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ISOMETRY OF RIEMANNIAN MANIFOLDS TO SPHERES

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1. Introduction

Let M be a differentiable connected Riemannian manifold of dimension n. We cover M by a system of coordinate neighborhoods $\{U; x^h\}$, where and in the sequel indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$, and denote by g_{ji}, ∇_j, $K_{kji}{}^h$, K_{ji} and K the metric tensor, the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of M respectively.

An infinitesimal transformation v^h on M is said to be conformal if it satisfies

$$\mathscr{L}_v g_{ji} = \nabla_j v_i + \nabla_i v_j = 2\rho g_{ji} \qquad (v_i = g_{ih} v^h) \tag{1.1}$$

for a certain function ρ on M, where $\mathscr{L}_v$ denotes the operator of Lie derivation with respect to the vector field v (see [6]). When we refer in the sequel to an infinitesimal conformal transformation v, we always mean by ρ the function appearing in (1.1). When ρ in (1.1) is a constant (respectively, zero), the infinitesimal transformation is said to be homothetic (respectively, isometric).

We also denote by $\mathscr{L}_{D\rho}$ the operator of Lie derivation with respect to the vector field ρ^i defined by

$$\rho^i = g^{ih} \rho_h = \nabla^i \rho \,, \tag{1.2}$$

where

$$\nabla^i = g^{ih} \nabla_h \,, \qquad \rho_h = \nabla_h \rho \,, \tag{1.3}$$

g^{ih} being contravariant components of the metric tensor. We use g_{ji} and g^{ih} to lower and raise the indices respectively.

The problem of finding conditions for a Riemannian manifold admitting an infinitesimal conformal transformation v to be isometric to a sphere has been extensively studied. For the history of this problem, see [7] and [8]. But in almost all the results on this problem the condition $K =$ constant or $\mathscr{L}_v K = 0$ has been assumed. As results in which the conditon $\mathscr{L}_v K = 0$ is not assumed, Sawaki and one of the present authors [12] (see also [11]) proved the following two theorems, in which and the remainder of this section, unless stated

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otherwise, M will always denote a compact oriented Riemannian manifold of dimension $n > 2$ admitting an infinitesimal nonhomothetic conformal transformation v.

Theorem A. *M is isometric to a sphere if v satisfies*

$$\mathscr{L}_v\left[\mathscr{L}_v\left(\|G\|^2 - \frac{n-2}{n+2}\Delta K\right) + \frac{2(n+1)(n-2)}{n(n+2)}\Delta\mathscr{L}_v K\right] = 0 , \tag{1.4}$$

where

$$G_{ji} = K_{ji} - \frac{1}{n}Kg_{ji} , \tag{1.5}$$

$$\|G\|^2 = G_{ji}G^{ji} , \tag{1.6}$$

$\Delta = g^{ji}\nabla_j\nabla_i$ denoting the Laplacian.

Theorem B. *M is isometric to a sphere if v satisfies*

$$\mathscr{L}_v\left[\mathscr{L}_v\left(\|Z\|^2 - \frac{4}{n+2}\Delta K\right) + \frac{8(n+1)}{n(n+2)}\Delta\mathscr{L}_v K\right] = 0 , \tag{1.7}$$

where

$$Z_{kji}{}^h = K_{kji}{}^h - \frac{1}{n(n-1)}K(\delta_k^h g_{ji} - \delta_j^h g_{ki}) , \tag{1.8}$$

$$\|Z\|^2 = Z_{kji}{}^h Z^{kji}{}_h . \tag{1.9}$$

Recently Amur and Hegde [2] (see also [3]) proved the following two theorems.

Theorem C. *M is conformal to a sphere if v satisfies $\mathscr{L}_{D\rho}\mathscr{L}_v K = 0$ and*

$$\int_M \left(G_{ji}\rho^j\rho^i + \frac{1}{n^2}\mathscr{L}_v\mathscr{L}_{D\rho}K\right)dV \geq 0 , \tag{1.10}$$

where $\mathscr{L}_{D\rho}$ denotes the operator of Lie derivation with respect to ρ^i and dV the volume element of M.

Theorem D. *M is conformal to a sphere if v satisfies $\mathscr{L}_{D\rho}\mathscr{L}_v K = 0$, $\mathscr{L}_v\mathscr{L}_{D\rho}K \geq 0$ and $\mathscr{L}_v\|G\|^2 = 0$.*

Very recently the present authors [9] proved the following two theorems.

Theorem E. *M is isometric to a sphere if v satisfies $\mathscr{L}_v\|G\|^2 = 0$ and*

$$\int_M K\rho_i\rho^i dV \geq \frac{1}{2n(n-1)}\int_M [2n\rho^2K^2 + (n+2)\rho K\mathscr{L}_v K + (\mathscr{L}_v K)^2]dV . \tag{1.11}$$

Theorem F. *M is isometric to a sphere if v satisfies* $\mathscr{L}_v \|Z\|^2 = 0$ *and* (1.11).

All the above theorems have been obtained by applying the following Theorem G of Tashiro [5].

The purpose of the present paper is to continue the joint work of the present authors [9] and to prove some propositions on isometry of Riemannian manifolds to spheres, in which the operator of Lie derivation $\mathscr{L}_{D\rho}$ plays an important role.

In the sequel, we need the following theorems.

Theorem G (*Tashiro* [5]). *If a complete Riemannian manifold M of dimension* $n > 2$ *admits a complete infinitesimal nonhomothetic conformal transformation v such that*

$$\nabla_j \rho_i - \frac{1}{n} \Delta \rho g_{ji} = 0 , \tag{1.12}$$

then M is isometric to a sphere.

Theorem H (*Yano and Obata* [10]. *See also Obata* [4]). *If a complete Riemannian manifold M of dimension* $n \geq 2$ *admits a nonconstant function* ρ *satisfying*

$$\nabla_j \rho_i - \frac{1}{n} \Delta \rho g_{ji} = 0 , \qquad \mathscr{L}_{D\rho} K = 0 , \tag{1.13}$$

then M is isometric to a sphere.

We remark here that if a Riemannian manifold M of dimension n is isometric to a sphere, then M admits not only an infinitesimal nonhomothetic conformal transformation v satisfying (1.1) and (1.12) but also a nonconstant function ρ satisfying (1.13).

2. Lemmas

In this section we prove some lemmas which we need in the next section. M is supposed to be a compact oriented Riemannian manifold of dimension n in all the lemmas except in Lemmas 4, 5, 6, 9 where M is supposed to be only a Riemannian manifold.

Lemma 1. *If M admits an infinitesimal conformal transformation v, then, for the function* ρ *appearing in* (1.1) *and for an arbitrary function f on M, we have*

$$\int_M \rho f dV = -\frac{1}{n} \int_M \mathscr{L}_v f dV . \tag{2.1}$$

Proof. Since $n\rho = \nabla_t v^t$, by Green's theorem (see [7]) we have

$$0 = \int_M \nabla_t(fv^t)dV = \int_M \mathcal{L}_v f dV + n\int_M \rho f dV ,$$

which proves (2.1).

Lemma 2. *In M we have*

$$\begin{aligned} \int_M \mathcal{L}_{Df} h dV &= \int_M \mathcal{L}_{Dh} f dV = \int_M (\nabla_i f)(\nabla^i h) dV \\ &= -\int_M f\Delta h dV = -\int_M h\Delta f dV \end{aligned} \tag{2.2}$$

for any functions f and h on M, where $\mathcal{L}_{Df}$ *denotes the operator of Lie derivation with respect to the vector field* $\nabla^i f$ *on M.*

Proof. This follows from

$$0 = \int_M \nabla_i(f\nabla^i h)dV = \int_M (\nabla_i f)(\nabla^i h)dV + \int_M f\Delta h dV ,$$

$$0 = \int_M \nabla_i(h\nabla^i f)dV = \int_M (\nabla_i h)(\nabla^i f)dV + \int_M h\Delta f dV .$$

Lemma 3. *In M we have*

$$\int_M \rho^2 \Delta K dV = -2\int_M \rho\rho^i \nabla_i K dV \tag{2.3}$$

for any function ρ *on M, K being the scalar curvature of M.*

Proof. We have (2.3) by putting $f = K$ and $h = \rho^2$ in (2.2).

Lemma 4 (*Yano* [7]). *For an infinitesimal conformal transformation v in M, we have*

$$\mathcal{L}_v K_{kji}{}^h = -\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} , \tag{2.4}$$

$$\mathcal{L}_v K_{ji} = -(n-2)\nabla_j \rho_i - \Delta\rho g_{ji} , \tag{2.5}$$

$$\mathcal{L}_v K = -2(n-1)\Delta\rho - 2\rho K . \tag{2.6}$$

Proof. We can prove these by using (1.1) and the following formulas on Lie derivatives:

$$\mathcal{L}_v\{{}_j{}^h{}_i\} = \delta_j^h \rho_i + \delta_i^h \rho_j - g_{ji}\rho^h ,$$

$$\mathcal{L}_v K_{kji}{}^h = \nabla_k \mathcal{L}_v\{{}_j{}^h{}_i\} - \nabla_j \mathcal{L}_v\{{}_k{}^h{}_i\} ,$$

$\{{}_j{}^h{}_i\}$ denoting Christoffel symbols formed with g_{ji}.

Lemma 5. *For an infinitesimal conformal transformation v in M, we have*

$$\mathcal{L}_v G_{ji} = -(n-2)\left(\nabla_j \rho_i - \frac{1}{n}\Delta\rho g_{ji}\right) , \tag{2.7}$$

$$\begin{aligned}(2.8)\qquad \mathscr{L}_v Z_{kji}{}^h = &-\delta_k^h \nabla_j \rho_i + \delta_j^h \nabla_k \rho_i - \nabla_k \rho^h g_{ji} + \nabla_j \rho^h g_{ki} \\ &+ \frac{2}{n} \Delta\rho(\delta_k^h g_{ji} - \delta_j^h g_{ki}) ,\end{aligned}$$

where G_{ji} *and* $Z_{kji}{}^h$ *are defined by* (1.5) *and* (1.8) *respectively.*

Proof. These follow from Lemma 4.

Lemma 6. *If* M *admits an infinitesimal conformal transformation* v, *then for any function* f *on* M *we have*

$$(2.9)\qquad \Delta \mathscr{L}_v f = \mathscr{L}_v \Delta f + 2\rho \Delta f - (n-2)\rho^i \nabla_i f .$$

Proof. For an infinitesimal conformal transformation v, we have (see [7])

$$(2.10)\qquad g^{kj}\nabla_k \nabla_j v^h + K_i{}^h v^i + \frac{n-2}{n}\nabla^h(\nabla_t v^t) = 0 .$$

Thus we obtain (2.9) by using (2.10) and the identity

$$g^{ji}\nabla_j \nabla_i \nabla_h f - K_h{}^i \nabla_i f = \nabla_h(\Delta f) ,$$

which holds for any function f on M.

Lemma 7. *If* M *admits an infinitesimal conformal transformation* v, *then*

$$\begin{aligned}(2.11)\qquad &\int_M \mathscr{L}_v \mathscr{L}_{D\rho} K dV \\ &= -\frac{n}{n+2}\int_M \rho \mathscr{L}_v \Delta K dV + \frac{n}{n+2}\int_M \rho \Delta \mathscr{L}_v K dV ,\end{aligned}$$

$$(2.12)\qquad \int_M \mathscr{L}_{D\rho} \mathscr{L}_v K dV = -\int_M \rho \Delta \mathscr{L}_v K dV ,$$

and consequently

$$\begin{aligned}(2.13)\qquad &\int_M \mathscr{L}_{[v,D\rho]} K dV \\ &= -\frac{n}{n+2}\int_M \rho \mathscr{L}_v \Delta K dV + \frac{2(n+1)}{n+2}\int_M \rho \Delta \mathscr{L}_v K dV ,\end{aligned}$$

where $D\rho$ *denotes the vector field* ρ^i, *and* $[v, D\rho]$ *the commutator of vector fields* v *and* $D\rho$.

Proof. Using Lemmas 1, 3 and 6, we have

$$\begin{aligned}\int_M \rho \mathscr{L}_v \Delta K dV &= \int_M \rho \Delta \mathscr{L}_v K dV - 2\int_M \rho^2 \Delta K dV + (n-2)\int_M \rho\rho^i \nabla_i K dV \\ &= \int_M \rho \Delta \mathscr{L}_v K dV + (n+2)\int_M \rho \mathscr{L}_{D\rho} K dV\end{aligned}$$

$$= \int_M \rho \Delta \mathcal{L}_v K dV - \frac{n+2}{n} \int_M \mathcal{L}_v \mathcal{L}_{D\rho} K dV ,$$

which proves (2.11). (2.12) follows immediately from Lemma 2.

Lemma 8. *In M we have, for any function ρ on M,*

$$\int_M K_{ji}\rho^j\rho^i dV = -\frac{1}{2}\int_M \rho(\mathcal{L}_{D\rho}K_{ji})g^{ji}dV , \tag{2.14}$$

$$\begin{aligned}\int_M K_{ji}\rho^j\rho^i dV \\ = -\frac{1}{4}\int_M \rho\mathcal{L}_{D\rho}KdV - \frac{1}{4}\int_M \rho(\mathcal{L}_{D\rho}K_{kjih})g^{kh}g^{ji}dV .\end{aligned} \tag{2.15}$$

Proof. From the definition of K it follows that

$$\begin{aligned}\int_M \rho\mathcal{L}_{D\rho}KdV &= \int_M \rho\mathcal{L}_{D\rho}(K_{ji}g^{ji})dV \\ &= \int_M \rho(\mathcal{L}_{D\rho}K_{ji})g^{ji}dV + \int_M \rho K_{ji}\mathcal{L}_{D\rho}g^{ji}dV .\end{aligned} \tag{2.16}$$

On the other hand, since ρ_i is a gradient, we have

$$\mathcal{L}_{D\rho}g_{ji} = 2\nabla_j\rho_i , \qquad \mathcal{L}_{D\rho}g^{ji} = -2\nabla^j\rho^i , \tag{2.17}$$

$$\nabla^j(\rho\rho^iK_{ji}) = K_{ji}\rho^j\rho^i + \rho K_{ji}\nabla^j\rho^i + \tfrac{1}{2}\rho\rho^i\nabla_iK , \tag{2.18}$$

where we have used $\nabla^jK_{ji} = \frac{1}{2}\nabla_iK$. Using (2.16), (2.17) and (2.18), we have (2.14). We also have

$$\begin{aligned}\int_M \rho\mathcal{L}_{D\rho}KdV &= \int_M \rho\mathcal{L}_{D\rho}(K_{kjih}g^{kh}g^{ji})dV \\ &= \int_M \rho(\mathcal{L}_{D\rho}K_{kjih})g^{kh}g^{ji}dV - 4\int_M \rho K_{ji}\nabla^j\rho^i dV ,\end{aligned} \tag{2.19}$$

from which and (2.18), (2.15) follows immediately.

Lemma 9. *In M we have, for any function ρ on M,*

$$\begin{aligned}K_{ji}\rho^j\rho^i + \frac{1}{n}(\Delta\rho)^2 + \mathcal{L}_{D\rho}\Delta\rho - \frac{1}{2}\Delta\mathcal{L}_{D\rho}\rho \\ = -\left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right) .\end{aligned} \tag{2.20}$$

Proof. Using Ricci formula we have

$$\Delta\mathcal{L}_{D\rho}\rho = g^{kj}\nabla_k\nabla_j(\rho_i\rho^i) = 2g^{kj}\nabla_k(\rho^i\nabla_j\rho_i)$$

$$= 2g^{kj}(\nabla_k\nabla_j\rho_i)\rho^i + 2(\nabla_j\rho_i)(\nabla^j\rho^i)$$
$$= 2g^{kj}(\nabla_i\nabla_k\rho_j - K_{kij}{}^h\rho_h)\rho^i + 2(\nabla_j\rho_i)(\nabla^j\rho^i) ,$$

from which we find (2.20).

Lemma 10. *In M we have, for any function ρ on M,*

$$\int_M K_{ji}\rho^j\rho^i dV + \frac{n-1}{n}\int_M \mathcal{L}_{D_\rho}\Delta\rho dV = -\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV , \tag{2.21}$$

or

$$\int_M K_{ji}\rho^j\rho^i dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV = -\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV . \tag{2.22}$$

Proof. These follow from Lemmas 2 and 9.

Lemma 11. *A sphere S^n of dimension $n \geq 2$ admits a nonconstant function ρ such that*

$$\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji} = 0 , \tag{2.23}$$

and consequently

$$\Delta^2\rho + \frac{1}{n-1}K\Delta\rho = 0 , \qquad \nabla_j\nabla_i\Delta\rho + \frac{1}{n-1}K\nabla_j\rho_i = 0 , \tag{2.24}$$

$$\nabla_j\nabla_i\Delta\rho - \frac{1}{n}\Delta^2\rho g_{ji} = 0 . \tag{2.25}$$

Proof. It is known [11] that S^n admits a nonconstant function ρ such that (2.23) holds. This shows that the vector field ρ^i defines an infinitesimal nonhomothetic conformal transformation on S^n with the associated function $(1/n)\Delta\rho$. Since K is a positive constant, using (2.6) in which v and ρ are replaced by ρ^h and $(1/n)\Delta\rho$ respectively we have the first equation of (2.24) and therefore $\Delta\rho + (1/(n-1))\rho K = c$ (c: constant), which implies the second equation of (2.24). From (2.23) and (2.24) we obtain (2.25).

3. Propositions

In this section, we prove a series of propositions in which the operator of Lie derivation $\mathcal{L}_{D_\rho}$ plays an important role. M is supposed to be a compact oriented Riemannian manifold of dimension n admitting an infinitesimal conformal transformation v in all the propositions and corollaries except: in Proposition 4 where M is supposed to be a complete Riemannian manifold of dimension $n \geq 2$, in Propositions 5, 7 and Corollary 5 where M is supposed to be a complete Riemannian manifold of dimension $n > 2$ admitting a complete infinitesimal nonhomothetic conformal transformation v, in Propositions 6, 12 and 13 where M is supposed to be only a Riemannian manifold, and in Propositions 8, 10 and Corollaries 1, 3 where M is supposed to be a compact oriented Riemannian manifold of dimension n.

Proposition 1. *For M we have*

$$(3.1)\quad \int_M G_{ji}\rho^j\rho^i dV + \frac{1}{n^2}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}K dV - \frac{1}{2n}\int_M \mathscr{L}_{D\rho}\mathscr{L}_v K dV \leq 0 .$$

The M of dimension $n > 2$ admits a nonhomothetic v such that the equality in (3.1) *holds if and only if M is isometric to a sphere.*

Proof. By using (1.5), (2.6), Lemmas 1 and 2 and the identity

$$(3.2)\quad \int_M \nabla_i(\rho\rho^i K)dV = \int_M K\rho_i\rho^i dV + \int_M \rho K\Delta\rho dV + \int_M \rho\rho^i\nabla_i K dV = 0 ,$$

we have

$$\begin{aligned}
&\int_M K_{ji}\rho^j\rho^i dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\
&= \int_M G_{ji}\rho^j\rho^i dV + \frac{1}{n}\int_M K\rho_i\rho^i dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\
&= \int_M G_{ji}\rho^j\rho^i dV - \frac{1}{n}\int_M \rho\mathscr{L}_{D\rho}K dV - \frac{1}{n}\int_M \rho K\Delta\rho dV \\
&\quad -\frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\
&= \int_M G_{ji}\rho^j\rho^i dV + \frac{1}{n^2}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}K dV + \frac{1}{2n}\int_M (\Delta\rho)\mathscr{L}_v K dV \\
&= \int_M G_{ji}\rho^j\rho^i dV + \frac{1}{n^2}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}K dV - \frac{1}{2n}\int_M \mathscr{L}_{D\rho}\mathscr{L}_v K dV .
\end{aligned}$$

Thus from Lemma 10 we obtain

$$(3.3)\quad \begin{aligned}
&\int_M G_{ji}\rho^j\rho^i dV + \frac{1}{n^2}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}K dV - \frac{1}{2n}\int_M \mathscr{L}_{D\rho}\mathscr{L}_v K dV \\
&\quad = -\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV ,
\end{aligned}$$

which implies (3.1). If the equality in (3.1) holds, then from (3.3) and Theorem G it follows that M is isometric to a sphere. Conversely, if M is isometric to a sphere, M admits an infinitesimal nonhomothetic conformal transformation v such that the equality in (3.1) holds because, for a sphere, $G_{ji} = 0$ and K is a positive constant.

Proposition 1 is a generalization of Theorem C.

Proposition 2. *If the dimension n of M is greater than 2, then*

$$\int_M \mathcal{L}_v \mathcal{L}_v \|G\|^2 dV - (n-2)\int_M \mathcal{L}_{[v,D\rho]} K dV \geq 0 . \tag{3.4}$$

The M of dimension $n > 2$ admits a nonhomothetic v such that the equality in (3.4) *holds if and only if M is isometric to a sphere.*

Proof. First of all we have

$$\mathcal{L}_v \|G\|^2 = 2(\mathcal{L}_v G_{ji})G^{ji} - 4\rho \|G\|^2 .$$

Substituting (2.7) in the above equation we find

$$\mathcal{L}_v \|G\|^2 = -2(n-2)G_{ji}\nabla^j \rho^i - 4\rho \|G\|^2 ,$$

because of $G_{ji}g^{ji} = 0$ or

$$K_{ji}\nabla^j \rho^i = -\frac{2}{n-2}\rho \|G\|^2 - \frac{1}{2(n-2)}\mathcal{L}_v \|G\|^2 + \frac{1}{n}K\Delta\rho . \tag{3.5}$$

Using (2.18) and (3.5) we have

$$\begin{aligned}\nabla^j(\rho\rho^i K_{ji}) = K_{ji}\rho^j\rho^i &- \frac{2}{n-2}\rho^2 \|G\|^2 \\ &- \frac{1}{2(n-2)}\rho\mathcal{L}_v \|G\|^2 + \frac{1}{2}\rho\mathcal{L}_{D\rho}K + \rho K\Delta\rho .\end{aligned}$$

Integrating both sides of the above equation over M and using (2.6) and Lemmas 1 and 2, we obtain

$$\begin{aligned}&\int_M K_{ji}\rho^j\rho^i dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\ &= \frac{2}{n-2}\int_M \rho^2 \|G\|^2 dV - \frac{1}{2n(n-2)}\int_M \mathcal{L}_v\mathcal{L}_v \|G\|^2 dV \\ &\quad + \frac{1}{2n}\int_M \mathcal{L}_v\mathcal{L}_{D\rho}K dV - \frac{1}{n}\int_M \rho K\Delta\rho dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\ &= \frac{2}{n-2}\int_M \rho^2 \|G\|^2 dV - \frac{1}{2n(n-2)}\int_M \mathcal{L}_v\mathcal{L}_v \|G\|^2 dV\end{aligned}$$

$$+ \frac{1}{2n}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}KdV - \frac{1}{2n}\int_M \mathscr{L}_{D\rho}\mathscr{L}_vKdV ,$$

or, by Lemma 10,

$$\begin{aligned}\int_M \mathscr{L}_v\mathscr{L}_v \|G\|^2 dV - (n-2)\int_M \mathscr{L}_{[v,D\rho]}KdV \\ = 2n(n-2)\int_M \Big(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\Big)\Big(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\Big)dV \\ + 4n\int_M \rho^2 \|G\|^2 dV ,\end{aligned}$$

which together with Theorem G gives the proposition.

Remark 1. Proposition 2 is a generalization of Theorem D. Using (2.13) and Lemma 1 we have

$$\begin{aligned}&\int_M \mathscr{L}_{[v,D\rho]}KdV \\ &= \frac{1}{n+2}\int_M \mathscr{L}_v\mathscr{L}_v\Delta KdV - \frac{2(n+1)}{n(n+2)}\int_M \mathscr{L}_v\Delta\mathscr{L}_vKdV .\end{aligned} \tag{3.6}$$

Therefore Proposition 2 is essentially equivalent to Theorem A. Using (2.6), (3.2) and Lemmas 1 and 2 we have

$$\begin{aligned}&\int_M \mathscr{L}_{[v,D\rho]}KdV = n\int_M K\rho_i\rho^i dV \\ &- \frac{1}{2(n-1)}\int_M [2n\rho^2K^2 + (n+2)\rho K\mathscr{L}_vK + (\mathscr{L}_vK)^2]dV ,\end{aligned} \tag{3.7}$$

which implies that Proposition 2 is essentially equivalent to Theorem E.

Proposition 3. *For M we have*

$$\int_M \mathscr{L}_v\mathscr{L}_v \|Z\|^2 dV - 4\int_M \mathscr{L}_{[v,D\rho]}KdV \geq 0 . \tag{3.8}$$

The M of dimension $n > 2$ admits a nonhomothetic v such that the equality in (3.8) *holds if and only if M is isometric to a sphere.*

Proof. First of all we have

$$\mathscr{L}_v \|Z\|^2 = 2(\mathscr{L}_vZ_{kji}{}^h)Z^{kji}{}_h - 4\rho \|Z\|^2 .$$

Substituting (2.8) in the above equation we find

$$\mathscr{L}_v \|Z\|^2 = -8G_{ji}\nabla^j\rho^i - 4\rho \|Z\|^2 ,$$

because of $Z_{kji}{}^k = G_{ji}$ and $G_{ji}g^{ji} = 0$, or

$$K_{ji}\nabla^j\rho^i = -\frac{1}{2}\rho\,\|Z\|^2 - \frac{1}{8}\mathscr{L}_v\,\|Z\|^2 + \frac{1}{n}K\Delta\rho\,. \tag{3.9}$$

Using (2.18) and (3.9) we have

$$\begin{aligned}\nabla^j(\rho\rho^i K_{ji}) &= K_{ji}\rho^j\rho^i - \frac{1}{2}\rho^2\,\|Z\|^2 \\ &\quad - \frac{1}{8}\rho\mathscr{L}_v\,\|Z\|^2 + \frac{1}{2}\rho\mathscr{L}_{D\rho}K + \frac{1}{n}\rho K\Delta\rho\,.\end{aligned}$$

Integrating both sides of the above equation over M and using (2.6) and Lemmas 1 and 2, we obtain

$$\begin{aligned}&\int_M K_{ji}\rho^j\rho^i dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\ &= \frac{1}{2}\int_M \rho^2\,\|Z\|^2\,dV - \frac{1}{8n}\int_M \mathscr{L}_v\mathscr{L}_v\,\|Z\|^2\,dV \\ &\quad + \frac{1}{2n}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}K dV - \frac{1}{2n}\int_M \mathscr{L}_{D\rho}\mathscr{L}_v K dV\,,\end{aligned}$$

or, by Lemma 10,

$$\begin{aligned}&\int_M \mathscr{L}_v\mathscr{L}_v\,\|Z\|^2\,dV - 4\int_M \mathscr{L}_{[v,D\rho]}K dV \\ &= 8n\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV + 4n\int_M \rho^2\,\|Z\|^2\,dV\,,\end{aligned}$$

which together with Theorem G gives Proposition 3.

Remark 2. Using (3.6), (3.7) and (3.8) we see that Proposition 3 is essentially equivalent to Theorems B and F.

Proposition 4. *M admits a nonconstant function ρ satisfying*

$$\mathscr{L}_{D\rho}g_{ji} = 2\varphi g_{ji}\,, \qquad \mathscr{L}_{D\rho}K = 0\,, \tag{3.10}$$

φ being a function on M, if and only if M is isometric to a sphere.

Proof. If M admits a nonconstant function ρ satisfying (3.10), then, by Theorem H, M is isometric to a sphere because (3.10) is equivalent to (1.13). Conversely if M is isometric to a sphere, then M admits a nonconstant function ρ satisfying (2.23) and hence (3.10) because K is a positive constant for a sphere.

Proposition 5. *M admits a transformation v such that*

$$\mathscr{L}_{D\rho}g_{ji} = 2\varphi g_{ji}\,,$$

φ being a function on M, if and only if M is isometric to a sphere.

Proof. This follows immediately from Theorem G.

Ackler and Hsiung [1] proved this proposition for a special case in which the manifold M is compact and oriented and both $\mathscr{L}_v K = 0$ and $\mathscr{L}_{D\rho} K = 0$ hold.

Proposition 6. *For any function ρ on M we have*

$$K_{ji}\rho^j\rho^i + \frac{1}{n}(\Delta\rho)^2 + \mathscr{L}_{D\rho}\Delta\rho - \frac{1}{2}\Delta\mathscr{L}_{D\rho}\rho \leq 0 \; . \tag{3.11}$$

The complete M of dimension $n \geq 2$ admits a nonconstant function ρ such that the equality in (3.11) *holds and $\mathscr{L}_{D\rho}K = 0$ if and only if M is isometric to a sphere.*

Proof. This follows from Theorem H and Lemma 9.

Proposition 7. *M admits a transformation v such that the equality in* (3.11) *holds if and only if M is isometric to a sphere.*

Proof. This follows from Theorem G and Lemma 9.

Proposition 8. *For any function ρ on M we have*

$$\int_M \rho(\mathscr{L}_{D\rho}K_{ji})g^{ji}dV + \frac{2(n-1)}{n}\int_M \rho\Delta^2\rho dV \geq 0 \; . \tag{3.12}$$

The M of dimension $n \geq 2$ admits a nonconstant function ρ such that $\mathscr{L}_{D\rho}K = 0$ and the equality in (3.12) *holds if and only if M is isometric to a sphere.*

Proof. Using Lemmas 2, 8 and 10 we have

$$\begin{aligned} &\int_M \rho(\mathscr{L}_{D\rho}K_{ji})g^{ji}dV + \frac{2(n-1)}{n}\int_M \rho\Delta^2\rho dV \\ &= 2\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV \; , \end{aligned} \tag{3.13}$$

which together with Theorem H gives Proposition 8.

Corollary 1. *M of dimension $n \geq 2$ admits a nonconstant function ρ such that $\mathscr{L}_{D\rho}K = 0$ and*

$$\mathscr{L}_{D\rho}K_{ji} = -\frac{2(n-1)}{n^2}\Delta^2\rho g_{ji} \; , \tag{3.14}$$

if and only if M is isometric to a sphere.

Proof. If M is isometric to a sphere, then M admits a nonconstant function ρ such that (2.23) holds. Therefore using (2.24) we have

$$\begin{aligned} \mathscr{L}_{D\rho}K_{ji} &= \frac{1}{n}K\mathscr{L}_{D\rho}g_{ji} = \frac{2}{n}K\nabla_j\rho_i \\ &= \frac{2}{n^2}K\Delta\rho g_{ji} = -\frac{2(n-1)}{n^2}\Delta^2\rho g_{ji} \; . \end{aligned}$$

The "only if" part of the corollary is an immediate consequence of Proposition 8.

Remark 3. By (2.25) in Lemma 11, (3.14) in Corollary 1 can be replaced by

$$\mathscr{L}_{D\rho}K_{ji} = -\frac{2(n-1)}{n}\nabla_j\nabla_i\Delta\rho \ . \tag{3.15}$$

Proposition 9. *For M we have* (3.12), *and the M of dimension* $n > 2$ *admits a nonhomothetic v such that the equality in* (3.12) *holds if and only if M is isometric to a sphere.*

Proof. This follows from (3.13) and Theorem G.

Corollary 2. *M of dimension* $n > 2$ *admits a nonhomothetic v such that* (3.14) *holds if and only if M is isometric to a sphere.*

Proof. This follows from Lemma 11 and Proposition 9.

Remark 4. By (2.25) in Lemma 11, (3.14) in Corollary 2 can be replaced by (3.15).

Proposition 10. *For any function* ρ *on M we have*

$$\begin{aligned}&\int_M \rho(\mathscr{L}_{D\rho}K_{kjih})g^{kh}g^{ji}dV + \int_M \rho\mathscr{L}_{D\rho}KdV \\ &\quad + \frac{4(n-1)}{n}\int_M \rho\Delta^2\rho dV \geq 0 \ .\end{aligned} \tag{3.16}$$

The M of dimension $n \geq 2$ *admits a nonconstant function* ρ *such that* $\mathscr{L}_{D\rho}K = 0$ *and the equality in* (3.16) *holds if and only if M is isometric to a sphere.*

Proof. Using Lemmas 2, 8 and 10, we have

$$\begin{aligned}&\int_M \rho(\mathscr{L}_{D\rho}K_{kjih})g^{kh}g^{ji}dV + \int_M \rho\mathscr{L}_{D\rho}KdV + \frac{4(n-1)}{n}\int_M \rho\Delta^2\rho dV \\ &= 4\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV \ ,\end{aligned} \tag{3.17}$$

which together with Theorem H gives the proposition.

Corollary 3. *M of dimension* $n \geq 2$ *admits a nonconstant function* ρ *such that* $\mathscr{L}_{D\rho}K = 0$ *and*

$$\mathscr{L}_{D\rho}K_{kjih} = -\frac{4}{n^2}\Delta^2\rho(g_{kh}g_{ji} - g_{jh}g_{ki}) \ , \tag{3.18}$$

if and only if M is isometric to a sphere.

Proof. If M is isometric to a sphere, then M admits a nonconstant function ρ such that (2.23) holds. Since K is a positive constant and

$$K_{kjih} = \frac{1}{n(n-1)} K(g_{kh}g_{ji} - g_{jh}g_{ki})$$

for a sphere, using (2.24) we obtain

$$\begin{aligned}\mathscr{L}_{D\rho}K_{kjih} &= \frac{2}{n(n-1)} K(\nabla_k\rho_h g_{ji} + g_{kh}\nabla_j\rho_i - \nabla_j\rho_h g_{ki} - g_{jh}\nabla_k\rho_i) \\ &= -\frac{2}{n}(\nabla_k\nabla_h\Delta\rho g_{ji} + g_{kh}\nabla_j\nabla_i\Delta\rho - \nabla_j\nabla_h\Delta\rho g_{ki} - g_{jh}\nabla_k\nabla_i\Delta\rho) \, ,\end{aligned}$$

which together with (2.25) gives (3.18). The "only if" part of the corollary is an immediate consequence of Proposition 10.

Remark 5. As is seen in the proof of Corollary 3, (3.18) in Corollary 3 can be replaced by

$$\begin{aligned}&\mathscr{L}_{D\rho}K_{kjih} \\ &= -\frac{2}{n}(\nabla_k\nabla_h\Delta\rho g_{ji} + g_{kh}\nabla_j\nabla_i\Delta\rho - \nabla_j\nabla_h\Delta\rho g_{ki} - g_{jh}\nabla_k\nabla_i\Delta\rho) \, .\end{aligned} \tag{3.19}$$

Proposition 11. *For M we have* (3.16). *The M of dimension $n > 2$ admits a nonhomothetic v such that the equality in* (3.16) *holds if and only if M is isometric to a sphere.*

Proof. This follows from (3.17) and Theorem G.

Corollary 4. *M of dimension $n > 2$ admits a nonhomothetic v such that*

$$\begin{aligned}&\mathscr{L}_{D\rho}K_{kjih} \\ &= -\frac{1}{n(n-1)}\left[\mathscr{L}_{D\rho}K + \frac{4(n-1)}{n}\Delta^2\rho\right](g_{kh}g_{ji} - g_{jh}g_{ki})\end{aligned} \tag{3.20}$$

holds if and only if M is isometric to a sphere.

Proof. This follows from Lemma 11 and Proposition 11.

Remark 6. In Corollary 4, we see, by using Lemma 11, that (3.20) can be replaced by

$$\begin{aligned}&\mathscr{L}_{D\rho}K_{kjih} \\ &= -\frac{1}{n(n-1)}(\mathscr{L}_{D\rho}K)(g_{kh}g_{ji} - g_{jh}g_{ki}) \\ &\quad -\frac{2}{n}(\nabla_k\nabla_h\Delta\rho g_{ji} + g_{kh}\nabla_j\nabla_i\Delta\rho - \nabla_j\nabla_h\Delta\rho g_{ki} - g_{jh}\nabla_k\nabla_i\Delta\rho) \, .\end{aligned} \tag{3.21}$$

Proposition 12. *If M of dimension $n \geq 2$ admits an infinitesimal conformal transformation v, then*

$$(\mathscr{L}_{D\rho}\mathscr{L}_v G_{ji})g^{ji} \leq 0 \, . \tag{3.22}$$

The complete M of dimension $n > 2$ admits a complete infinitesimal nonhomothetic conformal transformation v such that the equality in (3.22) *holds if and only if M is isometric to a sphere.*

Proof. By using (2.7) we have

$$(\mathscr{L}_v G_{ji})g^{ji} = 0 ,$$

and consequently

$$\begin{aligned}(\mathscr{L}_{D\rho}\mathscr{L}_v G_{ji})g^{ji} &= -(\mathscr{L}_v G_{ji})\mathscr{L}_{D\rho}g^{ji} = 2(\mathscr{L}_v G_{ji})\nabla^j\rho^i \\ &= -2(n-2)\Big(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\Big)\nabla^j\rho^i \\ &= -2(n-2)\Big(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\Big)\Big(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\Big) ,\end{aligned}$$

which together with Theorem G gives the proposition.

Proposition 13. *For M of dimension $n \geq 2$ we have*

$$(\mathscr{L}_{D\rho}\mathscr{L}_v Z_{kjih} - 2\rho\mathscr{L}_{D\rho}Z_{kjih})g^{kh}g^{ji} \leq 0 . \tag{3.23}$$

The complete M of dimension $n > 2$ admits a complete nonhomothetic v such that the equality in (3.23) *holds if and only if M is isometric to a sphere.*

Proof. From (2.8) it follows that

$$\begin{aligned}\mathscr{L}_v Z_{kjih} = &-g_{kh}\nabla_j\rho_i + g_{jh}\nabla_k\rho_i - \nabla_k\rho_h g_{ji} + \nabla_j\rho_h g_{ki} \\ &+\frac{2}{n}\Delta\rho(g_{kh}g_{ji} - g_{jh}g_{ki}) + 2\rho Z_{kjih} ,\end{aligned}$$

and therefore that

$$(\mathscr{L}_v Z_{kjih})g^{kh}g^{ji} = 0 .$$

Using this we obtain

$$\begin{aligned}&(\mathscr{L}_{D\rho}\mathscr{L}_v Z_{kjih})g^{kh}g^{ji} \\ &\quad = 4(\mathscr{L}_v Z_{kjih})g^{ji}\nabla^k\rho^h \\ &\quad = -4(n-2)\Big(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\Big)\Big(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\Big) \\ &\qquad + 8\rho Z_{kjih}g^{ji}\nabla^k\rho^h .\end{aligned}$$

On the other hand, since $Z_{kjih}g^{kh}g^{ji} = 0$ we have

$$(\mathscr{L}_{D\rho}Z_{kjih})g^{kh}g^{ji} = 4Z_{kjih}g^{ji}\nabla^k\rho^h .$$

Thus

$$(\mathscr{L}_{D\rho}\mathscr{L}_v Z_{kjih} - 2\rho\mathscr{L}_{D\rho}Z_{kjih})g^{kh}g^{ji} = -4(n-2)\Big(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\Big)\Big(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\Big) ,$$

which together with Theorem G gives the proposition.

Corollary 5. *M admits a transformation v such that*

$$\mathscr{L}_{D\rho}\mathscr{L}_v G_{ji} = 0 \tag{3.24}$$

or

$$\mathscr{L}_{D\rho}\mathscr{L}_v Z_{kjih} - 2\rho\mathscr{L}_{D\rho}Z_{kjih} = 0 , \tag{3.25}$$

if and only if M is isometric to a sphere.

Proof. This follows from Propositions 12 and 13.

Proposition 14. *For M we have*

$$\int_M \rho(\mathscr{L}_{D\rho}G_{ji})g^{ji}dV - \frac{1}{n}\int_M \mathscr{L}_{[v,D\rho]}KdV \geq 0 . \tag{3.26}$$

The M of dimension $n > 2$ admits a nonhomothetic v such that the equality in (3.26) *holds if and only if M is isometric to a sphere.*

Proof. We have, by using $G_{ji}g^{ji} = 0$,

$$\begin{aligned}\rho(\mathscr{L}_{D\rho}G_{ji})g^{ji} &= -\rho G_{ji}\mathscr{L}_{D\rho}g^{ji} = 2\rho G_{ji}\nabla^j\rho^i \\ &= 2\rho K_{ji}\nabla^j\rho^i - \frac{2}{n}\rho K\Delta\rho ,\end{aligned} \tag{3.27}$$

or, using (2.18),

$$\frac{1}{2}\rho(\mathscr{L}_{D\rho}G_{ji})g^{ji} = \nabla^j(\rho\rho^i K_{ji}) - K_{ji}\rho^j\rho^i - \frac{1}{2}\rho\mathscr{L}_{D\rho}K - \frac{1}{n}\rho K\Delta\rho .$$

Integrating both sides of the above equation over M and using (2.6), we find

$$\begin{aligned}&\int_M K_{ji}\rho^j\rho^i dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\ &= -\frac{1}{2}\int_M \rho(\mathscr{L}_{D\rho}G_{ji})g^{ji}dV - \frac{1}{2}\int_M \rho\mathscr{L}_{D\rho}KdV \\ &\quad -\frac{1}{n}\int_M \rho K\Delta\rho dV - \frac{n-1}{n}\int_M (\Delta\rho)^2 dV \\ &= -\frac{1}{2}\int_M \rho(\mathscr{L}_{D\rho}G_{ji})g^{ji}dV + \frac{1}{2n}\int_M \mathscr{L}_v\mathscr{L}_{D\rho}KdV\end{aligned}$$

$$+\frac{1}{2n}\int_M (\Delta\rho)\mathcal{L}_v K dV ,$$

or, by Lemmas 2 and 10,

$$\begin{aligned}&\int_M \rho(\mathcal{L}_{D\rho}G_{ji})g^{ji}dV - \frac{1}{n}\int_M \mathcal{L}_{[v,D\rho]}KdV \\ &= 2\int_M \left(\nabla_j\rho_i - \frac{1}{n}\Delta\rho g_{ji}\right)\left(\nabla^j\rho^i - \frac{1}{n}\Delta\rho g^{ji}\right)dV ,\end{aligned} \tag{3.28}$$

which together with Theorem G gives the proposition.

Corollary 6. *M of dimension $n > 2$ admits a nonhomothetic v such that*

$$\rho\mathcal{L}_{D\rho}G_{ji} = \frac{1}{n^2}(\mathcal{L}_{[v,D\rho]}K)g_{ji} , \tag{3.29}$$

if and only if M is isometric to a sphere.

Proof. This is an immediate consequence of Proposition 14.

Corollary 7. *M of dimension $n > 2$ admits a nonhomothetic v such that*

$$\mathcal{L}_{D\rho}G_{ji} = -\frac{1}{n(n+2)}\left[\mathcal{L}_v\Delta K - \frac{2(n+1)}{n}\Delta\mathcal{L}_v K\right]g_{ji} , \tag{3.30}$$

if and only if M is isometric to a sphere.

Proof. This follows from Lemma 7 and Proposition 14.

Proposition 15. *For M we have*

$$\int_M \rho(\mathcal{L}_{D\rho}Z_{kjih})g^{kh}g^{ji}dV - \frac{2}{n}\int_M \mathcal{L}_{[v,D\rho]}KdV \geq 0 . \tag{3.31}$$

The M of dimension $n > 2$ admits a nonhomothetic v such that the equality in (3.31) *holds if and only if M is isometric to a sphere.*

Proof. We have, by using $Z_{kjih}g^{kh} = G_{ji}$ and $G_{ji}g^{ji} = 0$,

$$\rho(\mathcal{L}_{D\rho}Z_{kjih})g^{kh}g^{ji} = -2\rho G_{ji}\mathcal{L}_{D\rho}g^{ji} ,$$

which together with

$$\rho(\mathcal{L}_{D\rho}G_{ji})g^{ji} = -\rho G_{ji}\mathcal{L}_{D\rho}g^{ji}$$

implies

$$\rho(\mathcal{L}_{D\rho}Z_{kjih})g^{kh}g^{ji} = 2\rho(\mathcal{L}_{D\rho}G_{ji})g^{ji} .$$

Integrating both sides of the above equation over M and using (3.28), we obtain

$$\int_M \rho(\mathcal{L}_{D\rho}Z_{kjih})g^{kh}g^{ji}dV - \frac{2}{n}\int_M \mathcal{L}_{[v,D\rho]}KdV$$

$$= 4 \int_M \left(\nabla_j \rho_i - \frac{1}{n} \Delta \rho g_{ji} \right) \left(\nabla^j \rho^i - \frac{1}{n} \Delta \rho g^{ji} \right) dV ,$$

which together with Theorem G gives the proposition.

Corollary 8. *M of dimension $n > 2$ admits a nonhomothetic v such that*

$$\rho \mathscr{L}_{D\rho} Z_{kjih} = \frac{2}{n^2(n-1)} (\mathscr{L}_{[v, D\rho]} K)(g_{kh} g_{ji} - g_{jh} g_{ki}) , \tag{3.32}$$

if and only if M is isometric to a sphere.

Proof. This is an immediate consequence of Proposition 15.

Corollary 9. *M of dimension $n > 2$ admits a nonhomothetic v such that*

$$\begin{aligned} &\mathscr{L}_{D\rho} Z_{kjih} \\ &= - \frac{2}{n(n-1)(n+2)} \left[\mathscr{L}_v \Delta K - \frac{2(n+1)}{n} \Delta \mathscr{L}_v K \right] (g_{kh} g_{ji} - g_{jh} g_{ki}) , \end{aligned} \tag{3.33}$$

if and only if M is isometric to a sphere.

Proof. This follows from Lemma 7 and Proposition 15.

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