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NONNEGATIVE CURVATURE OPERATORS: SOME NONTRIVIAL EXAMPLES

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1. Introduction

The object of this paper is to study the pointwise behavior of the Riemannian sectional curvature function.

More specifically, the Riemannian sectional curvature of a Riemannian manifold M is a real valued function σ on the Grassmann bundle of tangent 2-planes of M. Although there exist many theorems relating the curvature of M to various topological and geometric properties of M, there is little known of a general nature about the behavior of σ itself. In fact the critical point behavior of σ has been analyzed only in very special cases [1], [4].

Let G denote the Grassmann manifold of oriented tangent 2-planes at $m \in M$. G can be made, in a natural way, a submanifold of the vector space Λ^2 of 2-vectors at m. Furthermore, since G is a 2-fold covering space of the manifold of (unoriented) 2-planes at m, we may regard σ as a function on G. We will be interested in the description of the minimum and maximum sets of σ and in the question of characterizing positive sectional curvature in terms of the curvature tensor.

Since we are interested in the pointwise behavior of σ, we shall work in the setting of an arbitrary inner product space V. G is then the Grassmann manifold of oriented 2-planes in V. A curvature operator R is a self-adjoint linear transformation of $\Lambda^2(V)$ (e.g., the curvature tensor R of a Riemannian manifold M acting on $\Lambda^2(M_m)$, where M_m is the tangent space to M at m). For a curvature operator R, its sectional curvature $\sigma_R\colon G \to \mathbf{R}$ is given by $\sigma_R(P) = \langle RP, P\rangle$ for P in G.

For dimension $V \leqslant 4$, Thorpe has shown [3] that the minimum and maximum sets of σ_R are intersections with G of linear subspaces of $\Lambda^2(V)$, and he has given [2] a simple characterization of positive sectional curvature in terms of the curvature tensor. In fact, Thorpe [3] claimed that this

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description of the minimum and maximum sets of σ_R was true for all dimensions.

In what follows, we shall show that these results do not hold for higher dimensions. More specifically, for dimension $V \geqslant 5$ we exhibit a family of curvature operators with nonnegative sectional curvature each of whose members does not conform to the characterization suggested by Thorpe's result [2] for lower dimensions. Furthermore, it is shown that one member of this family has a zero set which is not the intersection with G of a linear subspace of $\Lambda^2(V)$ and so contradicts Thorpe's result in [3].

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2. Preliminaries

Let V be an n-dimensional real vector space with inner product $\langle\ ,\ \rangle$, and for $v \in V$ set $|v| = \sqrt{\langle v, v\rangle}$. For p an integer, $1 \leqslant p \leqslant n$, by $\Lambda^p(V)$ or Λ^p we mean the space of p-vectors of V. If $\{e_1, \cdots, e_n\}$ is a basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | i_1 < \cdots i_p\}$ is a basis for Λ^p, and it follows that Λ^p has dimension $\binom{n}{p}$. A p-vector ω is said to be decomposable if $\omega = v_1 \wedge \cdots \wedge v_p$ where $v_1, \cdots, v_p \in V$. Hence Λ^p has a basis of decomposable vectors. Thus when defining an inner product on Λ^p it suffices to specify its values on decomposable p-vectors. We set $\langle u_1 \wedge \cdots \wedge u_p,\ v_1 \wedge \cdots \wedge v_p\rangle = \det[\langle u_i, v_j\rangle]$ where $u_i, v_j \in V$. For $\xi \in \Lambda^2$ we set $\|\xi\| = \sqrt{\langle \xi, \xi\rangle}$. It follows that if $\{e_1, \cdots, e_n\}$ is an orthonormal basis for V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | i_1 < \cdots < i_p\}$ is an orthonormal basis for Λ^p. Let G denote the Grassmann manifold of oriented 2-dimensional subspaces of V; we identify G with the submanifold of Λ^2 consisting of decomposable 2-vectors of length 1 by $p \to u \wedge v$ where $\{u, v\}$ is any oriented orthonormal basis for P.

Let V be an n-dimensional real inner product space. A curvature operator R is a self-adjoint linear transformation of $\Lambda^2(V)$. The space $\mathfrak{R}$ of all curvature operators has dimension $[\binom{n}{2}^2 + \binom{n}{2}]/2$ and inner product given by $\langle R, T\rangle = \operatorname{trace} R \circ T$ where $R, T \in \mathfrak{R}$. Given $R \in \mathfrak{R}$ its sectional curvature is the function $\sigma_R\colon G \to \mathbf{R}$ defined by $\sigma_R(P) = \langle Rp, P\rangle$, $P \in G$. We define the zero set of R by $Z(R) = \{P \in G | \sigma_R(P) = 0\}$.

Let $\{e_1, \cdots, e_n\}$ be an oriented orthonormal basis for V. We define the *star operator*

$$*\colon \Lambda^p \to \Lambda^{n-p}$$

by

$$\langle *\alpha, \beta\rangle = \langle \alpha \wedge \beta, e_1 \wedge \cdots \wedge e_n\rangle,$$

where $\alpha \in \Lambda^p$ and $\beta \in \Lambda^{n-p}$. It is easily checked that this definition is independent of the choice of oriented orthonormal basis for V. It is also easily checked that $*^2 = (-1)^{p(n-p)}$ (identity) and so $*$ is nonsingular (see [5]).

If dimension $V = 4$ and $p = 2$, then $*: \Lambda^2 \to \Lambda^2$, and since $\alpha \wedge \beta = \beta \wedge \alpha$ for $\alpha, \beta \in \Lambda^2$, it follows that $*$ is symmetric.

By $\mathbf{R}$ we denote the set of all real numbers.

3. The Bianchi identity and the Grassmann quadratic 2-relations

In this section we examine the space $\mathfrak{S}$ complementary in $\mathfrak{R}$ to the subspace $\mathfrak{B} = \{R \in \mathfrak{R} \mid R$ satisfies the Bianchi identity$\}$. We recall that $\mathfrak{S}$ is naturally isomorphic to Λ^4, and we exhibit the relationship between $\mathfrak{S}$ and the Grassmann quadratic 2-relations which are necessary and sufficient conditions for decomposability of elements in Λ^2. These results are well-known and detailed proofs can be found in [3].

Given $R \in \mathfrak{R}$ we associate a 2-form on V with values in the vector space of skew symmetric endomorphisms of V by

$$\langle R(u, v)(w), x\rangle = \langle Ru \wedge v, w \wedge x\rangle, \quad u, v, w, x \in V.$$

It is easily checked that this "association" is a vector space isomorphism.

Using this identification we define the Bianchi map b: $\mathfrak{R} \to \mathfrak{R}$. Given $R \in \mathfrak{R}$ we set

$$[b(R)](u, v)(w) = R(u, v)(w) + R(v, w)(u) + R(w, u)(v).$$

It is easily checked that b is a linear map, and so its kernel is a linear subspace of $\mathfrak{R}$ which we will denote by $\mathfrak{B}$.

Let $\mathfrak{S} = \mathfrak{B}^{\perp}$, the orthogonal compliment of $\mathfrak{B}$ in $\mathfrak{R}$. For each $\varepsilon \in \Lambda^4$ we associate $S_\varepsilon \in \mathfrak{R}$ by $\langle S_\varepsilon \alpha, \beta\rangle = \langle \varepsilon, \alpha \wedge \beta\rangle$, where $\alpha, \beta \in \Lambda^2$.

Proposition 3.1. *The map $\varepsilon \to S_\varepsilon$ is an isomorphism of Λ^4 onto $\mathfrak{S}$. In fact $\varepsilon \to S_\varepsilon/\sqrt{6}$ is an isometry.*

Proposition 3.2. *Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V. For $1 \leq i \leq j \leq n$, set $S_{ijkl} = S_{e_i \wedge e_j \wedge e_k \wedge e_l}$. $\alpha \in \Lambda^2$ is decomposable if and only if $\langle S_{ijkl}\alpha, \alpha\rangle = 0$, $1 \leq i < j < k < l \leq n$.*

Corollary 3.3. *$\alpha \in \Lambda^2$ is decomposable if and only if $\alpha \wedge \alpha = 0$.*

Proof. Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V. Then

$$\alpha = \sum_{1 \leq i < j \leq n} a_{ij} e_i \wedge e_j,$$

$$\alpha \wedge \alpha = 2 \sum_{1 \leq i < j < k < l \leq n} (a_{ij}a_{kl} - a_{ik}a_{jl} + a_{il}a_{jk}) e_i \wedge e_j \wedge e_k \wedge e_l$$

$$= \sum_{1 \leq i < j < k < l \leq n} \langle S_{ijkl}\alpha, \alpha\rangle = 0,$$

and so by Proposition 3.2 if and only if α is decomposable.

Remark 1. The conditions $\langle S_{ijkl}\alpha, \alpha\rangle = 0$, $1 \leqslant i < j < k < l \leqslant n$, are known as the Grassmann quadratic 2-relations.

Remark 2. In view of Proposition 3.2 it is clear that each curvature operator $S \in \mathbb{S}$ has sectional curvature σ_S identically zero. Conversely, it is easily checked that this property characterizes $\mathbb{S}$.

4. Two results of Thorpe

In this section we restrict ourselves to the case where dimension $V = 4$, and state the two results of Thorpe which form the main concern of this paper.

Let $\mathfrak{R}^+\{R \in \mathfrak{R}: \langle RX, X\rangle \geqslant 0\ \forall X \in \Lambda^2\}$ and $\mathfrak{B}^+ = \{R \in \mathfrak{B}: \sigma_R \geqslant 0\}$. By definition of $\mathbb{S}$ and $\mathfrak{B}$, $\mathfrak{R} = \mathfrak{B} \otimes \mathbb{S}$, where $\oplus$ means orthogonal direct sum. We define π as orthogonal projection from $\mathfrak{R}$ into $\mathfrak{B}$. Since $\sigma_R = \sigma_{B+S} = \sigma_B$, it follows that $\pi(\mathfrak{R}^+) \subseteq \mathfrak{B}^+$, and so we can consider π as a map of $\mathfrak{R}^+$ into $\mathfrak{B}^+$.

Theorem 4.1. *If dimension $V = 4$, then the map*

$$\pi: \mathfrak{R}^+ \to \mathfrak{B}^+$$

is onto.

Theorem 4.2. *Let dimension $V = 4$, and suppose $R \in \mathfrak{R}$ is such that $\sigma_R \geqslant 0$ and $Z(R) \neq \varnothing$. Then there exists a unique $S \in \mathbb{S}$ such that $Z(R) = G \cap \operatorname{kernel}(R + S)$.*

Proofs of these theorems appear in [2] and [3] respectively.

Corollary 4.3. *Let dimension $V = 4$ and $R \in \mathfrak{R}$, and let λ denote the minimum (or maximum) value of σ_R. Then there exists a unique $S \in \mathbb{S}$ such that $\{P \in G | \sigma_R(P) = \lambda\} = G \cap \ker(R - \lambda I - S)$.*

Proof. This corollary follows from Theorem 4.2 by replacing R in that theorem by $R - \lambda I$ (or, when λ is the maximum value of σ_R, by $\lambda I - R$).

5. Dense subsets of G

In this section dimension $V = 5$. We describe a collection of dense subsets of the Grassmann manifold G of oriented two-dimensional subspaces of V. Specifically, given $P \in G$, we construct a dense subset of G which contains P. In the following sections this tool will greatly simplify our calculations.

Theorem 5.1. *Given $P \in G$, let $\{e_1, \cdots, e_5\}$ be an orthonormal basis of V such that $P = e_1 \wedge e_2$. If for $x_1, \cdots, x_5 \in \mathbf{R}$ we set $(x_1, x_2, x_3, x_4, x_5) =$*

$\Sigma_{i=1}^{5} x_i e_i$, *then*

$$Q = \left\{ \frac{(1, 0, x_3, x_4, x_5) \wedge (0, 1, y_3, y_4, y_5)}{\|(1, 0, x_3, x_4, x_5) \wedge (0, 1, y_3, y_4, y_5)\|} : x_3, x_4, x_5, y_3, y_4, y_5 \in R \right\}$$

is a dense subset of G which contains P.

To prove Theorem 5.1 we will need the following lemma.

Lemma 5.2. $G - Q = \{P \in G | \langle P, e_1 \wedge e_2 \rangle = 0\}$.

Proof. (Using the notation of Theorem 5.1.)

$$P \in G \Rightarrow P = \frac{(x_1, x_2, x_3, x_4, x_5) \wedge (y_1, y_2, y_3, y_4, y_5)}{\|(x_1, x_2, x_3, x_4, x_5) \wedge (y_1, y_2, y_3, y_4, y_5)\|}.$$

Now if

$$\langle P, e_1 \wedge e_2 \rangle = \frac{x_1 y_2 - x_2 y_1}{\|(x_1, x_2, x_3, x_4, x_5) \wedge (y_1, y_2, y_3, y_4, y_5)\|} \neq 0,$$

where for $i = 2, \cdots, 5$ we abusively denote x_i/x_1 by x_i. Replacing y_i by $y_i - x_i y_1$ we get

$$P = \frac{(1, x_2, x_3, x_4, x_5) \wedge (0, y_2, y_3, y_4, y_5)}{\|(1, x_2, x_3, x_4, x_5) \wedge (0, y_2, y_3, y_4, y_5)\|},$$

then either $x_1 \neq 0$ or $y_1 \neq 0$. We can assume $x_1 \neq 0$ (by interchanging x's and y's if necessary). Dividing each x_i by x_1 we get

$$P = \frac{(1, x_2, x_3, x_4, x_5) \wedge (y_1, y_2, y_3, y_4, y_5)}{\|(1, x_2, x_3, x_4, x_5) \wedge (y_1, y_2, y_3, y_4, y_5)\|},$$

where for $i = 2, \cdots, 5$ we abusively denote $y_i - x_i y_1$ by y_i. Since $0 \neq \langle P, e_1 \wedge e_2 \rangle = y_2$ (the new y_2) we can divide each y_i by y_2 to get

$$P = \frac{(1, x_2, x_3, x_4, x_5) \wedge (0, 1, y_3, y_4, y_5)}{\|(1, x_2, x_3, x_4, x_5) \wedge (0, 1, y_3, y_4, y_5)\|},$$

where for $i = 2, \cdots, 5$ we abusively denote y_i/y_2 by y_i. Finally by replacing x_i by $x_i - y_i x_2$ we get

$$P = \frac{(1, 0, x_3, x_4, x_5) \wedge (0, 1, y_3, y_4, y_5)}{\|(1, 0, x_3, x_4, x_5) \wedge (0, 1, y_3, y_4, y_5)\|},$$

where for $i = 3, 4, 5$ we abusively denote $x_i - y_i x_2$ by x_i. Hence we have shown $G - Q \subset \{P \in G | \langle P, e_1 \wedge e_2 \rangle = 0\}$.

It is clear that if $P \in Q \Rightarrow \langle P, e_1 \wedge e_2 \rangle \neq 0$, and so

$$\{P \in G | \langle P, e_1 \wedge e_2 \rangle = 0\} \subset G - Q.$$

Proof of Theorem **5.1.** Lemma 5.2 shows that the complement of Q in G is $G - Q = \{P \in G | \langle P, e_1 \wedge e_2 \rangle = 0\}$. Since the function $P \to \langle P, e_1 \wedge e_2 \rangle$ is a smooth function on G, it follows by the implicit function theorem that $G - Q$ has codimension one in G, and therefore Q is dense in G.

6. The curvature operator R_k

In this section we discuss the possibility of extending Theorems 4.1 and 4.2 to the case dimension $V \geqslant 5$.

Two claims are made and an example is presented. It will be the analysis of this example which occupies most of the remaining sections and results in a verification of these claims.

Claim 6.1. *When the dimension $V \geqslant 5$, the zero set of a curvature operator with nonnegative sectional curvature need not be the intersection with G of a linear subspace of Λ^2.*

Claim 6.2. *The map π, defined in §4, need not be onto. Indeed for dimension $V \geqslant 5$, there exist curvature operators with nonnegative sectional curvature which cannot be made positive semi-definite by adding an element of Λ^4.*

Until further notice, dimension $V = 5$. Let $\{e_1, \cdots, e_5\}$ be an orthonormal basis for V, and k a real number. Set $e_{ij} = e_i \wedge e_j$ and consider the following example.

Let R_k: $\Lambda^2 \to \Lambda^2$ be defined by

$$\begin{array}{lll} R_k e_{12} = e_{12} - e_{15} - e_{34}, & R_k e_{24} = R_k e_{35} = 0, & R_k e_{23} = k e_{23}, \\ R_k e_{15} = e_{15} - e_{12} - e_{34}, & R_k e_{13} = k e_{13}, & R_k e_{25} = k e_{25}, \\ R_k e_{34} = e_{34} - e_{12} - e_{15}, & R_k e_{14} = k e_{14}, & R_k e_{45} = k e_{45}. \end{array}$$

It is easily checked that R_k is self-adjoint. Let $\alpha = e_{12} + e_{15} + e_{34}$. Then $R\alpha = -\alpha$.

In the next section it will be shown that R_k has nonnegative sectional curvature.

7. The sectional curvature of R_k

In this section we will analyze sectional curvature on a dense subset of G containing the zero e_{24} of R_k. The sectional curvature of R_k will be shown to be nonnegative on this subset and so on all of G.

By Theorem 5.1

$$Q = \left\{ \frac{(\alpha, 1, \beta, 0, \gamma) \wedge (\delta, 0, \varepsilon, 1, \theta)}{\|(\alpha, 1, \beta, 0, \gamma) \wedge (\delta, 0, \varepsilon, 1, \theta)\|} : \alpha, \beta, \gamma, \delta, \varepsilon, \theta \in \mathbf{R} \right\}$$

is a dense subset of G containing e_{24}.

Let ζ be a typical element of Q. Since our goal is to show $\sigma_{R_k} \geqslant 0$, we can disregard the normalization factor. Let $\xi = \|\zeta\|\zeta$. Then

$$\begin{aligned}
\xi &= [\alpha e_1 + e_2 + \beta e_3 + \gamma e_5] \wedge [\delta e_1 + \varepsilon e_3 + e_4 + \theta e_5] \\
&= -\delta e_{12} + (\alpha\varepsilon - \beta\delta)e_{13} + \alpha e_{14} + (\alpha\theta - \gamma\delta)e_{15} \\
&\quad + \beta e_{34} + \varepsilon e_{23} + \theta e_{25} + e_{24} + (\beta\theta - \gamma\varepsilon)e_{35} - \gamma e_{45}, \\
R_k\xi &= (-\delta - \alpha\theta + \gamma\delta - \beta)e_{12} + (\delta + \alpha\theta - \gamma\delta - \beta)e_{15} \\
&\quad + (\delta - \alpha\theta + \gamma\delta + \beta)e_{34} \\
&\quad + k[(\alpha\varepsilon - \beta\delta)e_{13} + \alpha e_{14} + \varepsilon e_{23} + \theta e_{25} - \gamma e_{45}], \\
\langle R_k\xi, \xi\rangle &= (\delta + \beta)^2 - 2\gamma\delta^2 + 2\delta\alpha\theta - 2\alpha\theta\beta \\
&\quad + 2\beta\gamma\delta - 2\alpha\theta\gamma\delta + \gamma^2\delta^2 + \alpha^2\theta^2 \\
&\quad + k[(\alpha\varepsilon - \beta\delta)^2 + \alpha^2 + \varepsilon^2 + \theta^2 + \gamma^2] = (*).
\end{aligned}$$

For $k \geqslant 2$, we will write $(*)$ as the sum of squares of rational functions and hence conclude it is nonnegative.

Theorem 7.1.

$$\begin{aligned}
\langle R_k\xi, \xi\rangle &= (1 + \delta^2)\left[\left(\gamma + \frac{-\delta^2 + \alpha\varepsilon - \alpha\theta\delta}{1 + \delta^2}\right)^2 + \left(\beta + \frac{\delta - \alpha\varepsilon\delta - \alpha\theta}{1 + \delta^2}\right)^2\right] \\
&\quad + \frac{2(\alpha + \theta\delta)^2}{1 + \delta^2} + \frac{2\varepsilon^2}{1 + \delta^2} + \frac{2(\alpha + \varepsilon)^2\delta^2}{1 + \delta^2} + \frac{2\theta^2}{1 + \delta^2} \\
&\quad + (\alpha\varepsilon - \beta\delta - \gamma)^2 + (k - 2)[(\alpha\varepsilon - \beta\delta)^2 + \alpha^2 + \varepsilon^2 + \theta^2 + \gamma^2].
\end{aligned}$$

Proof. Expand the right-hand side and simplify to obtain $(*)$. It suffices to check this for $k = 2$ since

$$\langle R_k\xi, \xi\rangle = \langle R_2\xi, \xi\rangle + (k - 2)[(\alpha\varepsilon - \beta\delta)^2 + \alpha^2 + \varepsilon^2 + \theta^2 + \gamma^2].$$

Remark. From the above expression of $\langle R_k\xi, \xi\rangle$ as the sum of squares of rational functions it follows that $\langle R_2\xi, \xi\rangle = 0$ if and only if $\alpha = \varepsilon = \theta = 0$ and $\gamma = \delta^2/(1 + \delta^2)$, $\beta = -\delta/(1 + \delta^2)$. Normalizing, this gives a curve of zeroes, parametrized by δ, through e_{24}.

8. Some zeroes of R_2

In this section it is our goal to find two curves of zeroes of R_2 through the zero $(e_{12} + e_{15})/\sqrt{2}$. We will begin by examining a subset Q of G and finding a polynomial expression for $\langle R_2\xi, \xi\rangle$ for $\xi \in Q$ where $\xi/\|\xi\| \in G$. Let

$$\begin{aligned}
Q = \{\zeta \in \Lambda^2 | \zeta = (1, \gamma, \alpha, \beta, -\gamma) \wedge (0, 1 + \theta, \delta, \varepsilon, 1 - \theta); \\
\alpha, \beta, \gamma, \delta, \varepsilon, \theta \in \mathbf{R}\}.
\end{aligned}$$

Remark. It can be shown that normalizing makes Q into a dense subset of G. However, for what follows we only need to know that it contains $e_{12} + e_{15}$, which is obvious. Let

$$\begin{aligned}
\xi &= (1, \gamma, \alpha, \beta, -\gamma) \wedge (0, 1+\theta, \delta, \varepsilon, 1-\theta) \\
&= (1+\theta)e_{12} + \delta e_{13} + \varepsilon e_{14} + (1-\theta)e_{15} \\
&\quad + (-\alpha - \alpha\theta + \gamma\delta)e_{23} + (-\beta - \beta\theta + \gamma\varepsilon)e_{24} \\
&\quad + 2\gamma e_{25} + (\alpha\varepsilon - \beta\delta)e_{34} + (\alpha - \alpha\theta + \gamma\delta)e_{35} \\
&\quad + (\beta - \beta\theta + \gamma\varepsilon)e_{45}, \\
R_k\xi &= (2\theta - \alpha\varepsilon + \beta\delta)e_{12} + (-2\theta - \alpha\varepsilon + \beta\delta)e_{15} \\
&\quad + (-2 + \alpha\varepsilon - \beta\delta)e_{34} + k(-\alpha - \alpha\theta + \gamma\delta)e_{23} \\
&\quad + k(2\gamma)e_{25} + k(\beta - \beta\theta + \gamma\varepsilon)e_{45} \\
&\quad + k\delta e_{13} + k\varepsilon e_{14}, \\
\langle R_k\xi, \xi\rangle &= 4\theta^2 - 4(\alpha\varepsilon - \beta\delta) + (\alpha\varepsilon - \beta\delta)^2 \\
&\quad + k\big[(-\alpha - \alpha\theta + \gamma\delta)^2 + (\beta - \beta\theta + \gamma\varepsilon)^2 \\
&\quad + \delta^2 + \varepsilon^2 + 4\gamma^2\big] = (*).
\end{aligned}$$

Set $\alpha = \gamma = \varepsilon = 0$ and $k = 2$. Then

$$\langle R_2\xi, \xi\rangle = 4\theta^2 + 4\beta\delta + \beta^2\delta^2 + 2\beta^2(1-\theta)^2 + 2\delta^2.$$

For fixed B set

$$f(\theta, \delta) = 4\theta^2 + 4\beta\delta + \beta^2\delta^2 + 2\beta^2(1-\theta)^2 + 2\delta^2.$$

Now $\sigma_{R_2} \geqslant 0 \Rightarrow \langle R_2\xi, \xi\rangle \geqslant 0 \Rightarrow f(\theta, \delta) \geqslant 0$. Thus a zero of f is a minimum of f. But, at a minimum of f,

$$0 = \frac{\partial f}{\partial\theta} = 4(\beta^2 + 2)\theta - 4\beta^2, \quad 0 = \frac{\partial f}{\partial\delta} = 2(\beta^2 + 2)\delta + 4\beta.$$

Hence $\theta = \beta^2/(\beta^2 + 2)$ and $\delta = -2\beta/(\beta^2 + 2)$. It is easily checked that for these values of θ and δ, $f(\theta, \delta) = 0$.

Thus $\langle R_2\xi, \xi\rangle = 0$ if

$$\begin{aligned}
\xi = {}&\left(1 + \frac{\beta^2}{\beta^2+2}\right)e_{12} + \left(\frac{-2\beta}{\beta^2+2}\right)e_{13} + \left(1 - \frac{\beta^2}{\beta^2+2}\right)e_{15} \\
&+ (-\beta)\left(1 + \frac{\beta^2}{\beta^2+2}\right)e_{24} + \left(\frac{2\beta^2}{\beta^2+2}\right)e_{34} + (\beta)\left(1 - \frac{\beta^2}{\beta^2+2}\right)e_{45}.
\end{aligned}$$

Set

$$\xi^1 = (\beta^2 + 2)\xi = (2\beta^2 + 2)e_{12} + (-2\beta)e_{13} + 2e_{15} + (-\beta)(2\beta^2 + 2)e_{24} + (2\beta^2)e_{34} + (2\beta)e_{45}.$$

Then $\langle R_2\xi^1, \xi^1\rangle = 0$. Thus $\beta \to \xi^1(\beta)/\|\xi^1(\beta)\|$ is a curve of zeroes through $(e_{12} + e_{15})/\sqrt{2}$.

If in (*) we set $k = 2$ and $\delta = \beta = \gamma = 0$, we get $\langle R_2\xi, \xi\rangle = 4\theta^2 - 4\alpha\varepsilon + \alpha^2\varepsilon^2 + 2(\alpha + \alpha\theta)^2 + \varepsilon^2$. Following an approach identical to that above gives

$$\xi^2 = 2e_{12} + (2\alpha)e_{14} + (2\alpha^2 + 2)e_{15} - (2\alpha)e_{23} + (2\alpha^2)e_{34} + (\alpha)(2\alpha^2 + 2)e_{35}.$$

It is easily checked that $\xi^2/\|\xi^2\|$ is decomposable and that $\sigma_{R_2}(\xi^2/\|\xi^2\|) = 0$. Then $\alpha \to \xi^2(\alpha)/\|\xi^2(\alpha)\|$ is another curve of zeroes through $(e_{12} + e_{15})/\sqrt{2}$.

9. The zero set of R_k

In this section we prove Claims 6.1 and 6.2, and for each $k > 2$ we explicitly describe the zero set of R_k. Until further notice we set $k = 2$. Consider the following vectors.

$$\begin{aligned}
\alpha_1 &= \xi^1(1) = 4e_{12} - 2e_{13} + 2e_{15} - 4e_{24} + 2e_{34} + 2e_{45},\\
\alpha_2 &= \xi^1(-1) = 4e_{12} + 2e_{13} + 2e_{15} + 4e_{24} + 2e_{34} - 2e_{45},\\
\alpha_3 &= \xi^2(1) = 2e_{12} + 2e_{14} + 4e_{15} - 2e_{23} + 2e_{34} + 4e_{35},\\
\alpha_4 &= \xi^2(-1) = 2e_{12} - 2e_{14} + 4e_{15} + 2e_{23} + 2e_{34} - 4e_{35},\\
\alpha_5 &= -12e_{12} - 12e_{15}.
\end{aligned}$$

It is clear from the above construction of ξ^1 and ξ^2 that $\langle R_2\alpha_i, \alpha_i\rangle = 0$, $i = 1, \cdots, 5$, and thus $\beta_i = \alpha_i/\|\alpha_i\| \in Z(R_2)$ for $i = 1, \cdots, 5$. Let $\beta = \sum_{i=1}^5 \alpha_i$. It is easily checked that $\beta = 8e_{34}$ and so $\beta/8 \in G$. Now $\langle R_2\beta/8, \beta/8\rangle = \langle e_{34} - e_{12} - e_{15}, e_{34}\rangle = 1$.

We have found five zeros of R_2 whose linear span contains a 2-plane in G with nonzero sectional curvature. Let $L_2 = \pi(R_2)$. (To verify Claim 6.1 we need an example which satisfies the Bianchi identity.) Now by the remark at the end of §3, $\sigma_{L_2} = \sigma_{R_2}$, and so Claim 6.1 of §6 is verified.

Claim 6.2 is now easily verified. If there existed $S \in \Lambda^4$ such that $L_2 + S$ were positive semi-definite, then each $x \in Z(L_2)$ would be a minimum of $\langle (L_2 + S)\xi, \xi\rangle$ on the unit sphere in Λ^2, and so would be an eigenvector of $L_2 + S$ with zero eigenvalue. It would then follow that $Z(L_2)$ was the intersection with G of a linear subspace of Λ^2, namely the null space of $L_2 + S$. However, we have shown that this is not the case.

Lemma 9.1. *If*

$$Q = \left\{ P \in G | P = \frac{(\alpha, 1, \beta, 0, \gamma) \wedge (\delta, 0, \varepsilon, 1, \theta)}{\|(\alpha, 1, \beta, 0, \gamma) \wedge (\delta, 0, \varepsilon, 1, \theta)\|} : \alpha, \beta, \gamma, \delta, \varepsilon, \theta \in \mathbf{R} \right\},$$

then

$$G - Q = \{ P \in G | P = (\alpha, 0, \beta, 0, \gamma) \wedge (\delta, \mu, \varepsilon, \eta, \theta); \\ \alpha, \beta, \gamma, \delta, \mu, \varepsilon, \eta, \theta \in \mathbf{R} \}.$$

Proof. Replacing $e_1 \wedge e_2$ by $e_2 \wedge e_4$ in Lemma 5.2 shows that $G - Q = \{P \in G | \langle P, e_2 \wedge e_4 \rangle = 0\}$. Since

$$0 = \langle P, e_2 \wedge e_4 \rangle = -\langle P, *(e_1 \wedge e_3 \wedge e_5) \rangle \Rightarrow P \wedge e_1 \wedge e_3 \wedge e_5 = 0,$$

considering P as a 2-dimensional subspace of V and $e_1 \wedge e_3 \wedge e_5$ as a 3-dimensional subspace of V, we see that $P \cap (e_1 \wedge e_3 \wedge e_5) \neq (0)$, and so there exists $v \in P$ such that $|v| = 1$ and $v = (\alpha, 0, \beta, 0, \gamma)$. Choosing $w \in P$ such that $|w| = 1$ and $\langle w, v \rangle = 0$ we have that

$$P = v \wedge w = (\alpha, 0, \beta, 0, \gamma) \wedge (\delta, \mu, \varepsilon, \eta, \theta); \alpha, \beta, \gamma, \delta, \mu, \varepsilon, \eta, \theta \in \mathbf{R}.$$

Next we analyze the sectional curvature of $R_k (k \geqslant 2)$ on $G - Q$. Our goal being to explicitly describe $Z(R_k)(k > 2)$ we can desregard the normalization factor.

Let

$$\begin{aligned} \xi &= (\alpha, 0, \beta, 0, \gamma) \wedge (\delta, \mu, \varepsilon, \eta, \theta) \\ &= \alpha\mu e_{12} + (\alpha\varepsilon - \beta\delta)e_{13} + \alpha\eta e_{14} + (\alpha\theta - \gamma\delta)e_{15} \\ &\quad -\beta\mu e_{23} - \gamma\mu e_{25} + \beta\eta e_{34} + (\beta\theta - \gamma\varepsilon)e_{35} - \gamma\eta e_{45}, \\ R_k\xi &= (\alpha\mu - \alpha\theta + \gamma\delta - \beta\eta)e_{12} + (\alpha\theta - \gamma\delta - \alpha\mu - \beta\eta)e_{15} \\ &\quad + (\beta\eta - \alpha\theta + \gamma\delta - \alpha\mu)e_{34} \\ &\quad + k[(\alpha\varepsilon - \beta\delta)e_{13} + \alpha\eta e_{14} - \beta\mu e_{23} - \gamma\mu e_{25} - \gamma\eta e_{45}], \\ \langle R_k\xi, \xi \rangle &= \alpha^2\mu^2 - 2\alpha^2\mu\theta + 2\alpha\mu\gamma\delta - 2\alpha\mu\beta\eta \\ &\quad + (\alpha\theta - \gamma\delta)^2 - 2\alpha\theta\beta\eta + 2\gamma\delta\beta\eta + \beta^2\eta^2 \\ &\quad + k[(\alpha\varepsilon - \beta\delta)^2 + \alpha^2\eta^2 + \beta^2\mu^2 + \gamma^2\mu^2 + \gamma^2\eta^2] = (*). \end{aligned}$$

For $k \geqslant 2$ we will write $(*)$ as the sum of squares of polynomial functions.

Theorem 9.2. *For $k \geqslant 2$ and $\xi \in G - Q$,*

$$\langle R_k\xi, \xi \rangle = (-\beta\eta + \alpha\theta - \gamma\delta - \alpha\mu)^2 + 2(\beta\mu - \alpha\eta)^2 \\ + k[(\alpha\varepsilon - \beta\delta)^2 + \gamma^2\mu^2 + \gamma^2\eta^2] + (k - 2)(\alpha^2\eta^2 + \beta^2\mu^2).$$

Theorem 9.3. *For $k > 2$, $Z(R_k) = \{\pm(e_{12} + e_{15})/\sqrt{2}, \pm e_{24}, \pm e_{35}\}$.*

Proof. For $k > 2$ Theorem 7.1 implies that the only zeroes of R_k in Q are $\pm e_{24}$. For $k > 2$ and $\xi \in G - Q$, an analysis of the polynomial expression for $\langle R_k\xi, \xi\rangle$ given by Theorem 9.2 shows that $\langle R_k\xi, \xi\rangle = 0$ only if $\alpha^2\eta^2 + \beta^2\mu^2 = 0$. It is easily checked that this happens only when $\xi = \pm e_{35}$ or $\xi = \pm(e_{12} + e_{15}/\sqrt{2})$.

Proposition 9.4. *For $k \geqslant 2$, L_k is not the projection under π of a positive semi-definite operator on Λ^2.*

Proof. Suppose it is. Then for some $S \in \Lambda^4$, $R_k + S$ is a positive semi-definite operator on Λ^2. Let $\alpha = e_{12} + e_{15} + e_{34}$. Then $R_k\alpha = -\alpha$ and $\langle R_k\alpha, \alpha\rangle = -3$. Thus $\langle (R_k + s)\alpha, \alpha\rangle \geqslant 0$ implies that $\langle S\alpha, \alpha\rangle \geqslant 3$. Now since $S \in \Lambda^4$, it follows from Proposition 3.1 that

$$S = \sum_{1 \leqslant i<j<k<l \leqslant 5} \lambda_{ijkl}S_{ijkl}, \quad \lambda_{ijkl} \in \mathbf{R}.$$

Thus $\langle S\alpha, \alpha\rangle = 2\lambda_{1234} + 2\lambda_{1345}$, and since $\langle S\alpha, \alpha\rangle \geqslant 3$ it follows that $\lambda_{1234} + \lambda_{1345} \geqslant 3/2$.

Letting $w_1 = e_{13} + ke_{24}$ and $w_2 = e_{14} + ke_{35}$ we get $\langle (R_k + S)w_1, w_1\rangle = -k\lambda_{1234}$ and $\langle (R_k + S)w_2, w_2\rangle = -k\lambda_{1345}$. But this together with $\lambda_{1234} + \lambda_{1345} \geqslant 3/2$ implies that $\langle (R_k + S)w_1, w_2\rangle < 0$ or $\langle (R_k + S)w_2, w_2\rangle < 0$, thus contradicting the assumption that $R_k + S$ is positive semi-definite.

Theorem 9.5. *There exist curvature operators which satisfy the Bianchi identity, have nonnegative sectional curvature, and each of whose zero sets is the intersection with G of a linear subspace of Λ^2, but which are not the projection under π of a positive semi-definite operator on Λ^2.*

Proof. We claim that for $k > 2$, each curvature operator L_k is of this type. It follows by Theorem 9.4 that L_k is not the projection of a positive semi-definite operator, and by Theorem 7.1 that $\sigma_{L_k} \geqslant 0$. By Theorem 9.3, for $k > 2$ we see that

$$Z(L_k) = \{\pm(e_{12} + e_{15})/\sqrt{2}, \pm e_{24}, \pm e_{35}\}.$$

To complete the proof we verify that (for $k > 2$) $Z(L_k) = \operatorname{span} Z(L_k) \cap G$.

That $Z(L_k) \subset \operatorname{span} Z(L_k) \cap G$ is clear. If $\xi \in \operatorname{span} Z(L_k)$, then

$$\xi = a(e_{12} + e_{15})/\sqrt{2} + be_{24} + ce_{35}, \quad a, b, c, \in \mathbf{R}.$$

By Corollary 3.3, the following are equivalent:

(1) ξ is decomposable.

(2) $0 = \xi \wedge \xi$

$$= \left[\frac{a}{\sqrt{2}}(e_{12} + e_{15}) + be_{24} + ce_{35}\right] \wedge \left[\frac{a}{\sqrt{2}}(e_{12} + e_{15}) + be_{24} + ce_{35}\right]$$

$$= \frac{2ac}{\sqrt{2}} e_1 \wedge e_2 \wedge e_3 \wedge e_5 + \frac{2ab}{\sqrt{2}} e_1 \wedge e_2 \wedge e_4 \wedge e_5$$

$$+ 2bce_2 \wedge e_4 \wedge e_3 \wedge e_5.$$

(3) $ab = ac = bc = 0 \Rightarrow a = b = 0$ or $b = c = 0$ or $a = c = 0$.

(4) $\xi = \pm e_{35}$ or $\xi = \pm (e_{12} + e_{15})/\sqrt{2}$ or $\xi = \pm e_{24}$.

Theorem 9.6. *If dimension $V = n \geqslant 5$, then there exist curvature operators L_k^n which satisfy the Bianchi identity and have the following properties*:

1. *For $k \geqslant 2$, $\sigma_{L_k}^n \geqslant 0$.*
2. *For $k = 2$, $Z(L_k^n)$ is not the intersection with G of a linear subspace of Λ^2.*
3. *For $k \geqslant 2$, L_k^n is not the projection under π of a positive semi-definite operator on Λ^2.*

Proof. For $n \geqslant 5$ let $\{e_1, \cdots, e_n\}$ be an orthonormal basis for V, and let $W = \operatorname{span}\{e_1, e_2, e_3, e_4, e_5\}$. Since $W \subset V$, $\Lambda^2(W) \subset \Lambda^2(V)$. We define the linear map π_1: $\Lambda^2(V) \to \Lambda^2(W)$ by

$$\pi_1(\xi) = \sum_{1 < i < j < 5} a_{ij} e_{ij}$$

for $\xi = \sum_{1 \leqslant i < j \leqslant n} a_{ij} e_{ij} \in \Lambda^2(V)$. Note that if ξ is decomposable, then $\pi_1(\xi)$ is decomposable.

For k a real number and dimension $V = n \geqslant 5$, consider the following example: Let R_k^n: $\Lambda^2(V) \to \Lambda^2(V)$ be defined by

$$\begin{aligned} R_k^n e_{12} &= e_{12} - e_{15} - e_{34}, \\ R_k^n e_{15} &= e_{15} - e_{12} - e_{34}, \\ R_k^n e_{34} &= e_{34} - e_{12} - e_{15}, \\ R_k^n e_{24} &= R_k^n e_{35} = 0, \\ R_k^n e_{ij} &= k e_{ij} \quad \text{for remaining } e_{ij}. \end{aligned}$$

Note that for $k > 0$, $\langle R_k^n \xi, \xi \rangle \geqslant \langle R_k \pi_1(\xi), R_k \pi_1(\xi) \rangle$ for all $\xi \in \Lambda^2(V)$.

Let $L_k^n = \pi(R_k^n)$. Then L_k^n satisfies the Bianchi identity, and for $k \geqslant 2$

$$\sigma_{L_k^n}(\xi) = \sigma_{R_k^n}(\xi) = \langle R_k^n \xi, \xi \rangle \geqslant \langle R_k \pi_1(\xi), R_k \pi_1(\xi) \rangle \geqslant 0.$$

Thus L_k^n has Property 1.

To see that L_k^n has Property 2, lt $\beta_i (i = 1, \cdots, 5)$ and β be defined as above. Taking advantage of the natural inclusion of $\Lambda^2(W)$ in $\Lambda^2(V)$ we can consider β and β_i as elements of $\Lambda^2(V)$. Then

$$\begin{aligned} \sigma_{L_2^n}(\beta_i) &= \sigma_{R_2^n}(\beta_i) = \sigma_{R_2}(\beta_i) = 0, \\ \sigma_{L_2^n}(\beta/8) &= \sigma_{R_2^n}(\beta/8) = \sigma_{R_2}(\beta/8) = 1. \end{aligned}$$

Thus we have found five zeroes of L_2^n whose linear span contains a 2-plane in G with nonzero sectional curvature, and so $Z(L_2^n)$ is not the intersection with G of a linear subspace of $\Lambda^2(V)$.

Following an approach similar to that in the proof of Proposition 9.4 one can show that L_k^n has Property 3.

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