifold. We investigate here holomorphic foliations on M close to a fixed holomorphic foliation. We recall that a (real) codimension-$2q$ holomorphic foliation $\mathcal{F}$ is given by an open cover $\{U_\alpha\}_{\alpha\in A}$ of M, a collection of submersions $f_\alpha\colon U_\alpha \to \mathbf{C}^q$, and associated maps $\varphi^x_{\alpha\beta}$ for each $x \in U_\alpha \cap U_\beta$, which are local biholomorphic maps and satisfy $f_\beta(y) = \varphi^x_{\beta\alpha} \circ f_\alpha(y)$ for y near x. For the foliation to be global it is necessary that the collection $\{\varphi^x_{\alpha\beta}\}_{\alpha,\beta\in A}$ satisfy the cocycle condition $\varphi^x_{\alpha\gamma} = \varphi^x_{\alpha\beta} \circ \varphi^x_{\beta\gamma}$ for al α, β, γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$ and for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$. A collection $\{U_\alpha, f_\alpha, \{\varphi^x_{\alpha\beta}\}_{x\in U_\alpha\cap U_\beta}\}$ which is maximal with respect to the above properties is called a Haefliger cocycle with coefficients in $\Gamma^q_{\mathbf{C}}$, the pseudo-group of local biholomorphisms of $\mathbf{C}^q$. We set $\mathcal{H}_{\mathcal{F}} = \{f_\alpha\colon U_\alpha \to \mathbf{C}^q, \varphi^x_{\alpha\beta} \in \Gamma^q_{\mathbf{C}}, \alpha, \beta \in A\}$.

Associated to a codimension $-2q$ holomorphic foliation $\mathcal{F}$ is its tangent bundle L. Set $p = \dim L_x = n - 2q$. The normal bundle of $\mathcal{F}$ is the bundle $Q = TM/L$. There is an almost complex structure on Q obtained by lifting the standard complex structure on $\mathbf{C}^q$ to Q via the submersions f_α. We recall that Q is locally representable as the pull-back, along f_α, of the tangent bundle of $\mathbf{C}^q$.

The almost complex structure on Q induces a splitting of the complexified normal bundle in the standard way

$$Q^{\mathbf{C}} = Q^{(1,0)} \oplus Q^{(0,1)}. \tag{1.1}$$

We will need several short exact sequences of vector bundles associated to the foliation. Consider first the sequence defining Q:

$$0 \to L \xrightarrow{i_L} T_M \to \xrightarrow{\pi} Q \to 0. \tag{1.2}$$

Because of (1.1) we have an exact sequence of complex vector bundles

$$0 \to E \xrightarrow{i_E} T_M^{\mathbf{C}} \xrightarrow{\pi^{(1,0)}} Q^{(1,0)} \to 0, \tag{1.3}$$

where $\pi^{(1,0)}$ is defined to be the composition

$$T_M^{\mathbf{C}} \xrightarrow{\pi^{\mathbf{C}}} Q^{\mathbf{C}} \to Q^{(1,0)}$$

and $E = \operatorname{Ker} \pi^{(1,0)}$. We note that there is a noncanonical isomorphism $E \simeq L^{\mathbf{C}} \oplus Q^{(0,1)}$. We also consider the exact sequence of real vector bundles

$$0 \to L \xrightarrow{i_L} T_M \xrightarrow{\tau} Q^{(1,0)} \to 0. \tag{1.4}$$

The map τ in (1.4) is the composition

$$T_M \to T_M^{\mathbf{C}} \xrightarrow{\pi^{(1,0)}} Q^{(1,0)}.$$

To check that (1.4) is exact we need to check that $\operatorname{Ker} \tau = L$ and that τ is surjective. We do this in local coordinates. Let $U \subseteq M$ be open and let f:

$U \to \mathbf{C}^q$ be a submersion in $\mathcal{H}_{\mathfrak{F}}$. We can consider U to be contained in

$$\mathbf{R}^p \times \mathbf{C}^q = \{(x, z) | x = (x^1, \ldots, x^p), z = (z^{p+1}, \ldots, z^n)\},$$

and f to be the map $(x, z) \to z$. Such a coordinate system is said to be adapted to the holomorphic foliation $\mathfrak{F}$. Let $z^\alpha = u^\alpha + iv^\alpha$. Then T_M is spanned by the vector fields $\partial/\partial x^j$, $\partial/\partial u^\alpha$, $\partial/\partial v^\alpha$. Also $Q^{\mathbf{C}}$ is spanned by the vector fields

$$(1.5) \qquad \left[\frac{\partial}{\partial z^\alpha}\right] = \frac{1}{2}\left[\frac{\partial}{\partial u^\alpha} - i\frac{\partial}{\partial v^\alpha}\right], \quad \left[\frac{\partial}{\partial \bar{z}^\beta}\right] = \frac{1}{2}\left[\frac{\partial}{\partial u^\beta} + i\frac{\partial}{\partial v^\beta}\right],$$

and $Q^{(1,0)}$ is spanned by the vector fields $[\partial/\partial z^\alpha]$. Finally L is spanned by the vector fields $\partial/\partial x^\alpha$. We write $[\partial/\partial z^\alpha]$ for the equivalence class of $\partial/\partial z^\alpha$ under the projection $T_M^{\mathbf{C}} \to Q^{\mathbf{C}}$. Then for $X = X^j\partial/\partial x^j + U^\alpha\partial/\partial u^\alpha + V^\alpha\partial/\partial v^\alpha$ we have

$$(1.6) \qquad \tau(X) = (U^\alpha + iV^\alpha)\left[\frac{\partial}{\partial z^\alpha}\right] \in Q^{(1,0)}.$$

It is now clear that τ is surjective and $\operatorname{Ker} \tau = L$.

If $B \subseteq T_M^{\mathbf{C}}$ is a complex vector sub-bundle, we will denote its complex conjugate bundle by $\bar{B}$. With this definition we see that

$$(1.7) \qquad T_M^{\mathbf{C}} = E + \bar{E}.$$

We will need the following version of the complex Frobenius theorem of Nirenberg [11].

Theorem 1.8. *Let $E \subseteq T_M^{\mathbf{C}}$ be a sub-bundle of $T_M^{\mathbf{C}}$ of complex codimension q with $E + \bar{E} = T_M^{\mathbf{C}}$. Let $Q^{(1,0)} \equiv T_M^{\mathbf{C}}/E$ so that $Q^{(1,0)*} \subseteq T_M^{\mathbf{C}*}$. Then the following conditions are equivalent*:

(1) $[\underset{\sim}{E}, \underset{\sim}{E}] \subseteq \underset{\sim}{E}$.

(2) $d\underset{\sim}{Q}^{(1,0)*} \wedge \underset{\sim}{\Omega}_M^p \subseteq \underset{\sim}{Q}^{(1,0)*} \wedge \underset{\sim}{\Omega}_M^{p+1}$ *for all* $p \geqslant 0$.

(3) *E and $Q^{(1,0)}$ are obtained from a codimension $-2q$ holomorphic foliation as in the above discussion.*

Let $\rho: Q \to T_M$ be a splitting of (1.2). ρ induces a splitting of (1.3)

$$(1.9) \qquad 0 \to E \underset{\varphi}{\overset{i_{\mathbf{C}}}{\rightleftarrows}} T_M^{\mathbf{C}} \underset{\rho^{(1,0)}}{\overset{\pi^{(1,0)}}{\rightleftarrows}} Q^{(1,0)} \to 0.$$

The splitting (1.9) allows us to define a one-one correspondence between distributions (a distribution will be used to mean a sub-bundle of the tangent bundle and not a generalized function) near E and the space $\operatorname{Hom}_{\mathbf{C}}(E, Q^{(1,0)})$ in the following way. For $\varphi \in \operatorname{Hom}_{\mathbf{C}}(E, Q^{(1,0)})$ let $E_\varphi \subseteq T_M^{\mathbf{C}}$ be defined by $E_\varphi = \{i_\varphi(X) = i_{\mathbf{C}}(X) + \rho^{1,0} \circ \varphi(X) | X \in E\}$. Conversely let $E' \subseteq T_M^{\mathbf{C}}$ be a sub-bundle near E. Then the map

$$E' \subseteq T^{\mathbf{C}}_M \xrightarrow{\varphi} E$$

is an isomorphism. The inverse of this map is clearly of the form $X \mapsto i_{\mathbf{C}}(X) + \rho^{(1,0)} \circ \varphi(X)$ for a unique element $\varphi \in \mathrm{Hom}(E, Q^{(1,0)})$.

Remark 1.10. For $\varphi \in \mathrm{Hom}_{\mathbf{C}}(E, Q^{(1,0)})$ we have $E_\varphi + \bar{E}_\varphi = T^{\mathbf{C}}_M$.

Hence by Theorem (1.8) there is a one-to-one correspondence between holomorphic foliations near $\mathfrak{F}$ and the set

$$\mathrm{Fol}(\mathfrak{F}) = \{\varphi \in \mathrm{Hom}_{\mathbf{C}}(E, Q^{(1,0)}) | [\underset{\sim}{E}_\varphi, \underset{\sim}{E}_\varphi] \subseteq \underset{\sim}{E}_\varphi\}. \tag{1.11}$$

We wish to characterize $\mathrm{Fol}(\mathfrak{F})$ as an analytic subspace of a neighborhood in the first cohomology group of a certain sheaf on M. In order to do this it is necessary to define this sheaf and to construct a resolution of it by an elliptic complex. This will enable us to use the theory of elliptic partial differential equations.

We have the following short exact sequence of sheaves

$$0 \to \underset{\sim}{Q}^{(1,0)*} \wedge \underset{\sim}{\Omega}^{s-1}_M \to \underset{\sim}{\Omega}^s_M \xrightarrow{i^*_E} \underset{\sim}{E}^{*s} \to 0, \tag{1.12}$$

where E^{*s} is the sheaf of local sections of the bundle $\Lambda^s E^*$. So (1.12) is the exact sequence induced by the exact sequence

$$0 \to Q^{(1,0)*} \wedge \Lambda^{s-1} T^{\mathbf{C}}_M \to \Lambda^s T^{\mathbf{C}}_M \xrightarrow{i^*_{\mathbf{C}}} \Lambda^s E^* \to 0. \tag{1.13}$$

By Theorem 1.8 (2) we can define $\tilde{d}$: $\underset{\sim}{Q}^{(1,0)*} \wedge \underset{\sim}{\Omega}^s_M \to Q^{(1,0)*} \wedge \underset{\sim}{\Omega}^{s+1}_M$ as the restriction of the exterior derivative operator. Hence we can define d_ε as the unique operator which makes the diagram

$$\begin{array}{cccccccc} 0 \to & \underset{\sim}{Q}^{(1,0)*} & \wedge & \underset{\sim}{\Omega}^{s-1}_M & \to & \underset{\sim}{\Omega}^s_M & \xrightarrow{i^*_E} & \underset{\sim}{E}^{*s} & \to 0 \\ & & \downarrow \tilde{d} & & & & \downarrow d & & \downarrow d_\varepsilon \\ 0 \to & \underset{\sim}{Q}^{(1,0)*} & \wedge & \underset{\sim}{\Omega}^{s}_M & \to & \underset{\sim}{\Omega}^{s+1}_M & \xrightarrow{i^*_E} & \underset{\sim}{E}^{*s+1} & \to 0 \end{array} \tag{1.14}$$

commute. For $s = 0$, let $\underset{\sim}{E}^{*0} = \underset{\sim}{C}^\infty_M$, the sheaf of local complex-valued C^∞-functions and let

$$d_\varepsilon: \underset{\sim}{C}^\infty_M \xrightarrow{d} \underset{\sim}{\Omega}^1_M \xrightarrow{i^*_E} E^{*1}. \tag{1.15}$$

In adapted coordinates $(x, z) = (x^1, \cdots, x^p, z^{p+1}, \cdots, z^n)$ the sheaf $\underset{\sim}{E}^{*s}$ can be identified with the sheaf generated by the forms

$$[dx^1], \cdots, [dx^p], [d\bar{z}^{p+1}], \cdots, [d\bar{z}^n].$$

For $\varphi = \varphi_I[dx^{i_1} \wedge \cdots \wedge dx^{i_l}] \wedge [d\bar{z}^{i_{l+1}} \wedge \cdots \wedge d\bar{z}^{i_s}]$ in $\underset{\sim}{E}^{*s}$ we have

$$d_\varepsilon \varphi = d_\varepsilon \varphi_I \wedge [dx^{I'}] \wedge [d\bar{z}^{I''}], \tag{1.16}$$

where $I = (I'; I'') = (i_1, \cdots, i_t; i_{t+1}, \cdots, i_s)$ and

$$d_\varepsilon \varphi_I = \frac{\partial \varphi_I}{\partial x^j}[dx^j] + \frac{\partial \varphi_I}{\partial \bar{z}^\alpha}[d\bar{z}^\alpha], \tag{1.17}$$

$[dx^I]$ denoting the image of dx^I under i_E^*.

Let $\underset{\sim}{\mathcal{O}}_{\mathcal{F}} \subset \underset{\sim}{C}_M^\infty$ be the subsheaf of smooth complex-valued functions which are locally lifts, via the submersions in $\mathcal{H}_{\mathcal{F}}$, of holomorphic functions on $\mathbf{C}^q$. Specifically, let $f: U \to \mathbf{C}^q$ be a submersion in $\mathcal{H}_{\mathcal{F}}$ and define

$$\Gamma(U, \mathcal{O}_{\mathcal{F}}) = f^*\mathcal{O}_{\mathbf{C}^q} = \{ g \circ f \mid g \text{ is holomorphic on } f(U)\}. \tag{1.18}$$

Lemma 1.19. *The sequence*

$$0 \to \mathcal{O}_{\mathcal{F}} \to \underset{\sim}{C}_M^\infty \xrightarrow{d_\varepsilon} \underset{\sim}{E}^{*1} \xrightarrow{d_\varepsilon} \underset{\sim}{E}^{*2} \xrightarrow{d_\varepsilon} \cdots \tag{1.20}$$

is a resolution of the sheaf $\mathcal{O}_{\mathcal{F}}$.

Proof. We work in adapted coordinates. Note that if $d_\varepsilon f = 0$ for $f \in \mathcal{C}_M^\infty$; then $\partial f/\partial x_j = 0, j = 1, \cdots, p$. Hence, if $\pi: \mathbf{R}^p \times \mathbf{C}^q \to \mathbf{C}^q$ is the projection onto the second factor, we get $f = g \circ \pi$ where $g \in C^\infty(\mathbf{C}^q)$. Also $\partial g/\partial \bar{z}^\alpha = \partial f/\partial \bar{z}^\alpha \circ \pi = 0$ for $\alpha = 1, \cdots, q$. Hence g is holomorphic and $f \in \mathcal{O}_{\mathcal{F}}$. So we have that $\mathcal{O}_{\mathcal{F}} = \operatorname{Ker}(\underset{\sim}{C}_M^\infty \xrightarrow{d_\varepsilon} \underset{\sim}{E}^{*1})$.

The sheaves $\underset{\sim}{C}_M^\infty$, $\underset{\sim}{E}^{*s}$ being fine, our lemma will be proved once it is established that the complex (1.20) is exact.

The problem is local so we work on the open set $W = U \times V \subset \mathbf{R}^p \times \mathbf{C}^q$, where U is the unit ball and V is the unit polydisk. We assume that $\mathcal{F}|W$ is given by the fibers of the projection $U \times V \to V$ and we let $(x, z) = (x^1, \cdots, x^p, z^{p+1}, \cdots, z^n)$ be the local coordinate functions. The complex $(\Gamma(W, \underset{\sim}{E}^{**}), d_\varepsilon)$ is isomorphic to the double complex $(A^{s,t}, d_{\|} + \bar{\partial})$, $s, t \geqslant 0$, where $A^{s,t}$ is the space of $s + t$-forms on W of the form $\varphi = \varphi_{I,J}(x, z)[dx^I] \wedge [d\bar{z}^J]$, $|I| = s$, $|J| = t$ and where

$$d_{\|}\varphi = \frac{\partial \varphi_{I,J}}{\partial x^i}[dx^i \wedge dx^I] \wedge [d\bar{z}^J],$$

$$\bar{\partial}\varphi = \frac{\partial}{\partial \bar{z}^\alpha}\varphi_{I,J}[d\bar{z}^\alpha] \wedge [dx^I] \wedge [d\bar{z}^J] = (-1)^s \frac{\partial \varphi_{I,J}}{\partial \bar{z}^\alpha}[dx^I] \wedge [d\bar{z}^\alpha] \wedge [d\bar{z}^J].$$

Consider the spectral sequence associated to the second filtration on $A^{\cdot\cdot}$. Then

$$''E_2^{s,t} \cong H^s\big(H^t(A^{\cdot\cdot}, d_{\|}), \bar{\partial}\big) \Rightarrow H^\bullet(\Gamma(W, \underset{\sim}{E}^*), d_\varepsilon). \tag{1.21}$$

We must show that $''E_2^{s,t} = 0$ for $s + t > 0$. If we consider $A^{\cdot,t}$ as the de Rham complex of U parametrized by V, the proof of the Poincaré lemma [10] goes through to show that this sequence collapses to $''E_2^{s,0} \cong H^s(V, \underset{\sim}{\mathcal{O}}_V)$, where $\underset{\sim}{\mathcal{O}}_V$ is the sheaf of germs of holomorphic functions on V. But by Dolbeaut's lemma [10], this cohomology group is trivial for $s > 0$, so we are done.

Lemma 1.22. *The complex* $(E^{*\bullet}, d_\varepsilon)$ *is elliptic.*

Proof. We show that if $x \in M$ and $\theta \in T^*_{M_x}$ is a co-vector at x, the symbol sequence

$$\mathbf{C} \xrightarrow{\sigma_\theta} E^*_x \xrightarrow{\sigma_\theta} \Lambda^2 E^*_x \to \cdots \tag{1.23}$$

is exact. For $\beta \in \Lambda^s E^*_x$ let $\tilde{\beta} \in \Lambda^s(T^*_M \otimes \mathbf{C})$ be a form such that $i^*_E(\tilde{\beta}) = \beta$. It is easily seen that the symbol of d_ε at $\theta \in T^*_{M_x}$ is given by

$$\sigma_\theta(\beta) = i^*_E(\theta \wedge \tilde{\beta}). \tag{1.24}$$

To prove exactness pick a basis for $T^*_{M_x} \otimes \mathbf{C}$ of the form θ, $dz^1, \cdots, dz^q$, $\xi_{q+2}, \cdots, \xi_n$, and let $\beta \in E^*_x$ be such that $\sigma_\theta(\beta) = 0$. We will find $\tilde{\alpha} \in \Lambda^{s-1}(T^*_{M_x} \otimes \mathbf{C})$ for which $i^*_E(\theta \wedge \tilde{\alpha}) = \beta$. We proceed as follows: since $i^*_E(\theta \wedge \tilde{\beta}) = 0$ we can write $\theta \wedge \tilde{\beta}$ in the form $\theta \wedge \tilde{\beta} = dz^\alpha \wedge \gamma_\alpha$ where γ_α can be written in terms of θ, $dz^{\alpha+1}, \cdots, dz^q$, $\xi_{q+2}, \cdots, \xi_n$ for $j = 1, \cdots, q$. Since $\theta \wedge \theta \wedge \tilde{\beta} = 0$ we have $0 = \theta \wedge dz^\alpha \wedge \gamma_\alpha$, and since $\theta, dz^\alpha, \cdots$ is a basis it follows from the forms of the γ_α that $\theta \wedge \gamma_\alpha = 0$. Hence we can write $\gamma_\alpha = -\theta \wedge \delta_\alpha$, $\alpha = 1, \cdots, q$. Let $\delta = dz^\alpha \wedge \delta_\alpha$, then $\theta \wedge \delta = \theta \wedge \tilde{\beta}$ so

$$\theta \wedge (\tilde{\beta} - \delta) = 0. \tag{1.25}$$

But $i^*_E(\delta) = 0$ hence

$$i^*_E(\tilde{\beta} - \delta) = \beta. \tag{1.26}$$

By (1.25) there is an element $\tilde{\alpha} \in \Lambda^{s-1}(T^*_M \otimes \mathbf{C})$ for which $\theta \wedge \tilde{\alpha} = (\tilde{\beta} - \delta)$. Hence by (1.26) we have $\sigma_\theta(\alpha) = i^*_E(\theta \wedge \tilde{\alpha}) = i_E(\tilde{\beta} - \delta) = \beta$, where $\alpha = i^*_E(\tilde{\alpha})$. Thus the sequence is exact and the complex is elliptic. q.e.d.

We now define the notion of holomorphic vector field on a holomorphic foliation. Locally a holomorphic vector field is a lift, via a submersion in $\mathcal{H}_{\mathcal{F}}$, of a holomorphic vector field on $\mathbf{C}^q$. More precisely let $U \subseteq M$ be an open set such that there is a submersion $f\colon U \to \mathbf{C}^q, f \in \mathcal{H}_{\mathcal{F}}$. Then we define $\underset{\sim}{\theta}_{\mathcal{F}|U}$ as the pull-back $f^*(\underset{\sim}{\theta}_{\mathbf{C}^q})$ of the sheaf of germs of holomorphic vector fields on $\mathbf{C}^q$.

Remark. By a holomorphic vector field on $\mathbf{C}^q$ we mean a holomorphic section of the holomorphic tangent bundle.

Theorem 1.27. *The cohomology groups* $H^i(M, \mathcal{O}_{\mathcal{F}})$ *and* $H^i(M, \underset{\sim}{\theta}_{\mathcal{F}})$ *are finite dimensional.*

Proof. Since the resolution (1.20) is elliptic, it follows from the theory of elliptic complexes [14] that $H^i(M, \mathcal{O}_{\mathcal{F}})$ is finite dimensional. Similarly, to show that $H^i(M, \underset{\sim}{\theta}_{\mathcal{F}})$ is finite dimensional we will construct a resolution of $\underset{\sim}{\theta}_{\mathcal{F}}$ by an elliptic complex. Let

(1.28) $$\underset{\sim}{E}_Q^{*i} = \underset{\sim}{E}^{*i} \otimes_{\mathcal{O}_{\mathfrak{F}}} \underset{\sim}{\mathcal{O}}_{\mathfrak{F}} \cong E^{*i} \otimes_{\mathbf{C}} Q^{(1,0)},$$

and let $d_Q = d_E \otimes \mathrm{id}$. Since d_E is elliptic, so is d_Q and the required resolution is

(1.29) $$0 \to \underset{\sim}{\mathcal{O}}_{\mathfrak{F}} \to \underset{\sim}{E}_Q^{*0} \xrightarrow{d_Q} \underset{\sim}{E}_Q^{*1} \to \cdots .$$

This concludes the proof. q.e.d.

In adapted coordinates the operator d_Q is given by the formula

(1.30) $$d_Q\left(\varphi^\alpha \otimes \left[\frac{\partial}{\partial z^\alpha}\right]\right) = d_\varepsilon \varphi^\alpha \otimes \left[\frac{\partial}{\partial z^\alpha}\right],$$

where φ^α is in E^{*s}.

Remarks. The above discussion is an adaptation to holomorphic foliations of cohomology theories for C^∞-foliations as presented in [9]. See also [4], [5] and [6]. We summarize here results of theirs which we will need in §4, as they apply to a holomorphic foliation $\mathfrak{F}$, considered as a C^∞-foliation.

On the complex $\Lambda^\bullet L^*$ is a differential $d_{\|}$, which in adapted coordinates takes the form

(1.31) $$d_{\|}\varphi = \frac{\partial \varphi_I}{\partial x^j}\left[dx^j\right] \wedge dx^I,$$

where

(1.32) $$\varphi = \varphi_I\left[dx^I\right].$$

Let $\underset{\sim}{C}_{\mathfrak{F}}^\infty$ be the sheaf of complex-valued C^∞-functions, which are locally constant along the leaves of $\mathfrak{F}$. $(\Lambda^\bullet \underset{\sim}{L}^*, d_{\|})$ is a resolution of this sheaf [9].

Let $\underset{\sim}{Q}_{\mathfrak{F}}$, $\underset{\sim}{Q}_{\mathfrak{F}}^{(1,0)}$, etc., denote the sheaves of sections of Q, $Q^{(1,0)}$, etc, which are locally constant along the leaves of $\mathfrak{F}$. These are all modules over $\underset{\sim}{C}_{\mathfrak{F}}^\infty$, and tensoring over $C_{\mathfrak{F}}^\infty$ with $(\Lambda^\bullet \underset{\sim}{L}^*, d_{\|})$ gives resolutions of these sheaves. In particular [9]

(1.33)
$$H^\bullet\left(M, Q^{(0,s)} \otimes Q_{\mathfrak{F}}^{(1,0)}\right) \cong H^\bullet\left(\Gamma\left(\underbrace{Q^{*(0,s)} \otimes Q_{\mathfrak{F}}^{(1,0)}} \otimes_{\underset{\sim}{C}_{\mathfrak{F}}^\infty} \Lambda \underset{\sim}{L}^*\right), d_{Q\|}\right),$$

where $d_{Q\|} = \mathrm{id} \otimes d_{\|}$.

2. Infinitesimal deformations and the Spencer operator

As an application of Theorem (1.27) we will show that the space of infinitesimal deformations of a holomorphic foliation is finite dimensional. Let $\mathcal{P}_{\mathfrak{F}}$ be the pseudogroup of local diffeomorphisms of M which preserve the holomorphic foliation $\mathfrak{F}$. Specifically, $g: U \to V$ is in $\mathcal{P}_{\mathfrak{F}}$ if and only if for each submersion $f_\alpha: U_\alpha \to \mathbf{C}^q$ in $\mathcal{H}_{\mathfrak{F}}$ with $V \cap U_\alpha \neq \varnothing$, the submersion $f_\beta = f_\alpha \circ g: U \to \mathbf{C}^q$ is in $\mathcal{H}_{\mathfrak{F}}$, where $U_\beta = U_\alpha \cap g^{-1}(U_\alpha)$. It follows from the definition of a Haefliger cocycle that near each point $x \in U_\alpha \cap U_\beta$ there is a unique local biholomorphism $\tilde{g}^{x}_{\beta\alpha}$ of $\mathbf{C}^q$ with

$$f_\alpha \circ g = \tilde{g}^{x}_{\beta\alpha} \circ f_\alpha. \tag{2.1}$$

Now let $\eta_{\mathfrak{F}}$ be the sheaf of local vector fields whose flows lie in $\mathcal{P}_{\mathfrak{F}}$, and let $\underset{\sim}{L}$ be the sheaf of local vector fields in $L \subseteq T_M$. The following lemma follows easily from (2.1).

Lemma 2.2. *The exact sequence* (1.4) *induces an exact sequence*

$$0 \to \underset{\sim}{L} \to \eta_{\mathfrak{F}} \xrightarrow{\tau} \underset{\sim}{\theta}_{\mathfrak{F}} \to 0. \tag{2.3}$$

Theorem 2.4. *The space of infinitesimal deformations of the pseudogroup $\mathcal{P}_{\mathfrak{F}}$ is finite dimensional.*

Proof. Since $\underset{\sim}{L}$ is a finite sheaf, it follows that $H^j(M, \underset{\sim}{L}) = 0$ for $j \geqslant 1$. Using the long exact cohomology sequence associated to (2.3) we see that $H^1(M, \eta_{\mathfrak{F}}) \simeq H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}})$. By Spencer [13] the space of infinitesimal deformations of $\mathcal{P}_{\mathfrak{F}}$ is just $H^1(M, \eta_{\mathfrak{F}})$, which is finite dimensional by Theorem (1.27). q.e.d.

We will now define a nonlinear first order partial differential operator

$$D: \operatorname{Hom}(E, Q^{(1,0)}) \to \operatorname{Hom}(\Lambda^2 E, Q^{(1,0)}),$$

whose linearization is d_Q. We will call this the Spencer operator associated to $\mathfrak{F}$. This operator is defined in analogy with the operator $\bar{\partial} - [\ ,\ ]$ which is of fundamental importance in the study of deformations of complex structure on a complex manifold. For this see [8].

D will be of the form, $D = d_Q - [\ ,\ ]_Q$ where $[\ ,\ ]_Q$ is an operator to be defined below. We will show that

$$\operatorname{Fol}(\mathfrak{F}) = \{\varphi \in \operatorname{Hom}(E, Q^{(1,0)}): D\varphi = 0\}. \tag{2.5}$$

In §3 we will show how to realize $\operatorname{Fol}(\mathfrak{F})$, via (2.5), as an analytic subspace of $H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}})$.

Remark 2.6. Unfortunately, our techniques work only if we assume that the splitting (1.9) is induced by a foliation $\mathfrak{F}^\perp$ transverse to $\mathfrak{F}$. The foliation $\mathfrak{F}^\perp$ need not be holomorphic. We assume from this point on that $\mathfrak{F}^\perp$ is fixed and that $\rho: Q \to T_M$ is the tangent bundle to $\mathfrak{F}^\perp$. We can therefore think of all bundles as sub-bundles of the tensor algebra bundle of T_M. An adapted coordinate system will now be a chart (x, z) in $\mathbf{R}^p \times \mathbf{C}^q$ such that the projections $\mathbf{R}^p \times \mathbf{C}^q \to \mathbf{C}^q$ are in $\mathcal{H}_{\mathfrak{F}}$ and such that the leaves $\mathfrak{F}^\perp$ are locally given by the sets $\{x = \text{constant}\}$.

In these coordinates $d_Q\colon E_Q^{*s} \to E_Q^{*s+1}$ is given by the formula

$$
\begin{aligned}
&d_Q\left(\varphi_{JB}^{\alpha} dx^J \wedge d\bar{z}^B \otimes \frac{\partial}{dz^{\alpha}}\right) \\
(2.7)\qquad &= \left(\frac{\partial}{\partial x^i}\varphi_{JB}^{\alpha} dx^i + \frac{\partial}{\partial \bar{z}^{\beta}}\varphi_{JB}^{\alpha}\, d\bar{z}^{\beta}\right) \wedge dx^J \wedge d\bar{z}^B \otimes \frac{\partial}{\partial z^{\alpha}},
\end{aligned}
$$

where, as usual, $dx^J = dx^{j_1} \wedge \cdots \wedge dx^{j_l}$, $d\bar{z}^B = d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_k}$ and let $l + k = s$.

We now define $[\ ,\ ]_Q$, as a map $[\ ,\ ]_Q\colon \underset{\sim}{E}_q^{*r} \times \underset{\sim}{E}_Q^{*s} \to \underset{\sim}{E}_Q^{*r+s}$ as follows. Choose adapted coordinates, and let

$$
\varphi = \varphi_{JB}^{\alpha}\, dx^J \wedge d\bar{z}^B \otimes \frac{\partial}{\partial z^{\alpha}} \in E_Q^{*r},
$$

$$
\psi = \psi_{KG}^{\alpha}\, dx^K \wedge d\bar{z}^G \otimes \frac{\partial}{\partial z^{\alpha}} \in E_Q^{*\cdot}.
$$

Then

$$
\begin{aligned}
[\varphi, \psi]_Q = \frac{1}{2r!s!}\Bigg(&\varphi_{JB}^{\gamma}\frac{\partial}{\partial z^{\gamma}}\psi_{KG}^{\alpha} \\
(2.8)\qquad &+ (-1)^{rs+1}\psi_{KG}^{\gamma}\frac{\partial\varphi_{JB}^{\alpha}}{\partial z^{\gamma}}\Bigg) dx^J \wedge d\bar{z}^B \wedge dx^K \wedge d\bar{z}^G \otimes \frac{\partial}{\partial z^{\alpha}}.
\end{aligned}
$$

We now give a precise statement and proof of (2.5).

Proposition 2.9. *Given $\varphi \in \mathrm{Hom}(E, Q^{(1,0)})$, the distribution E_{φ} defines a holomorphic foliation if and only if $D\varphi = 0$.*

Proof. By the complex Frobenius theorem we must show that $[E_{\varphi}, E_{\varphi}] \subseteq E_{\varphi}$ if and only if $D\varphi = 0$.

Again we work in adapted coordinates. Suppose $D\varphi = 0$. Then we see by the definition of E_{φ} that E_{φ} is generated by the vector fields

$$
X_i = \frac{\partial}{\partial x^i} + \varphi_i^{\alpha}\frac{\partial}{\partial z^{\alpha}}, \qquad Y_{\bar{\beta}} = \frac{\partial}{\partial \bar{z}^{\beta}} + \varphi_{\bar{\beta}}^{\alpha}\frac{\partial}{\partial z^{\alpha}},
$$

where $\varphi = \varphi_i^{\alpha}\, dx^i \otimes \partial/\partial z^{\alpha} + \varphi_{\bar{\beta}}^{\alpha}\, dz^{\beta} \otimes \partial/\partial z^{\alpha}$. We need only show that for all i, j, α, β the vector fields $[X_i, X_j]$, $[X_i, Y_{\bar{\alpha}}]$, and $[Y_{\bar{\alpha}}, Y_{\bar{\beta}}]$ lie in E_{φ}.

It follows from (2.7) and (2.8) that

$$
\begin{aligned}
&[X_i, X_j] = (D\varphi)_{ij}^{\alpha}\frac{\partial}{\partial z^{\alpha}}, \\
(2.10)\qquad &[X_i, Y_{\bar{\beta}}] = (D\varphi)_{i\bar{\beta}}^{\alpha}\frac{\partial}{\partial z^{\alpha}}, \\
&[Y_{\bar{\beta}}, Y_{\bar{\gamma}}] = (D\varphi)_{\bar{\beta}\bar{\gamma}}^{\alpha}\frac{\partial}{\partial z^{\alpha}},
\end{aligned}
$$

where

$$D\varphi = (D\varphi)^{\alpha}_{ij}\, dx^i \wedge dx^j \otimes \frac{\partial}{\partial z^{\alpha}} + (D\varphi)^{\alpha}_{i\bar{\beta}}\, dx^i \wedge d\bar{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}}$$
$$+ (D\varphi)^{\alpha}_{\bar{\beta}\bar{\gamma}}\, d\bar{z}^{\beta} \wedge d\bar{z}^{\gamma} \otimes \frac{\partial}{\partial z^{\alpha}}.$$

Hence all brackets are zero, and the distribution E_{φ} is involutive.

Conversely suppose E_{φ} is involutive. Then the brackets $[X_i, X_j]$, $[X_i, Y_{\bar{\alpha}}]$, $[Y_{\bar{\alpha}}, Y_{\bar{\beta}}]$ lie in E_{φ}. But by (2.10) this is impossible unless all brackets are zero. Again, by (2.10) this is impossible unless $D\varphi = 0$. q.e.d.

We conclude this section with a list of properties of $[\ ,\ ]_Q$ which will be needed in the construction of solutions of $D\varphi = 0$. They are easily verified.

(2.11) $$[\ ,\ ]_Q \text{ is bilinear.}$$

If $\varphi \in \underset{\sim}{E}^{*r}_{Q}$, $\psi \in \underset{\sim}{E}^{*s}_{Q}$ and $\tau \in \underset{\sim}{E}^{*t}_{Q}$, then

(2.12) $$[\varphi, \psi]_Q = (-1)^{rs}[\psi, \varphi]_Q,$$

(2.13) $$d_Q[\varphi, \psi]_Q = [d_Q\varphi, \psi]_Q + (-1)^r[\varphi, d_Q\psi]_Q,$$

(2.14) $$(-1)^{st}[\varphi, [\psi, \tau]_Q]_Q + (-1)^{rs}[\psi, [\tau, \varphi]_Q]_Q + (-1)^{rt}[\tau, [\varphi, \psi]_Q]_Q = 0.$$

3. The Kuranishi family of a holomorphic foliation

In this section we extend Kuranishi's theorem [8] on the existence of locally complete families of complex analytic structures to the case of a holomorphic foliation for which there is a transverse foliation. More specifically, we will prove the following theorem.

Theorem 3.1. *Let $\mathfrak{F}_0$ be a holomorphic foliation on a compact C^{∞}-manifold M, and let $\mathfrak{F}^{\perp}$ be a C^{∞}-foliation transverse to $\mathfrak{F}_0$. Then there are a local analytic subset $B \subseteq H^1(M, \underset{\sim}{\theta}_{\mathfrak{F}_0})$ and a holomorphic map*

(3.2) $$B \to \mathfrak{F}\mathrm{ol}(\mathfrak{F}_0) \subseteq \mathrm{Hom}(E, Q^{(1,0)}): t \to \mathfrak{F}_t,$$

which defines a locally complete family of holomorphic foliations in the sense that if $\tilde{\mathfrak{F}}$ is a holomorphic foliation sufficiently close to $\mathfrak{F}_0$, then $\tilde{\mathfrak{F}}$ is conjugate to a foliation of the form $\mathfrak{F}_t$ via a diffeomorphism of M close to the identity. Furthermore, given a Riemannian metric respecting the local product structure on M induced by $\mathfrak{F}_0$ and $\mathfrak{F}^{\perp}$ this diffeomorphism can be unambiguously defined.

Remarks 3.3. This theorem is a generalization of Kuranishi's theorem in the following sense. A complex manifold M can be thought of as the holomorphic foliation on M given by points. The foliation $\mathcal{F}^{\perp}$ is just the codimension -0 foliation of M whose single leaf is M itself.

The proof of Theorem 3.1 is an adaptation of Kuranishi's proof [8]. In fact, if the following substitutions are made, the proofs are almost identical: replace the Dolbault complex by $(\underset{\sim}{E}_Q^*, d_Q)$ and replace the bracket operation of Kuranishi by $[\ ,\ ]_Q$. In place of the operator $\bar\partial - [\ ,\ ]$ substitute the operator $D = d_Q - [\ ,\ ]_Q$. The proof of Theorem 3.1 proceeds in two steps. We first construct the family $\mathcal{F}_t$ as solutions of a certain system of equations. Then we show that any holomorphic foliation close to $\mathcal{F}_0$ is conjugate to $\mathcal{F}_t$ for some t.

Step 1. The construction of $\mathcal{F}_t$. We will now construct a map from elements of a certain analytic subset B of $H^1(M, \underset{\sim}{\theta}_{\mathcal{F}})$ near zero to solutions of the system of equations

$$d_Q\varphi = [\varphi, \varphi]_Q, \qquad \delta_Q\varphi = 0 \tag{3.4}$$

with $\varphi \in \Gamma(E_Q^{*1}) \cong \mathrm{Hom}(E, Q^{1,0})$ having small norm. Here δ_Q denotes the adjoint of the operator d_Q: $\Gamma(E_Q^{*s}) \to \Gamma(E_Q^{*s+1})$ with respect to the inner product induced by the Riemannian metric on M associated to an $SO(p) \times U(q)$ reduction of the tangent bundle of M which is compatible with the local product structure on M and the complex structure on Q.

Recall that, by the Hodge decomposition theorem for elliptic complexes [14], there is a Green's operator

$$G_Q: \Gamma(E_Q^{*r}) \to \Gamma(E_Q^{*r}),\ r \geq 0 \tag{3.5}$$

with the property that

$$I = H_Q + \Delta_Q \circ G_Q, \tag{3.6}$$

where $\Delta_Q = d_Q\delta_Q + \delta_Q d_Q$, and H_Q: $\Gamma(E_Q^{*r}) \to H^r(M, \underset{\sim}{\theta}_{\mathcal{F}})$ is projection onto $\mathrm{Ker}\, \Delta_Q$, which by Lemma 1.19 we can identify with $H^r(M, \underset{\sim}{\theta}_{\mathcal{F}})$.

Let $\|\ \|_s$ denote the Sobolov norm on $H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}})$ induced by the metric on M. Pick a basis $\varphi_1, \varphi_2, \cdots, \varphi_m$ for $H^1(M, \underset{\sim}{\theta}_{\mathcal{F}})$. Given $\varphi_0 = \sum_{i=1}^m t_i\varphi_i \in H^1(M, \underset{\sim}{\theta}_{\mathcal{F}})$ with $\|\varphi_0\|_s$ small, say $< \varepsilon$, we wish to solve the equation

$$\varphi = \varphi_0 + \varphi_Q G_Q[\varphi, \varphi]_Q, \tag{3.7}$$

and show that the solution $\varphi(t)$ depends holomorphically on $t = (t_1, \cdots, t_m) \in \mathbf{C}^m$. To do this we need two estimates:

$$\|[\varphi_1, \varphi_2]_Q\|_s \leq C\|\varphi_1\|_{s+1} \cdot \|\varphi_2\|_{s+1}, \tag{3.8}$$

and

$$\|\delta_Q G_Q\varphi\|_s \leq C\|\varphi\|_{s-1}, \tag{3.9}$$

$$\|H_Q\varphi\|_s \leq C\|\varphi\|_s. \tag{3.10}$$

The first estimate follows trivially from the definition of $[\ ,\ ]_Q$, and the second and third follow from the fact that the d_Q-complex is elliptic. The solution of (3.7) and its holomorphic dependence follow, verbatim as in [7] using the implicit function theorem or a power series expansion.

We can now solve the system (3.4) using the above result. Begin by assuming that φ is a solution of (3.7). We will soon see that for $\|\varphi\|_s$ sufficiently small this assumption is redundant. Note that, by the Hodge decomposition (3.6),

$$[\varphi, \varphi]_Q = H_Q[\varphi, \varphi]_Q + d_Q\delta_Q G_Q[\varphi, \varphi]_Q + \delta_Q d_Q G_Q[\varphi, \varphi]_Q, \tag{3.11}$$

and that, since $d_Q\varphi_0 = 0$,

$$d_Q\varphi = d_Q\delta_Q G_Q[\varphi, \varphi]_Q. \tag{3.12}$$

Combining (3.11) and (3.12) yields

$$-d_Q\varphi + [\varphi, \varphi]_Q = H_Q[\varphi, \varphi]_Q + \delta_Q d_Q G_Q[\varphi, \varphi]_Q,$$

and therefore

$$-d_Q\varphi + [\varphi, \varphi]_Q = H_Q[\varphi, \varphi]_Q + \delta_Q G_Q d_Q[\varphi, \varphi]_Q, \tag{3.13}$$

since $d_Q G_Q = G_Q d_Q$. Since the terms on the right are orthogonal, φ is a solution of (3.4) if and only if the equations

$$H_Q[\varphi, \varphi]_Q = 0, \tag{3.14}$$

$$\delta_Q G_Q d_Q[\varphi, \varphi]_Q = 0 \tag{3.15}$$

are satisfied. However, (3.15) is a consequence of (3.14) by the following argument. First observe that

$$\delta_Q G_Q d_Q[\varphi, \varphi]_Q = 2\delta_Q G_Q[d_Q\varphi, \varphi]_Q \tag{3.16}$$

by (2.12) and (2.13). If $H_Q[\varphi, \varphi]_Q = 0$, then by (3.13) we can write (3.16) as

$$\begin{aligned} \delta_Q G_Q d_Q[\varphi, \varphi]_Q &= 2\delta_Q G_Q[[\varphi, \varphi]_Q, \varphi]_Q - 2\delta_Q G_Q[\delta_Q G_Q d_Q[\varphi, \varphi]_Q, \varphi]_Q \\ &= -2\delta_Q G_Q[\delta_Q G_q d_Q[\varphi, \varphi]_Q, \varphi]_Q \end{aligned} \tag{3.17}$$

by the Jacobi identity (2.14). Hence by (3.8) and (3.9) we have the inequality

$$\|\delta_Q G_Q d_Q[\varphi, \varphi]_Q\|_s \leqslant C\|\delta_Q G_Q d_Q[\varphi, \varphi]_Q\|_s\|\varphi\|_s.$$

So, for $\|\varphi\|_s$ sufficiently small, (3.15) holds.

We can now construct the space B of the theorem. Let

$$B = \{\varphi_0 \in H^1(M, \underset{\sim}{\theta}_{\mathscr{F}}) \mid \|\varphi_0\| < \varepsilon, \quad H_Q[\varphi(t), \varphi(t)]_Q = 0\}, \tag{3.18}$$

where ε is to be chosen as in Lemma 3.23. This is an analytic subset of $H^1(M, \underset{\sim}{\theta}_{\mathscr{F}})$. Furthermore, by the above argument, the elements $\varphi(t)$ for

$\sum t_i\varphi_i \in B$ are solutions of the equation $D\varphi \equiv d_Q\varphi - [\varphi, \varphi]_Q = 0$, and therefore define holomorphic foliations.

Note that if ψ is a solution of (3.4) of sufficiently small norm, then $\psi = \varphi(t)$ for a unique element $\varphi_0 = \sum t_i\varphi_i \in B$. To see this, notice that since $D\psi = 0$ and $\delta_Q\psi = 0$, we have

$$\Delta_Q\psi = \delta_Q[\psi, \psi]_Q. \tag{3.19}$$

Hence

$$\psi - H_Q\psi = G_Q\delta_Q[\psi, \psi]_Q. \tag{3.20}$$

Set $\varphi_0 = H_Q\psi$. Then from (3.20)

$$\psi = \varphi_0 + \delta_Q G_Q[\psi, \psi]_Q. \tag{3.21}$$

By (3.10)

$$\|\varphi_0\|_s = \|H_Q\psi\|_s \leqslant c\|\varphi\|_s. \tag{3.22}$$

Therefore there is a number $\eta > 0$ with the property that if $\|\psi\|_s < \eta$, then $\|\varphi_0\|_s < \varepsilon$. Hence $\psi = \varphi(t)$ for $\varphi_0 = \sum t_i\varphi_i \in B$ by the following lemma.

Lemma 3.23. *The set* $\{\varphi(t) | \sum t_i\varphi_i \in B\}$ *comprises all solutions of* (3.7) *of small norm, and these solutions are unique.*

Proof. Fix φ_0 with $\|\varphi_0\|_s$ small, and let $\varphi(t)$ be the solution obtained by power series. Suppose φ is another solution. Let $\omega = \varphi - \varphi(t)$. Then

$$\begin{aligned}\omega &= \delta_Q G_Q([\varphi, \varphi]_Q - [\varphi(t), \varphi(t)]_Q) \\ &= \delta_Q G_Q([\omega, \varphi(t)]_Q + [\varphi(t), \omega]_Q + [\omega, \omega]_Q) \\ &= \delta_Q G_Q(2[\omega, \varphi(t)]_Q + [\omega, \omega]_Q).\end{aligned}$$

Hence by (3.8)

$$\|\omega\|_s \leqslant c\|\omega\|_s(\|\varphi(t)\|_s + \|\omega\|_s).$$

For $\|\varphi(t)\|_s$ sufficiently small say $< \varepsilon$, this can only happen if $\omega = 0$. q.e.d.

At this point we have shown that every solution of the equations $D\varphi = 0$ and $\delta_Q\varphi = 0$ is of the form $\varphi(t)$ for $\varphi_0 = \sum t_i\varphi_i \in B$.

Step 2. Suppose now that the norm of φ is small, and that $D\varphi = 0$, but that $\delta_Q\varphi \neq 0$. We wish to show that the corresponding foliation $\mathfrak{F}_\varphi$ is conjugate to one of the form $\mathfrak{F}_{\varphi(t)}$ for $\sum t_i\varphi_i \in B$. As in Kuranishi [8], we do this using diffeomorphisms generated by geodesics.

We just examine the action of diffeomorphisms of M near the identity on holomorphic foliations, or more precisely their associated distributions. Let $\varphi \in \mathrm{Hom}(E, Q^{(1,0)})$, and denote the distribution associated to φ by $E_\varphi \subseteq T_M^{\mathbf{C}}$. Let f be a diffeomorphism of M close to the identity in the C^∞-topology. Then the Jacobian map f_* maps E_φ to a bundle $f_*(E_\varphi)$, and there is a unique element $\psi \in \operatorname{Hom}(E, Q^{(1,0)})$ with $E_\psi = f_*(E_\varphi)$. Denote this element by $f_*\varphi$.

We wish to find a formula for $f_*\varphi$ in terms of f and φ in adapted coordinates. Let $(x, z) = (x^1, \cdots, x^p, z^{p+1}, \cdots, z^n)$ be adapted coordinates. Then

$$\varphi = \varphi_i^\alpha \, dx^i \otimes \frac{\partial}{\partial z^\alpha} + \varphi_{\bar\beta}^\alpha \, d\bar z^\beta \otimes \frac{\partial}{\partial z^\alpha}, \tag{3.24}$$

and E_φ is spanned locally by the vector fields

$$X_i^\varphi = \frac{\partial}{\partial x^i} + \varphi_i^\alpha \frac{\partial}{\partial z^\alpha}, \qquad X_{\bar\beta}^\varphi = \frac{\partial}{\partial \bar z^\beta} + \varphi_{\bar\beta}^\alpha \frac{\partial}{\partial z^\alpha}. \tag{3.25}$$

Hence locally $f_*(E_\varphi)$ is spanned by the entries of the $(p + q) \times 1$ matrix

$$\begin{bmatrix} f_*(X_i^\varphi) \\ f_*(X_{\bar\beta}^\varphi) \end{bmatrix} = \begin{pmatrix} M_{ij} & M_{i\alpha} \\ M_{\bar\beta j} & M_{\bar\beta\alpha} \end{pmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x^j} \\ \dfrac{\partial}{\partial \bar z^\alpha} \end{bmatrix} + \begin{pmatrix} N_{i\alpha} \\ N_{\bar\beta\alpha} \end{pmatrix} \left(\frac{\partial}{\partial z^\alpha} \right), \tag{3.26}$$

where

$$\begin{aligned} M_{ij} &= \left(\frac{\partial f^j}{\partial x^i} + \varphi_i^\gamma \frac{\partial f^j}{\partial z^\gamma} \right), & M_{i\alpha} &= \left(\frac{\partial \bar f^\alpha}{\partial x^i} + \varphi_i^\gamma \frac{\partial \bar f^\alpha}{\partial z^\gamma} \right), \\ M_{\bar\beta j} &= \left(\frac{\partial f^j}{\partial \bar z^\beta} + \varphi_{\bar\beta}^\gamma \frac{\partial f^j}{\partial z^\gamma} \right), & M_{\bar\beta\alpha} &= \left(\frac{\partial \bar f^\alpha}{\partial \bar z^\beta} + \varphi_{\bar\beta}^\gamma \frac{\partial \bar f^\alpha}{\partial z^\gamma} \right), \\ N_{i\alpha} &= \left(\frac{\partial f^\alpha}{\partial x^i} + \varphi_i^\gamma \frac{\partial f^\alpha}{\partial z^\gamma} \right), & N_{\bar\beta\alpha} &= \left(\frac{\partial f^\alpha}{\partial \bar z^\beta} + \varphi_{\bar\beta}^\gamma \frac{\partial f^\alpha}{\partial z^\gamma} \right), \end{aligned} \tag{3.27}$$

and $f = (f^1, \cdots, f^n)$. Setting

$$\psi = f_*(\varphi) = \psi_i^\alpha \, dx^i \otimes \frac{\partial}{\partial z^\alpha} + \psi_{\bar\beta}^\alpha \, d\bar z^\beta \otimes \frac{\partial}{\partial z_\alpha}, \tag{3.28}$$

we see that $f_*(E_\varphi)$ is spanned locally by the vectors of the matrix

$$\begin{bmatrix} \dfrac{\partial}{\partial x^i} \\ \dfrac{\partial}{\partial \bar z^\beta} \end{bmatrix} + \begin{pmatrix} \psi_i^\gamma \\ \psi_{\bar\beta}^\gamma \end{pmatrix} \left(\frac{\partial}{\partial z^\gamma} \right). \tag{3.29}$$

Since f is near the identity, the matrix

$$M = \begin{pmatrix} M_{ij} & M_{i\alpha} \\ M_{\bar\beta j} & M_{\bar\beta\alpha} \end{pmatrix} \tag{3.30}$$

is invertible. Combining (3.26) and (3.29) we see that

$$\begin{pmatrix} \psi_i^\gamma \\ \psi_{\bar\beta}^\gamma \end{pmatrix} = M^{-1} \circ N, \tag{3.31}$$

where

$$N = \begin{pmatrix} N_{i\alpha} \\ N_{\bar{\beta}\alpha} \end{pmatrix}. \tag{3.32}$$

We summarize our results in the following lemma.

Lemma 3.33. *Let $\varphi \in \mathrm{Hom}(E, Q^{(1,0)})$, and let f be a diffeomorphism of M near the identity in the Whitney C^∞-topology. Then in adapted coordinates $\psi = f_*(\varphi)$ is given by* (3.31).

We will now apply Lemma 3.33 to diffeomorphisms associated to geodesics. Considering $Q^{(1,0)}$ as a real vector bundle, we see that the map τ of (1.4) induces an isomorphism $Q \xrightarrow{\tau} Q^{(1,0)}$. Use τ to identity $Q^{(1,0)}$ with $Q \subseteq T_M$. See [26]. Let $X \in \Gamma(Q^{(1,0)}) \subseteq \Gamma(T_M)$ be a vector field close to zero in the C^∞-topology. Since M is compact, it is complete in our metric. Consider the map $f(X, \cdot)\colon M \to M$ defined by

$$f(X, y) \equiv \gamma(X, y, 1), \tag{3.34}$$

where $t \to \gamma(X, \gamma, t)$ is the geodesic with initial conditions

$$\gamma(X, y, 0) = y, \quad \gamma'(X, y, 0) = X(y). \tag{3.35}$$

For X small, $f(X,\cdot)$ is a diffeomorphism of M. We wish to express $f(X,\cdot)$ locally as a Taylor series in the components of X, and use this expansion to represent (3.31) in terms of the components of X. In adapted coordinates $f(X, x, z) = (f^j(X, x, z), f^\alpha(X, x, z))$ and since $f(tX, x, z) = \gamma(X, (x, z), t)$ the equations

$$\begin{aligned} X^\alpha \frac{\partial f^i}{\partial X^\alpha}(0, x, z) + \bar{X}^\alpha \frac{\partial f^j}{\partial \bar{X}^\alpha}(0, x, z) &= \frac{d}{dt}\gamma^j(X, (x, z), 0) = 0, \\ X^\alpha \frac{\partial f^\beta}{\partial X^\alpha}(0, x, z) + \bar{X}^\alpha \frac{\partial f^\beta}{\partial \bar{X}^\alpha}(0, x, z) &= \frac{d}{dt}\gamma^\beta(X, (x, z), 0) = X^\beta \end{aligned} \tag{3.36}$$

are satisfied, where $X = X^\alpha \partial/\partial z^\alpha$. Therefore

$$\frac{\partial f^j}{\partial X^\alpha} = \frac{\partial f^j}{\partial \bar{X}^\alpha} = \frac{\partial f^\beta}{\partial \bar{X}^\alpha} = 0, \tag{3.37}$$

$$\frac{\partial f^\beta}{\partial X^\alpha} = \delta^\beta_\alpha. \tag{3.38}$$

Hence f is of the form

$$\begin{aligned} f^j(X, x, z) &= x^j + X^\alpha X^\beta r^j_{\alpha\beta}(X, x, z), \\ f^\alpha(X, x, z) &= z^\alpha + X^\alpha + X^\beta X^\gamma r^\alpha_{\beta\gamma}(X, x, z). \end{aligned} \tag{3.39}$$

Now for X close to zero, the matrix M can be written in the form $I + A_{tX}$, where $A_{tX} = t\tilde{A}_{(X,t)}$, and $\tilde{A}_{(X,t)}$ is a matrix-valued C^∞-function in X^α,

$\partial X^\alpha/\partial x^i$, $\partial X^\alpha/\partial z^\beta$, $\partial X^\alpha/\partial \bar{z}^\beta$, φ_i^α, $\varphi_{\bar{\beta}}^{\underline{\alpha}}$ and t. Hence

$$M_{tX,\varphi}^{-1} = \sum_{l=0}^{\infty} (-1)^l A_{tX,\varphi}^l = I + H_{tX,\varphi}, \tag{3.40}$$

where H is C^∞ in the variables X^α, $\partial X^\alpha/\partial x^i$, etc. Also N can be expressed in the form

$$N_{tX} = \begin{pmatrix} \varphi_i^\alpha \\ \varphi_{\bar{\beta}}^{\underline{\alpha}} \end{pmatrix} + t \begin{bmatrix} \dfrac{\partial X^\alpha}{\partial x^i} \\ \dfrac{\partial X^\alpha}{\partial \bar{z}^\beta} \end{bmatrix} + tK_{tX,\varphi}, \tag{3.41}$$

where $K_{tX,\varphi}$ is C^∞ in the variables X^α, $\partial X^\alpha/\partial x_i$, etc. (3.40) and (3.41) allow us to write (3.31) in the form:

$$\begin{pmatrix} \psi_i^\alpha \\ \psi_{\bar{\beta}}^{\underline{\alpha}} \end{pmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x^i}(X^\alpha) \\ \dfrac{\partial}{\partial \bar{z}^\beta}(X^\alpha) \end{bmatrix} + \begin{pmatrix} \varphi_i^\alpha \\ \varphi_{\bar{\beta}}^{\underline{\alpha}} \end{pmatrix} + R(X, \varphi), \tag{3.42}$$

where $R(tX, t\varphi) = t^2 R_1(X, \varphi, t)$, and R_1 is C^∞ in t, φ, X and their derivatives. In invariant form, (3.42) reads

$$f_* \varphi = d_Q X + \varphi + R(\psi, X), \tag{3.43}$$

where $R(t\psi, tX) = t^2 R_1(\psi, X, t)$, and R_1 is C^∞ in t, X, φ and their derivatives.

We will now use (3.43) to show that if $\varphi \in \operatorname{Hom}(E, Q^{(1,0)})$ is a solution of the equation $D\varphi = 0$ with $\|\varphi\|_s$ sufficiently small, then there is a unique element $\varphi(t)$ with $\sum t_i \varphi_i \in B$ and a unique vector field $X \in \Gamma(Q^{(1,0)})$ with $f_*(X, \cdot)(\varphi(t)) = \varphi$. This will complete Step 2 and the proof of Theorem 3.1.

Proposition 3.44. *Let $H^\perp$ be the orthogonal complement of the space $\Gamma(\underline{\theta}_{\mathcal{F}})$ of $\mathcal{F}$-invariant holomorphic vector fields in $\Gamma(Q^{1,0})$. Then there is a neighborhood U of the origin of $H^\perp$ and a neighborhood V_1 of the origin of $\Gamma(E_Q^{*1}) =$* $\operatorname{Hom}(E, Q^{(1,0)})$ *such that for any element $\varphi \in V$ satisfying the equation $D\varphi = 0$, there is a unique element $X \in U$ with $f_*(X, \cdot)\varphi = \varphi(t)$ for $\sum t_i \varphi_i \in B$.*

Proof. Set $f = f(X, \cdot)$. Then $f_* \varphi$ is of the required form, provided only that $\delta_Q(f_* \varphi) = 0$. This follows from Step 1. But $\delta_Q(f_* \varphi) = 0$ if and only if

$$\delta_Q d_Q X + \delta_Q \varphi + \delta_Q R(\varphi, X) = 0 \tag{3.45}$$

by (3.43). Since $X \in H^\perp$ it satisfies the equation

$$X = G_Q \Delta_Q X \equiv G_Q \delta_Q d_Q X. \tag{3.46}$$

Hence $\delta_Q(f_* \varphi) = 0$ if and only if

$$(3.47) \qquad G_Q\big(\delta_Q d_Q X + \delta_Q \varphi + \delta_Q R(\varphi, X)\big) = 0,$$

or

$$(3.48) \qquad X + G_Q \delta_Q \varphi + G_Q \delta_Q R(\varphi, x) = 0.$$

We will use the implicit function theorem to find such an X. Define a map

$$(3.49) \qquad h\colon U_1 \times V_1 \subseteq H^{\perp} \times \Gamma\big(E_Q^{*1}\big) \to H^{\perp}$$

by

$$h(X, \varphi) = X + G_Q \delta_Q + G_Q \delta_Q R(\varphi, X),$$

where U_1 and V_1 have been chosen so that R is defined. If U_1, V_1 and $H^{\perp}$ are given the topology induced by the Sobolev norm, then h is continuous and the Frechet derivative $\partial h/\partial X|_{(0,0)}$ is the identity map. Hence, by the implicit function theorem, there is a C^{∞}-function $g\colon V \to U$ such that (3.48) holds if and only if $X = g(\varphi)$ for $\varphi \in V$. To see that X is smooth, note that it satisfies the second order elliptic equation with C^{∞} coefficients

$$\Delta_Q X + \delta_Q R(\varphi, X) + \delta_Q = 0.$$

Hence X is smooth by the regularity theorem.

4. Computation of $H^{\bullet}(M, \underset{\sim}{\theta}_{\mathcal{F}})$

We now investigate the cohomology groups $H^{\bullet}(M, \underset{\sim}{\theta}_{\mathcal{F}})$. We begin by defining a filtration on the complex (1.28).

Let $\underset{\sim}{Q}^{(p,q)*}$ denote the sheaf of germs of sections of the bundle $\Lambda^p Q^{(1,0)*} \otimes \Lambda^q Q^{(1,\bar{0})*}$. Then the differential complex (1.28) is filtered as follows. For $s \geqslant 0$ let

$$(4.1) \qquad F^s \underset{\sim}{E}_Q^{*\bullet} = \underset{\sim}{Q}^{(0,s)*} \wedge \underset{\sim}{E}_Q^{*\bullet - s}.$$

Observe that $d_Q(F^s \underset{\sim}{E}_Q^{*\bullet}) \subset F^s \underset{\sim}{E}_Q^{*\bullet}$, as can easily be seen from the formulas (1.16) and (1.30) for d_{ε} and d_Q. Associated to this filtration is a spectral sequence converging to $H^{\bullet}(M, \underset{\sim}{\theta}_{\mathcal{F}})$. The edge terms of this spectral sequence are of particular interest to us. Let $E_{Q,\mathcal{F}}^{*s}$ be the subsheaf of $\underset{\sim}{E}_Q^{*s}$ consisting of sections which in adapted coordinates are of the form

$$(4.2) \qquad \varphi = \varphi_{\beta,\alpha}(z)\, d\bar{z}^{\beta} \otimes \left[\frac{\partial}{\partial z^{\alpha}}\right].$$

Such sections are invariant under Lie differentiation with respect to vector fields tangent to $\mathcal{F}$ and are therefore called $\mathcal{F}$-invariant sections. The restriction of d_Q to $E_{Q,\mathcal{F}}^{*\bullet}$ is denoted by $\bar{\partial}$ and applied to a section φ as in (4.2) is of the form

$$(4.3)\qquad \bar{\partial}\varphi = \frac{\partial \varphi_{\beta,\alpha}}{\partial \bar{z}^{\gamma}}\, d\bar{z}^{\gamma} \wedge d\bar{z}^{\beta} \otimes \left[\frac{\partial}{\partial z^{\alpha}}\right].$$

Clearly $\bar{\partial}(E^{*\bullet}_{Q,\mathfrak{F}}) \subseteq E^{*\bullet}_{Q,\mathfrak{F}}$ and there is a complex

$$(4.4)\qquad 0 \to \underset{\sim}{Q}^{(1,0)}_{\mathfrak{F}} \to E^{*0}_{Q,\mathfrak{F}} \xrightarrow{\bar{\partial}} E^{*1}_{Q,\mathfrak{F}} \xrightarrow{\bar{\partial}} \cdots.$$

Since $E = L^{C} \oplus Q^{(1,0)}$, there is an exact sequence

$$0 \to Q^{(0,1)*} \to E^{*} \to L^{C*} \to 0,$$

which induces exact sequences

$$(4.5)\qquad 0 \to F^{p+1}\underset{\sim}{E}{}^{*\bullet}_{Q} \to F^{p}\underset{\sim}{E}{}^{*\bullet}_{Q} \xrightarrow{\tau} \Lambda^{\bullet - p}\underset{\sim}{L}{}^{C*} \otimes_{C^{\infty}_{F}} Q^{(0,p)*} \otimes_{\mathfrak{F}} Q^{(1,0)} \to 0.$$

Now $\tau^{\bullet} d_{Q} = d_{Q_{\|}}{}^{\bullet}\tau$, hence by (1.33) we get the following result.

Lemma 4.6. $H^{\bullet}(gr^{p}(\underset{\sim}{E}{}^{*\bullet}_{Q}), gr(d_{Q})) \simeq H^{\bullet}(M, Q^{(0,p)*} \otimes Q^{(1,0)}_{\mathfrak{F}})$.

The next proposition follows from (4.4) and (4.6).

Proposition 4.7. *The spectral sequence induced by the filtration* $F^{\bullet}$ *of* $E^{*\bullet}_{Q}$ *converges to* $H^{\bullet}(M, \underset{\sim}{\theta}_{\mathfrak{F}})$. *More specifically,* $E^{s,t}_{1} = (H^{t}(M, Q^{(0,s)*} \otimes Q^{(1,0)}_{\mathfrak{F}})) \Rightarrow H^{s+t}(M, \underset{\sim}{Q}^{(1,0)}_{\mathfrak{F}})$ *and* $E^{s,0}_{2} = H^{s}(\Gamma(\underset{\sim}{E}{}^{*\bullet}_{Q,\mathfrak{F}}), \bar{\partial})$.

Recall that a V-manifold is an analytic space which locally has the structure of the orbit space defined by a finite group action on an open disc in $\mathbf{C}^{q}$ where the group acts by biholomorphisms. By [3], if $\mathfrak{F}$ is a Hausdorff foliation, the leaf space $M/\mathfrak{F}$ has the structure of a V-manifold of the complex dimension q of the normal bundle to $\mathfrak{F}$. For details concerning V-manifolds, see Satake [12]. In case $\mathfrak{F}$ has no holonomy, then $M/\mathfrak{F}$ is non-singular and $M \to M/\mathfrak{F}$ is a fibration. A V-manifold N has a Dolbeaut complex defined on it and a holomorphic tangent bundle $\underset{\sim}{\theta}_{N}$. Bailey [1] has shown that the cohomology groups $H^{\bullet}(N, \underset{\sim}{\theta}_{N})$ are finite dimensional. From the definition of the holomorphic tangent bundle of a V-manifold we have the following proposition.

Proposition 4.8. *If* $\mathfrak{F}$ *is Hausdorff, then* $E^{s,0}_{2} \cong H^{s}(M/\mathfrak{F}, \underset{\sim}{\theta}_{M/\mathfrak{F}})$ *and this space is finite dimensional. Furthermore, if* S *denotes the generic leaf of* $\mathfrak{F}$ *and* $H^{1}(S, \mathbf{R}) = 0$, *then* $H^{1}(M, \underset{\sim}{\theta}_{\mathfrak{F}}) \cong H^{1}(M/\mathfrak{F}, \underset{\sim}{\theta}_{M/\mathfrak{F}})$.

Proof. The first part of the proposition is immediate from the definitions.

To prove the second part of the proposition observe that

$$E^{0,1}_{1} = H^{1}(M, \underset{\sim}{Q}^{(1,0)}_{\mathfrak{F}}) = H^{1}(M, \underset{\sim}{Q}_{\mathfrak{F}}),$$

since $Q \cong Q^{(1,0)}$ by (1.2) and (1.4). Since Hamilton [4] has shown that $H^{1}(S, \mathbf{R}) = 0$ implies that $H^{1}(M, \underset{\sim}{Q}_{\mathfrak{F}}) = 0$, we have $E^{0,1}_{2} = 0$ and $E^{1,0}_{2} \cong H^{1}(M/\mathfrak{F}, \underset{\sim}{\theta}_{M/\mathfrak{F}})$ from which the result follows.

At this point we wish to present some cases where the groups $H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}})$ can be computed explicitly. The computations use standard techniques in sheaf theory and are quite similar to those of Mostow [9]. Therefore we will be brief.

We begin by considering the trivial example of a product foliation. Suppose that N is a complex manifold and that K is a compact C^∞-manifold with $\dim_{\mathbf{C}} N = q$, $\dim_{\mathbf{R}} K = p$. Now let $M = N \times K$ and define $\mathcal{F}$ to be the foliation on M given by the fibers of the projection $M \xrightarrow{\pi} N$. Then $\underset{\sim}{\theta}_{\mathcal{F}} = \pi^*(\underset{\sim}{\theta}_N)$. By Bredon [2] we obtain the next lemma.

Lemma 4.9. *$H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}}) \cong H^\bullet(N, \underset{\sim}{\theta}_N) \otimes \mathbf{C}\, H^\bullet_{DR}(K\colon \mathbf{C})$. In particular, if N is Stein $H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}}) \cong \Gamma(N, \underset{\sim}{\theta}_N) \otimes_{\mathbf{C}} H_{DR}(K, \mathbf{C})$.*

If N is compact this implies the following corollary.

Corollary 4.10. *The set of holomorphic foliations near the holomorphic foliation $\mathcal{F}$, given as above, is a local analytic subset of the complex vector space $H^1(N, \underset{\sim}{\theta}_N) \oplus H^1_{DR}(K, \mathbf{C}) \otimes \Gamma(N, \underset{\sim}{\theta}_N)$.*

Assume that M is compact and that $\mathcal{F}$ is a Hausdorff holomorphic foliation transverse to the fibers of a fibration $N \to M \to X$. Then N is a compact complex manifold and $M \simeq \tilde{X} \times N/G$, where $\tilde{X}$ is a finite cover of X and $\tilde{M} = \tilde{X} \times N$ is a G manifold for G a finite group of deck transformations of $\tilde{M}$ which acts biholomorphically on N. Further, $\mathcal{F}$ is the foliation $\tilde{\mathcal{F}}/G$ for $\tilde{\mathcal{F}}$ the product foliation $\tilde{X} \times N \to N$. In this case G acts on $H^\bullet(N, \underset{\sim}{\theta}_N)$ and on $H^\bullet_{DR}(\tilde{X}, \mathbf{C})$ and we have the following proposition.

Proposition 4.11. *$H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}}) \cong H^\bullet_{DR}(\tilde{X}, \mathbf{C})^G \otimes_{\mathbf{C}} H^\bullet(N, \underset{\sim}{\theta}_N)^G$ where $(\ )^G$ denotes the space of G-invariant elements.*

Proof. Consider the resolution (1.29) applied to $\tilde{\mathcal{F}}$ on $\tilde{X}$, i.e.,

$$0 \to \underset{\sim}{\tilde{Q}}^{(1,0)} \to \underset{\sim}{\tilde{E}}^0_Q \xrightarrow{d_{\tilde{Q}}} \underset{\sim}{\tilde{E}}^1_Q \to \cdots.$$

Then since G is finite $H^\bullet(\Gamma(\tilde{E}_Q)^G, d_{\tilde{Q}}) = H^\bullet(\Gamma(\tilde{E}_Q), d_{\tilde{Q}})^G$, and $\Gamma(\tilde{E}_Q)^G$ is isomorphic to the complex

$$0 \to \Gamma(\underset{\sim}{\theta}_{\mathcal{F}}) \to \Gamma\big(E^0_Q\big) \to \cdots$$

associated to the resolution of $\underset{\sim}{\theta}_{\mathcal{F}}$. Therefore

$$H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}}) \cong H^\bullet\left(\Gamma\big(\tilde{E}_Q\big)^G, \tilde{d}_{\tilde{Q}}\right) \simeq H^\bullet(M, \underset{\sim}{\theta}_{\mathcal{F}})^G.$$

Note the above computation applies to the case where $\mathcal{F}$ is given by the suspension via a biholomorphism $\varphi\colon N \to N$, where N is a compact complex manifold and φ has finite period.

References

[1] W. Bailey, *The decomposition theorem for V manifolds*, Amer. J. Math. **78** (1956) 862–888.
[2] G. Bredon, *Sheaf theory*, McGraw-Hill, New York, 1967.
[3] D. B. A. Epstein, *Foliations with all leaves compact*, Dynamical Systems-Warwick, Lecture Notes in Math. Vol. 468, Springer, Berlin, 1974.
[4] R. S. Hamilton, *Deformation theory for foliations*, preprint.
[5] J. L. Heitsch, *A cohomology for foliated manifolds*, Comment. Math. Helv. **50** (1975) 197–218.
[6] F. Kamber & Ph. Tondeur, *Invariant differential operators and the cohomology of Lie algebra sheaves*, Mem. Amer. Math. Soc. No. 113, 1971.
[7] K. Kodaira, L. Nirenberg & D. Spencer, *On the existence of deformations of complex analytic structures*, Ann. of Math. **68** (1958) 450–459.
[8] M. Kuranishi, *New proof for the existence of locally complete families of complex analytic structures*, Proc. Conf. Complex Analysis (Minneapolis), Springer, Berlin, 1965, 142–154.
[9] M. Mostow, *Continuous cohomology of spaces with two topologies*, Mem. Amer. Math. Soc. **7** (1976) 1975.
[10] R. Narasimhan, *Analysis on real and complex manifolds*, North Holland, Amsterdam, 1968.
[11] L. Nirenberg, *A complex Frobenius theorem*, Seminar on Analytic Functions, Institute for Advanced Study, Princeton, 1957, 172–189.
[12] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956) 359–363.
[13] D. C. Spencer, *Deformations of structures on manifolds defined by transitive continuous pseudogroups*. I, II, Ann. of Math. **76** (1962) 306–445.
[14] R. O. Wells, Jr., *Differential analysis on complex manifolds*, Prentice-Hall, Englewood Cliffs, N. J. 1973.
[15] I. Vaisman, *Variétés Riemannienne feuilletées*, Czechoslovak Math. J. **21** (1971) 46–75.

UNIVERSITY OF UTAH