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# THE HOLONOMY RING ON THE LEAVES OF FOLIATED MANIFOLDS

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## Introduction

Let $F$ be a foliation on a manifold $M$, and let $\nu F$ denote the normal bundle of $F$. In [1] and [2], Bott exploited the existence of certain special foliation connections on $\nu F$ to prove the following results:

**Theorem 1** (*The vanishing theorem for foliations*). *If $F$ is a $q$-codimensional foliation, then the real Pontryagin ring of $\nu F$ is trivial in dimensions greater than $2q$.*

**Theorem 2** (*The obstruction theorem for foliations*). *Let $M$ be an $m$-dimensional manifold, and let $E$ be a subbundle of the tangent bundle $T(M)$ of dimension $m - q$. Then a necessary condition for $E$ to be isomorphic to a $q$-codimensional foliation on $M$ is that the real Pontryagin ring of the quotient bundle $T(M)/E$ must be trivial in dimensions greater than $2q$.*

Now let $L$ be a leaf of the foliation $F$, and let $\nu L$ denote the normal bundle of $L$ in $M$. By pulling back the foliation connections of the bundle $\nu F$ to the bundle $\nu L$, we obtain a unique natural connection on $\nu L$, which we shall call the leaf connection on $\nu L$. It can be shown that if $K^L$ is the curvature of the leaf connection on $\nu L$, then $K^L = 0$. Therefore parallel to Bott's theorems for foliations, we have the following results for the leaves of foliated manifolds:

**Theorem 3** (*The vanishing theorem for leaves*). *If $L$ is a leaf of a foliation, then the real Pontryagin ring of $\nu L$ is trivial.*

**Theorem 4** (*The obstruction theorem for leaves*). *Let $N$ be a connected manifold and let $j: N \to M$ be a 1-1 immersion. Then a necessary condition for $N$ to be an integral manifold of a foliation on $M$ is that the real Pontryagin ring of the normal $\nu N$ of $N$ in $M$ must be trivial. In particular, a necessary condition for $N$ to be a leaf of a foliation on $M$ is that the real Pontryagin ring of the normal bundle $\nu N$ must be trivial.*

The vanishing of the real Pontryagin ring of $\nu F$ in high dimensions led to the construction of certain secondary characteristic classes for foliations [2, p. 68]. Similarly, the vanishing of the real Pontryagin ring of $\nu L$ leads to the construction of certain secondary characteristic classes, called the holonomy ring, on the leaves of foliations. In fact, a unified construction for both types

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of secondary characteristic classes is given by Kamber and Tondeur in [15], [16] by using the more general concept of a foliated bundle.

Nevertheless, there is one very important difference between the secondary foliation classes and the holonomy classes. The secondary foliation classes of a foliation $F$ are invariants of the homotopy class of $F$ [2, p. 69]. However even though the holonomy classes depend only on the choice of a foliation $F$ and a leaf $L$ of $F$, these classes are not invariants of the homotopy class of $F$ because the leaves of $F$ themselves are not homotopy invariants. Hence the holonomy classes are often more sensitive than the secondary foliation classes since they can distinguish between homotopic foliations.

Still, the holonomy ring and the secondary foliation classes are quite intimately related as is shown by Shulman and Tischler in [23]. Moreover let $Tq$ denote the pseudogroup of all local diffeomorphisms of $\boldsymbol{R}^q$, and $a(Tq)$ the Lie algebra of formal $Tq$ vector fields. If $F$ is a $q$-codimensional foliation with a trivialized normal bundle on a manifold $M$, then there is a natural homomorphism $\lambda_F^*: H^*(a(Tq)) \to H^*(M)$, and the secondary foliation classes of $F$ are just the classes in the image of $\lambda_F^*$, [3], [9]. Similarly, if $L$ is a leaf of $F$, then there is a natural homomorphism $\phi_{F,L}^*: H^*(gl(q, \boldsymbol{R})) \to H^*(L)$, and the holonomy ring of $L$ is just the image of $\phi_{F,L}^*$ in $H^*(L)$. Moreover, $gl(q, \boldsymbol{R})$ can be naturally embedded as a Lie algebra in $a(Tq)$, and the diagram

$$\begin{array}{ccc} H^*(a(Tq)) & \xrightarrow{\lambda_F^*} & H^*(M) \\ \downarrow & & \downarrow \\ H^*(gl(q, \boldsymbol{R})) & \xrightarrow{\phi_{F,L}^*} & H^*(L) \end{array}$$

commutes. Since the homomorphism $H^*(a(Tq)) \to H^*(gl(q, \boldsymbol{R}))$ is actually the zero homomorphism, all of the secondary foliation classes vanish when restricted to a leaf.

In general, characteristic classes on the leaves of foliations resemble characteristic classes on foliations in yet another way. Let $\Gamma_q$ denote the groupoid of all the germs of all the local diffeomorphisms of $\boldsymbol{R}^q$. Put the sheaf topology on $\Gamma_q$. Then since $\Gamma_q$ is a topological groupoid, there is a space $B\Gamma_q(F\Gamma_q)$ which classifies $\Gamma_q$-structures (with trivialized normal bundles) [4], [8]. Now every $q$-codimensional foliation on a manifold $M$ induces a $\Gamma_q$-structure on $M$. Moreover in [3] Bott and Haefliger state the following result.

**Theorem 5.** *There is a* 1-1 *correspondence between* $H^*(B\Gamma_q)(H^*(F\Gamma_q))$ *and the collection of characteristic classes on foliations* (*with trivialized normal bundles*).

Similarly, let $\tilde{\Gamma}_q$ denote the groupoid $\Gamma_q$ with the discrete topology. Again since $\tilde{\Gamma}_q$ is a topological groupoid, there is a space $B\tilde{\Gamma}_q(F\tilde{\Gamma}_q)$ which classifies $\tilde{\Gamma}_q$-structures (with trivialized normal bundles). Moreover, if $L$ is a leaf of a foliation $F$, then $F$ induces a $\tilde{\Gamma}_q$-structure on $L$. Now in [6], [7] the following theorems are proved.

**Theorem 6.** *The universal $\Gamma_q$-structure on the classifying space $B\Gamma_q(F\Gamma_q)$ has only one leaf and this leaf is homeomorphic to $B\tilde{\Gamma}_q(F\tilde{\Gamma}_q)$.*

**Theorem 7.** *There is a 1-1 correspondence between $H^*(B\tilde{\Gamma}_q)(H^*(F\tilde{\Gamma}_q))$ and the collection of characteristic classes on the leaves of foliations (with trivialized normal bundles).*

In [6] and [7] we study $H^*(F\tilde{\Gamma}_q)$ directly and show by a spectral sequence argument that there are many nonzero classes in $H^*(F\tilde{\Gamma}_q)$. Thus we are able to conclude that there are many nontrivial characteristic classes on the leaves of foliations with trivialized normal bundles.

There are still many unsolved problems concerning characteristic classes on the leaves of foliated manifolds. First, it is not yet known whether the holonomy ring exhausts the collection of these characteristic classes. Second, the topological significance of the holonomy classes is not well understood. Finally, the algebraic relationship between $H^*(B\Gamma_q)$ and $H^*(B\tilde{\Gamma}_q)$ has not yet been completely explored.

The primary purpose of this paper is to give several equivalent constructions of the holonomy ring on the leaves of foliated manifolds. These constructions though different are nevertheless equivalent to the construction of characteristic classes for flat bundles given by Kamber and Tondeur in [15] and [16]. In §§ 1 and 2 we present most of the basic concepts and notation used throughout this paper. In § 3 we introduce the leaf connection and derive some of its intrinsic properties. Next, in § 4, we make use of the leaf connection to prove the vanishing and obstruction theorems for the leaves of foliations.

With these preliminaries accomplished, we are in a position to begin our investigation of the holonomy ring. We shall give three distinct constructions of this ring. Indeed, let $F$ be a $q$-codimensional foliation and let $L$ be a leaf of $F$ with trivialized normal bundle $\nu L$. In § 5 we construct a natural homomorphism $\phi^*_{F,L}: E(h_1, h_2, \cdots, h_q) \to H^*(L)$ by comparing leaf connections to flat connections. The homomorphism $\phi^*_{F,L}$ is called the holonomy homomorphism, and the image of $\phi^*_{F,L}$ in $H^*(L)$ is called the holonomy ring of the leaf $L$ with respect to the foliation $F$. Unfortunately, even though this construction is easy to develop, it adds little to our intuitive understanding of the holonomy classes. However as we show in § 6, this construction does lead to a better understanding of the secondary foliation classes.

In § 7 we use the connection form of the leaf connection to construct a natural homomorphism $\phi^*_{F,L}: H^*(gl(q, \boldsymbol{R})) \to H^*(L)$. Moreover, we prove that the image of this homomorphism is actually the same as the image of the holonomy homomorphism constructed in § 5. The appearance of the Lie algebra $gl(q, \boldsymbol{R})$ hints at the essentially linear nature of the holonomy classes. Furthermore in § 8 we are able to use this homomorphism to construct several elementary examples of foliations which have leaves with nontrivial holonomy rings. In addition, in § 9 we use this homomorphism to derive a product formula for the holonomy classes. This product formula leads to certain necessary conditions, in terms of leaf invariants, for a $(q_1 + q_2)$-codimensional foliation to be the intersection of a $q_1$-codimensional foliation and a $q_2$-codimensional foliation.

The theory of characteristic classes on the leaves of foliations is related to the theory of characteristic classes for vector bundles with discrete structure group. Indeed, if $L$ is a leaf of a foliation, then the normal bundle $\nu L$ has a discrete structure group. Hence to construct characteristic classes on the leaves of foliations, one need only construct characteristic classes for vector bundles with discrete structure group. A construction of characteristic classes for these bundles is given by Kamber and Tondeur in [15] and [16]. However, we shall adopt a somewhat different approach which ties in more readily with our other constructions.

In § 10 we discuss the relationship between the category of vector bundles with discrete structure group and the category of the leaves of foliations. Then in § 11 we construct a holonomy ring for all vector bundles with discrete structure group and show that when a vector bundle is the normal bundle of a leaf, this holonomy ring coincides with the holonomy ring of the leaf. Thus, since the holonomy classes of a leaf depend only on the structure of the normal bundle of the leaf, we can conclude that the holonomy classes are essentially linear invariants.

Throughout this paper all manifolds are differentiable $C^\infty$-manifolds, all maps are smooth $C^\infty$-maps, and all cohomology is understood to have real coefficients.

Some of the results in this paper first appeared in the author's thesis which was written while the author was attending the University of Maryland on a leave of absence from Johns Hopkins University. The author would like to express his thanks to the people at the University of Maryland for granting sanctuary to an exile from the house of the philistines, and would also especially like to thank his advisor Bruce Reinhart for rescuing him from silence and the void.

## 1. Foliation categories

To begin, we recall some basic concepts associated with foliations. Let $M$ be an $m$-dimensional manifold, and let $T(M)$ denote the tangent bundle of $M$. A subbundle $E$ of $T(M)$ is said to be integrable if $[X, Y]$ is a vector field in $E$ whenever both $X$ and $Y$ are vector fields in $E$.

**Definition.** A $q$-codimensional foliation on $M$ is an integrable subbundle of $T(M)$ of dimension $m - q$.

Let $E$ be a subbundle, not necessarily integrable, of $T(M)$. The normal bundle $\nu E$ is the quotient bundle $T(M)/E$. In particular, if $E$ is a $q$-codimensional foliation on $M$, then $\nu E$ is a $q$-dimensional vector bundle over $M$. When the normal bundle $\nu E$ is isomorphic to the trivial vector bundle over $M$, we shall say that $E$ has a trivial normal bundle. However, if a specific trivialization of the normal bundle has been selected, then we shall say that $E$ has a trivialized normal bundle.

To construct foliation categories, we must introduce the notion of foliation maps. Let $F$ be a $q$-codimensional foliation on $M$, and $N$ an $n$-dimensional manifold. A map $g: N \to M$ is said to be transverse to $F$ if $(dg)^{-1}(F)$ is an $(n-q)$-dimensional subbundle of $T(N)$. In this case we shall denote $(dg)^{-1}(F)$ by $g^*(F)$. The bundle $g^*(F)$ is a $q$-codimensional foliation on $N$.

**Definition.** Let $F_i$ be a $q$-codimensional foliation on a manifold $M_i$, $i = 1, 2$. A map $g: M_1 \to M_2$ is said to be a foliation map from $F_1$ to $F_2$ if and only if $g$ is transverse to $F_2$ and $F_1 = g^*(F_2)$. We shall write $g: F_1 \to F_2$ to denote that $g$ is a foliation map from $F_1$ to $F_2$.

Now we shall adopt the following notation:

1. $CF_q$ will denote the category whose objects are $q$-codimensional foliations and whose morphisms are foliation maps.

2. $CF_q^0$ will denote the category whose objects are $q$-codimensional foliations with trivialized normal bundles and whose morphisms are foliation maps compatible with the given trivializations.

Characteristic classes have been constructed on these foliation categories by Bott in [2], by Haefliger in [9], and by Kamber and Tondeur in a series of papers [11], [12], [13], [14], [15], and [16]. In § 6 of this paper we shall briefly review Bott's construction of these secondary foliation classes.

## 2. Leaf categories

Now we are going to introduce three categories associated with the leaves of foliations. Let $F$ be a $q$-codimensional foliation on an $m$-dimensional manifold $M$. An $(m-q)$-dimensional manifold $L$ is said to be an integral manifold of $F$ if there is a 1-1 immersion $j: L \to M$ such that $dj(T(L))$ is contained in $F$. That is, $L$ is an integral manifold of $F$ if $F$ coincides with the tangent space of $L$ at every point in the image of $j$.

**Definition.** Let $F$ be a foliation. A leaf of $F$ is a maximal connected integral manifold of $F$.

Let $j: L \to M$ be a 1-1 immersion. The normal bundle $\nu L$ is the quotient bundle $T(M)/dj(T(L))$ restricted to $L$. If $F$ is a foliation on $M$, and $L$ is a leaf of $F$, then $\nu L = \nu F|_L$ by construction. Again we shall adopt the distinction made in § 1 between a trivial normal bundle and a trivialized normal bundle.

The objects of the leaf categories will be pairs $(F, L)$, where $F$ is a foliation and $L$ is a leaf of $F$. In order to describe the morphisms of these categories, we must introduce the notion of a map of pairs.

**Definition.** Let $F_i$ be a $q$-codimensional foliation on a manifold $M_i$, $i = 1, 2$. In addition, let $L_i$ be a leaf of $F_i$ and let $j_i: L_i \to M_i$ be the immersion of $L_i$ in $M_i$, $i = 1, 2$. A map $g: M_1 \to M_2$ is called a map of pairs from $(F_1, L_1)$ to $(F_2, L_2)$ if and only if for a map $g: F_1 \to F_2$, $g \cdot j_1(L_1)$ is contained in $j_2(L_2)$. We shall write $g: (F_1, L_1) \to (F_2, L_2)$ to denote that $g$ is a map of pairs from $(F_1, L_1)$ to $(F_2, L_2)$.

Note that if $g: (F_1, L_1) \to (F_2, L_2)$ is a map of pairs, then $F_1 = g^*(F_2)$. Moreover, there is a map $\bar{g}: L_1 \to L_2$ induced by $g$ such that the diagram

$$\begin{tikzcd} L_1 \arrow[r, "\bar{g}"] \arrow[d, "j_1"'] & L_2 \arrow[d, "j_2"] \\ M_1 \arrow[r, "g"] & M_2 \end{tikzcd}$$

commutes.

We shall adopt the following notation:

1. $CF_qL$ will denote the category whose objects are pairs $(F, L)$ and whose morphisms are maps of pairs, where $F$ is a $q$-codimensional foliation and $L$ is a leaf of $F$.

2. $CF_qL^0$ will denote the category whose objects are pairs $(F, L)$ and whose morphisms are maps of pairs compatible with the given trivializations, where $F$ is a $q$-codimensional foliation, $L$ is a leaf of $F$, and $\nu L$ is a trivialized bundle.

3. $CF_q^0L$ will denote the category whose objects are pairs $(F, L)$, and whose morphisms are maps of pairs compatible with the given trivializations, where $F$ is a $q$-codimensional foliation, $L$ is a leaf of $F$, and $\nu F$ is a trivialized bundle.

Much of this paper is devoted to techniques for constructing characteristic classes on these three categories.

## 3. The leaf connection

There are special connections associated with the normal bundle of a foliation and the normal bundle of a leaf. By examining the curvature forms of these connections, we can extract information about the real Pontryagin rings of these vector bundles. In fact, the vanishing and obstruction theorems, both for foliations and leaves, are a consequence of the existence of these special connections. For a detailed discussion of connections and the Pontryagin ring, the reader is referred to [2, § 5] and [17, Vol. II, Chap. 12].

In this section we shall introduce an especially useful connection on the normal bundle of a leaf. This connection is rather intimately related to the foliation connections investigated by Bott in [2, p. 33]. Therefore we shall begin our discussion by recalling the definition of a foliation connection.

Let $\eta: E \to M$ be a vector bundle over a manifold $M$. We shall write $\Gamma(E)$ to denote the collection of cross sections of $\eta$. In particular, $\Gamma(T(M))$ is the collection of all vector fields on $M$.

**Definition.** Let $F$ be a foliation on a manifold $M$, and $\Pi: T(M) \to \nu F$ the projection map. A connection $\nabla$ on the normal bundle $\nu F$ is called a foliation connection if and only if $\nabla_X(Z) = \Pi([X, \tilde{Z}])$ for every $X$ in $\Gamma(F)$, where $\tilde{Z}$ is any vector field on $M$ for which $\Pi(\tilde{Z}) = Z$.

Let $F$ be a foliation on a manifold $M$. In general there exist many foliation connections on $\nu F$, [2, p. 33], [25]. However, if $\nabla$ is a foliation connection on $\nu F$, and $g\colon N \to M$ is a map transverse to $F$, then the pullback $g^*(\nabla)$ is a foliation connection on $\nu g^*(F)$, [2, p. 69]. Moreover, all foliation connections share the following properties:

**Lemma 3.1.** *Let $\nabla^i$ be a foliation connection on $\nu F$ with local connection form $\theta^{i\alpha}$, $i = 1, 2$. If $X$ is in $\Gamma(F)$, then $\theta^{1\alpha}(X) = \theta^{2\alpha}(X)$.*

*Proof.* Let $s_1^\alpha, \cdots, s_q^\alpha$ be a basis for the sections of $\nu F$ over $U^\alpha$, and let $\Pi\colon T(M) \to F$ be the projection map. If $X$ is in $\Gamma(F)$, then by the definition of a foliation connection

$$\sum_j \theta_{ij}^{1\alpha}(X) s_j^\alpha = \nabla_X^1(s_i^\alpha) = \Pi([X, \tilde{s}_i^\alpha]) = \nabla_X^2(s_i^\alpha) = \sum_j \theta_{ij}^{2\alpha}(X) s_j^\alpha ,$$

where $\tilde{s}_i^\alpha$ is any vector field on $M$ such that $\Pi(\tilde{s}_i^\alpha) = s_i^\alpha$. Hence $\theta_{ij}^{1\alpha}(X) = \theta_{ij}^{2\alpha}(X)$.

**Lemma 3.2.** *Let $\nabla^i$ be a foliation connection on $\nu F$ with local connection form $\theta^{i\alpha}$, $i = 1, 2$. If $L$ is a leaf of $F$, then $\theta^{1\alpha}|_L = \theta^{2\alpha}|_L$.*

*Proof.* This result is an immediate consequence of Lemma 3.1.

**Definition.** Let $F$ be a foliation on $M, L$ a leaf of $F$, and $j\colon L \to M$ the immersion of $L$ in $M$. A connection $\nabla$ on $\nu L$ is called a leaf connection if and only if there is a foliation connection $\bar{\nabla}$ on $\nu F$ such that $\nabla = j^*(\bar{\nabla})$.

**Theorem 3.3.** *Let $F$ be a foliation on $M$, and $L$ a leaf of $F$. Then there is one and only one leaf connection on $\nu L$.*

*Proof.* The existence of leaf connections follows immediately from the existence of foliation connections. To prove uniqueness, suppose that $\nabla^1$ and $\nabla^2$ are two leaf connections on $\nu L$ and that $j\colon L \to M$ is the immersion of $L$ in $M$. Then there are foliation connections $\bar{\nabla}^1$ and $\bar{\nabla}^2$ on $\nu F$ such that $\nabla^1 = j^*(\bar{\nabla}^1)$ and $\nabla^2 = j^*(\bar{\nabla}^2)$. Let $\theta^{i\alpha}$ be the local connection form of $\bar{\nabla}^i$ over $U^\alpha$, $i = 1, 2$. Then $j^*(\theta^{i\alpha}) = \theta^{i\alpha}|_L$ is the local connection form of $\nabla^i$ over $j^{-1}(U^\alpha)$, $i = 1, 2$. However by Lemma 3.2, $\theta^{1\alpha}|_L = \theta^{2\alpha}|_L$. Since this equality holds on every neighborhood $j^{-1}(U^\alpha)$, we can conclude that $\nabla^1 = \nabla^2$. Therefore there is one and only one leaf connection on $\nu L$. q.e.d.

Let $L$ be a leaf of a foliation. Since the leaf connection on $\nu L$ is unique, we will always denote this connection by $\nabla^L$ and the curvature of $\nabla^L$ by $K^L$. Now the leaf connection satisfies the following naturality property.

**Proposition 3.4.** *Let $F_i$ be a $q$-codimensional foliation on a manifold $M_i$, and $L_i$ a leaf of $F_i$, $i = 1, 2$. In addition, let $g\colon (F_1, L_1) \to (F_2, L_2)$ be a map of pairs, and $\bar{g}\colon L_1 \to L_2$ the map induced by $g$. Then $\nabla^{L_1} = \bar{g}^*(\nabla^{L_2})$.*

*Proof.* Let $\nabla$ be a foliation connection on $\nu F_2$. Since $g\colon (F_1, L_1) \to (F_2, L_2)$ is a map of pairs, $F_1 = g^*(F_2)$. Therefore $g^*(\nabla)$ is a foliation connection on $F_1$. Now let $\alpha_i\colon L_i \to M_i$ be the immersion of $L_i$ in $M_i$, $i = 1, 2$. Then the diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{\bar{g}} & L_2 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\ M_1 & \xrightarrow{g} & M_2 \end{array}$$

commutes. Hence $\nabla^{L_1} = \alpha_1^* g^*(\nabla) = \bar{g}^* \alpha_2^*(\nabla) = \bar{g}^*(\nabla^{L_2})$. q.e.d.

While the uniqueness of the leaf connection will help to simplify many of the proofs in this paper, the existence and naturality of the leaf connection are really the crucial points in most of our arguments.

We shall close this section by deriving an important alternate expression for the local connection form of the leaf connection. Let $F$ be a $q$-codimensional foliation on a manifold $M$, and $L$ a leaf of $F$. Since the normal bundle $\nu F$ is locally trivial, there is a local basis $Z_1^\alpha, \cdots, Z_q^\alpha$ of sections for $\nu F$. Let $\Pi: T(M) \to \nu F$ be the projection map, and let $\tilde{Z}_1^\alpha, \cdots, \tilde{Z}_q^\alpha$ be local vector fields on $M$ such that $\Pi(\tilde{Z}_i^\alpha) = Z_i^\alpha$, $1 \le i \le q$. Finally, let $\omega_1^\alpha, \cdots, \omega_q^\alpha$ be the local 1-forms on $M$ dual to the local vector fields $\tilde{Z}_1^\alpha, \cdots, \tilde{Z}_q^\alpha$, that is, the local 1-forms such that $\omega_i^\alpha(\tilde{Z}_j^\alpha) = \delta_{ij}$. Then the 1-forms $\omega_1^\alpha, \cdots, \omega_q^\alpha$ are annihilated by $F$. If $X, Y$ are in $\Gamma(F)$, then by the integrability condition

$$d\omega_i^\alpha(X, Y) = \tfrac{1}{2}(X\omega_i^\alpha(Y) - Y\omega_i^\alpha(X) - \omega_i^\alpha[X, Y]) = 0 \, .$$

Therefore there exist local 1-forms $\tau_{ji}^\alpha$ on $M$ such that

$$d\omega_i^\alpha = \sum_j \omega_j^\alpha \wedge \tau_{ji}^\alpha \, .$$

Let $r: L \to M$ be the immersion of $L$ in $M$, and let $\tau_{ji}^{L,\alpha} = r^*(\tau_{ji}^\alpha)$. In addition, let $\tau^\alpha$ denote the $q \times q$ matrix $(\tau_{ji}^\alpha)$, and $\tau^{L,\alpha}$ the $q \times q$ matrix $(\tau_{ji}^{L,\alpha})$. We shall show that $\tau^{L,\alpha}$ is actually the local connection form of the leaf connection on $\nu L$ with respect to the local basis of sections $Z_1^\alpha \circ r, \cdots, Z_q^\alpha \circ r$.

**Proposition 3.5.** *Let $\nabla$ be a foliation connection on $\nu F$, and $\theta^\alpha$ the local connection form of $\nabla$ with respect to the local basis $Z_1^\alpha, \cdots, Z_q^\alpha$. If $X$ is a vector field in $\Gamma(F)$, then $\tau^\alpha(X) = \theta^\alpha(X)$.*

*Proof.* Let $F^*$ denote the dual of $F$, and let $\Pi^*: T^*(M) \to T^*(M)/F^* \cong (\nu F)^*$ be the projection map. Then for any vector field $\tilde{Z}$ on $M$

$$\omega_i^\alpha(\tilde{Z}) = \Pi^*(\omega_i^\alpha)(\Pi(\tilde{Z})) \, .$$

If $X$ is a vector field in $\Gamma(F)$, then

$$\begin{aligned} \tfrac{1}{2}\tau_{ji}^\alpha(X) &= (\omega_j^\alpha \wedge \tau_{ji}^\alpha)(\tilde{Z}_j^\alpha, X) = \Big(\sum_k \omega_k^\alpha \wedge \tau_{ki}^\alpha\Big)(\tilde{Z}_j^\alpha, X) \\ &= d\omega_i^\alpha(\tilde{Z}_j^\alpha, X) = \tfrac{1}{2}(\tilde{Z}_j^\alpha\omega_i^\alpha(X) - X\omega_i^\alpha(\tilde{Z}_j^\alpha) - \omega_i^\alpha[\tilde{Z}_j^\alpha, X]) \\ &= \tfrac{1}{2}(-X(\delta_{ij}) - \omega_i^\alpha[\tilde{Z}_j^\alpha, X]) = \tfrac{1}{2}\omega_i^\alpha[X, \tilde{Z}_j^\alpha] \end{aligned}$$

$$= \tfrac{1}{2}\Pi^*(\omega_i^\alpha)(\Pi[X, \tilde{Z}_j^\alpha]) = \tfrac{1}{2}\Pi^*(\omega_i^\alpha)\nabla_X(Z_j^\alpha)$$
$$= \tfrac{1}{2}\Pi^*(\omega_i^\alpha)\left(\sum_k \theta_{jk}^\alpha(X)Z_k^\alpha\right) = \tfrac{1}{2}\theta_{ji}^\alpha(X) \ .$$

Hence $\tau^\alpha(X) = \theta^\alpha(X)$.

**Proposition 3.6.** *Let $\theta^{L,\alpha}$ denote the local connection form of the leaf connection on $\nu L$ with respect to the local basis of sections $Z_1^\alpha \circ r, \cdots, Z_q^\alpha \circ r$. Then $\tau^{L,\alpha} = \theta^{L,\alpha}$.*

*Proof.* This result is an immediate consequence of Proposition 3.5.

If the normal bundle $\nu F$ is a trivialized bundle, then the forms $\tau_{ji}^\alpha$ are global forms on $M$. Therefore in this case we shall write $\tau_{ji}$ in place of $\tau_{ji}^\alpha$, $\tau$ in place of $\tau^\alpha$, $\tau_{ji}^L$ in place of $\tau_{ji}^{L,\alpha}$, and $\tau^L$ in place of $\tau^{L,\alpha}$.

**Proposition 3.7.** *Let $F$ be a $q$-codimensional foliation with a trivialized normal bundle $\nu F$, and $L$ a leaf of $F$. Then $\tau^L$ is the global connection form of the leaf connection on $\nu L$ with respect to the given trivialization.*

*Proof.* This result follows immediately from Proposition 3.6.

## 4. The vanishing and obstruction theorems

We shall now take a close look at the curvatures of the foliation and leaf connections, and shall use the special properties of these curvature forms to prove the vanishing and obstruction theorems.

Let $F$ be a $q$-codimensional foliation on $M, L$ a leaf of $F$, and $j : L \to M$ the immersion of $L$ in $M$.

**Lemma 4.1.** *If $K$ is the curvature of a foliation connection on $\nu F$, and $X, Y$ are vector fields in $\Gamma(F)$, then $K(X, Y) = 0$.*

*Proof.* Let $\Pi : T(M) \to \nu F$ be the projection map, $Z$ a section of $\nu F$, and $\tilde{Z}$ a vector field such that $\Pi(\tilde{Z}) = Z$. If $K$ is the curvature of the foliation connection $\nabla$ on $\nu F$, and $X, Y$ are vector fields in $\Gamma(F)$, then

$$\begin{aligned} K(X,Y)(Z) &= \nabla_X\nabla_Y(Z) - \nabla_Y\nabla_X(Z) - \nabla_{[X\ Y]}(Z) \\ &= \nabla_X(\Pi([Y,\tilde{Z}])) - \nabla_Y(\Pi([X,\tilde{Z}])) - \Pi([[X,Y],\tilde{Z}]) \\ &= \Pi([X,[Y,\tilde{Z}]]) - \Pi([Y,[X,\tilde{Z}]]) - \Pi([[X,Y],\tilde{Z}]) \\ &= -\Pi([[Y,\tilde{Z}],X] + [[\tilde{Z},X],Y] + [[X,Y],\tilde{Z}]) = 0 \end{aligned}$$

by the Jacobi identity,

**Lemma 4.2.** *If $K^\alpha = (K_{ji}^\alpha)$ is the local curvature form over $U^\alpha$ of a foliation connection on $\nu F$, then there is an ideal $I_\alpha$ of forms on $U^\alpha$ such that $I_\alpha^{q+1} = 0$ and $K_{ji}^\alpha$ is in $I_\alpha$ for all $j, i$.*

*Proof.* Let $X_1^\alpha, \cdots, X_{m-q}^\alpha$ be a basis for the sections of $F$ over $U^\alpha$. Then this basis can be extended to a basis $X_1^\alpha, \cdots, X_{m-q}^\alpha, Y_1^\alpha, \cdots, Y_q^\alpha$ for all vector fields over $U^\alpha$. Let $\omega_s^\alpha$ be the 1-form dual to $Y_s^\alpha$, and $\theta_t^\alpha$ the 1-form dual to $X_t^\alpha$. If $I_\alpha$ denotes the ideal of forms over $U^\alpha$ generated by $\omega_1^\alpha, \cdots, \omega_q^\alpha$, then certainly

$I_\alpha^{q+1} = 0$. Moreover, since $K_{ji}^\alpha$ is a 2-form over $U^\alpha$,

$$K_{ji}^\alpha = \sum a_{pr}\theta_p^\alpha \wedge \theta_r^\alpha + \Psi^\alpha$$

for some $\Psi^\alpha$ in $I_\alpha$. However, by Lemma 4.1, $a_{pr} = K_{ji}^\alpha(X_p^\alpha, X_r^\alpha) = 0$. Therefore $K_{ji}^\alpha$ is in $I_\alpha$ for all $j, i$.

**Lemma 4.3.** *If $K^L$ is the curvature of the leaf connection on $\nu L$, then $K^L = 0$.*

*Proof.* This result is an immediate consequence of Lemma 4.1.

**Theorem 4.4** (*The vanishing theorem for foliations*). *If $F$ is a $q$-codimensional foliation, then the real Pontryagin ring of $\nu F$ is trivial in dimensions greater then $2q$.*

*Proof.* Let $K$ be the curvature of a foliation connection on $\nu F$. If $\gamma$ is a class in the real Pontryagin ring of $\nu F$ such that degree $\gamma > 2q$, then $\gamma$ is cohomologous to a form of the type $\sum r_{j_1\cdots j_t} c_{j_1}(K) \cdots c_{j_t}(K)$ where $r_{j_1\cdots j_t}$ is in $\boldsymbol{R}$, $c_j$ is the $j$th chern polynomial, and $\sum j_s > q$. Let $K^\alpha = (K_{ji}^\alpha)$ be the local curvature form of $K$ over $U^\alpha$. By Lemma 4.2 there is an ideal $I_\alpha$ of forms on $U^\alpha$ such that $I_\alpha^{q+1} = 0$, and $K_{ji}^\alpha$ is in $I_\alpha$ for all $j, i$. Hence $c_{j_1}(K^\alpha) \cdots c_{j_t}(K^\alpha)$ is in $I_\alpha^{\Sigma j_s}$. However $\sum j_s > q$; therefore $I_\alpha^{\Sigma j_s} = 0$ so $c_{j_1}(K^\alpha) \cdots c_{j_t}(K^\alpha) = 0$. Hence $\gamma|_{U^\alpha} = 0$. Since this last equality holds on every neighborhood $U^\alpha$, we can conclude that $\gamma = 0$.

**Theorem 4.5** (*The vanishing theorem for leaves*). *If $L$ is a leaf of a foliation, then the real Pontryagin ring of $\nu L$ is trivial.*

*Proof.* This result follows immediately from Lemma 4.3.

**Theorem 4.6** (*The obstruction theorem for foliations*). *Let $M$ be an $m$-dimensional manifold, and $E$ a subbundle of the tangent bundle $T(M)$ of dimension $m - q$. Then a necessary condition for $E$ to be isomorphic to a $q$-codimensional foliation on $M$ is that the real Pontryagin ring of the quotient bundle $T(M)/E$ must be trivial in dimensions greater than $2q$.*

*Proof.* This result is an immediate consequence of Theorem 4.4.

**Theorem 4.7** (*The obstruction theorem for leaves*). *Let $N$ be a connected manifold and let $j: N \to M$ be a 1-1 immersion. Then a necessary condition for $N$ to be an integral manifold of a foliation on $M$ is that the real Pontryagin ring of the normal bundle $\nu N$ must be trivial. In particular, a necessary condition for $N$ to be a leaf of a foliation on $M$ is that the real Pontryagin ring of $\nu N$ must be trivial.*

*Proof.* This result is an immediate consequence of Theorem 4.5.

The vanishing and obstruction theorems for foliations were first proved by Bott in [1] and [2]. We have added nothing new to his techniques. Rather we have included these theorems here in order to stress the similarities between these theorems for foliations and the corresponding theorems for the leaves of foliations. Two general principles emerge when we examine these similarities. First, given a theorem pertaining to foliations, one can often find an analogous theorem pertaining to the leaves of foliations. Second, the propositions referring to leaves are usually simpler than the corresponding propositions referring to foliations. This second principle is to be expected because foliations are much more complex than their individual leaves. We shall see the second principle in operation again in § 6 when we discuss secondary characteristic classes. Indeed, the secondary leaf classes form only one algebraic component of the secondary foliation classes.

## 5. The holonomy homomorphism and special connections

A characteristic class on the category $CF_qL$ ($CF_qL^0$, $CF_q^0L$) is a natural transformation which assigns to each pair $(F, L)$ in $CF_qL$ ($CF_qL^0$, $CF_q^0L$) a class in $H^*(L)$. In this section we shall use the leaf connection along with other special connections to construct a collection of characteristic classes, called the holonomy ring, on the category $CF_qL$ ($CF_qL^0$, $CF_q^0L$). We begin by recalling a technique used by Bott in [2] for comparing connections.

Let $M$ be a manifold, and let $A^*(M)$ denote the collection of differential forms on $M$. In addition, let $\eta: E \to M$ be a vector bundle over $M$ and let $\nabla^0, \cdots, \nabla^n$ be connections on $E$. Finally, let $c_j$ denote the $j$th Chern polynomial and define

$$\lambda(\nabla^0, \cdots, \nabla^n)(c_j) = \Pi_*[c_j(K)|_{M\times\Delta^n}] ,$$

where

$$\Delta^n = \{(t_0, \cdots, t_n) \,|\, t_i \geq 0 \text{ and } \textstyle\sum t_i = 1\}$$

is the standard $n$-simplex, $K$ is the curvature of the connection $(1 - t_1 - \cdots - t_n)\nabla^0 + t_1\nabla^1 + \cdots + t_n\nabla^n$ on the vector bundle $E \times \boldsymbol{R}^n \to M \times \boldsymbol{R}^n$, $\Pi_*: A^p(M \times \Delta^n) \to A^{p-n}(M)$ denotes integration along $\Delta^n$.

In particular, if $K^0$ is the curvature of $\nabla^0$, then $\lambda(\nabla^0)(c_j) = c_j(K^0)$. Moreover $\lambda$ has the following useful properties.

**Lemma 5.1.** $d\lambda(\nabla^0, \cdots, \nabla^n)(c_j) = \sum (-1)^i \lambda(\nabla^0, \cdots, \hat{\nabla}^i, \cdots, \nabla^n)(c_j)$.

*Proof.* See [2, p. 65].

**Lemma 5.2.** *If $f: N \to M$, then $f^*\lambda(\nabla^0, \cdots, \nabla^n)(c_j) = \lambda(f^*(\nabla^0), \cdots, f^*(\nabla^n))(c_j)$.*

*Proof.* Let $\alpha_N: N \times \Delta^n \to N \times \boldsymbol{R}^n$ and $\alpha_M: M \times \Delta^n \to M \times \boldsymbol{R}^n$ be the inclusion maps. Then the diagram

$$\begin{array}{ccc} N \times \Delta^n & \xrightarrow{f\times \mathrm{id}} & M \times \Delta^n \\ {\scriptstyle \alpha_N}\downarrow & & \downarrow{\scriptstyle \alpha_M} \\ N \times \boldsymbol{R}^n & \xrightarrow{f\times \mathrm{id}} & M \times \boldsymbol{R}^n \end{array}$$

commutes. Let $\nabla$ denote the connection $(1 - t_1 - \cdots - t_n)\nabla^0 + t_1\nabla^1 + \cdots + t_n\nabla^n$ on the vector bundle $E \times \boldsymbol{R}^n \to M \times \boldsymbol{R}^n$, and let $K$ be the curvature of $\nabla$. Then $(f \times \mathrm{id})^*(\nabla) = (1 - t_1 - \cdots - t_n)f^*(\nabla^0) + t_1 f^*(\nabla^1) + \cdots + t_n f^*(\nabla^n)$, and the curvature of $(f \times \mathrm{id})^*(\nabla)$ is $(f \times \mathrm{id})^*(K)$. Now directly from the definition of $\lambda$ we have

$$\begin{aligned}
\lambda(f^*(\nabla^0), \cdots, f^*(\nabla^n))(c_j) &= \Pi_*[\alpha_N^* c_j[(f \times \mathrm{id})^*(K)]] \\
&= \Pi_*[\alpha_N^* \circ (f \times \mathrm{id})^* c_j(K)] \\
&= \Pi_*[(f \times \mathrm{id})^* \circ \alpha_M^* c_j(K)] \\
&= f^* \Pi_*[\alpha_M^* c_j(K)] \\
&= f^* \lambda(\nabla^0, \cdots, \nabla^n)(c_j) \,. \quad \text{q.e.d.}
\end{aligned}$$

Call a connection on a trivial vector bundle flat if its connection form is zero with respect to some trivialization of the vector bundle. Let $\nabla$ be a connection on a trivial vector bundle with connection form $\theta$ and curvature form $K$. Then by [2, p. 25], $K = d\theta - \theta^2$. Thus, if $\nabla$ is a flat connection, then the curvature of $\nabla$ is also zero.

Two flat connections on the same trivial vector bundle will be said to be homotopic if the trivializations to which they correspond are homotopic. Since the collection of all homotopic trivializations forms a convex set, the collection of all homotopic flat connections also forms a convex set. Therefore we have

**Lemma 5.3.** *If $\nabla^0, \cdots, \nabla^n$ are homotopic flat connections, then $\lambda(\nabla^0, \cdots, \nabla^n)(c_j) = 0$.*

*Proof.* If $\nabla^0, \cdots, \nabla^n$ are homotopic flat connections, then $\nabla = (1 - t_1 - \cdots - t_n)\nabla^0 + t_1\nabla^1 + \cdots + t_n\nabla^n$ is also a flat connection.

If $K$ denotes the curvature of $\nabla$, then $K = 0$. Hence $\lambda(\nabla^0, \cdots, \nabla^n)(c_j) = \Pi_*[c_j(K)|_{M \times \Delta^n}] = 0$. q.e.d.

With these preliminary results in hand, we are about ready to construct characteristic classes on the leaves of foliated manifolds.

**Definition.** A characteristic class on the category $CF_qL$ ($CF_qL^0, CF_q^0L$) is a transformation $\gamma$ which associates to each pair $(F, L)$ in $CF_qL$ ($CF_qL^0, CF_q^0L$) a class $\gamma(F, L)$ in $H^*(L)$ such that if $g\colon (F_1, L_1) \to (F, L)$ is a map of pairs (compatible with the given trivializations), and $\bar{g}\colon L_1 \to L$ is the map induced by $g$, then $\gamma(F_1, L_1) = \bar{g}^*\gamma(F, L)$.

The collection of characteristic classes on the category $CF_qL$ forms a ring which we shall denote by $R(CF_qL)$. If $(F, L)$ is a pair in the category $CF_qL$, then there is a ring homomorphism $R_{F,L}\colon R(CF_qL) \to H^*(L)$ given by $R_{F,L}(\gamma) = \gamma(F, L)$. Similarly, there are rings $R(CF_qL^0)$ and $R(CF_q^0L)$ with analogous homomorphisms. Moreover, it is known that the ring $R(CF_qL^0)$, is isomorphic to the ring $R(CF_q^0L)$, [17]. However we shall not attempt to prove this result in this paper.

Now let $F$ be a $q$-codimensional foliation, and $L$ a leaf of $F$ with a trivialized normal bundle $\nu L$. As usual, $\nabla^L$ will denote the leaf connection on $\nu L$. In addition, $\nabla^0$ will denote the flat connection on $\nu L$ corresponding to the given trivialization. By construction, the form $\lambda(\nabla^0, \nabla^L)(c_j)$ is a $(2j-1)$-form on $L$. Moreover we have

**Proposition 5.4.** *The form $\lambda(\nabla^0, \nabla^L)(c_j)$ has the following properties:*

1. *$\lambda(\nabla^0, \nabla^L)(c_j)$ is a closed form,*
2. *the cohomology class of $\lambda(\nabla^0, \nabla^L)(c_j)$ depends only on the homotopy class of the original trivialization.*

*Proof.* 1. Let $K^L$ denote the curvature of $\nabla^L$, and $K^0$ the curvature of $\nabla^0$. Since $\nabla^0$ is a flat connection, $K^0 = 0$. Moreover by Lemma 4.3, $K^L = 0$. Hence by Lemma 5.1

$$d\lambda(\nabla^0, \nabla^L)(c_j) = \lambda(\nabla^L)(c_j) - \lambda(\nabla^0)(c_j) = c_j(K^L) - c_j(K^0) = 0 \ .$$

2. Let $\bar{\nabla}^0$ be a flat connection on $\nu L$ homotopic to $\nabla^0$. Then by Lemma 5.3, $\lambda(\bar{\nabla}^0, \nabla^0)(c_j) = 0$. Therefore by Lemma 5.1, $d\lambda(\bar{\nabla}^0, \nabla^0, \nabla^L)(c_j) = \lambda(\nabla^0, \nabla^L)(c_j) - \lambda(\bar{\nabla}^0, \nabla^L)(c_j)$. Thus $\lambda(\nabla^0, \nabla^L)(c_j)$ is cohomologous to $\lambda(\bar{\nabla}^0, \nabla^L)(c_j)$, and hence the cohomology class of $\lambda(\nabla^0, \nabla^L)(c_j)$ depends only on the homotopy class of the original trivialization. q.e.d.

Let $E(h_1, h_2, \cdots, h_q)$ denote the exterior algebra on generators $h_1, h_2, \cdots, h_q$. Define a ring homomorphism

$$\phi^*_{F,L} : E(h_1, h_2, \cdots, h_q) \to H^*(L)$$

by letting

$$\phi^*_{F,L}(h_j) = \{\lambda(\nabla^0, \nabla^L)(c_j)\} \ .$$

Then by Proposition 5,4 the homomorphism $\phi^*_{F,L}$ is well-defined and depends only on the homotopy class of the original trivialization. The homomorphism $\phi^*_{F,L}$ is called the holonomy homomorphism, and the image of $\phi^*_{F,L}$ in $H^*(L)$ is called the holonomy ring of the leaf $L$ with respect to the foliation $F$. Let

$$h_j(F, L) = \{\lambda(\nabla^0, \nabla^L)(c_j)\} \ .$$

The class $h_j(F, L)$ is called the $j$th holonomy class of the leaf $L$ with respect to the foliation $F$. By construction, the degree of the class $h_j(F, L)$ is $2j-1$. Now to show that the holonomy classes are indeed characteristic classes on the category $CF_qL^0$, we must demonstrate that they satisfy the required naturality property.

**Proposition 5.5.** *Let $F_i$ be a $q$-codimensional foliation, and $L_i$ a leaf of $F_i$ with a trivialized normal bundle $\nu L_i$, $i = 1, 2$. In addition, let $g : (F_1, L_1) \to (F_2, L_2)$ be a map of pairs compatible with the given trivializations, and $\bar{g} : L_1 \to L_2$ the map induced by $g$. Then the diagram*

$$\begin{tikzcd} E(h_1, h_2, \cdots, h_q) \arrow[r, "\phi^*_{F_2,L_2}"] \arrow[dr, "\phi^*_{F_1,L_1}"'] & H^*(L_2) \arrow[d, "\bar{g}^*"] \\ & H^*(L_1) \end{tikzcd}$$

*commutes.*

*Proof.* If $\nabla^0$ is the flat connection corresponding to the given trivialization on $\nu L_2$, then the connection $\bar{g}(\nabla^0)$ is the flat connection corresponding to the given trivialization on $\nu L_1$ since the map $g$ is compatible with the given trivializations. Moreover by Proposition 3.4 if $\nabla^{L_2}$ is the leaf connection on $\nu L_2$, then $\bar{g}^*(\nabla^{L_2})$ is the leaf connection on $\nu L_1$. Therefore by Lemma 5.2

$$\begin{aligned} \phi^*_{F_1,L_1}(h_j) &= \{\lambda(\bar{g}^*(\nabla^0), \bar{g}^*(\nabla^{L_2}))(c_j)\} \\ &= \{\bar{g}^*\lambda(\nabla^0, \nabla^{L_2})(c_j)\} = \bar{g}^*\phi^*_{F_2,L_2}(h_j) . \end{aligned}$$

**Theorem 5.6.** *There is a homomorphism $\phi^*_q : E(h_1, h_2, \cdots, h_q) \to R(CF_qL^0)$ such that if $(F, L)$ is a pair in the category $CF_qL^0$, then the diagram*

$$\begin{tikzcd} E(h_1, h_2, \cdots, h_q) \arrow[r, "\phi^*_q"] \arrow[dr, "\phi^*_{F,L}"'] & R(CF_qL^0) \arrow[d, "R_{F,L}"] \\ & H^*(L) \end{tikzcd}$$

*commutes.*

*Proof.* This result is an immediate consequence of Proposition 5.5.

The image of the homomorphism $\phi^*_q : E(h_1, h_2, \cdots, h_q) \to R(CF_qL^0)$ is called the holonomy ring of $R(CF_qL^0)$. The holonomy ring on the category $CF^0_qL$ can be constructed in much the same manner. However to construct the holonomy ring on the category $CF_qL$ we must modify our procedure somewhat.

Let $\eta : E \to M$ be a vector bundle, and $\langle\ ,\ \rangle$ a smooth inner product on the fibres of $E$. Call a connection $\nabla^R$ on a vector bundle $\eta : E \to M$ a Riemannian connection if and only if for every $X$ in $\Gamma(T(M))$ and all $s_1, s_2$ in $\Gamma(E)$

$$X\langle s_1, s_2\rangle = \langle \nabla^R_X s_1, s_2\rangle + \langle s_1, \nabla^R_X s_2\rangle .$$

If $K^R$ is the curvature of a Riemannian connection, then $c_{2j-1}(K^R) = 0$ [2, p. 28]. Therefore the construction in this section remains valid if we replace flat connections by Riemannian connections, the ring $E(h_1, h_2, \cdots, h_q)$ by the ring $E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1})$, and the category $CF_qL^0$ by the category $CF_qL$. Moreover, the proofs are essentially the same.

Now if the pair $(F, L)$ is in the category $CF_qL$, then the class $h_{2j-1}(F, L)$ depends only on the foliation $F$ and the leaf $L$ because the collection of all Riemannian connections forms a convex set (see Lemmas 5.3, 5.4). Therefore even when the pair $(F, L)$ is in the category $CF_qL^0$, the characteristic class

$h_{2j-1}(F, L)$ is completely independent of the choice of the trivialization of $\nu L$. Thus, if $(F, L)$ is a pair in $CF_q L^0$, then $h_{2j}(F, L)$ depends on $F, L$ and the homotopy class of the trivialization of $\nu L$, but $h_{2j-1}(F, L)$ depends only on $F$ and $L$.

The characteristic classes in the holonomy ring of a leaf $L$ are called the secondary leaf classes or the secondary leaf invariants because they exist due to the vanishing of the real Pontryagin ring of $\nu L$. In § 11 we shall show that the homomorphism $\phi_q^*$ is a monomorphism. Thus every characteristic class in the holonomy ring is a nontrivial characteristic class. However, it is not yet known whether the holonomy ring actually exhausts the collection of characteristic classes on the leaves of foliations.

The construction of characteristic classes which we have just developed is actually valid for the category whose objects are the (trivialized) vector bundles with a connection whose curvature is zero. We shall not attempt to pursue this subject any further at this time. However a detailed discussion of characteristic classes for these vector bundles is given in [15] and [16].

## 6. The secondary foliation classes

A characteristic class on the category $CF_q^0$ is a transformation $\gamma$ which associates, to each $q$-codimensional foliation $F$ with a trivialized normal bundle on a manifold $M$, a class $\gamma(F)$ in $H^*(M)$ such that if $g$ is a map transverse to $F$, then $\gamma(g^*(F)) = g^*\gamma(F)$. We shall denote the ring of characteristic classes on the category $CF_q^0$ by $R(CF_q^0)$. If $F$ is a $q$-codimensional foliation with a trivialized normal bundle on a manifold $M$, then there is a ring homomorphism $R_F\colon R(CF_q^0) \to H^*(M)$ given by $R_F(\gamma) = \gamma(F)$. In [2] Bott constructed a collection of characteristic classes in $R(CF_q^0)$ by comparing foliation connections to flat connections. We shall briefly review this construction.

Let $F$ be a $q$-codimensional foliation with a trivialized normal bundle on a manifold $M$. Let $\nabla^B$ be a foliation connection on $\nu F$ with curvature $K^B$, and $\nabla^0$ the flat connection on $\nu F$ corresponding to the given trivialization. Define $W_q$ to be the cochain complex

$$E(h_1, h_2, \cdots, h_q) \otimes \boldsymbol{R}[c_1, c_2, \cdots, c_q]/\text{degree } p > 2q ,$$

where degree $c_j = 2j$, $dc_j = 0$, and $dh_j = c_j$. Then there is a ring homomorphism $\lambda_F\colon W_q \to A^*(M)$ given by

$$\lambda_F(h_j) = \lambda(\nabla^0, \nabla^B)(c_j) , \qquad \lambda_F(c_j) = c_j(K^B) .$$

Using Lemma 5.1, one can easily show that $\lambda_F$ is a cochain map. Now the cochain map $\lambda_F$ depends on the original choices of the foliation connection and the trivialization. However by employing the techniques used in § 5, one can show that the homomorphism

$$\lambda_F^* : H^*(W_q) \to H^*(M)$$

induced by the cochain map $\lambda_F$ depends only on the homotopy class of the original trivialization. Moreoever, if $g: N \to M$ is a map transverse to $F$, then by Lemma 5.2 the diagram

$$\begin{tikzcd} H^*(W_q) \arrow[r, "\lambda_F^*"] \arrow[dr, "\lambda_{g^*(F)}^*"'] & H^*(M) \arrow[d, "g^*"] \\ & H^*(N) \end{tikzcd}$$

commutes.

Therefore to summarize, we have the following theorem.

**Theorem 6.1.** *There is a homomorphism $\lambda_q^* : H^*(W_q) \to R(CF_q^0)$ such that if $F$ is a $q$-codimensional foliation with a trivialized normal bundle on a manifold $M$, then the diagram*

$$\begin{tikzcd} H^*(W)_q \arrow[r, "\lambda_q^*"] \arrow[dr, "\lambda_F^*"'] & R(CF_q^0) \arrow[d, "R_F"] \\ & H^*(M) \end{tikzcd}$$

*commutes.*

The classes in the image of the homomorphism $\lambda_F^* : H^*(W_q) \to H^*(M)$ are called the secondary foliation classes of $F$. Similarly, the classes in the image of the homomorphism $\lambda_q^* : H^*(W_q) \to R(CF_q^0)$ are called the secondary classes of $CF_q^0$, because their existence is due to the vanishing of the real Pontryagin ring of $\nu F$ in high dimensions.

Now make $E(h_1, \cdots, h_q)$ into a cochain complex by letting $dh_j = 0$. Then the ring homomorphism

$$\mu_q : W_q \to E(h_1, \cdots, h_q)$$

given by

$$\mu_q(h_j) = h_j , \qquad \mu_q(c_j) = 0$$

is a cochain map. Let $L$ be a leaf of the foliation $F$ and let $\alpha: L \to M$ be the immersion of $L$ in $M$. Then the trivialization on the normal bundle $\nu F$ induces a trivialization on the normal bundle $\nu L$. Now the ring homomorphism

$$\phi_{F,L} : E(h_1, \cdots, h_q) \to A^*(L)$$

given by

$$\phi_{F,L}(h_j) = \lambda(\alpha^*(\nabla^0), \nabla^L)(c_j)$$

is a cochain map. Moreover, since $\alpha^*(\nabla^0)$ is the flat connection on $\nu L$ with respect to the induced trivialization, the homomorphism $\phi^*_{F,L}$ induced by the cochain map $\phi_{F,L}$ is exactly the same as the homomorphism $\phi^*_{F,L}$ defined in § 5.

**Proposition 6.2.** *The diagram*

$$\begin{array}{ccc} W_q & \xrightarrow{\lambda_F} & A^*(M) \\ {\scriptstyle \mu_q}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ E(h_1, \cdots, h_q) & \xrightarrow{\phi_{F,L}} & A^*(L) \end{array}$$

*commutes.*

*Proof.* It is enough to verify this result on the generators of $W_q$. First we have

$$\begin{aligned} \alpha^*\lambda_F(h_j) &= \alpha^*\lambda(\nabla^0, \nabla^B)(c_j) = \lambda(\alpha^*(\nabla^0), \alpha^*(\nabla^B))(c_j) \\ &= \lambda(\alpha^*(\nabla^0), \nabla^L)(c_j) = \phi_{F,L}(h_j) = \phi_{F,L}\mu_q(h_j) \ . \end{aligned}$$

Moreover we also have

$$\alpha^*\lambda_F(c_j) = \alpha^*c_j(K^B) = c_j(\alpha^*(K^B)) = c_j(K^L) = 0 = \phi_{F,L}\mu_q(c_j) \ .$$

Therefore $\alpha^*\lambda_F = \phi_{F,L}\mu_q$.

**Corollary 6.3.** *The diagram*

$$\begin{array}{ccc} H^*(W_q) & \xrightarrow{\lambda_F^*} & H^*(M) \\ {\scriptstyle \mu_q^*}\downarrow & & \downarrow{\scriptstyle \alpha^*} \\ E(h_1, \cdots, h_q) & \xrightarrow{\phi^*_{F,L}} & H^*(L) \end{array}$$

*commutes.*

*Proof.* This result follows immediately from Proposition 6.2.

**Corollary 6.4.** $h_j(F, L) = \{\lambda_F(h_j)|_L\}$.

*Proof.* This result follows immediately from Proposition 6.2.

**Lemma 6.5.** *The homomorphism $\mu_q^*: H^*(W_q) \to E(h_1, \cdots, h_q)$ is the zero homomorphism.*

*Proof.* Every cocycle in $W_q$ is the sum of elements of the form $h_Ic_J$ where $h_I$ is an element in the ring $E(h_1, \cdots, h_q)$ and $c_J$ is a nonconstant element in the ring $\boldsymbol{R}[c_1, \cdots, c_q]/\text{degree } p > 2q$. However $\mu_q(c_J) = 0$. Thus $\mu_q^*(h_Ic_J) = 0$, and hence $\mu_q^*$ is the zero homomorphism.

**Corollary 6.6.** *The homomorphism $\alpha^*\lambda_F^*: H^*(W_q) \to H^*(L)$ is the zero homomorphism. Therefore, if $\gamma$ is a secondary foliation class, then $\gamma(F)|_L = 0$.*

*Proof.* This result follows immediately from Corollary 6.3 and Lemma 6.5.

To construct the secondary classes on the category $CF_q$, one proceeds exactly as for the category $CF_q^0$ except that flat connections are replaced by Riemannian connections and the cochain complex $W_q$ is replaced by the cochain complex

$$WO_q = E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1}) \otimes \boldsymbol{R}[c_1, \cdots, c_q]/\text{degree } p > 2q ,$$

where degree $c_j = 2j$, $dc_j = 0$, and $dh_{2j-1} = c_{2j-1}$. Therefore the results in this section remain valid if we replace the category $CF_q^0$ by the category $CF_q$, the cochain complex $W_q$ by the cochain complex $WO_q$, and the ring $E(h_1, \cdots, h_q)$ by the ring $E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1})$. Moreover the proofs are essentially the same. Now Corollary 6.6 can be interpreted as a generalization of the vanishing theorem for leaves. Indeed this result states that not only the Pontryagin classes but also all of the other secondary classes of a foliation vanish on the leaves.

In light of Proposition 6.2 we can give the following analysis of the cochain complex $W_q(WO_q)$. The ring $\boldsymbol{R}[c_1, \cdots, c_q]$ represents the ring of Chern classes on the normal bundle of a foliation; the truncation represents the vanishing theorem for foliations; and the ring $E(h_1, \cdots, h_q)$ $(E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1}))$ represents the holonomy ring on the category $CF_q^0L(CF_qL)$. In fact by Corollary 6.4, $h_j(F, L) = \{\lambda_F(h_j)|_L\}$. Hence the holonomy classes of the leaves can be represented by the pullback of forms on the manifold in which they are immersed. Thus the secondary foliation classes on $CF_q^0(CF_q)$ and the secondary leaf classes on $CF_q^0L$ $(CF_qL)$ are intimately related.

We close this section with a warning. Two foliations $F_0, F_1$ on $M$ are said to be homotopic if there is a foliation $F$ on $M \times I$ such that the inclusion maps $i_0: M \times 0 \to M \times I$ and $i_1: M \times 1 \to M \times I$ are transverse to $F$ and $F_0 = i_0^*(F)$ and $F_1 = i_1^*(F)$. The secondary foliation classes are homotopy invariants [2, p. 69]. That is, if $\gamma$ is a secondary foliation class and $F_0, F_1$ are homotopic foliations, then $\gamma(F_0) = \gamma(F_1)$. However, even though the secondary leaf invariants depend only on the foliation and the choice of leaf, these classes are not homotopy invariants of the foliation. This is true because the leaves themselves are not homotopy invariants; indeed, even if $F_0$ and $F_1$ are homotopic foliations, the leaves of $F_0$ and the leaves of $F_1$ may be quite different spaces.

## 7. The holonomy homomorphism and special differential forms

In this section we shall give an alternate construction of the holonomy homomorphism. Let $F$ be a $q$-codimensional foliation, and let $L$ be a leaf of $F$ with a trivialized normal bundle $\nu L$. In addition, let $\theta^L$ denote the connection form of the leaf connection with respect to the given trivialization on $\nu L$. We shall show that the holonomy ring of $L$ consists of the cohomology classes of certain polynomials in $\theta^L$.

Recall that by [2, p. 25] the local curvature form $K^\alpha$ and the local connection form $\theta^\alpha$ are related by the matrix equation

$$K^\alpha = d\theta^\alpha - (\theta^\alpha)^2 . \tag{7.1}$$

However by Lemma 4.3, $K^L = 0$. Therefore

$$d\theta^L = (\theta^L)^2 , \tag{7.2}$$

or in terms of the coordinates of the matrix $\theta^L$

$$d\theta^L_{ji} = \sum_k \theta^L_{jk} \wedge \theta^L_{ki} . \tag{7.3}$$

Let $A^*(gl(q, \boldsymbol{R}))$ denote the collection of left-invariant forms on $GL(q, \boldsymbol{R})$, and let $\sigma = (\sigma_{ji})$ be the $q \times q$ matrix whose entries are the canonical generators of $A^*(gl(q, \boldsymbol{R}))$. Since the equations represented by (7.3) have the same form as the Maurer-Cartan equations for $GL(q, \boldsymbol{R})$, we can define a cochain map

$$\phi_{F,L} \colon A^*(gl(q, \boldsymbol{R})) \to A^*(L)$$

by letting

$$\phi_{F,L}(\sigma_{ji}) = \theta^L_{ki} . \tag{7.4}$$

We shall show that the image of the homomorphism

$$\phi^*_{F,L} \colon H^*(gl(q, \boldsymbol{R})) \to H^*(L)$$

is the holonomy ring of $L$ with respect to the foliation $F$. This result is reasonable since $H^*(gl(q, \boldsymbol{R}))$ is isomorphic to the ring $E(h_1, \cdots, h_q)$.

To begin, let $A$ be a $q \times q$ matrix and let $c_j$ denote the $j$th Chern polynomial. Define the polynomials $e_{j,s}$, $0 \leq j, s \leq q$, by the equation

$$c_j(xA + A^2) = \sum_s x^s e_{j,s}(A) .$$

In addition, let

$$\frac{1}{N_j} = (2j - 1)\binom{2j-2}{j-1} ,$$

and let

$$e_{2j-1}(A) = N_j e_{j,1}(A) .$$

Then the degree of $e_{2j-1}$ is $2j - 1$. Moreover the ring homomorphism

$$e \colon E(h_1, \cdots, h_q) \to H^*(gl(q, \boldsymbol{R}))$$

defined by

$$e(h_j) = \{e_{2j-1}(\sigma)\}$$

is an isomorphism.

**Proposition 7.1.** *Let $Z_1, \cdots, Z_q$ be a global basis of sections for $\nu L$. If $\theta^L$ is the connection form of $\nabla^L$ with respect to $Z_1, \cdots, Z_q$, and $\nabla^0$ is the flat connection on $\nu L$ with respect to $Z_1, \cdots, Z_q$, then $\lambda(\nabla^0, \nabla^L)(c_j) = e_{2j-1}(\theta^L)$.*

*Proof.* Consider the connection $\nabla = (1-t)\nabla^0 + t\nabla^L$ on the trivial bundle $\nu L \times \boldsymbol{R} \to L \times \boldsymbol{R}$. Let $K$ be the curvature form of $\nabla$ with respect to the global basis of sections $Z_1, \cdots, Z_q, d/dt$, and let $\theta$ be the connection form of $\nabla$ with respect to this same global basis of sections. If $\theta^0$ is the connection form of $\nabla^0$ with respect to the basis $Z_1, \cdots, Z_q$, then $\theta = (1-t)\theta^0 + t\theta^L$. However $\nabla^0$ is the flat connection on $\nu L$ with respect to the basis $Z_1, \cdots, Z_q$. Therefore $\theta^0 = 0$ and $\theta = t\theta^L$. Now by [2, p. 25]

$$\begin{aligned} K = d\theta - \theta^2 &= d(t\theta^L) - (t\theta^L)^2 = dt\, \theta^L + t\, d\theta^L - t^2(\theta^L)^2 \\ &= dt\, \theta^L + t\, d\theta^L - t(\theta^L)^2 + t(\theta^L)^2 - t^2(\theta^L)^2 \\ &= dt\, \theta^L + tK^L + (t - t^2)(\theta^L)^2 \\ &= dt\, \theta^L + t(1-t)(\theta^L)^2 , \end{aligned}$$

since $K^L = 0$ by Lemma 4.3. Therefore

$$\begin{aligned} c_j(K) &= c_j(dt\, \theta^L + t(1-t)(\theta^L)^2) \\ &= t^{j-1}(1-t)^{j-1}\, dt\, e_{j,1}(\theta^t) + \text{terms not involving } dt . \end{aligned}$$

Now integrating by parts, we get

$$\int_0^1 t^{j-1}(1-t)^{j-1}\, dt = N_j .$$

Therefore integrating along $I$, we obtain

$$\lambda(\nabla^0, \nabla^L)(c_j) = \Pi_*[c_j(K)_{L\times I}] = N_j e_{j,1}(\theta^L) = e_{2j-1}(\theta^L) .$$

**Theorem 7.2.** *The diagram*

$$\begin{tikzcd} E(h_1, \cdots, h_q) \arrow[r, "e"] \arrow[dr, "\phi^*_{F,L}"'] & H^*(gl(q, \boldsymbol{R})) \arrow[d, "\phi^*_{F,L}"] \\ & H^*(L) \end{tikzcd}$$

*commutes.*

*Proof.* Let $Z_1, \cdots, Z_q$, $\theta^L$, and $\nabla^0$ be as in Proposition 7.1; then

$$\begin{aligned}\phi^*_{F,L} \circ e(h_j) &= \phi^*_{F,L}\{e_{2j-1}(\sigma)\} = \{e_{2j-1}(\theta^L)\} \\ &= \{\lambda(\nabla^0, \nabla^L)(c_j)\} = \phi^*_{F,L}(h_j) \ .\end{aligned}$$

Thus $\phi^*_{F,L} \circ e = \phi^*_{F,L}$. q.e.d.

We shall call the homomorphism $\phi^*_{F,L}\colon H^*(gl(q, \boldsymbol{R})) \to H^*(L)$ the holonomy homomorphism. By Theorem 7.2 there is no substantial ambiguity in this terminology. However the appearance of $H^*(gl(q, \boldsymbol{R}))$ in place of the abstract ring $E(h_1, \cdots, h_q)$ suggests that the holonomy classes are essentially linear invariants. In § 11 we will show that this suggestion is in fact true.

If the normal bundle $\nu F$ is a trivialized bundle, then by Proposition 3.7, $\theta^L = \tau^L$. Since the form $\tau^L$ is easily accessible, the holonomy classes were first studied by other authors in the case of foliations with trivialized normal bundles. For codimension 1, the class $h_1(F, L) = \{\tau^L_{1,1}\}$ was first introduced and studied by Reeb in [19, p. 115]. Sacksteder then generalized this class to arbitrary codimensions by examining the forms $\sum_j \tau^L_{jj}$, [21], [22]. Finally the entire holonomy ring on the category $CF^0_qL$ was constructed by Reinhart in [20]. However in order to simplify his computations, Reinhart considered the polynomials

$$P_{2j-1}(\tau^L) = \text{trace}\,[(\tau^L)^{2j-1}]$$

rather than the polynomials $e_{2j-1}(\tau^L)$.

It has long been known that the characteristic class $h_1$ on the category $CF^0_qL$ is related to the holonomy of the leaves, [19]. Moreover, the following theorem of Reeb demonstrates that the nonvanishing of $h_1(F, L)$ has topological implications concerning the structure of the foliation $F$ near the leaf $L$.

**Theorem 7.3.** *Let $F$ be a 1-codimensional foliation with a trivial normal bundle $\nu F$ on a manifold $M$. Suppose further that $L$ is a compact leaf of $F$ and that $h_1(F, L) \neq 0$. Then there exists a neighborhood $U$ of the leaf $L$ in the manifold $M$ such that if $L_1$ is any leaf of $F$ which intersects $U$, then the closure of $L_1$ contains $L$.*

*Proof.* See [19, p. 117].

The topological significance of the other holonomy classes is still an open problem.

The holonomy classes on the category $CF_qL$ can be constructed in a similar manner. Let $F$ be a $q$-codimensional foliation, and let $L$ be a leaf of $F$. In addition, let $A^*(gl(q, \boldsymbol{R}), 0(q))$ denote the collection of all left-invariant $0(q)$-basic forms on $GL(q, \boldsymbol{R})$. That is, a form $\omega$ in $A^*(gl(q, \boldsymbol{R}))$ belongs to $A^*(gl(q, \boldsymbol{R}), 0(q))$ if and only if $\omega$ is $0(q)$-invariant and $\omega|_{0(q)} = 0$. We shall construct a ring homomorphism $\phi^*_{F,L}\colon H^*(gl(q, \boldsymbol{R}), 0(q)) \to H^*(L)$.

**Lemma 7.6.** *Let $(g_{\beta\alpha})$ denote the transition functions of the normal bundle $\nu L$, and let $\theta^L_\alpha$ denote the local connection form of the leaf connection on $\nu L$. Then*

$$\theta_\beta^L = dg_{\beta\alpha}\cdot g_{\beta\alpha}^{-1} + g_{\beta\alpha}\theta_\alpha^L g_{\beta\alpha}^{-1} .$$

*Proof.* Let $s_i^1, \cdots, s_i^q$ be a local basis of sections for $\nu L$, and let $s_i = \begin{bmatrix} s_i^1 \\ \vdots \\ s_i^q \end{bmatrix}$, $i = \alpha, \beta$. Then locally, $s_\beta = g_{\beta\alpha}\cdot s_\alpha$. Now let $\nabla^L$ denote the leaf connection on $\nu L$. Again locally we have

$$\begin{aligned}[dg_{\beta\alpha}(X) + g_{\beta\alpha}\cdot\theta_\alpha^L(X)]\cdot s_\alpha &= [X(g_{\beta\alpha}) + g_{\beta\alpha}\theta_\alpha^L(X)]\cdot s_\alpha \\ &= \nabla_X^L(g_{\beta\alpha}\cdot s_\alpha) = \nabla_X^L(s_\beta) \\ &= \theta_\beta^L(X)\cdot s_\beta = (\theta_\beta^L(X)\cdot g_{\beta\alpha})\cdot s_\alpha ,\end{aligned}$$

and therefore $\theta_\beta^L = dg_{\beta\alpha}\cdot g_{\beta\alpha}^{-1} + g_{\beta\alpha}\cdot\theta_\alpha^L\cdot g_{\beta\alpha}^{-1}$ since $s_\alpha^1, \cdots, s_\alpha^q$ is a local basis of sections. q.e.d.

Now since every $q$-dimensional vector bundle is reducible to an $0(q)$-bundle, we can assume that the normal bundle $\nu L$ is already an $0(q)$-bundle. Hence the transition functions $(g_{\beta\alpha})$ map open sets in $L$ into $0(q)$. Let $x$ denote the $q \times q$ matrix whose entries are the coordinate functions of $GL(q, \boldsymbol{R})$, and let $\sigma$ denote the $q \times q$ matrix whose entries are the canonical generators of $A^*(gl(q, \boldsymbol{R}))$. Then

$$\sigma = dx\cdot x^{-1} . \tag{7.5}$$

Moreover, $A^*(gl(q, \boldsymbol{R}), 0(q))$ is generated by linear $0(q)$-invariant polynomials in $\sigma$ which vanish on $0(q)$. Let $p$ be such a polynomial. Then by Lemma 7.4 and (7.5)

$$\begin{aligned} p(\theta_\beta^L) &= p(dg_{\beta\alpha}\cdot g_{\beta\alpha}^{-1} + g_{\beta\alpha}\theta_\alpha^L g_{\beta\alpha}^{-1}) = p(dg_{\beta\alpha}\cdot g_{\beta\alpha}^{-1}) + p(g_{\beta\alpha}\theta_\alpha^L g_{\beta\alpha}^{-1}) \\ &= p(g_{\beta\alpha}^*(\sigma)) + p(\theta_\alpha^L) = g_{\beta\alpha}^* p(\sigma) + p(\theta_\alpha^L) = p(\theta_\alpha^L) .\end{aligned} \tag{7.6}$$

Let $\phi_{F,L}(p(\sigma))$ denote the form in $A^*(L)$ which is given locally by $p(\theta_\alpha^L)$. Then by (7.6), $\phi_{F,L}(p(\sigma))$ is a well-defined form on $L$. Hence there is a cochain map $\phi_{F,L}\colon A^*(gl(q, \boldsymbol{R}), 0(q)) \to A^*(L)$. Now all of the results in this section remain valid if we replace $H^*(gl(q, \boldsymbol{R}))$ by $H^*(gl(q, \boldsymbol{R}), 0(q))$, $E(h_1, \cdots, h_q)$ by $E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1})$, and flat connections by Riemannian connections. Moreover the proofs are essentially the same.

## 8. Examples

In this section we shall use the holonomy homomorphism $\phi_{F,L}^*\colon H^*(gl(q, \boldsymbol{R})) \to H^*(L)$ to construct foliations which have leaves with nontrivial holonomy rings. Moreover we shall actually compute the holonomy rings of these leaves with respect to these foliations.

To begin, suppose that $\omega_1, \cdots, \omega_q$ are $q$ global independent 1-forms on a manifold $M$. Suppose further that there exist 1-forms $\tau_{ji}$ such that

$$d\omega_i = \sum_j \omega_j \wedge \tau_{ji} \ .$$

Let $F$ be the subbundle of $T(M)$ which consists of all the tangent vectors of $M$ which are annihilated by $\omega_1, \cdots, \omega_q$. If $X, Y$ are in $\Gamma(F)$, then

$$\begin{aligned}\omega_i[X, Y] &= X\omega_i(Y) - Y\omega_i(X) - 2d\omega_i(X, Y) \\ &= -2(\sum \omega_j \wedge \tau_{ji})(X, Y) = 0 \ .\end{aligned}$$

Hence $[X, Y]$ is also in $\Gamma(F)$. Therefore $F$ is a foliation on $M$. We shall call $F$ the foliation defined by the 1-forms $\omega_1, \cdots, \omega_q$. Let $\tilde{Z}_1, \cdots, \tilde{Z}_q$ be the vector fields on $M$ dual to the 1-forms $\omega_1, \cdots, \omega_q$. Then $\tilde{Z}_1, \cdots, \tilde{Z}_q$ are a global basis of sections for the normal bundle $\nu F$. Thus the normal bundle of $F$ is trivialized. We shall use this construction in the proof of the following theorem.

**Theorem 8.1.** *Let $L$ be a connected manifold and let $\phi\colon A^*(gl(q, \boldsymbol{R})) \to A^*(L)$ be a cochain map. Then there is a $q$-codimensional foliation $F(\phi)$ with a trivialized normal bundle on the manifold $L \times \boldsymbol{R}^q$ such that*:

a. *$L$ is a leaf of $F(\phi)$*;

b. $\phi_{F(\phi),L} = \phi$;

c. *the holonomy ring of $L$ with respect to $F(\phi)$ is the image of $\phi^*$ in $H^*(L)$.*

*Proof.* Let $\sigma = (\sigma_{ji})$ be the $q \times q$ matrix whose entries are the canonical generators of $A^*(gl(q, \boldsymbol{R}))$. Then the Maurer-Cartan equations are expressed by the matrix equation

$$d\sigma = \sigma^2 \ . \tag{8.1}$$

Let $\tau_{ji} = \phi(\sigma_{ji})$ and $\tau = (\tau_{ji})$. Then

$$d\tau = \tau^2 \ . \tag{8.2}$$

Now for $1 \leq i \leq q$ let $\omega_i$ be the 1-form on $L \times \boldsymbol{R}^q$ given by

$$\omega_i = dx_i + \sum_k x_k \tau_{ki} \ . \tag{8.3}$$

We shall show that

$$d\omega_i = \sum_j \omega_j \wedge \tau_{ji} \ . \tag{8.4}$$

Indeed to prove (8.4) let $\omega = (\omega_1, \cdots, \omega_q)$ and $x = (x_1, \cdots, x_q)$. Then we can express (8.3) by the matrix equation

$$\omega = dx + x \cdot \tau \ . \tag{8.5}$$

Now by differentiating (8.5) and applying (8.2) we obtain

$$d\omega = dx \cdot \tau + x \cdot d\tau = dx \cdot \tau + x \cdot \tau^2 = (dx + x \cdot \tau) \cdot \tau = \omega \cdot \tau \ , \tag{8.6}$$

or equivalently for $1 \leq i \leq q$

$$d\omega_i = \sum \omega_j \wedge \tau_{ji} . \tag{8.7}$$

Hence the 1-forms $\omega_1, \cdots, \omega_q$ define a $q$-codimensional foliation $F(\phi)$ with a trivialized normal bundle on the manifold $L \times R^q$. Moreover $\omega_1, \cdots, \omega_q$ vanish on $L$, so that $L$ is a leaf of $F(\phi)$. Now by Proposition 3.7, $\tau$ is the connection form of the leaf connection of $\nu L$ with respect to the given trivialization. Thus by construction the cochain map $\phi_{F(\phi),L}: A^*(gl(q, R)) \to A^*(L)$ is defined by

$$\phi_{F(\phi),L}(\sigma_{ji}) = \tau_{ji} . \tag{8.8}$$

which implies that $\phi_{F(\phi),L} = \phi$. Hence the holonomy ring of $L$ with respect to $F(\phi)$ is the image of $\phi^*$ in $H^*(L)$. q.e.d.

Thus to create examples of foliations which have leaves with nontrivial holonomy rings, we need only construct cochain maps $\phi: A^*(gl(q, R)) \to A^*(L)$ such that the image of the homomorphism $\phi^*: H^*(gl(q, R)) \to H^*(L)$ is nontrivial.

**Example 8.2.** Let $\mu$ be a closed 1-form in $A^*(S^1)$ which is a representative of a nonzero class in $H^1(S^1)$, and let $\sigma$ be the canonical generator of $A^*(gl(1, R))$. Define a cochain map $\phi: A^*(gl(1, R)) \to A^*(S^1)$ by letting $\phi(\sigma) = \mu$. Then the homomorphism $\phi^*: H^*(gl(1, R)) \to H^*(S^1)$ is an isomorphism. Hence $F(\phi)$ is a 1-codimensional foliation on $S^1 \times R$, $S^1$ is a leaf of $F(\phi)$, and the holonomy ring of $S^1$ with respect to the foliation $F(\phi)$ is isomorphic to $E(h_1)$.

**Example 8.3.** Let $GL^+(q, R)$ denote the subgroup of $GL(q, R)$ which consists of all the matrices which have positive determinants. Then

$$\begin{aligned} H^*(GL^+(q, R)) &= E(h_2, h_4, \cdots, h_{q-2}, e_q) \qquad \text{for even } q \\ &= E(h_2, h_4, \cdots, h_{q-1}) \qquad \text{for odd } q , \end{aligned}$$

where degree $h_{2j} = 4j - 1$ and degree $e_q = q - 1$. Let $\psi: A^*(gl(q, R)) \to A^*(GL^+(q, R))$ be the inclusion map, and let $e: E(h_1, \cdots, h_q) \to H^*(gl(q, R))$ be the isomorphism described in § 7. Then $\psi^* \circ e: E(h_2, h_4, \cdots, h_{2[(q-1)/2]}) \to H^*(GL^+(q, R))$ is a monomorphism and $\psi^* \circ e: E(h_1, h_3, \cdots, h_{2[(q+1)/2]-1}) \to H^*(GL^+(q, R))$ is the zero map. Hence $F(\psi)$ is a $q$-codimensional foliation with a trivialized normal bundle on $GL^+(q, R) \times R^q$, $GL^+(q, R)$ is a leaf of $F(\psi)$, and the holonomy ring of $GL^+(q, R)$ with respect to the foliation $F(\psi)$ is isomorphic to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$. We shall denote the foliation $F(\psi)$ by $F(GL^+(q, R))$.

**Theorem 8.4.** *Let $G$ be a connected Lie subgroup of $GL^+(q, R)$ and let $i: G \to GL^+(q, R)$ be the inclusion map. Then there is a $q$-codimensional foliation $F(G)$ with a trivialized normal bundle on the manifold $G \times R^q$ such that $G$ is a leaf of $F(G)$, and the holonomy ring of $G$ with respect to the* *foliation $F(G)$ is equal to the image of the homomorphism $i^*: H^*(GL^+(q, \boldsymbol{R})) \to H^*(G)$ restricted to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$.*

*Proof.* The inclusion $i \times \mathrm{id}: G \times \boldsymbol{R}^q \to GL^+(q, \boldsymbol{R}) \times \boldsymbol{R}^q$ is transverse to the foliation $F(GL^+(q, \boldsymbol{R}))$. Let $F(G) = (i \times \mathrm{id})^*[F(GL^+(q, \boldsymbol{R}))]$. Then by construction $F(G)$ is a $q$-codimensional foliation with a trivialized normal bundle on $G \times \boldsymbol{R}^q$, and $G$ is a leaf of $F(G)$. Now by Proposition 5.5 the holonomy ring of $G$ with respect to the foliation $F(G)$ is equal to the image of the homomorphism $i^*: H^*(GL^+(q, \boldsymbol{R})) \to H^*(G)$ restricted to the holonomy ring of $GL^+(q, \boldsymbol{R})$ with respect to the foliation $F(GL^+(q, \boldsymbol{R}))$. Moreover by Example 8.3 the holonomy ring of $GL^+(q, \boldsymbol{R})$ with respect to the foliation $F(GL^+(q, \boldsymbol{R}))$ is isomorphic to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$ of $H^*(GL^+(q, \boldsymbol{R}))$. Therefore the holonomy ring of $G$ with respect to the foliation $F(G)$ is equal to the image of the homomorphism $i^*: H^*(GL^+(q, \boldsymbol{R})) \to H^*(G)$ restricted to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$.

**Example 8.5.** The inclusion $i: SO_q \to GL^+(q, \boldsymbol{R})$ is a homotopy equivalence. Hence the image of the homomorphism $i^*: H^*(GL^+(q, \boldsymbol{R})) \to H^*(SO^q)$ restricted to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$ of $H^*(GL^+(q, \boldsymbol{R}))$ is isomorphic to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$ of $H^*(SO_q)$. Therefore by Theorem 8.4, $F(SO_q)$ is a $q$-codimensional foliation with a trivialized normal bundle on $SO_q \times \boldsymbol{R}^q$, $SO_q$ is a leaf of $F(SO_q)$, and the holonomy ring of $SO_q$ with respect to the foliation $F(SO_q)$ is isomorphic to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$ of $H^*(SO_q)$.

**Example 8.6.** There is an inclusion $i: U(q) \to GL^+(2q, \boldsymbol{R})$. Moreover $H^*(U(q))$ is isomorphic to the ring $E(h_1, \cdots, h_q)$, and the image of the homomorphism $i^*: H^*(GL^+(2q, \boldsymbol{R})) \to H^*(U(q))$ restricted to the subring $E(h_2, h_4, \cdots, h_{2q-2})$ of $H^*(GL^+(2q, \boldsymbol{R}))$ is isomorphic to the subring $E(h_2, h_4, \cdots, h_{2[q/2]})$ of $H^*(U(q))$. Therefore by Theorem 8.4, $F(U(q))$ is a $2q$-codimensional foliation with a trivialized normal bundle on $U(q) \times \boldsymbol{R}^{2q}$, $U(q)$ is a leaf of $F(U(q))$, and the holonomy ring of $U(q)$ with respect to the foliation $F(U(q))$ is isomorphic to the subring $E(h_2, h_4, \cdots, h_{2[q/2]})$ of $H^*(U(q))$.

By Examples 8.3 and 8.5, the homomorphism $\phi_q^*: E(h_1, \cdots, h_q) \to R(CF_q^0L)$ restricted to the subring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$ is a monomorphism. Thus every class in the ring $E(h_2, h_4, \cdots, h_{2[(q-1)/2]})$ represents a nontrivial characteristic class on the category $CF_q^0L$. In § 11 we shall prove that the homomorphism $\phi_q^*$ is actually a monomorphism. Hence we shall be able to conclude that every characteristic class in the holonomy ring of $CF_q^0L$ is a nontrivial characteristic class.

## 9. The product formula

Let $F_i$ be a $q_i$-codimensional foliation on a manifold $M_i$, $i = 1, 2$. In addition, let $L_i$ be a leaf of $F_i$ with a trivialized normal bundle $\nu L_i$, $i = 1, 2$. Then $F_1 \times F_2$ is a $(q_1 + q_2)$-codimensional foliation on the manifold $M_1 \times M_2$.

Moreover $L_1 \times L_2$ is a leaf of $F_1 \times F_2$, and the normal bundle $\nu(L_1 \times L_2) = \nu L_1 \times \nu L_2$ is also a trivialized bundle. The product formula for holonomy classes asserts that

$$h_j(F_1 \times F_2, L_1 \times L_2) = h_j(F_1, L_1) \otimes 1 + 1 \otimes h_j(F_2, L_2) \ .$$

Here it is tacitly understood that $h_n(F, L) = 0$ whenever the codimension of $F$ is less than $n$.

To prove the product formula, we need a few preparatory lemmas. Let $A$ be a $p \times p$ matrix, and $B$ a $q \times q$ matrix. Define $A \times B$ to be the $(p+q) \times (p+q)$ matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

**Lemma 9.1.** $c_n(A \times B) = \sum_{i+j=n} c_i(A)c_j(B)$.

*Proof.* Let $I_m$ denote the $m \times m$ identity matrix. Then directly from the definition of the polynomial $c_n$, we have

$$\begin{aligned} 1 + \sum_n t^n c_n(A \times B) &= \det [I_{p+q} + t(A \times B)] \\ &= \det [(I_p + tA) \times (I_q + tB)] \\ &= \det [([I_p + tA] \times I_q) \cdot (I_p \times [I_q + tB]) \\ &= \det (I_p + tA) \cdot det\, (I_q + tB) \\ &= \Big[1 + \sum_i t^i c_i(A)\Big]\Big[1 + \sum_j t^j c_j(B)\Big] \\ &= 1 + \sum_n \sum_{i+j=n} t^n c_i(A)c_j(B) \ . \end{aligned}$$

Hence $c_n(A \times B) = \sum_{i+j=n} c_i(A)c_j(B)$.

**Lemma 9.2.** $e_{n,t}(A \times B) = \sum_{\substack{i+j=n \\ r+s=t}} e_{i,r}(A)e_{j,s}(B)$.

*Proof.* Directly from the definition of $e_{n,t}$ we have

$$\begin{aligned} \sum_t y^t e_{n,t}(A \times B) &= c_n[y(A \times B) + (A \times B)^2] \\ &= c_n[(yA + A^2)] \times (yB + B^2)] \\ &= \sum_{i+j=n} c_i(yA + A^2)c_j(yB + B^2) \\ &= \sum_{i+j=n} \Big[\Big(\sum_r y^r e_{i,r}(A)\Big)\Big(\sum_s y^s e_{j,s}(B)\Big)\Big] \\ &= \sum_t \sum_{\substack{i+j=n \\ r+s=t}} y^t e_{i,r}(A)e_{j,s}(B) \ . \end{aligned}$$

Hence $e_{n,t}(A \times B) = \sum_{\substack{i+j=n \\ r+s=t}} e_{i,r}(A)e_{j,s}(B)$. q.e.d.

Now suppose that $F_i$ is a $q_i$-codimensional foliation on $M_i$, and that $L_i$ is a leaf of $F_i$ with a trivialized normal bundle $\nu L_i$, $i = 1, 2$. If $\nabla_i^B$ is a foliation connection on $\nu F_i$, $i = 1, 2$, then it is not hard to show that $\nabla_1^B \times \nabla_2^B$ is a foliation connection on $\nu(F_1 \times F_2)$, [5]. Let $\alpha_i \colon L_i \to M_i$ be the immersion of $L_i$ in $M_i$; then

$$\nabla^{L_1 \times L_2} = (\alpha_1 \times \alpha_2)^*(\nabla_1^B \times \nabla_2^B) = \alpha_1^*(\nabla_1^B) \times \alpha_2^*(\nabla_2^B) = \nabla^{L_1} \times \nabla^{L_2} .$$

Therefore, if $\theta^{L_1}, \theta^{L_2}$ and $\theta^{L_1 \times L_2}$ are the global connection forms of the leaf connections on the bundles $\nu L_1$, $\nu L_2$, and $\nu(L_1 \times L_2)$ with respect to the given trivializations, then $\theta^{L_1 \times L_2} = \theta^{L_1} \times \theta^{L_2}$.

**Proposition 9.3.** $e_{2n-1}(\theta^{L_1 \times L_2}) = e_{2n-1}(\theta^{L_1}) \otimes 1 + 1 \otimes e_{2n-1}(\theta^{L_2})$ $+$ *coboundary.*

*Proof.* Recall from § 7 that

$$d\theta^{L_i} = (\theta^{L_i})^2 , \qquad i = 1, 2 .$$

Therefore

$$e_{j,0}(\theta^{L_i}) = c_j[(\theta^{L_i})^2] = c_j(d\theta^{L_i}) , \qquad i = 1, 2 .$$

Now by Lemma 9.2 we have

$$\begin{aligned} e_{2n-1}(\theta^{L_1 \times L_2}) &= e_{2n-1}(\theta^{L_1} \times \theta^{L_2}) = N_n e_{n,1}(\theta^{L_1} \times \theta^{L_2}) \\ &= N_n e_{n,1}(\theta^{L_1}) \otimes 1 + 1 \otimes N_n e_{n,1}(\theta^{L_2}) \\ &\quad + \sum_{i+j=n} [N_n e_{i,1}(\theta^{L_1}) \otimes e_{j,0}(\theta^{L_2}) + e_{i,0}(\theta^{L_1}) \otimes N_n e_{j,1}(\theta^{L_2})] \\ &= e_{2n-1}(\theta^{L_1}) \otimes 1 + 1 \otimes e_{2n-1}(\theta^{L_2}) \\ &\quad + \sum_{i+j=n} \Big[ \frac{N_n}{N_i} e_{2i-1}(\theta^{L_1}) \otimes c_j(d\theta^{L_2}) \\ &\qquad + c_i(d\theta^{L_1}) \otimes \frac{N_n}{N_i} e_{2j-1}({}_j\theta^{L_2}) \Big] . \end{aligned}$$

However by Propositions 5.4 and 7.1, $e_{2j-1}(\theta^{L_1})$ is a closed form. Moreover, $c_j(d\theta^{L_2})$ is exact. Therefore $e_{2i-1}(\theta^{L_1}) \otimes c_j(d\theta^{L_2})$ is a coboundary. Similarly, $c_i(d\theta^{L_1}) \otimes e_{2j-1}(\theta^{L_2})$ is a coboundary. Hence $e_{2n-1}(\theta^{L_1 \times L_2}) = e_{2n-1}(\theta^{L_1}) \otimes 1 + 1 \otimes e_{2n-1}(\theta^{L_2}) +$ coboundary.

**Theorem 9.4.** $h_j(F_1 \times F_2, L_1 \times L_2) = h_j(F_1, L_1) \otimes 1 + 1 \otimes h_j(F_2, L_2)$.

*Proof.* By Theorem 7.2 and Proposition 9.3 we have

$$\begin{aligned} h_j(F_1 \times F_2, L_1 \times L_2) &= \{e_{2j-1}(\theta^{L_1 \times L_2})\} \\ &= \{e_{2j-1}(\theta^{L_1}) \otimes 1 + 1 \otimes e_{2j-1}(\theta^{L_2})\} \\ &= h_j(F_1, L_1) \otimes 1 + 1 \otimes h_j(F_2, L_2) . \quad \text{q.e.d.} \end{aligned}$$

Now let $F_i$ be a $q_i$-codimensional foliation on a manifold $M$, $i = 1, 2$. In addition, let $L_i$ be a leaf of $F_i$ with a trivialized normal bundle $\nu L_i$, and let $\alpha_i: L_i \to M$ be the immersion of $L_i$ in $M$, $i = 1, 2$. If the diagonal map $\Delta: M \to M \times M$ is transverse to the foliation $F_1 \times F_2$, then $\Delta^*(F_1 \times F_2)$ is a $(q_1 + q_2)$-codimensional foliation on $M$. We shall call the foliation $\Delta^*(F_1 \times F_2)$ the intersection or the sum of $F_1$ and $F_2$, and we shall denote this foliation by $F_1 \oplus F_2$. A leaf $L$ of $F_1 \oplus F_2$ is mapped by $\Delta$ into $L_1 \times L_2$ if and only if $L$ is immersed in $\alpha_1(L_1) \cap \alpha_2(L_2)$. We shall denote a leaf of $F_1 \oplus F_2$ which is immersed in $\alpha_1(L_1) \cap \alpha_2(L_2)$ by $L_1 \cap_\Delta L_2$. Let $\Delta_1: L_1 \cap_\Delta L_2 \to L_1 \times L_2$ denote the map induced by $\Delta$ and let $l_i: L_1 \cap_\Delta L_2 \to L_i$ denote the map induced by the immersion of $L_1 \cap_\Delta L_2$ in $M$. If $\Pi_i: L_1 \times L_2 \to L_i$ is the projection map, then $\Pi_i \circ \Delta_1 = l_i$, $i = 1, 2$.

**Corollary 9.5.** $h_j(F_1 \oplus F_2, L_1 \cap_\Delta L_2) = l_1^* h_j(F_1, L_1) + l_2^* h_j(F_2, L_2)$.

*Proof.* By the naturality of the class $h_j$, we have

$$\begin{aligned} h_j(F_1 \oplus F_2, L_1 \cap_\Delta L_2) &= h_j(\Delta^*(F_1 \times F_2), L_1 \cap_\Delta L_2) \\ &= \Delta_1^* h_j(F_1 \times F_2, L_1 \times L_2) \\ &= \Delta_1^*[h_j(F_1, L_1) \otimes 1 + 1 \otimes h_j(F_2, L_2)] \\ &= \Delta_1^* \circ \Pi_1^* h_j(F_1, L_1) + \Delta_1^* \circ \Pi_2^* h_j(F_2, L_2) \\ &= l_1^* h_j(F_1, L_1) + l_2^* h_j(F_2, L_2) \,. \end{aligned}$$

**Corollary 9.6.** *Let $F_i$ be a $q_i$-codimensional foliation with a trivialized normal bundle $i = 1, 2$, and let $q = \max(q_1, q_2)$. If $L$ is a leaf of $F_1 \otimes F_2$ and if $j > q$, then $h_j(F_1 \oplus F_2, L) = 0$, so that the holonomy ring of each leaf of $F_1 \oplus F_2$ vanishes in dimensions greater than $q^2$.*

*Proof.* If $L$ is a leaf of $F_1 \oplus F_2$, then there are leaves $L_i$ of $F_i$, $i = 1, 2$ such that $L = L_1 \cap_\Delta L_2$. Therefore this result follows immediately from Corollary 9.5.

**Corollary 9.7.** *Let $F$ be a $(q_1 + q_2)$-codimensional foliation with a trivialized normal bundle, and let $q = \max(q_1, q_2)$. If there exist a leaf $L$ of $F$ and an integer $j > q$ for which $h_j(F, L) \neq 0$, then $F$ is not the intersection of a $q_1$-codimensional foliation with a trivialized normal bundle and a $q_2$-codimensional foliation with a trivialized normal bundle.*

*Proof.* This result follows immediately from Corollary 9.5.

The results in this section remain valid if we replace the holonomy classes $h_j$ on the category $CF_qL^0$ $(CF_q^0L)$ by the holonomy classes $h_{2j-1}$ on the category $CF_qL$. Moreover the proofs are essentially the same.

Corollaries 9.6 and 9.7 provide necessary conditions in terms of the holonomy invariants of the leaves for a foliation to be the intersection of two other foliations of lesser codimensions. However it must be stressed that these results apply only to the specific foliation and not necessarily to its homotopy class. Indeed it may happen that $F$ is homotopic to a foliation which is the intersection of a $q_1$-codimensional foliation and a $q_2$-codimensional foliation even though $F$ itself cannot be formed by such an intersections. Such a foliation may exist because the holonomy classes are invariants of a foliation but not not of its homotopy class (see the closing remark of § 6).

## 10. Vector bundles with discrete structure group

A $q$-dimensional vector bundle $\eta$ over a manifold $L$ is a pair $(U_i, h_{ji})$ such that:

1. $(U_i)$ is an open cover of $L$.
2. $h_{ji}\colon U_i \cap U_j \to GL(q, \boldsymbol{R})$ are smooth maps.
3. $h_{ki}(x) = h_{kj}(x) \cdot h_{ji}(x)$ for every $x$ in $U_i \cap U_j \cap U_k$.

The total space $E(\eta)$ is the space $\bigcup_i U_i \times \boldsymbol{R}^q \times i / \sim$ where $(x, t, i) \sim (x', t', j)$ if and only if $x' = x$ and $t' = h_{ji}(x) \cdot t$.

Let $\tilde{G}L(q, \boldsymbol{R})$ denote $GL(q, \boldsymbol{R})$ with the discrete topology. A vector bundle $(U_i, h_{ji})$ is said to have discrete structure group if the functions $\tilde{h}_{ji}\colon U_i \cap U_j \to \tilde{G}L(q, \boldsymbol{R})$ induced by the maps $h_{ji}\colon U_i \cap U_j \to GL(q, \boldsymbol{R})$ are continuous. We shall adopt the following notation:

a. $\widetilde{VBM}_q$ will denote the category whose objects are $q$-dimensional vector bundles with discrete structure group over connected manifolds and whose morphisms are bundle maps.

b. $\widetilde{VBM}_q^0$ will denote the category whose objects are the $q$-dimensional vector bundles with discrete structure group over connected manifolds which are trivialized as vector bundles (but are not necessarily trivialized as vector bundles with discrete structure group) and whose morphisms are bundle maps compatible with the given trivializations. Presently, we shall show that the categories $\widetilde{VBM}_q$ and $CF_qL$ are intimately related.

Let $F$ be a $q$-codimensional foliation on a manifold $M$. By Frobenius' theorem [5, pp. 88–94], [19, pp. 132–135] there is an indexed triple $(U, f_U, H_{VU})$ with the following properties:

(a) $(U)$ is an open cover of $M$,
(b) $f_U\colon U \to \boldsymbol{R}^q$ and the rank of $df_U$ is $q$,
(c) $H_{VU}$ is a local diffeomorphism of $\boldsymbol{R}^q$,
(d) $f_V = H_{VU} \circ f_U$ on $U \cap V$,
(e) $f_{U|L}$ is locally constant on each leaf of $F$,
(f) $dH_{VU|f_U}$ are the transition functions of the normal bundle $\nu F$.

Conversely, given an indexed triple $(U, f_U, H_{VU})$ which satisfies properties (a)-(d), there exists a unique $q$-codimensional foliation $F$ on the manifold $M$ with respect to which properties (e) and (f) are valid. If the foliation $F$ corresponds to the indexed triple $(U, f_U, H_{VU})$, then we shall often abuse notation and write $F = (U, f_U, H_{VU})$. Now let $\tilde{\boldsymbol{R}}^q$ denote $\boldsymbol{R}^q$ with the discrete topology, and let $F = (U, f_U, H_{VU})$ be a $q$-codimensional foliation on a manifold $M$. The leaf topology on $M$ with respect to $F$ is the coarsest topology on $M$ which contains the original topology on $M$ and which also satisfies the condition that each function $f_U: U \to \tilde{R}^q$ is continuous. Let $\tilde{M}$ denote $M$ with the leaf topology with respect to $F$. Then the leaves of $F$ are the connected components of $\tilde{M}$ [5, p. 92].

Now let $F = (U, f_U, H_{VU})$ be a $q$-codimensional foliation on a manifold $M$, and let $L$ be a leaf of $F$. Then by properties (e) and (f) above, the transition functions of the normal bundle $\nu L$ are locally constant. Hence the normal bundle $\nu L$ has discrete structure group. Therefore there is a natural transformation

$$\tilde{\tilde{\nu}}_q : CF_qL \to \widetilde{VBM}_p$$

given by

$$\tilde{\tilde{\nu}}_q(F, L) = \nu L \ .$$

On the other hand, if $\eta = (U_i, h_{ji})$ is a vector bundle with discrete structure group over a connected manifold $L$, then there is a natural foliation on the total space $E(\eta)$. Indeed, let $W_i$ denote the image of $U_i \times \boldsymbol{R}^q \times i$ in $E(\eta)$ and define $p_i: W_i \to \boldsymbol{R}^q$ by $p_i(x, t, i) = t$. In addition since the map $h_{ji}$ is locally constant, we can define a linear map $H_{ji}: \boldsymbol{R}^q \to \boldsymbol{R}^q$ by letting $H_{ji}(t) = h_{ji} \cdot t$. Now the indexed triple $(W_i, p_i, H_{ji})$ satisfies properties (a)-(d) above. We shall call the foliation which corresponds to this triple the horizontal foliation on $E(\eta)$ and we shall denote this foliation by $H(E(\eta))$. The foliation $H(E(\eta))$ has the following properties:

1. the zero section immerses $L$ in $E(\eta)$ as a leaf of $H(E(\eta))$,
2. the normal bundle of $L$ in $E(\eta)$ is isomorphic to $\eta$,
3. if $f: L_1 \to L$ is a smooth map, and $\bar{f}: E(f^*(\eta)) \to E(\eta)$ is the map induced by $f$, then $H(E(f^*(\eta))) = \bar{f}^* H(E(\eta))$.

Hence there is a natural embedding

$$T_q : \widetilde{VBM}_q \to CF_qL$$

given by

$$T_q(\eta) = (H(E(\eta)), L) \ .$$

Moreover the diagram

$$\begin{tikzcd} \widetilde{VBM}_q \arrow[r, "T_q"] \arrow[dr, "\mathrm{id}"'] & CF_qL \arrow[d, "\tilde{\tilde{\nu}}_q"] \\ & \widetilde{VBM}_q \end{tikzcd}$$

commutes.

**Theorem 10.1.** *If $\eta$ is a vector bundle with discrete structure group over a connected manifold, then the real Pontryagin ring is $\eta$ is trivial.*

*Proof.* Let $\eta$ be a vector bundle with discrete structure group over a connected manifold $L$, and let $\nu L$ denote the normal bundle of $L$ in the total space $E(\eta)$. If $p_j$ is the $j$th Pontryagin class, then $p_j(\nu L) = 0$ by the vanishing theorem for leaves. However, $\eta$ is isomorphic to $\nu L$; hence $p_j(\eta) = 0$. q.e.d.

Of course, the preceding result is well-known; see Milnor [18] and Kamber and Tondeur [10]. Since the normal bundle of a leaf has discrete structure group, Theorem 10.1 is actually equivalent to the vanishing theorem for leaves. In § 11 we shall see that the vanishing of the real Pontryagin ring leads directly to a construction of secondary characteristic classes on the category $\widetilde{VBM}_q$ ($\widetilde{VBM}^0_q$).

**Definition.** A characteristic class on the category $\widetilde{VBM}_q$ ($\widetilde{VBM}^0_q$) is a transformation $\gamma$ which associates to each $q$-dimensional vector bundle $\eta$ with discrete structure group (which is trivialized as a vector bundle) over a connected manifold $L$ a class $\gamma(\eta)$ in $H^*(L)$ such that if $L_1$ is a connected manifold and if $f: L_1 \to L$ is a smooth map, then $\gamma(f^*(\eta)) = f^*\gamma(\eta)$.

As usual, the ring of characteristic classes on the category $\widetilde{VBM}_q$ ($\widetilde{VBM}^0_q$) will be denoted by $R(\widetilde{VBM}_q)$ ($R(\widetilde{VBM}^0_q)$). If $\eta$ is a $q$-dimensional vector bundle with discrete structure group over a connected manifold $L$, then there is a ring homomorphism $R_\eta: R(\widetilde{VBM}_q) \to H^*(L)$ given by $R_\eta(\gamma) = \gamma(\eta)$.

**Theorem 10.2.** *The homomorphism $\tilde{\nu}^*_q: R(\widetilde{VBM}_q) \to R(CF_qL)$ is a monomorphism.*

*Proof.* The diagram

$$\begin{tikzcd} R(\widetilde{VBM}_q) \arrow[r, "\tilde{\nu}^*_q"] \arrow[dr, "\mathrm{id}"'] & R(CF_qL) \arrow[d, "T^*_q"] \\ & R(\widetilde{VBM}_q) \end{tikzcd}$$

commutes. Therefore $\tilde{\nu}^*_q$ must be a monomorphism. q.e.d.

Of course, Theorem 10.2 remains valid if we replace $\widetilde{VBM}_q$ by $\widetilde{VBM}^0_q$ and $CF_qL$ by $CF_qL^0$. Hence to construct characteristic classes on the category $CF_qL$ ($CF_qL^0$), we need only construct characteristic classes on the category $\widetilde{VBM}_q$ ($\widetilde{VBM}^0_q$). In the following section we shall give an explicit construction of such classes.

## 11. The holonomy homomorphism for vector bundles with discrete structure group

In this section we shall construct characteristic classes on the categories $\widetilde{VBM}_q^0$ and $\widetilde{VBM}_q$. In addition, we shall show that when a vector bundle is the normal bundle of a leaf these classes coincide with the holonomy classes of the leaf. Thus, since the holonomy classes of a leaf depend only on its normal bundle, we can conclude that the holonomy classes of a leaf are essentially linear invariants. The technique which we shall employ here is similar to the method used by Haefliger in [9] to construct characteristic classes for a $K$-fibré $G$-feuilleté, Another equivalent construction of characteristic classes for vector bundles with discrete structure group is given by Kamber and Tondeur in [15] and [16].

Let $\eta = (U_i, h_{ji})$ be a $q$-dimensional vector bundle with discrete structure group over a connected manifold $L$. Suppose further that $\eta$ is a trivialized vector bundle. Then there exist maps $f_i\colon U_i \to GL(q, \boldsymbol{R})$ such that

$$f_j(x) = h_{ji}(x)\cdot f_i(x) \tag{11.1}$$

for every $x$ in $U_i \cap U_j$. The collection of functions $(f_i)$ is also called a trivialization of the vector bundle $\eta$. Now the transition functions $(h_{ji})$ of $\eta$ are locally constant; that is, there is an element $h_{ji}$ of $GL(q, \boldsymbol{R})$ such that locally $h_{ji}(x) = h_{ji}$. Therefore, if $l_g\colon GL(q, \boldsymbol{R}) \to GL(q, \boldsymbol{R})$ denotes left multiplication by an element $g$ of $GL(q, \boldsymbol{R})$, then locally

$$f_j = l_{h_{ji}} \circ f_i \ . \tag{11.2}$$

Let $A^*(gl(q, \boldsymbol{R}))$ denote the collection of all left-invariant forms of $GL(q, \boldsymbol{R})$, and let $\omega$ be a form in $A^*(gl(q, \boldsymbol{R}))$. Then by (11.2) we have

$$f_j^*(\omega) = f_i^*(\omega) \tag{11.3}$$

on $U_i \cap U_j$. Let $\phi_\eta(\omega)$ denote the form in $A^*(L)$ which is given locally by $f_i^*(\omega)$. By (11.3), $\phi_\eta(\omega)$ is a well-defined form on $L$. Therefore there is a cochain map $\phi_\eta\colon A^*(gl(q, \boldsymbol{R})) \to A^*(L)$. We shall show that the image of the homomorphism

$$\phi_\eta^*\colon H^*(gl(q, \boldsymbol{R})) \to H^*(L)$$

induced by the cochain map $\phi_\eta$ is actually the holonomy ring of $L$ with respect to the horizontal foliation on $E(\eta)$ and the trivialization $(f_i)$.

**Proposition 11.1.** *Let $\sigma$ be the $q \times q$ matrix whose entries are the canonical generators of $A^*(gl(q, \boldsymbol{R}))$. Then the matrix $\phi_\eta(\sigma)$ is the connection form of the leaf connection on $\nu L$ with respect to the horizontal foliation on $E(\eta)$ and the trivialization $(f_i)$.*

*Proof.* Since $\eta = (U_i, h_{ji})$ and since $(f_i)$ is a trivialization of $\eta$, we have

$$f_j = h_{ji}\cdot f_i \ . \tag{11.4}$$

Moreover by construction

$$\phi_\eta(\sigma)|_{U_i} = f_i^*(\sigma) \ . \tag{11.5}$$

Now as in § 1, let $H(E(\eta)) = (W_i, p_i, H_{ji})$ be the horizontal foliation on $E(\eta)$. Then

$$p_j = H_{ji}\circ p_i \ . \tag{11.6}$$

Define $\Pi_i\colon W_i \to U_i$ by letting $\Pi_i(x, t, i) = x$; then

$$dH_{ji}|_{p_i} = h_{ji}\circ \Pi_i \ . \tag{11.7}$$

Let $r_k\colon \boldsymbol{R}^q \to \boldsymbol{R}$ be the projection on the $k$th factor. In addition, let $p_i^k = r_k\circ p_i$ and $dp_i = (dp_i^1, \cdots, dp_i^q)$, and let ${}^tA$ denote the transpose of the matrix $A$. Then applying the chain rule to (11.6) and substituting into (11.7), we obtain

$$dp_j = (dp_i)\,{}^t(dH_{ji}|_{p_i}) = (dp_i)\,{}^t(h_{ji}\circ \Pi_i) \ . \tag{11.8}$$

Let $\omega = (\omega_1, \cdots, \omega_q)$ be the row vector of 1-forms given locally by

$$\omega|_{U_i} = (dp_i)\,{}^t[(f_i\circ \Pi_i)^{-1}] \ . \tag{11.9}$$

Then from (11.4), (11.8), and (11.9) we have

$$\begin{aligned} \omega|_{U_j} &= (dp_j)\,{}^t[(f_j\circ \Pi_j)^{-1}] = (dp_i)\,{}^t(h_{ji}\circ \Pi_i)\,{}^t[(f_j\circ \Pi_j)^{-1}] \\ &= (dp_i)\,{}^t[f_j\circ \Pi_j)^{-1}\cdot (h_{ji}\circ \Pi_j)] = (dp_i)\,{}^t[(f_i\circ \Pi_i)^{-1}] = \omega|_{U_i} \end{aligned} \tag{11.10}$$

on $U_i \cap U_j$. Hence $\omega$ is indeed well-defined globally. Moreover, since $p_i$ is locally constant on the leaves of $H(E(\eta))$, $\omega$ vanishes on the leaves of $H(E(\eta))$. Hence the foliation $H(E(\eta))$ is defined by the 1-forms $\omega_1, \cdots, \omega_q$. Now differentiating (11.9), locally we have

$$\begin{aligned} d\omega|_{U_i} &= (dp_i)\,{}^t[d(f_i\circ \Pi_i)^{-1}] \\ &= (dp_i)\,{}^t[-(f_i\circ \Pi_i)^{-1}\cdot d(f_i\circ \Pi_i)\cdot (f_i\circ \Pi_i)^{-1}] \\ &= [(dp_i)\cdot{}^t[(f_i\circ \Pi_i)^{-1}]]\cdot{}^t[-(f_i\circ \Pi_i)^{-1}\cdot d(f_i\circ \Pi_i)] \\ &= \omega|_{U_i}\cdot{}^t[-(f_i\circ \Pi_i)^{-1}\cdot d(f_i\circ \Pi_i)] \ . \end{aligned} \tag{11.11}$$

Let $\theta^L$ denote the connection form of the leaf connection on $\nu L$ with respect to the horizontal foliation on $E(\eta)$ and the trivialization $(f_i)$. The by (11.11) and Proposition 3.6, (11.12) $$\theta^L|_{U_i} = {}^t[-(f_i \circ \Pi_i)^{-1} \cdot d(f_i \circ \Pi_i)]|_L = {}^t(-f_i^{-1} \cdot df_i) ,$$

since $L$ is identified with the zero section in $E(\eta)$. Finally, let $x$ denote the $q \times q$ matrix whose entries are the coordinate functions of $GL(q, \boldsymbol{R})$. Then

(11.13) $$\sigma = {}^t(-x^{-1}dx) .$$

Therefore from (11.2), (11.12) and (10), we get

(11.14) $$\phi_\eta(\sigma)|_{U_i} = f_i^*(\sigma) = f_i^*[{}^t(-x^{-1}dx)] = {}^t(-f_i^{-1}df_i) = \theta^L|_{U_i} .$$

Hence $\phi_\eta(\sigma)$ is the connection form of the leaf connection on $\nu L$ with respect to the horizontal foliation on $E(\eta)$ and the trivialization $(f_i)$.

**Theorem 11.2.** *Let $\eta$ be a vector bundle with discrete structure group over a connected manifold $L$, and let $(f_i)$ be a trivialization of $\eta$. Then $\phi_\eta^* = \phi_{H(E(\eta)),L}^*$.*

*Proof.* This result follows immediately from Proposition 11.1 and the construction of the homomorphism $\phi_{H(E(\eta)),L}^*$ given in § 7.

The homomorphism $\phi_\eta^*: H^*(gl(q, \boldsymbol{R})) \to H^*(L)$ is called the holonomy homomorphism, and the image of $\phi_\eta^*$ in $H^*(L)$ is called the holonomy ring of $\eta$. If $\gamma$ is a class in $H^*(gl(q, \boldsymbol{R}))$, then we shall let $\gamma(\eta) = \phi_\eta^*(\gamma)$. Now it is easy to show either directly from the definition of $\gamma(\eta)$ or from Theorem 11.2 that $\gamma(g^*(\eta)) = g^*\gamma(\eta)$. Therefore the holonomy classes actually represent characteristic classes on the category $\widetilde{VBM}_q^0$. To summarize, we have the following theorem.

**Theorem 11.3.** *There is a homomorphism $\phi_q^*: H^*(gl(q, \boldsymbol{R})) \to R(\widetilde{VBM}_q^0)$ such that if $\eta$ is a vector bundle over $L$ in the category $\widetilde{VBM}_q^0$ then the diagram*

$$\begin{tikzcd} H^*(gl(q, \boldsymbol{R})) \arrow[r, "\phi_q^*"] \arrow[dr, "\phi_\eta^*"'] & R(\widetilde{VBM}_q^0) \arrow[d, "R_\eta"] \\ & H^*(L) \end{tikzcd}$$

*commutes.*

Let $\eta = (U_i, h_{ji})$ be a vector bundle with discrete structure group over a connected manifold $L$, and let $(f_i)$ be a trivialization of $\eta$. In addition, let $H(E(\eta)) = (W_i, p_i, H_{ji})$ be the horizontal foliation on $E(\eta)$, and define $\Pi_i: W_i \to U_i$ by $\Pi_i(x, t, i) = x$. Then $dH_{ji}|_{p_i} = h_{ji} \circ \Pi_i$; hence the maps $(h_{ji} \circ \Pi_i)$ are the transition functions of the normal bundle $\nu H(E(\eta))$. Therefore $(f_i \circ \Pi_i)$ is a trivialization of $\nu H(E(\eta))$. Thus the natural transformation $T_q: \widetilde{VBM}_q \to CF_qL$ defined by $T_q(\eta) = (H(E(\eta)), L)$ induces a natural transformation $T_q: \widetilde{VBM}_q^0 \to CF_q^0L$. Moreover by Theorem 11.2, $\phi_\eta^* = \phi_{T_q^*(\eta)}^*$.

**Proposition 11.4.** *The diagram*

$$\begin{tikzcd}[column sep=small] & H^*(gl(q,\boldsymbol{R})) \arrow[dl, "\phi_q^*"'] \arrow[dr, "\phi_q^*"] & \\ R(CF_q^0L) \arrow[rr, "T_q^*"] & & R(\widetilde{VBM_q^0}) \end{tikzcd}$$

*commutes.*

*Proof.* Let $\gamma$ be a class in $H^*(gl(q,\boldsymbol{R}))$, and let $\eta$ be a vector bundle in the category $\widetilde{VBM_q^0}$. Then

$$\begin{aligned} T_q^*\phi_q^*(\gamma)(\eta) &= \phi_q^*(\gamma)[T_q(\eta)] = R_{T_q(\eta)}\phi_q^*(\gamma) \\ &= \phi_{T_q(\eta)}^*(\gamma) = \phi_\eta^*(\gamma) = R_\eta\phi_q^*(\gamma) = \phi_q^*(\gamma)(\eta) \ . \end{aligned}$$

Hence $T_q^*\phi_q^* = \phi_q^*$. q.e.d·

The homomorphism $\phi_q^* : H^*(gl(q,\boldsymbol{R})) \to R(\widetilde{VBM_q^0})$ is called the holonomy homomorphism, and the image of $\phi_q^*$ in $R(\widetilde{VBM_q^0})$ is called the holonomy ring of $R(\widetilde{VBM_q^0})$. By Proposition 11.4 the homomorphism $T_q^* : R(CF_q^0L) \to R(\widetilde{VBM_q^0})$ maps the holonomy ring of $R(CF_q^0L)$ into the holonomy ring of $R(\widetilde{VBM_q^0})$.

**Theorem 11.5.** *The homomorphism $\phi_q^* : H^*(gl(q,\boldsymbol{R})) \to R(\widetilde{VBM_q^0})$ is a monomorphism. Hence every characteristic class in the holonomy ring of $R(\widetilde{VBM_q^0})$ is a nontrivial characteristic class.*

*Proof.* See [9].

**Theorem 11.6.** *The homomorphism $\phi_q^* : H^*(gl(q,\boldsymbol{R})) \to R(CF_q^0L)$ is a monomorphism, so that every characteristic class in the holonomy ring of $R(CF_q^0L)$ is a nontrivial characteristic class.*

*Proof.* This result follows immediately from Proposition 11.4 and Theorem 11.5.

Finally, we shall close our discussion of the holonomy ring by showing that the holonomy classes in $R(CF_q^0L)$ are essentially linear invariants.

**Theorem 11.7.** *Let $F$ be a $q$-codimensional foliation with a trivialized normal bundle $\nu F$, and let $L$ be a leaf of $F$. Then $\phi_{F,L}^* = \phi_{\nu L}^*$.*

*Proof.* The proof of this theorem is similar to that of Proposition 11.1. Indeed, let $F = (U, f_U, H_{VU})$, and let $(G_U)$ be the trivialization of the normal bundle $\nu F$. Then

$$G_V = dH_{VU}|_{f_U} \cdot G_U \ , \tag{11.15}$$

$$f_V = H_{VU} \circ f_U \tag{11.16}$$

on $U \cap V$. Now let $\sigma = (\sigma_{ji})$ be the $q \times q$ matrix whose entries are the canonical generators of $A^*(gl(q,\boldsymbol{R}))$, and let $G_{U\cap L} = G_U|_L$. Then $(G_{U\cap L})$ is the trivialization of the normal bundle $\nu L$. Hence by the construction given at the start of this section, we can define a cochain map $\phi\colon A^*(gl(q, \boldsymbol{R})) \to A^*(L)$ by letting $\phi(\sigma_{ji})$ be the 1-form in $A^*(L)$ which is given locally by $G_{U\cap L}(\sigma_{ji})$, that is,

$$\phi(\sigma_{ji})|_{U\cap L} = G^*_{U\cap L}(\sigma_{ji}) \ . \tag{11.17}$$

The form $\phi(\sigma_{ji})$ is globally well-defined, and by construction

$$\phi^*_{\nu L} = \phi^* \ . \tag{11.18}$$

Now we shall show that $\phi^*_{F,L} = \phi^*$. Let $\Pi_j\colon \boldsymbol{R}^q \to \boldsymbol{R}$ be the projection on the $j$th factor. In addition, let $f^j_U = \Pi_j \circ f_U$ and $df_U = (df^1_U, \cdots, df^q_U)$, and let ${}^tA$ denote the transpose of the matrix $A$. Applying the chain rule to (11.16), we obtain

$$df_V = (df_U)\cdot {}^t(dH_{VU}|_{f_U}) \ . \tag{11.19}$$

Let $\omega = (\omega_1, \cdots, \omega_q)$ be the row vector of 1-forms given locally by

$$\omega|_U = (df_U)\cdot {}^t(G_U^{-1}) \ . \tag{11.20}$$

Then from (11.15), (11.19), and (11.20) we have

$$\begin{aligned} \omega|_V &= (df_V)\cdot {}^t(G_V^{-1}) = (df_U)\cdot {}^t(dH_{VU}|_{f_U})^t(G_V^{-1}) \\ &= (df_U)\cdot {}^t(G_V^{-1}\cdot dH_{VU}|_{f_U}) = (df_U)\cdot {}^t(G_U^{-1}) = \omega|_U \end{aligned} \tag{11.21}$$

on $U \cap V$. Hence $\omega$ is indeed well-defined globally. Moreover, since $f_U$ is locally constant on the leaves of $F$, $\omega$ vanishes on the leaves of $F$. Hence the foliation $F$ is defined by the 1-forms $\omega_1, \cdots, \omega_q$. Now differentiating (11.20), locally we have

$$\begin{aligned} d\omega|_U &= (df_U)\cdot {}^t[d(G_U^{-1})] = (df_U)^t(-G_U^{-1}\cdot dG_U\cdot G_U^{-1}) \\ &= [(df_U)\cdot {}^t(G_U^{-1})]\cdot {}^t[-G_U^{-1}\cdot dG_U] = \omega|_U\cdot {}^t(-G_U^{-1}\cdot dG_U) \ . \end{aligned} \tag{11.22}$$

Let $\theta^L$ denote the connection form of the leaf connection on $\nu L$ with respect to the trivialization $G_{U\cap L}$. Then by (11.23) and Proposition 3.6

$$\theta^L|_{U\cap L} = {}^t(-G_U^{-1}\cdot dG_U)|_L = {}^t(-G_{U\cap L}^{-1}\cdot dG_{U\cap L}) \ . \tag{11.23}$$

Finally, let $x$ denote the $q \times q$ matrix whose entries are the coordinate functions of $GL(q, \boldsymbol{R})$. Then

$$\sigma = {}^t(-x^{-1}dx) \ . \tag{11.24}$$

Therefore from (11.17), (11.23), and (11.24) we get

$$(11.25)\qquad \begin{aligned} \phi(\sigma)|_{U\cap L} &= G^*_{U\cap L}(\sigma) = G^*_{U\cap L}[{}^t(-x^{-1}dx)] \\ &= {}^t(-G^{-1}_{U\cap L}\cdot dG_{U\cap L}) = \theta^L|_{U\cap L} \,. \end{aligned}$$

Hence $\phi(\sigma)$ is the connection form of the leaf connection on $\nu L$ with respect to the trivialization $G_{U\cap L}$. Therefore by the construction given in § 7

$$(11.26)\qquad \phi^*_{F,L} = \phi^* \,.$$

Hence from (11.18) and (11.26) it follows that

$$(11.27)\qquad \phi^*_{F,L} = \phi^*_{\nu L} \,. \qquad \text{q.e.d.}$$

Let $F = (U, f_U, H_{VU})$ be a $q$-codimensional foliation with a trivialized normal bundle $\nu F$, and let $L$ be a leaf of $F$. If $\gamma$ is a class in $H^*(gl(q, \boldsymbol{R}))$, then $\gamma(F, L) = \gamma(\nu L)$ by Theorem 11.7, so that all of the information about the holonomy ring of $L$ with respect to the foliation $F$ is contained in the normal bundle $\nu L$. Thus the holonomy invariants of the leaves of $F$ depend only on $dH_{VU}$, that is, on the linear part of $H_{VU}$. Hence the holonomy invariants of the leaves of $F$ are essentially linear invariants of the foliation $F$.

**Corollary 11.8.** *The diagram*

$$\begin{tikzcd} & H^*(gl(q, \boldsymbol{R})) \arrow[dl, "\phi^*_q"'] \arrow[dr, "\phi^*_q"] & \\ R(\widetilde{VBM}^0_q) \arrow[rr, "\tilde{\tilde{\nu}}^*_q"] & & R(CF^0_qL) \end{tikzcd}$$

*commutes. Hence $\tilde{\tilde{\nu}}^*_q$ is an isomorphism from the holonomy ring of $R(\widetilde{VBM}^0_q)$ to the holonomy ring of $R(CF^0_qL)$.*

*Proof.* Let $F$ be a $q$-codimensional foliation with a trivialized normal bundle $\nu F$, and let $L$ be a leaf of $F$. Then for any class $\gamma$ in $H^*(gl(q, \boldsymbol{R}))$, we have

$$\begin{aligned} \tilde{\tilde{\nu}}^*_q\phi^*_q(\gamma)(F, L) &= \phi^*_q(\gamma)[\tilde{\tilde{\nu}}_q(F, L)] = \phi^*_q(\gamma)(\nu L) = R_{\nu L}\phi^*_q(\gamma) \\ &= \phi^*_{\nu L}(\gamma) = \phi^*_{F,L}(\gamma) = R_{F,L}\phi^*_q(\gamma) = \phi^*_q(\gamma)(F, L) \,. \end{aligned}$$

Hence $\tilde{\tilde{\nu}}^*_q\phi^*_q = \phi^*_q$.

**Corollary 11.9.** *The diagram*

$$\begin{tikzcd} & H^*(gl(q, \boldsymbol{R})) \arrow[dl, "\phi^*_q"'] \arrow[dr, "\phi^*_q"] & \\ R(CF^0_qL) \arrow[rr, "(T_q\circ\tilde{\tilde{\nu}}_q)^*"] & & R(CF^0_qL) \end{tikzcd}$$

*commutes, so that $(T_q\circ\tilde{\tilde{\nu}}_q)^*$ is the identity homomorphism on the holonomy ring of $R(CF^0_qL)$.*

*Proof.* By Proposition 11.4 and Corollary 11.8,

$$(T_q \circ \tilde{\tilde{\nu}}_q)^* \phi_q^* = \tilde{\tilde{\nu}}_q^* T_q^* \phi_q^* = \tilde{\tilde{\nu}}_q^* \phi_q^* = \phi_q^* . \qquad \text{q.e.d.}$$

Characteristic classes can be constructed on the category $\widetilde{VBM}_q$ by a technique similar to that used to construct the holonomy ring on the category $\widetilde{VBM}_q^0$. In fact, let $\eta = (U_i, h_{ji})$ be a $q$-dimensional vector bundle with discrete structure group over a connected manifold $L$. Then $\eta$ is reducible to an $0(q)$-bundle $\tau = (V_\alpha, g_{\beta\alpha})$. Hence there are functions $f_{i\beta}\colon U_i \cap V_\beta \to GL(q, \boldsymbol{R})$ such that

$$f_{j\alpha}(x) = h_{ji}(x) f_{i\beta}(x) g_{\beta\alpha}(x) \tag{11.28}$$

for every $x$ in $U_i \cap U_j \cap V_\alpha \cap V_\beta$. Let $A^*(gl(q, \boldsymbol{R}), 0(q))$ denote the collection of all left-invariant $0(q)$-basic forms on $GL(q, \boldsymbol{R})$, and let $\omega$ be a form in $A^*(gl(q, \boldsymbol{R}), 0(q))$. Then from (11.28) it follows that

$$f_{j\alpha}^*(\omega) = f_{i\beta}^*(\omega) \tag{11.29}$$

on $U_i \cap U_j \cap V_\alpha \cap V_\beta$. Let $\phi_\eta(\omega)$ be the form in $A^*(L)$ which is given locally by $f_{i\beta}^*(\omega)$. Then by (11.29), $\phi_\eta(\omega)$ is a well-defined form on $L$. Therefore there is a cochain map $\phi_\eta\colon A^*(gl(q, \boldsymbol{R}), 0(q)) \to A^*(L)$. As before, the homomorphism $\phi_\eta^*$ is called the holonomy homomorphism, and the image of $\phi_\eta^*$ in $H^*(L)$ is called the holonomy ring of $\eta$. Now all of the results proved in this section remain valid if we replace $H^*(gl(q, \boldsymbol{R}))$ by $H^*(gl(q, \boldsymbol{R}), 0(q))$, $\widetilde{VBM}_q^0$ by $\widetilde{VBM}_q$, and $CF_q^0 L$ by $CF_q L$. Moreover the proofs are essentially the same.

Finally since $\phi_\eta^* = \phi_{T(\eta)}^*$, the results in §§ 8 and 9 remain valid if we replace pairs in $CF_q L$ ($CF_q L^0$) by vector bundles in $\widetilde{VBM}_q$ ($\widetilde{VBM}_q^0$). In particular,

$$h_j(\eta_1 \oplus \eta_2) = h_j(\eta_1) \otimes 1 + 1 \otimes h_j(\eta_2) .$$

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