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POSITIVELY CURVED COMPLEX SUBMANIFOLDS IMMERSED IN A COMPLEX PROJECTIVE SPACE. III

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1. Statement of results

Let $P_{n+p}(C)$ be an $(n+p)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and let M be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. Denote the sectional curvature and the holomorphic sectional curvature of M by K and H respectively. Then it is natural to conjecture the following (cf. [1]):

(I) If $H > \frac{1}{2}$, then M is totally geodesic.
(II) If $K > 0$ and $p < \frac{1}{2}n(n+1)$, then M is totally geodesic.
(III) If $K > \frac{1}{8}$ and $n \geq 2$, then M is totally geodesic.

There have been several partial solutions to these conjectures (cf. [1]). Recently S. T. Yau [2] proved the following.

Proposition Y. *If* $K > \dfrac{n(2p-1)+8p-3}{4n(4p-1)}$, *then* M *is totally geodesic.*

The purpose of this paper is to prove some results in the same direction.

Theorem 1. *If* $K > \dfrac{n+3}{8n}$, *then* M *is totally geodesic.*

Theorem 2. *If* $K > \frac{1}{8}$ *and* $H > \frac{1}{2}$, *then* M *is totally geodesic.*

It is easily seen that Theorem 1 is an improvement of Proposition Y.

2. Basic lemmas

We use the same notation and terminologies as in [1] unless otherwise stated. It is well-known (cf. [1]) that the second fundamental form of the immersion satisfies a differential equation of Simons type:

$$(1)\qquad \begin{aligned} \tfrac{1}{2}\Delta\,\|\sigma\|^2 = \|\nabla'\sigma\|^2 &+ \sum_{\lambda,i,j,k,l}(h^\lambda_{ij}h^\lambda_{kl}R_{lijk} + h^\lambda_{ij}h^\lambda_{il}R_{lkjk}) \\ &- 4\,\mathrm{tr}\Big(\sum_\alpha A_\alpha^2\Big)^2 - \tfrac{1}{2}\,\|\sigma\|^2\,. \end{aligned}$$

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On the one hand, using the equation of Gauss we obtain

$$(2)\qquad \sum_{\lambda,i,j,k,l}(h^\lambda_{ij}h^\lambda_{kl}R_{lijk} + h^\lambda_{ij}h^\lambda_{il}R_{lkjk}) = \frac{n+3}{2}\|\sigma\|^2 - 4\,\mathrm{tr}\left(\sum_\alpha A^2_\alpha\right)^2 - \sum_{\lambda,\mu}(\mathrm{tr}\,A_\lambda A_\mu)^2 .$$

On the other hand, Yau's idea can be applied as follows: For each α, let $h^\alpha_1, \cdots, h^\alpha_n, -h^\alpha_1, \cdots, -h^\alpha_n$ be the eigenvalues of A_α. Then we have

$$\begin{aligned}\sum_{i,j,k,l}&(h^\alpha_{ij}h^\alpha_{kl}R_{lijk} + h^\alpha_{ij}h^\alpha_{il}R_{lkjk} + h^{\alpha^*}_{ij}h^{\alpha^*}_{kl}R_{lijk} + h^{\alpha^*}_{ij}h^{\alpha^*}_{il}R_{lkjk})\\ &= 4\sum_{a,b}\{(h^\alpha_a)^2(R_{abab} + R_{ab^*ab^*}) - h^\alpha_a h^\alpha_b(R_{abab} - R_{ab^*ab^*})\}\\ &= 2\sum_{a,b}\{(h^\alpha_a - h^\alpha_b)^2R_{abab} + (h^\alpha_a + h^\alpha_b)^2R_{ab^*ab^*}\} .\end{aligned}$$

Therefore, if $K \geq \delta_K$ and $H \geq \delta_H$, then we have

$$\begin{aligned}\sum_{i,j,k,l}&(h^\alpha_{ij}h^\alpha_{kl}R_{lijk} + h^\alpha_{ij}h^\alpha_{il}R_{lkjk} + h^{\alpha^*}_{ij}h^{\alpha^*}_{kl}R_{lijk} + h^{\alpha^*}_{ij}h^{\alpha^*}_{il}R_{lkjk})\\ &\geq 2\sum_{a\neq b}\{(h^\alpha_a - h^\alpha_b)^2\delta_K + (h^\alpha_a + h^\alpha_b)^2\delta_K\} + 8\sum_a (h^\alpha_a)^2\delta_H\\ &= 8\{(n-1)\delta_K + \delta_H\}\sum_a (h^\alpha_a)^2 = 4\{(n-1)\delta_K + \delta_H\}\,\mathrm{tr}\,A^2_\alpha ,\end{aligned}$$

from which it follows that

$$(3)\qquad \sum_{\lambda,i,j,k,l}(h^\lambda_{ij}h^\lambda_{kl}R_{lijk} + h^\lambda_{ij}h^\lambda_{il}R_{lkjk}) \geq 2\{(n-1)\delta_K + \delta_H\}\|\sigma\|^2 .$$

From (1), (2) and (3) we have

Lemma 1. *If $K \geq \delta_K$ and $H \geq \delta_H$, then*

$$\begin{aligned}\tfrac{1}{2}\Delta\|\sigma\|^2 \geq \|\nabla'\sigma\|^2 &+ 2(1+a)\{(n-1)\delta_K + \delta_H\}\|\sigma\|^2\\ &+ a\sum_{\lambda,\mu}(\mathrm{tr}\,A_\lambda A_\mu)^2 + 4(a-1)\,\mathrm{tr}\left(\sum_\alpha A^2_\alpha\right)^2 - \frac{1}{2}\{1 + (n+3)a\}\|\sigma\|^2\end{aligned}$$

for any real number a (≥ -1).

The following lemma is purely algebraic.

Lemma 2. $8\,\mathrm{tr}\left(\sum_\alpha A^2_\alpha\right)^2 \leq (n+1)\sum_{\lambda,\mu}(\mathrm{tr}\,A_\lambda A_\mu)^2$.

Proof. It is easily seen that

$$8\,\mathrm{tr}\left(\sum_\alpha A^2_\alpha\right)^2 = (n+1)(\|\sigma\|^2 - \rho) + 2\|S\|^2$$

$$\sum_{\lambda,\mu}(\mathrm{tr}\,A_\lambda A_\mu)^2 = \|\sigma\|^2 - \rho + \tfrac{1}{2}\|R\|^2 ,$$

where $\|R\|$ and $\|S\|$ denote the length of the curvature tensor and the Ricci tensor of M respectively. Hence we have

$$(n+1)\sum_{\lambda,\mu}(\operatorname{tr} A_\lambda A_\mu)^2 - 8\operatorname{tr}\Big(\sum_\alpha A_\alpha^2\Big)^2 = \tfrac{1}{2}(n+1)\|R\|^2 - 2\|S\|^2 \geq 0\,.$$

The last inequality is obtained by considering the length of the tensor field with local complex components $R^a_{bc\bar{d}} - \frac{1}{2(n+1)}(\delta^a_c R_{b\bar{d}} + \delta^a_b R_{c\bar{d}})$, where $R_{a\bar{b}}$ are the local complex components of S.

3. Proof of theorems

From Lemma 1 and Lemma 2 it follows that

$$\frac{1}{2}\Delta\|\sigma\|^2 \geq 2(1+a)\{(n-1)\delta_K + \delta_H\}\|\sigma\|^2 + 8\Big(\frac{a}{n+1} + \frac{a-1}{2}\Big)\operatorname{tr}\Big(\sum_\alpha A_\alpha^2\Big)^2 - \frac{1}{2}\{1+(n+3)a\}\|\sigma\|^2$$

for any real number $a \geq 0$. In particular, putting $a = \frac{n+1}{n+3}$, we obtain

$$\frac{1}{2}\Delta\|\sigma\|^2 \geq (n+2)\Big[\frac{4}{n+3}\{(n-1)\delta_K + \delta_H\} - \frac{1}{2}\Big]\|\sigma\|^2\,.$$

Hence we have

Proposition. *If $(n-1)K + H > \frac{1}{8}(n+3)$, then M is totally geodesic.*

Theorems 1 and 2 follow immediately from the above Proposition.

References

[1] K. Ogiue, *Differential geometry of Kaehler submanifolds*, Advances in Math. **13** (1974) 73–114.

[2] S. T. Yau, *Submanifolds with constant mean curvature*. II, Amer. J. Math. **96** (1975) 76–100.

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