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ANTI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD WITH VANISHING CONTACT BOCHNER CURVATURE TENSOR

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0. Introduction

In 1949, by using a complex coordinate system Bochner [3] (see also Yano and Bochner [23]) introduced, as an analogue of the Weyl conformal curvature tensor in a Riemannian manifold, what we now call the Bochner curvature tensor in a Kaehlerian manifold. In 1967 Tachibana [13] gave a tensor expression of this curvature tensor in a real coordinate system. Since then the tensor has been studied by Chen [5], Ishihara [25], Liu [14], Matsumoto [10], Sato [17], Tachibana [14], Takagi [15], Watanabe [15], Yamaguchi [17], and the present author [5], [19], [20], [21], [22], [25].

Let M^{2m} be a real $2m$-dimensional Kaehlerian manifold with the almost complex structure F, and M^n an n-dimensional Riemannian manifold isometrically immersed in M^{2m}. If $T_x(M^n) \perp FT_x(M^n)$, where $T_x(M^n)$ denotes the tangent space to M^n at a point x of M^n and is identified with its image under the differential of the immersion, then we call M^n a *totally real* or *anti-invariant* submanifold of M^{2m}. Since the rank of F is $2m$, we have $n \leq 2m - n$, that is, $n \leq m$.

The totally real submanifolds of a Kaehlerian manifold have been studied by Chen [4], Houh [6], Kon [7], [26], [27], Ludden [8], [9], Ogiue [4], Okumura [8], [9] and the present author [8], [9], [21], [22], [26], [27].

As a theorem connecting the Weyl conformal curvature tensor and the Bochner curvature tensor, Blair [1] proved

Theorem A. *Let M^{2n}, $n \geq 4$, be a Kaehlerian manifold with vanishing Bochner curvature tensor, and M^n a totally geodesic, totally real submanifold of M^{2n}. Then M^n is conformally flat.*

Generalizing this theorem of Blair, the present author [21] established the following theorems.

Theorem B. *Let M^n, $n \geq 4$, be a totally umbilical, totally real submanifold of a Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. Then M^n is conformally flat.*

Theorem C. *Let M^3 be a totally geodesic, totally real submanifold of a*

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Kaehlerian manifold M^{2m} with vanishing Bochner curvature tensor. Then M^3 is conformally flat.

Theorem D. *Let M^n, $n \geq 4$, be a totally real submanifold of a Kaehlerian manifold M^{2n} with vanishing Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is conformally flat.*

The main purpose of the present paper is to obtain theorems, analogous to the above theorems, for anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor. For anti-invariant submanifolds of a Sasakian manifold, see Blair and Ogiue [2], Yamaguchi, Kon and Ikawa [16], Yano and Kon [28], [29], and for the contact Bochner curvature tensor see Matsumoto and Chūman [11].

First of all, in § 1 we recall the definition and the fundamental properties of a Sasakian manifold. In § 2 we define a curvature tensor in a Sasakian manifold which is called the contact Bochner curvature tensor and corresponds to the Bochner curvature tensor in a Kaehlerian manifold.

§ 3 is devoted to general discussions on anti-invariant submanifolds of a Sasakian manifold, and § 4 to the study of anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor.

In the last two sections (§§ 5 and 6) we study Sasakian manifolds with vanishing contact Bochner curvature tensor regarded as fibred spaces with invariant Riemannian metric (see Yano and Ishihara [24]).

1. Sasakian manifolds

We first of all recall the definition and the fundamental properties of almost contact manifolds for the later use. Let M^{2m+1} be a $(2m+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U; x^\kappa\}$ in which there are given a tensor field $\varphi_\lambda{}^\kappa$ of type (1,1), a vector field ξ^κ and a 1-form η_λ satisfying

$$(1.1) \qquad \varphi_\lambda{}^\kappa\varphi_\mu{}^\lambda = -\delta_\mu^\kappa + \eta_\mu\xi^\kappa\,, \quad \varphi_\lambda{}^\kappa\xi^\lambda = 0\,, \quad \eta_\lambda\varphi_\mu{}^\lambda = 0\,, \quad \eta_\lambda\xi^\lambda = 1\,,$$

where and in the sequel the indices $\alpha, \beta, \cdots, \kappa, \lambda, \mu, \cdots$ run over the range $\{1, 2, \cdots, 2m+1\}$. Such a set (φ, ξ, η) consisting of a tensor field φ, a vector field ξ and a 1-form η is called an *almost contact structure*, and a manifold with an almost contact structure an *almost contact manifold* (see Sasaki [12]).

If the Nijenhuis tensor

$$(1.2) \qquad N_{\mu\lambda}{}^\kappa = \varphi_\mu{}^\alpha\partial_\alpha\varphi_\lambda{}^\kappa - \varphi_\lambda{}^\alpha\partial_\alpha\varphi_\mu{}^\kappa - (\partial_\mu\varphi_\lambda{}^\alpha - \partial_\lambda\varphi_\mu{}^\alpha)\varphi_\alpha{}^\kappa$$

formed with $\varphi_\lambda{}^\kappa$ satisfies

$$(1.3) \qquad N_{\mu\lambda}{}^\kappa + (\partial_\mu\eta_\lambda - \partial_\lambda\eta_\mu)\xi^\kappa = 0\,,$$

where $\partial_\mu = \partial/\partial x^\mu$, then the almost contact structure is said to be *normal* and the manifold is called a *normal almost contact manifold*.

Suppose that in an almost contact manifold there is given a Riemannian metric $g_{\mu\lambda}$ such that

$$g_{\gamma\beta}\varphi_\mu{}^\gamma\varphi_\lambda{}^\beta = g_{\mu\lambda} - \eta_\mu\eta_\lambda\,, \qquad \eta_\lambda = g_{\lambda\kappa}\xi^\kappa\,, \tag{1.4}$$

then the almost contact structure is said to be *metric*, and the manifold is called an *almost contact metric manifold*. In view of the second equation of (1.4) we shall write ξ_λ instead of η_λ in the sequel. In an almost contact metric manifold, the tensor field $\varphi_{\mu\lambda} = \varphi_\mu{}^\alpha g_{\alpha\lambda}$ is skew-symmetric.

If an almost contact metric structure satisfies

$$\varphi_{\mu\lambda} = \tfrac{1}{2}(\partial_\mu\xi_\lambda - \partial_\lambda\xi_\mu)\,, \tag{1.5}$$

then the almost contact metric structure is called a *contact structure*. A manifold with a normal contact structure is called a *Sasakian manifold*.

It is well known that in a Sasakian manifold we have

$$\nabla_\lambda\xi^\kappa = \varphi_\lambda{}^\kappa\,, \tag{1.6}$$

$$\nabla_\mu\varphi_\lambda{}^\kappa = -g_{\mu\lambda}\xi^\kappa + \delta_\mu^\kappa\xi_\lambda\,, \tag{1.7}$$

where ∇_λ denotes the operator of covariant differentiation with respect to $g_{\mu\lambda}$. (1.6) written as $\nabla_\lambda\xi_\kappa = \varphi_{\lambda\kappa}$ shows that ξ^κ is a Killing vector field.

(1.6), (1.7) and the Ricci identity

$$\nabla_\nu\nabla_\mu\xi^\kappa - \nabla_\mu\nabla_\nu\xi^\kappa = K_{\nu\mu\lambda}{}^\kappa\xi^\lambda\,,$$

where $K_{\nu\mu\lambda}{}^\kappa$ is the curvature tensor, give

$$K_{\nu\mu\lambda}{}^\kappa\xi^\lambda = \delta_\nu^\kappa\xi_\mu - \delta_\mu^\kappa\xi_\nu\,, \tag{1.8}$$

or

$$K_{\nu\mu\lambda}{}^\kappa\xi_\kappa = \xi_\nu g_{\mu\lambda} - \xi_\mu g_{\nu\lambda}\,. \tag{1.9}$$

From (1.9) by contraction we have

$$K_{\mu\lambda}\xi^\lambda = 2m\xi_\mu\,, \tag{1.10}$$

where $K_{\mu\lambda} = K_{\alpha\mu\lambda}{}^\alpha$ is the Ricci tensor.

(1.6), (1.7) and the Ricci identity

$$\nabla_\nu\nabla_\mu\varphi_\lambda{}^\kappa - \nabla_\mu\nabla_\nu\varphi_\lambda{}^\kappa = K_{\nu\mu\alpha}{}^\kappa\varphi_\lambda{}^\alpha - K_{\nu\mu\lambda}{}^\alpha\varphi_\alpha{}^\kappa$$

imply

$$K_{\nu\mu\alpha}{}^\kappa\varphi_\lambda{}^\alpha - K_{\nu\mu\lambda}{}^\alpha\varphi_\alpha{}^\kappa = -\varphi_\nu{}^\kappa g_{\mu\lambda} + \varphi_\mu{}^\kappa g_{\nu\lambda} - \delta_\nu^\kappa\varphi_{\mu\lambda} + \delta_\mu^\kappa\varphi_{\nu\lambda}\,, \tag{1.11}$$

from which, by contraction, it follows that

$$K_{\mu\alpha}\varphi_{\lambda}{}^{\alpha} + K_{\beta\mu\lambda\alpha}\varphi^{\beta\alpha} = -(2m-1)\varphi_{\mu\lambda}\,, \tag{1.12}$$

where $\varphi^{\beta\alpha} = g^{\beta\lambda}\varphi_{\lambda}{}^{\alpha}$, $g^{\beta\lambda}$ being contravariant components of the metric tensor. Since $K_{\beta\mu\lambda\alpha}\varphi^{\beta\alpha}$ is skew-symmetric in μ and λ, we have from (1.12)

$$K_{\mu\alpha}\varphi_{\lambda}{}^{\alpha} + K_{\lambda\alpha}\varphi_{\mu}{}^{\alpha} = 0\,. \tag{1.13}$$

From (1.12) we also find

$$K_{\beta\alpha\mu\lambda}\varphi^{\beta\alpha} = 2K_{\mu\alpha}\varphi_{\lambda}{}^{\alpha} + 2(2m-1)\varphi_{\mu\lambda}\,. \tag{1.14}$$

2. Contact Bochner curvature tensor

As an analogue of the Bochner curvature tensor in a Kaehlerian manifold, we define the contact Bochner curvature tensor in a Sasakian manifold by

$$\begin{aligned} B_{\nu\mu\lambda}{}^{\kappa} = K_{\nu\mu\lambda}{}^{\kappa} &+ (\delta_{\nu}^{\kappa} - \xi_{\nu}\xi^{\kappa})L_{\mu\lambda} - (\delta_{\mu}^{\kappa} - \xi_{\mu}\xi^{\kappa})L_{\nu\lambda} + L_{\nu}{}^{\kappa}(g_{\mu\lambda} - \xi_{\mu}\xi_{\lambda}) \\ &- L_{\mu}{}^{\kappa}(g_{\nu\lambda} - \xi_{\nu}\xi_{\lambda}) + \varphi_{\nu}{}^{\kappa}M_{\mu\lambda} - \varphi_{\mu}{}^{\kappa}M_{\nu\lambda} + M_{\nu}{}^{\kappa}\varphi_{\mu\lambda} - M_{\mu}{}^{\kappa}\varphi_{\nu\lambda} \\ &- 2(\varphi_{\nu\mu}M_{\lambda}{}^{\kappa} + M_{\nu\mu}\varphi_{\lambda}{}^{\kappa}) + (\varphi_{\nu}{}^{\kappa}\varphi_{\mu\lambda} - \varphi_{\mu}{}^{\kappa}\varphi_{\nu\lambda} - 2\varphi_{\nu\mu}\varphi_{\lambda}{}^{\kappa})\,, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} L_{\mu\lambda} &= \frac{1}{2(m+2)}[-K_{\mu\lambda} - (L+3)g_{\mu\lambda} + (L-1)\xi_{\mu}\xi_{\lambda}]\,, \\ L_{\mu}{}^{\kappa} &= L_{\mu\alpha}g^{\alpha\kappa}\,, \end{aligned} \tag{2.2}$$

$$L = g^{\mu\lambda}L_{\mu\lambda}\,, \tag{2.3}$$

$$M_{\mu\lambda} = -L_{\mu\alpha}\varphi_{\lambda}{}^{\alpha}\,, \qquad M_{\nu}{}^{\kappa} = M_{\nu\alpha}g^{\alpha\kappa}\,. \tag{2.4}$$

From (2.2) and (2.3) it follows that

$$L = -\frac{K + 2(3m+2)}{4(m+1)}\,, \tag{2.5}$$

where K is the scalar curvature of the manifold.

Using (1.10) we have, from (2.2),

$$L_{\mu\lambda}\xi^{\lambda} = -\xi_{\mu}\,, \tag{2.6}$$

which, together with the first equation of (2.4), yields

$$M_{\mu\alpha}\varphi_{\lambda}{}^{\alpha} = L_{\mu\lambda} + \xi_{\mu}\xi_{\lambda}\,. \tag{2.7}$$

We can easily verify that the contact Bochner curvature tensor satisfies the following identities:

(2.8) $$B_{\nu\mu\lambda}{}^{\kappa} = -B_{\mu\nu\lambda}{}^{\kappa}\,,\quad B_{\nu\mu\lambda}{}^{\kappa} + B_{\mu\lambda\nu}{}^{\kappa} + B_{\lambda\nu\mu}{}^{\kappa} = 0\,,\quad B_{\alpha\mu\lambda}{}^{\alpha} = 0\,,$$

(2.9) $$B_{\nu\mu\lambda\kappa} = -B_{\nu\mu\kappa\lambda}\,,\qquad B_{\nu\mu\lambda\kappa} = B_{\lambda\kappa\nu\mu}\,,$$

where $B_{\nu\mu\lambda\kappa} = B_{\nu\mu\lambda}{}^{\alpha} g_{\alpha\kappa}$ and

(2.10) $$B_{\nu\mu\lambda}{}^{\kappa}\xi_{\kappa} = 0\,,\quad B_{\nu\mu\alpha}{}^{\kappa}\varphi_{\lambda}{}^{\alpha} = B_{\nu\mu\lambda}{}^{\alpha}\varphi_{\alpha}{}^{\kappa}\,,\quad B_{\nu\mu\lambda}{}^{\kappa}\varphi^{\nu\mu} = 0\,.$$

3. Anti-invariant submanifolds of a Sasakian manifold

We consider an n-dimensional Riemannian manifold M^n, $n > 1$, covered by a system of coordinate neighborhoods $\{V; y^h\}$ and isometrically immersed in a Sasakian manifold M^{2m+1}, and denote the immersion by

(3.1) $$x^{\kappa} = x^{\kappa}(y^h)$$

where and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1', 2', \cdots, n'\}$. We put

(3.2) $$B_i{}^{\kappa} = \partial_i x^{\kappa} \qquad (\partial_i = \partial/\partial y^i)\,,$$

and denote $2m + 1 - n$ mutually orthogonal unit vectors normal to M^n by $C_y{}^{\kappa}$, where and in the sequel the indices x, y, z run over the range $\{(n + 1)', \cdots, (2m + 1)'\}$.

Then the metric tensor g_{ji} of M^n and that of the normal bundle are respectively given by

(3.3) $$g_{ji} = g_{\mu\lambda} B_{ji}^{\mu\lambda}\,,$$

(3.4) $$g_{zy} = g_{\mu\lambda} C_{zy}^{\mu\lambda}\,,$$

where $B_{ji}^{\mu\lambda} = B_j{}^{\mu} B_i{}^{\lambda}$ and $C_{zy}^{\mu\lambda} = C_z{}^{\mu} C_y{}^{\lambda}$.

If the transform by $\varphi_{\lambda}{}^{\kappa}$ of any vector tangent to M^n is orthogonal to M^n, we say that the submanifold M^n is *anti-invariant* in M^{2m+1}. Since the rank of $\varphi_{\lambda}{}^{\kappa}$ is $2m$, we have $n - 1 \leq 2m + 1 - n$, that is, $n \leq m + 1$.

For an anti-invariant submanifold M^n in M^{2m+1}, we have equations of the form

(3.5) $$\varphi_{\lambda}{}^{\kappa} B_i{}^{\lambda} = -f_i{}^x C_x{}^{\kappa}\,,$$

(3.6) $$\varphi_{\lambda}{}^{\kappa} C_y{}^{\lambda} = f_y{}^i B_i{}^{\kappa} + f_y{}^x C_x{}^{\kappa}\,,$$

(3.7) $$\xi^{\kappa} = \xi^i B_i{}^{\kappa} + \xi^x C_x{}^{\kappa}\,.$$

Using $\varphi_{\mu\lambda} = -\varphi_{\lambda\mu}$ we have, from (3.5) and (3.6),

$$f_{iy} = f_{yi} , \tag{3.8}$$

where $f_{iy} = f_i{}^z g_{zy}$ and $f_{yi} = f_y{}^j g_{ji}$ and

$$f_{yx} = -f_{xy} , \tag{3.9}$$

where $f_{yx} = f_y{}^z g_{zx}$.

Applying φ to (3.5), (3.6) and (3.7) and using (1.1), (3.8), (3.9) we find

$$\begin{array}{ll} \text{(i)} & f_i{}^y f_y{}^h = \delta_i^h - \xi_i \xi^h , \\ \text{(ii)} & f_i{}^y f_y{}^x = -\xi_i \xi^x , \\ \text{(iii)} & f_y{}^z f_z{}^h = \xi_y \xi^h , \\ \text{(iv)} & f_y{}^z f_z{}^x = -\delta_y^x + \xi_y \xi^x + f_y{}^i f_i{}^x , \\ \text{(v)} & f_x{}^i \xi^x = 0 , \\ \text{(vi)} & f_i{}^x \xi^i = f_y{}^x \xi^y , \\ \text{(vii)} & \xi_i \xi^i + \xi_y \xi^y = 1 , \end{array} \tag{3.10}$$

where $\xi_i = g_{ih}\xi^h$ and $\xi_y = g_{yx}\xi^x$, (vii) being a consequence of $\xi_\lambda \xi^\lambda = 1$.

Differentiating (3.5), (3.6) and (3.7) covariantly over M^n and using (1.6), (1.7), (3.10), equations of Gauss

$$\nabla_j B_i{}^\kappa = h_{ji}{}^x C_x{}^\kappa \tag{3.11}$$

and those of Weingarten

$$\nabla_j C_y{}^\kappa = -h_j{}^i{}_y B_i{}^\kappa , \tag{3.12}$$

where ∇_j denotes the operator of covariant differentiation over M^n, and $h_{ji}{}^x$ and $h_j{}^i{}_y = h_{jt}{}^z g^{ti} g_{zy}$ are the second fundamental tensors of M^n with respect to the normals $C_x{}^\kappa$, we find

$$\begin{array}{ll} \text{(i)} & -g_{ji}\xi^h + \delta_j^h \xi_i = -h_{ji}{}^x f_x{}^h + h_j{}^h{}_x f_i{}^x , \\ \text{(ii)} & \nabla_j f_i{}^x = g_{ji}\xi^x - h_{ji}{}^y f_y{}^x , \\ \text{(iii)} & \nabla_j f_y{}^h = \delta_j^h \xi_y + h_j{}^h{}_x f_y{}^x , \\ \text{(iv)} & \nabla_j f_y{}^x = -h_{ji}{}^x f_y{}^i + h_j{}^i{}_y f_i{}^x , \\ \text{(v)} & \nabla_j \xi^h = h_j{}^h{}_y \xi^y , \\ \text{(vi)} & \nabla_j \xi^x = -f_j{}^x - h_{ji}{}^x \xi^i . \end{array} \tag{3.13}$$

I. The case in which ξ^κ is tangent to M^n. Suppose that $n = m + 1$. Then the codimension of M^n is $2m + 1 - n = n - 1$, and consequently $[f_y{}^h, \xi^h]$ and $\begin{bmatrix} f_i{}^y \\ \xi_i \end{bmatrix}$ are both square matrices and satisfy

$$[f_y{}^h, \xi^h]\begin{bmatrix} f_i{}^y \\ \xi_i \end{bmatrix} = \text{unit matrix}$$

because of (3.10) (i). Thus we have

$$\begin{bmatrix} f_i{}^x \\ \xi_i \end{bmatrix}[f_y{}^i, \xi^i] = \text{unit matrix}\ ,$$

from which it follows that

$$f_i{}^x f_y{}^i = \delta_y^x\ , \quad f_i{}^x \xi^i = 0\ , \quad \xi_i f_y{}^i = 0\ , \quad \xi_i \xi^i = 1\ . \tag{3.14}$$

By remembering that $\xi_i\xi^i + \xi_x\xi^x = 1$, we further find $\xi^x = 0$ and hence ξ^κ is tangent to M^n.

In general suppose that ξ^κ is tangent to M^n, that is, $\xi^x = 0$. Then (3.10) becomes

$$(3.15)\qquad \begin{aligned} &(\mathrm{i}) && f_i{}^y f_y{}^h = \delta_i^h - \xi_i\xi^h\ , \\ &(\mathrm{ii}) && f_i{}^y f_y{}^x = 0\ , \\ &(\mathrm{iii}) && f_y{}^z f_z{}^h = 0\ , \\ &(\mathrm{iv}) && f_y{}^z f_z{}^x = -\delta_y^x + f_y{}^i f_i{}^x\ , \\ &(\mathrm{v}) && f_i{}^x \xi^i = 0\ , \\ &(\mathrm{vi}) && \xi_i\xi^i = 1\ . \end{aligned}$$

From (3.15)(iii) and (iv) we see that $f_y{}^x$ defines a so-called *f-structure* in the normal bundle (see Yano [18]). In this case (3.13) becomes

$$(3.16)\qquad \begin{aligned} &(\mathrm{i}) && -g_{ji}\xi^h + \delta_j^h\xi_i = -h_{ji}{}^x f_x{}^h + h_j{}^h{}_x f_i{}^x\ , \\ &(\mathrm{ii}) && \nabla_j f_i{}^x = -h_{ji}{}^y f_y{}^x\ , \\ &(\mathrm{iii}) && \nabla_j f_y{}^h = h_j{}^h{}_x f_y{}^x\ , \\ &(\mathrm{iv}) && \nabla_j f_y{}^x = -h_{ji}{}^x f_y{}^i + h_j{}^i{}_y f_i{}^x\ , \\ &(\mathrm{v}) && \nabla_j \xi^h = 0\ , \\ &(\mathrm{vi}) && h_{ji}{}^x \xi^i + f_j{}^x = 0\ . \end{aligned}$$

(3.16)(v) shows that whenever the vector field ξ^κ is tangent to an anti-invariant submanifold of a Sasakian manifold, it is parallel over the submanifold.

(3.16)(i) shows that an anti-invariant submanifold tangent to ξ^κ cannot be totally umbilical or totally contact umbilical. For, if $h_{ji}{}^x$ is of the form $(\alpha g_{ji} + \beta\xi_j\xi_i)h^x$, then from (3.16)(i) we have

$$-g_{ji}\xi^h + \delta_j^h\xi_i = -(\alpha g_{ji} + \beta\xi_j\xi_i)h^x f_x{}^h + (\alpha\delta_j^h + \beta\xi_j\xi^h)h_x f_i{}^x\ ,$$

from which, by contracting with respect to h and j and using (3.15)(v) we obtain

$$(n-1)\xi_i = (n-1)\alpha h_x f_i{}^x + \beta h_x f_i{}^x ,$$

and consequently transvecting with ξ^i and using (3.15)(v) give $(n-1)\xi_i\xi^i = 0$, which is a contradiction for $n > 1$.

We now come back to the case $n = m + 1$. In this case, from the first equation of (3.14) and (3.15)(iv), we have $f_y{}^z f_z{}^x = 0$ or $f_{yx}f^{yx} = 0$ because $f_{yx} = f_y{}^z g_{zx}$ is skew-symmetric and $f_y{}^x = 0$. Thus (3.16)(ii) becomes

$$\nabla_j f_i{}^x = 0 , \tag{3.17}$$

from which, using the Ricci identity we obtain

$$K_{kji}{}^h f_h{}^x - K_{kjy}{}^x f_i{}^y = 0 , \tag{3.18}$$

where $K_{kji}{}^h$ (respectively, $K_{kjy}{}^x$) is the curvature tensor of M^n (respectively, the normal bundle of M^n).

From (3.18) we have, taking account of the first equation of (3.14) and (3.15)(i),

$$K_{kjy}{}^x f_i{}^y f_x{}^h = K_{kji}{}^h , \tag{3.19}$$

$$K_{kji}{}^h f_y{}^i f_h{}^x = K_{kjy}{}^x , \tag{3.20}$$

because of $K_{kji}{}^h\xi^i = 0$ derived from (3.16)(v). (3.19) and (3.20) show that $K_{kji}{}^h = 0$ and $K_{kjy}{}^x = 0$ are equivalent.

II. The case in which ξ^κ is normal to M^n. Now suppose that ξ^κ is normal to M^n, that is, $\xi^h = 0$. Then (3.10) becomes

$$(3.21)\quad \begin{array}{ll}
(\mathrm{i}) & f_i{}^y f_y{}^h = \delta_i^h , \\
(\mathrm{ii}) & f_i{}^y f_y{}^x = 0 , \\
(\mathrm{iii}) & f_y{}^z f_z{}^h = 0 , \\
(\mathrm{iv}) & f_y{}^z f_z{}^x = -\delta_y^x + \xi_y\xi^x + f_y{}^i f_i{}^x , \\
(\mathrm{v}) & f_x{}^i\xi^x = 0 , \\
(\mathrm{vi}) & f_y{}^x\xi^y = 0 , \\
(\mathrm{vii}) & \xi_y\xi^y = 1 .
\end{array}$$

(3.21) (iiii), (iv) and (vi) show that $f_y{}^x$ defines an f-structure in the normal bundle. In this case, (3.13) becomes

$$(3.22)\quad \begin{array}{ll}
(\mathrm{i}) & -h_{ji}{}^x f_x{}^h + h_j{}^h{}_x f_i{}^x = 0 , \\
(\mathrm{ii}) & \nabla_j f_i{}^x = g_{ji}\xi^x - h_{ji}{}^y f_y{}^x , \\
(\mathrm{iii}) & \nabla_j f_y{}^h = \delta_j^h\xi_y + h_j{}^h{}_x f_y{}^x , \\
(\mathrm{iv}) & \nabla_j f_y{}^x = -h_{ji}{}^x f_y{}^i + h_j{}^i{}_y f_i{}^x ,
\end{array}$$

$$\begin{array}{ll} \text{(v)} & h_j{}^h{}_y \xi^y = 0 , \\ \text{(vi)} & \nabla_j \xi^x = -f_j{}^x . \end{array}$$

From (3.21)(i) it follows that $f_{iy}f^{yi} = n$, and consequently by (3.21)(iv) and (vii) we find

$$-f_{zy}f^{zy} = -(2m + 1 - n) + 1 + n = -2(m - n) .$$

Thus, if $n = m$, then we have $f_y{}^x = 0$, and (3.21) and (3.22) become respectively

$$(3.23)\quad \begin{array}{ll} \text{(i)} & f_i{}^y f_y{}^h = \delta_i^h , \\ \text{(ii)} & f_i{}^x f_y{}^i = \delta_y^x - \xi_y \xi^x , \\ \text{(iii)} & f_x{}^h \xi^x = 0 , \\ \text{(iv)} & \xi_x \xi^x = 1 ; \end{array}$$

$$(3.24)\quad \begin{array}{ll} \text{(i)} & -h_{ji}{}^x f_x{}^h + h_j{}^h{}_x f_i{}^x = 0 , \\ \text{(ii)} & \nabla_j f_i{}^x = g_{ji} \xi^x , \\ \text{(iii)} & \nabla_j f_y{}^h = \delta_j^h \xi_y , \\ \text{(iv)} & -h_{ji}{}^x f_y{}^i + h_j{}^i{}_y f_i{}^x = 0 , \\ \text{(v)} & h_j{}^h{}_y \xi^y = 0 , \\ \text{(vi)} & \nabla_i \xi^x = -f_i{}^x . \end{array}$$

Suppose that M^n is totally umbilical, and put $h_{ji}{}^x = g_{ji}h^x$. Then from (3.24)(i) we have

$$-g_{ji}h^x f_x{}^h + \delta_j^h h_x f_i{}^x = 0 ,$$

which implies $h^x f_x{}^h = 0$ for $n > 1$. From (3.24)(iv) it follows that

$$-h^x f_{yj} + h_y f_j{}^x = 0 ,$$

from which, by transvecting with h^y and using $f_{yj}h^y = 0$ we have $h_y h^y f_j{}^x = 0$, and consequently $h_y h^y = 0$ and hence $h_y = 0$. Thus M^n must be totally geodesic.

By (3.24)(ii) and (vi), we find

$$\nabla_j \nabla_i \xi^x = -g_{ji}\xi^x ,$$

from which, using the Ricci identity we obtain

$$K_{kjy}{}^x \xi^y = 0 .$$

On the other hand, from (3.24)(ii) and (vi), we have, using the Ricci identity,

$$-K_{kji}{}^h f_h{}^x + K_{kjy}{}^x f_i{}^y = -f_k{}^x g_{ji} + f_j{}^x g_{ki} ,$$

which, together with (3.23)(i), implies that

$$K_{kji}{}^h = K_{kjy}{}^x f_i{}^y f_x{}^h + \delta_k^h g_{ji} - \delta_j^h g_{ki} \tag{3.25}$$

and that, in consequence of $K_{kjy}{}^x \xi^y = 0$ and (3.23)(ii),

$$K_{kjy}{}^x = K_{kji}{}^h f_y{}^i f_h{}^x + f_{yk} f_j{}^x - f_{yj} f_k{}^x . \tag{3.26}$$

(3.25) and (3.26) show that M^n is of constant curvature 1 if and only if the connection induced in the normal bundle is of zero curvature.

4. Anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor

We first of all remember that the equations of Gauss, Codazzi and Ricci are respectively

$$K_{kjih} = K_{\nu\mu\lambda\kappa} B_{kjih}^{\nu\mu\lambda\kappa} + h_{khx} h_{ji}{}^x - h_{jhx} h_{ki}{}^x , \tag{4.1}$$

$$0 = K_{\nu\mu\lambda\kappa} B_{kji}^{\nu\mu\lambda} C_y{}^\kappa - (\nabla_k h_{jiy} - \nabla_j h_{kiy}) , \tag{4.2}$$

$$K_{kjyx} = K_{\nu\mu\lambda\kappa} B_{kj}^{\nu\mu} C_{yx}^{\lambda\kappa} - (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) , \tag{4.3}$$

where $K_{\nu\mu\lambda\kappa}$, K_{kjih} and K_{kjyx} are the covariant components of the curvature tensors of M^{2m+1}, M^n and the normal bundle respectively, $B_{kjih}^{\nu\mu\lambda\kappa} = B_k{}^\nu B_j{}^\mu B_i{}^\lambda B_h{}^\kappa$ and $B_{kji}^{\nu\mu\lambda} = B_k{}^\nu B_j{}^\mu B_i{}^\lambda$.

We assume that the contact Bochner curvature tensor of M^{2m+1} vanishes identically. Then from (2.1) we have

$$\begin{aligned} &K_{\nu\mu\lambda\kappa} + (g_{\nu\kappa} - \xi_\nu \xi_\kappa) L_{\mu\lambda} - (g_{\mu\kappa} - \xi_\mu \xi_\kappa) L_{\nu\lambda} + L_{\nu\kappa}(g_{\mu\lambda} - \xi_\mu \xi_\lambda) \\ &\quad - L_{\mu\kappa}(g_{\nu\lambda} - \xi_\nu \xi_\lambda) + \varphi_{\nu\kappa} M_{\mu\lambda} - \varphi_{\mu\kappa} M_{\nu\lambda} + M_{\nu\kappa} \varphi_{\mu\lambda} - M_{\mu\kappa} \varphi_{\nu\lambda} \\ &\quad - 2(\varphi_{\nu\mu} M_{\lambda\kappa} + M_{\nu\mu} \varphi_{\lambda\kappa}) + (\varphi_{\nu\kappa} \varphi_{\mu\lambda} - \varphi_{\mu\kappa} \varphi_{\nu\lambda} - 2\varphi_{\nu\mu} \varphi_{\lambda\kappa}) = 0 , \end{aligned} \tag{4.4}$$

from which, by using $g_{\mu\lambda} B_{ji}^{\mu\lambda} = g_{ji}$, $\varphi_{\mu\lambda} B_{ji}^{\mu\lambda} = 0$, $\varphi_{\mu\lambda} B_j{}^\mu C_y{}^\lambda = -f_{jy}$, $\varphi_{\mu\lambda} C_{yx}^{\mu\lambda} = f_{yx}$, $\xi_\nu B_k{}^\nu = \xi_k$ and $\xi_\nu C_y{}^\nu = \xi_y$, we find

$$\begin{aligned} &K_{\nu\mu\lambda\kappa} B_{kjih}^{\nu\mu\lambda\kappa} + (g_{kh} - \xi_k \xi_h) L_{ji} - (g_{jh} - \xi_j \xi_h) L_{ki} \\ &\quad + L_{kh}(g_{ji} - \xi_j \xi_i) - L_{jh}(g_{ki} - \xi_k \xi_i) = 0 , \end{aligned} \tag{4.5}$$

$$\begin{aligned} &K_{\nu\mu\lambda\kappa} B_{kji}^{\nu\mu\lambda} C_y{}^\kappa - \xi_k \xi_y L_{ji} + \xi_j \xi_y L_{ki} + L_{ky}(g_{ji} - \xi_j \xi_i) \\ &\quad - L_{jy}(g_{ki} - \xi_k \xi_i) - f_{ky} M_{ji} + f_{jy} M_{ki} + 2M_{kj} f_{iy} = 0 , \end{aligned} \tag{4.6}$$

$$\begin{aligned} &K_{\nu\mu\lambda\kappa} B_{kj}^{\nu\mu} C_{yx}^{\lambda\kappa} - \xi_k \xi_x L_{jy} + \xi_j \xi_x L_{ky} - L_{kx} \xi_j \xi_y + L_{jx} \xi_k \xi_y - f_{kx} M_{jy} \\ &\quad + f_{jx} M_{ky} - M_{kx} f_{jy} + M_{jx} f_{ky} - 2M_{kj} f_{yx} + (f_{kx} f_{jy} - f_{jx} f_{ky}) = 0 , \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
&L_{ji} = L_{\mu\lambda}B_{ji}^{\mu\lambda}\,, \qquad L_{ky} = L_{\mu\lambda}B_k{}^\mu C_y{}^\lambda\,,\\
&M_{ji} = M_{\mu\lambda}B_{ji}^{\mu\lambda}\,, \qquad M_{ky} = M_{\mu\lambda}B_k{}^\mu C_y{}^\lambda\,.
\end{aligned} \tag{4.8}
$$

Since $M_{\mu\lambda} = -L_{\mu\alpha}\varphi_\lambda{}^\alpha$, we have

$$M_{ji} = -L_{\mu\alpha}\varphi_\lambda{}^\alpha B_{ji}^{\mu\lambda} = L_{\mu\alpha}B_j{}^\mu f_i{}^x C_x{}^\alpha\,,$$

that is,

$$M_{ji} = L_{jx}f_i{}^x\,, \tag{4.9}$$

and also

$$M_{ky} = -L_{\mu\alpha}\varphi_\lambda{}^\alpha B_k{}^\mu C_y{}^\lambda = -L_{\mu\alpha}B_k{}^\mu(f_y{}^i B_i{}^\alpha + f_y{}^x C_x{}^\alpha)\,,$$

that is,

$$M_{ky} = -L_{ki}f_y{}^i - L_{kx}f_y{}^x\,. \tag{4.10}$$

Thus (4.1), (4.2) and (4.3) can be written respectively as

$$
\begin{aligned}
&K_{kjih} + (g_{kh} - \xi_k\xi_h)L_{ji} - (g_{jh} - \xi_j\xi_h)L_{ki} + L_{kh}(g_{ji} - \xi_j\xi_i)\\
&\quad - L_{jh}(g_{ki} - \xi_k\xi_i) - (h_{khx}h_{ji}{}^x - h_{jhx}h_{ki}{}^x) = 0\,,
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
&(\xi_k L_{ji} - \xi_j L_{ki})\xi_y - L_{ky}(g_{ji} - \xi_j\xi_i) + L_{jy}(g_{ki} - \xi_k\xi_i)\\
&\quad + f_{ky}M_{ji} - f_{jy}M_{ki} - 2M_{kj}f_{iy} - (\nabla_k h_{jiy} - \nabla_j h_{kiy}) = 0\,,
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&K_{kjyx} - (\xi_k L_{jy} - \xi_j L_{ky})\xi_x - (L_{kx}\xi_j - L_{jx}\xi_k)\xi_y\\
&\quad + M_{ky}f_{jx} - M_{jy}f_{kx} + f_{ky}M_{jx} - f_{jy}M_{kx} - 2M_{kj}f_{yx}\\
&\quad + (f_{kx}f_{jy} - f_{jx}f_{ky}) + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) = 0\,.
\end{aligned} \tag{4.13}
$$

I. The case in which the vector field ξ^κ is tangent to M^n. We now assume that $n = m + 1$. Then the vector field ξ^κ is tangent to M^n and $f_y{}^x = 0$. Thus (4.13) becomes

$$
\begin{aligned}
&K_{kjyx} - f_{kx}M_{jy} + f_{jx}M_{ky} - M_{kx}f_{jy} + M_{jx}f_{ky}\\
&\qquad + (f_{kx}f_{jy} - f_{jx}f_{ky}) + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) = 0\,,
\end{aligned}
$$

from which, by transvecting with $f_i{}^y f_h{}^x$ and using $f_{jx}f_i{}^x = g_{ji} - \xi_j\xi_i$ derived from (3.15)(i), we find

$$
\begin{aligned}
&K_{kjyx}f_i{}^y f_h{}^x - (g_{kh} - \xi_k\xi_h)M_{jy}f_i{}^y + (g_{jh} - \xi_j\xi_h)M_{ky}f_i{}^y\\
&\quad - M_{kx}f_h{}^x(g_{ji} - \xi_j\xi_i) + M_{jx}f_h{}^x(g_{ki} - \xi_k\xi_i)\\
&\quad + (g_{kh} - \xi_k\xi_h)(g_{ji} - \xi_j\xi_i) - (g_{jh} - \xi_j\xi_h)(g_{ki} - \xi_k\xi_i)\\
&\quad + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx})f_i{}^y f_h{}^x = 0\,.
\end{aligned} \tag{4.14}
$$

We now assume that the second fundamental tensors are commutative. Then from (3.19) and (4.14) we have

$$\begin{aligned}(4.15)\qquad & K_{kjih} + (g_{kh} - \xi_k\xi_h)N_{ji} - (g_{jh} - \xi_j\xi_h)N_{ki} \\ & + N_{kh}(g_{ji} - \xi_j\xi_i) - N_{jh}(g_{ki} - \xi_k\xi_i) \\ & + (g_{kh} - \xi_k\xi_h)(g_{ji} - \xi_j\xi_i) - (g_{jh} - \xi_j\xi_h)(g_{ki} - \xi_k\xi_i) = 0 ,\end{aligned}$$

where $N_{ji} = -M_{jy}f_i{}^y$.

Now since the vector ffeld ξ^h is parallel, the Riemannian manifold M^n is locally a product of M^{n-1} and M^1 generated by ξ^h, and M^{n-1} is totally geodesic in M^n. We represent M^{n-1} in M^n by parametric equations $y^h = y^h(z^a)$ $(a, b, c, d, \cdots = 1'', 2'', \cdots, (n-1)'')$, and put $B_b{}^h = \partial y^h/\partial z^b$. Then we have $\xi_i B_b{}^i = 0$, and the curvature tensor K_{dcba} of M^{n-1} is given by

$$(4.16)\qquad K_{dcba} = K_{kjih}B_{dcba}^{kjih} ,$$

where $B_{dcba}^{kjih} = B_d{}^kB_c{}^jB_b{}^iB_a{}^h$. Thus transvecting (4.15) with B_{dcba}^{kjih}, we obtain

$$(4.17)\qquad K_{dcba} + g_{da}C_{cb} - g_{ca}C_{db} + C_{da}g_{cb} - C_{ca}g_{db} = 0 ,$$

where $g_{cb} = g_{ji}B_c{}^jB_b{}^i$ is the metric tensor of M^{n-1} and

$$C_{cb} = N_{ji}B_c{}^jB_b{}^i + \tfrac{1}{2}g_{cb} .$$

(4.17) shows that the Weyl conformal curvature tensor of M^{n-1} vanishes, and M^{n-1} is conformally flat if $n - 1 \geq 4$. Thus we have

Theorem 4.1. *Let M^n, $n \geq 5$, be an anti-invariant submanifold of a Sasakian manifold M^{2n-1} with vanishing contact Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is locally a product of a conformally flat Riemannian space and a 1-dimensional space.*

II. The case in which the vector field ξ^κ is normal to M^n. We now consider the case in which the vector field ξ^κ is normal to the anti-invariant submanifold M^n, so that $\xi^h = 0$. Then from (4.11) we obtain

$$\begin{aligned}(4.18)\qquad & K_{kjih} + g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} \\ & - (h_{khx}h_{ji}{}^x - h_{jhx}h_{ki}{}^x) = 0 .\end{aligned}$$

If M^n is umbilical, that is, if $h_{jix} = g_{ji}h_x$, then we can write (4.18) in the form

$$\begin{aligned}(4.19)\qquad & K_{kjih} + g_{kh}(L_{ji} - \tfrac{1}{2}h_xh^xg_{ji}) - g_{jh}(L_{ki} - \tfrac{1}{2}h_xh^xg_{ki}) \\ & + (L_{kh} - \tfrac{1}{2}h_xh^xg_{kh})g_{ji} - (L_{jh} - \tfrac{1}{2}h_xh^xg_{jh})g_{ki} = 0 ,\end{aligned}$$

which shows that the Weyl conformal curvature tensor of M^n vanishes. Thus we have

Theorem 4.2. *Let M^n, $n \geq 4$, be a totally umbilical anti-invariant submanifold normal to the structure vector field ξ^κ of a Sasakian manifold M^{2m+1} with vanishing contact Bochner curvature tensor. Then M^n is conformally flat.*

Next from (4.13) we obtain

$$
\begin{aligned}
K_{kjyx} &+ M_{ky}f_{jx} - M_{jy}f_{kx} + f_{ky}M_{jx} - f_{jy}M_{kx} + 2M_{kj}f_{yx} \\
&+ (f_{kx}f_{jy} - f_{jx}f_{ky}) + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) = 0 \,.
\end{aligned} \tag{4.20}
$$

If $n = m$, which implies that $f_y{}^x = 0$, and the second fundamental tensors of M^n commute, then from (4.20) we have

$$
\begin{aligned}
K_{kjyx} &- f_{kx}M_{jy} + f_{jx}M_{ky} - M_{kx}f_{jy} + M_{jx}f_{ky} \\
&+ (f_{kx}f_{jy} - f_{jx}f_{ky}) = 0 \,,
\end{aligned} \tag{4.21}
$$

from which, by transvecting with $f_i{}^y f_h{}^x$ and using (3.23)(i), we find

$$
\begin{aligned}
K_{kjyx}f_i{}^y f_h{}^x &- g_{kh}M_{jy}f_i{}^y + g_{jh}M_{ky}f_i{}^y - M_{ky}f_h{}^y g_{ji} + M_{jy}f_h{}^y g_{ki} \\
&+ (g_{kh}g_{ji} - g_{jh}g_{ki}) = 0 \,.
\end{aligned} \tag{4.22}
$$

Substituting (4.22) in (3.25) yields

$$
K_{kjih} - g_{kh}M_{jy}f_i{}^y + g_{jh}M_{ky}f_i{}^y - M_{ky}f_h{}^y g_{ji} + M_{jy}f_h{}^y g_{ki} = 0 \,, \tag{4.23}
$$

which shows that the Weyl conformal curvature tensor of M^n vanishes. Thus we have

Theorem 4.3. *Let M^n, $n \geq 4$, be an anti-invariant submanifold normal to the structure vector field ξ^κ of a Sasakian manifold M^{2n+1} with vanishing contact Bochner curvature tensor. If the second fundamental tensors commute, then M^n is conformally flat.*

5. Sasakian manifolds as fibred spaces with invariant Riemannian metric

It is well known that in a Sasakian manifold we have

$$
\mathscr{L}g_{\mu\lambda} = 0 \,, \quad \mathscr{L}\varphi_\lambda{}^\kappa = 0 \,, \quad \mathscr{L}\xi_\lambda = 0 \,, \tag{5.1}
$$

where $\mathscr{L}$ denotes the operator of Lie derivation with respect to the structure vector field ξ^κ. Thus, assuming that ξ^κ is regular, we can regard a Sasakian manifold M^{2m+1} as a fibred space with invariant Riemannian metric (see Yano and Ishihara [24]). Denoting $2m$ functionally independent solutions of

$$
\xi^\lambda \partial_\lambda u = 0
$$

by $u^h(x)$, we see that u^h are local coordinates of the base space M^{2m}. We put

$$
E_\lambda{}^h = \partial_\lambda u^h \,, \quad E_\lambda = \xi_\lambda \,, \quad E^\kappa = \xi^\kappa \,, \tag{5.2}
$$

where and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1', 2', \cdots, (2m)'\}$. Then we have

$$E^\lambda E_\lambda{}^h = 0 , \qquad E^\lambda E_\lambda = 1 .$$

Since $E_\lambda{}^h$ and E_λ are linearly independent, we put

$$\begin{bmatrix} E_\lambda{}^h \\ E_\lambda \end{bmatrix}^{-1} = [E^\lambda{}_i, E^\lambda] .$$

Then we have

$$E_\lambda{}^h E^\lambda{}_i = \delta_i^h , \quad E_\lambda{}^h E^\lambda = 0 , \quad E_\lambda E^\lambda{}_i = 0 , \quad E_\lambda E^\lambda = 1 , \tag{5.3}$$

$$E_\lambda{}^i E^\kappa{}_i + E_\lambda E^\kappa = \delta_\lambda^\kappa . \tag{5.4}$$

For the Lie derivatives of E's we have

$$\mathscr{L} E_\lambda{}^h = 0 , \quad \mathscr{L} E_\lambda = 0 , \quad \mathscr{L} E^\kappa{}_i = 0 , \qquad \mathscr{L} E^\kappa = 0 . \tag{5.5}$$

Thus using $\mathscr{L} g_{\mu\lambda} = 0$ and (5.5) we see that

$$g_{ji} = g_{\mu\lambda} E^\mu{}_j E^\lambda{}_i \tag{5.6}$$

is the metric tensor of the base space M^{2m}. From (5.6) we have

$$g_{\mu\lambda} = g_{ji} E_\mu{}^j E_\lambda{}^i + E_\mu E_\lambda . \tag{5.7}$$

It will be easily verified that

$$E^\kappa{}_i = E_\lambda{}^j g^{\lambda\kappa} g_{ji} , \quad E^\kappa = E_\lambda g^{\lambda\kappa} , \quad E_\lambda{}^h = E^\mu{}_i g_{\mu\lambda} g^{ih} , \quad E_\lambda = E^\mu g_{\mu\lambda} , \tag{5.8}$$

where g^{ih} are contravariant components of the metric tensor g_{ji} of the base space M^{2m}. Also using $\mathscr{L}\varphi_\lambda{}^\kappa = 0$ and (5.5) we see that

$$F_i{}^h = \varphi_\lambda{}^\kappa E^\lambda{}_i E_\kappa{}^h \tag{5.9}$$

is a tensor field of type $(1, 1)$ of the base space M^{2m} and defines an almost complex structure of M^{2m}. From (5.6) and (5.9) we easily find

$$g_{ts} F_j{}^t F_i{}^s = g_{ji} , \tag{5.10}$$

which shows that g_{ji} is a Hermitian metric with respect to this almost complex structure. Thus the base space M^{2m} is an almost Hermitian manifold.

From (5.9) it follows that

$$\varphi_\lambda{}^\kappa E^\lambda{}_i = F_i{}^h E^\kappa{}_h , \quad \varphi_\lambda{}^\kappa E_\kappa{}^h = F_i{}^h E_\lambda{}^i , \quad \varphi_\lambda{}^\kappa = F_i{}^h E_\lambda{}^i E^\kappa{}_h . \tag{5.11}$$

For a function $f(u(x))$ on the base manifold M^{2m} we have

$$(5.12) \qquad \partial_\lambda f = E_\lambda{}^i \partial_i f\ , \qquad \partial_i f = E^\lambda{}_i \partial_\lambda f\ ,$$

where $\partial_i = \partial/\partial u^i$.

Now using (5.7) we compute the Christoffel symbols $\{{}_\mu{}^\kappa{}_\lambda\}$ formed with $g_{\mu\lambda}$ and find

$$(5.13) \qquad \begin{aligned} \{{}_\mu{}^\kappa{}_\lambda\} = \{{}_j{}^h{}_i\}E_\mu{}^j E_\lambda{}^i E^\kappa{}_h + (\partial_\mu E_\lambda{}^h)E^\kappa{}_h + \tfrac{1}{2}(\partial_\mu E_\lambda + \partial_\lambda E_\mu)E^\kappa \\ + E_\mu \varphi_\lambda{}^\kappa + E_\lambda \varphi_\mu{}^\kappa\ , \end{aligned}$$

where $\{{}_j{}^h{}_i\}$ are Christoffel symbols formed with g_{ji}. From (5.13) we have, in consequence of (5.11),

$$(5.14) \qquad \partial_\mu E_\lambda{}^h - \{{}_\mu{}^\kappa{}_\lambda\}E_\kappa{}^h + \{{}_j{}^h{}_i\}E_\mu{}^j E_\lambda{}^i = -(E_\mu E_\lambda{}^i + E_\lambda E_\mu{}^i)F_i{}^h\ .$$

Putting

$$(5.15) \qquad \nabla_\mu E_\lambda{}^h = \partial_\mu E_\lambda{}^h - \{{}_\mu{}^\kappa{}_\lambda\}E_\kappa{}^h + \{{}_j{}^h{}_i\}E_\mu{}^j E_\lambda{}^i\ ,$$

we have, from (5.14),

$$(5.16) \qquad \nabla_\mu E_\lambda{}^h = -(E_\mu E_\lambda{}^i + E_\lambda E_\mu{}^i)F_i{}^h\ .$$

Thus putting $\nabla_j = E^\mu{}_j \nabla_\mu$ we find

$$(5.17) \qquad \nabla_j E_\lambda{}^h = -F_j{}^h E_\lambda\ ,$$

from which it follows that

$$(5.18) \qquad \nabla_j E^\kappa{}_i = -F_{ji} E^\kappa\ ,$$

where $F_{ji} = F_j{}^t g_{ti}$. Thus by (5.9), (5.17) and (5.18) we obtain

$$(5.19) \qquad \nabla_j F_i{}^h = 0\ ,$$

which shows that the base manifold M^{2m} is Kaehlerian.

From (5.16) and the Ricci identity

$$\nabla_\nu \nabla_\mu E_\lambda{}^h - \nabla_\mu \nabla_\nu E_\lambda{}^h = -K_{\nu\mu\lambda}{}^\kappa E_\kappa{}^h + K_{kji}{}^h E_\nu{}^k E_\mu{}^j E_\lambda{}^i\ ,$$

we find

$$(5.20) \qquad \begin{aligned} K_{kji}{}^h E_\nu{}^k E_\mu{}^j E_\lambda{}^i = K_{\nu\mu\lambda}{}^\kappa E_\kappa{}^h - (E_\nu E_\mu{}^h - E_\mu E_\nu{}^h)E_\lambda \\ + (E_\nu{}^i \varphi_{\mu\lambda} - E_\mu{}^i \varphi_{\nu\lambda} - 2\varphi_{\nu\mu} E_\lambda{}^i)F_i{}^h\ , \end{aligned}$$

which implies that

$$(5.21) \qquad K_{kjih} = K_{\nu\mu\lambda\kappa} E^{\nu\mu\lambda\kappa}_{kjih} + (F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih})\ ,$$

where $E^{\nu\mu\lambda\kappa}_{kjih} = E^\nu{}_k E^\mu{}_j E^\lambda{}_i E^\kappa{}_h$.

6. Sasakian manifolds with vanishing contact Bochner curvature tensor as a fibred space with invariant Riemannian metric

We now assume that the contact Bochner curvature tensor of the Sasakian manifold M^{2m+1} vanishes identically. Then transvecting (4.4) with $E^{\nu\mu\lambda\kappa}_{kjih}$ we find

$$\begin{aligned} &K_{\nu\mu\lambda\kappa}E^{\nu\mu\lambda\kappa}_{kjih} + g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} \\ &\quad + F_{kh}M_{ji} - F_{jh}M_{ki} + M_{kh}F_{ji} - M_{jh}F_{ki} \\ &\quad - 2(F_{kj}M_{ih} + M_{kj}F_{ih}) + (F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih}) = 0 , \end{aligned} \tag{6.1}$$

where

$$L_{ji} = L_{\mu\lambda}E^{\mu}{}_{j}E^{\lambda}{}_{i} , \qquad M_{ji} = M_{\mu\lambda}E^{\mu}{}_{j}E^{\lambda}{}_{i} .$$

Thus we have

$$M_{ji} = -L_{\mu\alpha}\varphi_{\lambda}{}^{\alpha}E^{\mu}{}_{j}E^{\lambda}{}_{i} = -L_{\mu\alpha}E^{\mu}{}_{j}F_{i}{}^{t}E^{\alpha}{}_{t} ,$$

that is,

$$M_{ji} = -L_{jt}F_{i}{}^{t} , \tag{6.2}$$

which implies that

$$L_{ji} = M_{jt}F_{i}{}^{t} . \tag{6.3}$$

Substituting (6.1) in (5.21) we find

$$\begin{aligned} &K_{kjih} + g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} + F_{kh}M_{ji} - F_{jh}M_{ki} \\ &\quad + M_{kh}F_{ji} - M_{jh}F_{ki} - 2(F_{kj}M_{ih} + M_{kj}F_{ih}) = 0 , \end{aligned} \tag{6.4}$$

from which, by transvecting with g^{kh} and using (6.2), we find

$$K_{ji} = -2(m+2)L_{ji} - Lg_{ji} , \tag{6.5}$$

where $L = g^{ji}L_{ji}$, from which transvecting with g^{ji} gives

$$K = -4(m+1)L \quad \text{or} \quad L = -\frac{1}{4(m+1)}K . \tag{6.6}$$

Substituting (6.6) in (6.5) we find

$$L_{ji} = -\frac{1}{2(m+2)}K_{ji} + \frac{1}{8(m+1)(m+2)}Kg_{ji} . \tag{6.7}$$

Thus (6.4) shows that the Bochner curvature tensor of the base space M^{2m} vanishes. Hence we have

Theorem 6.1. *Let M^{2m+1} be a Sasakian manifold with vanishing contact Bochner curvature tensor regarded as a fibred space with invariant Riemannian metric. Then the Bochner curvature tensor of the Kaehlerian base space vanishes.*

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