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SECONDARY CHARACTERISTIC CLASSES FOR RIEMANNIAN FOLIATIONS

CONNOR LAZAROV & JOEL PASTERNACK

Introduction

Riemannian foliations are an interesting special class of smooth foliations which were introduced by Reinhart in 1956 (cf. [14]), and in recent years it has been of interest to specialize to Riemannian foliations the results of Bott, Haefliger, Thurston, et. al, from the rapidly developing theory of smooth foliations. The general theory of foliations and Haefliger structures developed by Haefliger [4] implies the existence of of a classifying space $BR\Gamma_q$ for q-codimensional Riemannian Haefliger structures. A basic problem is to understand the topology of the classifying spaces and to find invariants which distinguish between Riemannian foliations which are not equivalent in some appropriate sense (homotopic, cobordant, etc.). In this paper we develop invariants for Riemannian foliations with framed normal bundle and as a consequence begin the study of the algebraic topology of $\overline{BR\Gamma_q}$, the classifying space for foliations of this type. The invariants are a specialization of the secondary characteristic classes of smooth foliations developed by Bott in [2]. Our theory is also a special case of the theory of characteristic classes for foliated bundles developed by Kamber and Tondeur [8].

In § 1, an abstract real cochain complex RW_q is constructed (analogous to W_q in [2] and $W'_{q/2}$ in [10]) having the property that given a manifold admitting a smooth Riemannian foliation with framed normal bundle then there is a natural map from $H^*(RW_q)$ into $H^*(M; R)$; the image in $H^*(M; R)$ is the set of secondary characteristic classes for the given foliation. A coset foliation of a compact Lie group yields a Riemannian foliation with framed normal bundle and, in § 2, examples of such foliations are given which have nonzero secondary characteristic classes. Moreover, as in [2], one has a map $\delta_q^*: H^*(RW_q) \to H^*(\overline{BR\Gamma_q}; R)$ and the examples given show that $\delta_2^{(3)}, \delta_3^{(3)}, \delta_4^{(7)}, \delta_4^{(10)}$ are nonzero.

The secondary classes depend upon the choice of framing of the normal bundle and, in § 4, the precise dependence is given by a formula involving the transgression map $\tau: H^*(BSO_q; R) \to H^*(SO_q; R)$.

In § 5, the behavior of the secondary characteristic classes with repsect to continuous deformations of Riemannian foliation is considered and, inparticular, following Heitsch [5] it is shown that the classes which are rigid are generated

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in cohomology dimensions greater than $q + 1$. It is also shown that the classes do vary continuously in some examples in dimension q for q odd and $q + 1$ for q even. As in [2] the examples show that $\pi_3(\overline{BR\Gamma_2})$ and $\pi_3(\overline{BR\Gamma_3})$ are uncountable groups. It is also shown in § 5 that in cohomology dimension greater than q the secondary classes are smooth foliation invariants for Riemannian foliations with framed normal bundle independent of the particular Riemannian structure.

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1. The cochain complex RW_q

In this section we construct the cochain complex RW_q and a natural map from the cohomology of RW_q into the de Rham cohomology of a manifold on which is defined a smooth Riemannian foliation with trivial bundle.

1.1. Riemannian foliations. We will begin with a brief discussion of Riemannian foliations (compare [14], [12]). Suppose that $\mathscr{F}$ is a smooth foliation of a manifold M, and g is a Riemannian metric on the normal bundle $\nu(\mathscr{F})$. The pair $(\mathscr{F}, g)$, denoted $\mathscr{F}_g$, is a Riemannian foliation if g is preserved by the natural parallelism of $\nu(\mathscr{F})$ along the leaves of $\mathscr{F}$ (the metric g is called a "preserved" metric). Let $q = \operatorname{codim}(\mathscr{F})$. If U is an open set in M, and $f: U \to R^q$ is a submersion whose fibres are the local leaves of $\mathscr{F}$, then there is a unique Riemannian metric $\langle\, , \rangle$ on R^q so that $f^{-1}(\langle\, , \rangle) = g \,|\, U$ where one recalls that $f^{-1}(T(R^q)) \cong \nu(\mathscr{F}) \,|\, U$. Furthermore there is a unique metric preserving connection ∇_g on $\nu(\mathscr{F}_g)$ defined by

$$\nabla_g \,|\, U = f^{-1}(\nabla_{\langle,\rangle}) \,, \tag{1.1}$$

where $\nabla_{\langle,\rangle}$ is the unique torsion free connection on the Riemannian manifold $(R^q, \langle\, , \rangle)$.

Example 1.1. A basic example of a Riemannian foliation is as follows. Suppose $(M, \langle\, , \rangle)$ is a Riemannian manifold and a Lie group acts by isometries with all the orbits of the same dimension. Then M is foliated by these orbits, and the induced Riemannian metric on the normal vectors to the orbits yields a Riemannian foliation.

1.2. The differential forms $\Delta_P(\nabla_g, D_{\mathscr{S}})$. For a Lie group G denote by $I^*(G)$ the graded ring of multilinear, symmetric, ad (G)-invariant real valued functions on the Lie algebra of G. In this paper G will be either GL_q or SO_q.

Suppose that D_1 and D_2 are smooth connections on a q-dimensional vector bundle V over M. For each $P \in I^{(r)}(GL_q)$ we recall, following [2], the definition of the differential $(2r - 1)$-form $\Delta_P(D_1, D_2)$ on M. Consider the projection $\Pi: M \times [0, 1] \to M$ and define a connection $\mathscr{D}$ on $\Pi^{-1}(V)$ by

$$\mathscr{D} = tD_1 + (1 - t)D_2 \,. \tag{1.2}$$

The definition of $\Delta_P(D_1, D_2)$ is

$$(1.3) \qquad \Delta_P(D_1, D_2) = \Pi_*(\underbrace{P(K(\mathscr{D}), \cdots, K(\mathscr{D}))}_{r}) ,$$

where $K(\mathscr{D})$ is the curvature of $\mathscr{D}$, and Π_* is "integration over the fibre" of Π. The essential property of $\Delta_P(D_1, D_2)$ is

$$(1.4) \quad d\Delta_P(D_1, D_2) = P(K(D_1), \cdots, K(D_1)) - P(K(D_2), \cdots, K(D_2)) ,$$

where again and in the sequel $K(\cdot)$ denotes the curvature of the connection.

Now, let $\mathscr{F}_g$ be a smooth Riemannian foliation of codimension q on a manifold M. Suppose that the normal bundle $\nu(\mathscr{F})$ is trivial, and $\mathscr{S} = \{s_1, s_2, \cdots, s_q\}$ is a given framing. Let $D_{\mathscr{S}}$ be the connection on $\nu(\mathscr{F})$ which is flat with respect to $\mathscr{S}$, that is, $D_{\mathscr{S}} s_i = 0$ for $i = 1, 2, \cdots, q$. Since $K(D_{\mathscr{S}}) \equiv 0$ on M, (1.4) yields

$$(1.5) \qquad d\Delta_P(\nabla_g, D_{\mathscr{S}}) = P(K(\nabla_g), \cdots, K(\nabla_g)) .$$

Furthermore since ∇_g is locally pulled back from R^q,

$$(1.6) \qquad d\Delta_P(\nabla_g, D_{\mathscr{S}}) = 0$$

in case $r > [\frac{1}{2}q]$. Finally, observe that if $\mathscr{S}$ is orthonormal, then the curvature matrices $K(\mathscr{D})$ and $K(\nabla)$ are skew symmetric with respect to orthonormal framings, and the above formulas hold for $P \in I^*(SO_q)$.

Remark 1.7. The forms $\Delta_P(\nabla_g, D_{\mathscr{S}})$ are related to the Chern-Simons TP forms [3] as follows: If E is the principal bundle of $\nu(\mathscr{F}_g)$ with connection ∇_g, and $\sigma: M \to E$ is the global section defined by the framing $\mathscr{S}$, then

$$\Delta_P(\nabla_g, D_{\mathscr{S}}) = \sigma^*(TP(\nabla)) .$$

1.3. The cochain complex RW_q. In defining RW_q we distinguish the cases: q even and q odd.

Case 1: q even. It is well known [8] that in this case $I^*(SO_q)$ is generated as a ring by homogeneous polynomials $c_\chi, c_2, c_4, \cdots, c_{q-2}$ where degree $(c_j) = j$, degree $(c_\chi) = \frac{1}{2}q$ and

$$(1.8) \quad \begin{aligned} &\text{(i)} \quad c_j(\underbrace{A, \cdots, A}_{j}) = \operatorname{trace}(\Lambda^j(A)) , \\ &\text{(ii)} \quad (c_\chi(\underbrace{A, \cdots, A}_{\frac{1}{2}q}))^2 = \det(A) , \end{aligned}$$

A being a skew symmetric matrix, that is, an element of the Lie algebra so_q, and $\Lambda^j(A)$ being the j-th exterior power. Recall that $(2\pi)^{-\frac{1}{2}q}c_\chi$ corresponds under the Weil homomorphism to the Euler class.

As in [2] the algebra RW_q is defined as a tensor product of a polynomial algebra with an exterior algebra.

Definition 1.9.

$$RW_q = R[c_\chi, c_2, \cdots, c_{q-2}]/\{P \mid \deg P > \tfrac{1}{2}q\} \otimes \Lambda(h_\chi, h_2, \cdots, h_{q-2}) ,$$

where $R[c_\chi, c_2, \cdots, c_{q-2}]$ is the real polynomial algebra, and $\Lambda(h_\chi, h_2, \cdots, h_{q-2})$ is the real exterior algebra on the indicated indeterminants. In grading RW_q we let

$$(1.10)\quad \begin{aligned} &(\mathrm{i})\ \ \dim(c_j) = 2j && \text{for } j = 2, 4, \cdots, q-2 , \\ &(\mathrm{ii})\ \ \dim(c_\chi) = q , && \\ &(\mathrm{iii})\ \ \dim(h_j) = 2j - 1 && \text{for } j = 2, 4, \cdots, q-2 , \\ &(\mathrm{iv})\ \ \dim(h_\chi) = q - 1 . && \end{aligned}$$

The differential $d: RW_q \to RW_q$ is the anti-derivation of degree 1 satisfying

$$(1.11)\quad \begin{aligned} &(\mathrm{i})\ \ dc_j = 0 && \text{for } j = \chi, 2, 4, \cdots, q-2 , \\ &(\mathrm{ii})\ \ dh_j = c_j && \text{for } j = \chi, 2, 4, \cdots, \tfrac{1}{2}q , \\ &(\mathrm{iii})\ \ dh_j = 0 && \text{for } j > \tfrac{1}{2}q . \end{aligned}$$

Case 2: q odd. In this case $I^*(SO_q)$ is generated by homogeneous polynomials $c_2, c_4, \cdots, c_{q-1}$ where c_j satisfies (1.8) (i). Now as in Definition 1.9, we have

Definition 1.12.

$$RW_q = R[c_2, c_4, \cdots, c_{q-1}]/\{P \mid \deg P > \tfrac{1}{2}q\} \otimes \Lambda(h_2, h_4, \cdots, h_{q-1}) .$$

The grading here on RW_q is as in (1.10), and the differential is as in (1.11).

Suppose that $\mathscr{F}_g$ is a smooth Riemannian foliation on a manifold M, and $\mathscr{S}$ is an orthonormal framing of $\nu(\mathscr{F}_g)$. Let $A^*(M)$ be the de Rham complex of smooth differential forms on M. Comparing (1.5) with (1.11) (ii) and (1.6) with (1.11) (iii) we can define, as in [3] and [10], a map of differential complexes $\delta_{\mathscr{F},\mathscr{S}}: RW_q \to A^*(M)$ by

Definition 1.13.

(i) $\delta_{\mathscr{F}_g,\mathscr{S}}(c_j) = c_j(K(\nabla_g), \cdots, K(\nabla_g))$ for $j = 2, 4, \cdots, q-2$ or χ for q even,

(ii) $\delta_{\mathscr{F}_g,\mathscr{S}}(h_j) = \Delta_{c_j}(\nabla_g, D_{\mathscr{S}})$ for $j = 2, 4, \cdots, q-2$, or χ in case q is even, and for $j = 2, 4, \cdots, q-1$ in case q is odd.

This map passes to a map in cohomology

$$(1.14)\qquad \delta^*_{\mathscr{F}_g,\mathscr{S}}: H^*(RW_q) \to H^*_{\text{de Rham}}(M) .$$

We will call elements of the image of $\delta^*_{\mathscr{F}_g,\mathscr{S}}$ secondary characteristic classes of the foliation $(\mathscr{F}_g, \mathscr{S})$ and refer to $H^*(RW_q)$ as the algebra of universal secondary characteristic classes.

Remark 1.15. If $\mathscr{S}$ is not orthonormal, then $\delta_{\mathscr{F},\mathscr{S}}$ can be defined exactly as in Definition 1.13 for those elements of RW_q not involving c_χ or h_χ.

In § 2 we will give examples of Riemannian foliations for which the map $\delta^*_{\mathscr{F}_g,\mathscr{S}}$ is nontrivial. In § 4 we discuss the dependence of $\delta^*_{\mathscr{F},\mathscr{S}}$ on the framing $\mathscr{S}$, and in § 5 we discuss the dependence of $\delta^*_{\mathscr{F}_g,\mathscr{S}}$ on the metric g and the behavior of $\delta^*_{\mathscr{F}_g,\mathscr{S}}$ with respect to continuous deformations.

Remark 1.16. Continuing Remark 1.7 it should be observed:

(i) For $j > \frac{1}{2}q$, $Tc_j(\nabla)$ defines a de Rham cohomology class, and

$$\delta^*_{\mathscr{F}_g,\mathscr{S}}(\{h_j\}) = \sigma^*\{Tc_j(\nabla)\} ,$$

where here and in the sequel $\{\cdot\}$ denotes cohomology class.

(ii) If $\gamma \in H^*(RW_q)$ is represented by a monomial containing only a single h_j, specifically

$$\gamma = \{c_{i_1}c_{i_2} \cdots c_{i_p}h_j\}$$

where per force $\dim(c_{i_1}c_{i_2} \cdots c_{i_p})) + \dim(c_j) > q$, then

$$\delta^*_{\mathscr{F}_g,\mathscr{S}}(\gamma) = \sigma^*\{Tc_{i_1}c_{i_2} \cdots c_{i_p}c_j(\nabla)\} .$$

(See [3] where Proposition 3.7 states that $P(K(\nabla)) \wedge TQ(\nabla) = TPQ(\nabla) +$ exact.)

2. Examples

In this section examples of Riemannian foliations are given for which certain secondary characteristic classes are not zero.

Most of the examples in this paper are of the type described in Example 1.1. We now give a useful explicit formula for the unique Riemannian connection ∇_g on $\nu(\mathscr{F}_g)$ for a Riemannian foliation generated by an isometric Lie group action on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ as in Example 1.1. Let T be the tangent bundle of M, and E the subbundle of tangents to the orbits, $E^\perp$ the orthogonal complement of E, and let $\Pi_1\colon T \to E$ and $\Pi_2\colon T \to E^\perp$ be the orthogonal projections. Let $D_{\langle,\rangle}$ be the unique torsion free Riemannian connection on $(M, \langle\ ,\ \rangle)$. Identify $E^\perp$ with $\nu(\mathscr{F}_g)$; then for a vector field Y, a cross section of $E^\perp$ viewed as a cross section of $\nu(\mathscr{F}_g)$, and an arbitrary vector field X we have [10],

$$(2.1) \qquad (\nabla_g)_X Y = \Pi_2([\Pi_1 X, Y] + (D_{\langle,\rangle})_{\Pi_2 X} Y) .$$

Moreover, here and in the sequel we have the following fairly standard point of view towards connection forms. In general, if D is a connection on a vector bundle V, and $\mathscr{S} = \{s_1, s_2, \cdots, s_q\}$ is a (local) framing for V, then the connection matrix $\|\mathcal{O}_{ij}\|$ of D with respect to $\mathscr{S}$ is given by

$$Ds_i = \sum_{j=1}^{q} \mathcal{O}_{ij} \otimes s_j . \tag{2.2}$$

Finally, we recall [7] that if $\langle , \rangle$ is a bi-invariant metrix on a Lie group G, then for left invariant vector fields X, Y the unique Riemannian connection $D_{\langle,\rangle}$ satisfies

$$(D_{\langle,\rangle})_X Y = \tfrac{1}{2}[X, Y] . \tag{2.3}$$

Example 2.1. Let $M = S^3$ viewed as the Lie group $SU(2)$ of special unitary 2×2 matrices, and let X, Y_1, Y_2 be the left invariant vector fields on $SU(2)$ represented by the Lie algebra elements

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

respectively. Let $\langle , \rangle$ be the unique bi-invariant metric on $SU(2)$ for which $\{X, Y_1, Y_2\}$ is an orthonormal framing. Consider the free isometric action of S^1 on $(SU(2), \langle , \rangle)$ given by

$$\sigma \to \sigma \cdot \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \qquad \text{for } \sigma \in SU(2) , \qquad 0 \le t \le 2\pi . \tag{2.4}$$

Compare with Example 1.1, and let $\mathscr{F}_g$ be the induced Riemannian foliation of $SU(2)$. The leaves of $\mathscr{F}_g$ are the integral curves of X, and let $\mathscr{S} = \{y_1, y_2\}$ be the framing of $\nu(\mathscr{F}_g)$ induced by the framing $\{Y_1, Y_2\}$ of the bundle of normal vectors to the leaves. It is clear that $\mathscr{S}$ is an orthonormal framing.

Proposition 2.1. *For the element* $\{\chi h_\chi\} \in H^3(RW_2)$

$$\delta^*_{\mathscr{F}_g,\mathscr{S}}(\{\chi h \chi\}) \neq 0 .$$

In fact, if $\{\alpha, \beta_1, \beta_2\}$ *are the dual left invariant* 1*-forms to* $\{X, Y_1, Y_2\}$ *then*

$$\delta^*_{\mathscr{F}_g,\mathscr{S}}(\{\chi h_\chi\}) = 8\{\beta_2\beta_1\alpha\} .$$

Proof. By (2.1), $\cdots$, (2.3) the connection matrix of ∇_g with respect to $\mathscr{S}$ is

$$\begin{pmatrix} 0 & 2\alpha \\ -2\alpha & 0 \end{pmatrix} , \tag{2.5}$$

and the matrix of $\mathscr{D}$ as in (1.2) is

$$\begin{pmatrix} 0 & 2t\alpha \\ -2t\alpha & 0 \end{pmatrix} . \tag{2.6}$$

Using the formula for the curvature K: $K = d\Theta - \Theta \wedge \Theta$, we have

$$K(\nabla) = \begin{pmatrix} 0 & 4\beta_2\beta_1 \\ -4\beta_2\beta_1 & 0 \end{pmatrix} , \tag{2.7}$$

$$K(\mathscr{D}) = \begin{pmatrix} 0 & (4t\beta_2\beta_1 + 2dt\alpha) \\ -(4t\beta_2\beta_1 + 2dt\alpha) & 0 \end{pmatrix} . \tag{2.8}$$

Finally, the polynomial $c_\chi \in I^*(SO(2))$ is given by

$$c_\chi\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a . \tag{2.9}$$

Comparing (2.7), $\cdots$, (2.9) with Definition 1.13 yields

$$\delta_{\mathscr{F}_g,\mathscr{S}}(\chi h_\chi) = 8\beta_2\beta_1\alpha . \qquad \text{q.e.d.}$$

The next example is due to Chern-Simons [3].

Example 2.2. Let $(S^3, \langle , \rangle)$ be a Riemannian manifold as in Example 1.1, and $\mathscr{F}$ the 3-codimensional foliation of S^3 by points. Clearly $\mathscr{F}_{\langle , \rangle}$ is a Riemannian foliation, $\nu(\mathscr{F}) = T(S^3)$, and $\nabla_{\langle , \rangle} = D_{\langle , \rangle}$ where $D_{\langle , \rangle}$ is given by (2.3). Let $\mathscr{S} = \{X, Y_1, Y_2\}$ be an orthonormal framing of $\nu(\mathscr{F}_{\langle , \rangle})$.

Proposition 2.2, [3]. *For the element* $\{h_2\} \in H^3(RW_3)$

$$\delta^*_{\mathscr{F}_{\langle , \rangle},\mathscr{S}}(\{h_2\}) \neq 0 .$$

In fact

$$\delta_{\mathscr{F}_{\langle , \rangle},\mathscr{S}}(h_2) = 4\beta_2\beta_1\alpha .$$

In comparing the above with [3], note that $\delta_{\mathscr{F},\mathscr{S}}(h_2)$ is a Chern-Simons *TP* form as described in Remark 1.18 (i).

Example 2.3. Let $M = SO(5)$. For $1 \leq j < i \leq 5$, let Y_{ij} be the left invariant vector field on $SO(5)$ represented in the Lie algebra by the skew-symmetric matrix with $+1$ in the i-th row and j-th column, -1 in the j-th row and i-th column, and 0 elsewhere. Let $\langle , \rangle$ be the bi-invariant metric on $SO(5)$ for which $\{Y_{ij}\}$ is an orthonormal framing. Consider the isometric action of $SO(4)$ on $SO(5)$ given by

$$\sigma \longrightarrow \sigma \cdot \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right) \tag{2.10}$$

for $A \in SO(4)$.

As in Examples 1.1 and 2.1, let $\mathscr{F}_g$ be the induced Riemannian foliation of $SO(5)$, and let $\mathscr{S} = \{y_1, y_2, y_3, y_4\}$ be the orthonormal framing of $\nu(\mathscr{F}_g)$ induced by the framing $\{Y_{51}, Y_{52}, Y_{53}, Y_{54}\}$ of the bundle of normal vectors to the orbits.

Proposition 2.3. *Consider the element $\{\chi h_\chi h_2\}) \in H^{10}(RW_4)$. For the above foliation*

$$\delta^*_{\mathscr{F}_g,\mathscr{S}}(\{\chi h_\chi h_2\}) \neq 0 \ .$$

In fact, if $\{\alpha_{ij} \mid 1 \leq j < i \leq 5\}$ is the dual basis to $\{Y_{ij} \mid 1 \leq j < i \leq 5\}$, then

$$\delta_{\mathscr{F}_g,\mathscr{S}}(\{\chi h_\chi h_2\}) = 6\alpha_{54}\alpha_{53}\alpha_{52}\alpha_{51}\alpha_{21}\alpha_{31}\alpha_{41}\alpha_{32}\alpha_{42}\alpha_{43} \ .$$

The proof of the proposition is by a direct computation similar to (though more complicated than) the proof of Proposition 2.1. In doing the computation one finds that $\delta_{\mathscr{F}_g,\mathscr{S}}(h_2)$ is closed, and thus $\delta^*_{\mathscr{F}_g,\mathscr{S}}(\{\chi h_\chi\}) \neq 0$.

Remark 2.4. Examples 2.1, 2.2, 2.3 are of the following type: Suppose $(N, \langle, \rangle)$ is a Riemannian manifold, and $f: M \to N$ is a submersion. Then the fibres of f foliate M and $f^{-1}(\langle, \rangle)$ is a preserved metric on the normal bundle. The reader should compare Corollary 3.3 in the next section.

Remark 2.5. The computations in Example 2.1, 2.2, 2.3 are entirely Lie algebra computations, and in fact using (2.3) it is clear that given a Lie subalgebra $\mathfrak{H}$ of a compact Lie algebra $\mathfrak{G}$ one has a map from $H^*(RW_q)$ into the Lie algebra cohomology of $\mathfrak{G}$ where $q = \dim \mathfrak{G} - \dim h$.

3.1. Basic properties of the secondary characteristic classes

In order to state the naturality property of the secondary characteristic classes for Riemannian foliations with trivial normal bundle, we first observe that these secondary classes are in fact defined more generally for smooth Riemannian Haefliger structures with trivial normal bundle. A Riemannian Haefliger structure is called an $\overline{R\Gamma_q}$-structure, and it is called an $R\Gamma_q$-structure if the normal bundle is trivial.

For a precise definition of $R\Gamma_q$-structures see [4] or [13]. Suffice it to recall that if $\mathscr{H}$ is an $R\Gamma_q$-structure on a manifold M, then associated to the normal bundle $\nu(\mathscr{H})$ are a unique Riemannian metric g and a unique Riemannian connection ∇. Each point $m \in M$ is contained in an open set U for which there are

$$(3.1) \quad \begin{array}{ll} \text{(i)} & \text{a smooth map } f: U \to R^q \ , \\ \text{(ii)} & \text{a Riemannian metric } \langle, \rangle \text{ on } R^q \end{array}$$

so that

(i) $f^{-1}(T(R^q)) = \nu(\mathscr{H})\,|\,U$ and $f^{-1}(\langle\,,\ \rangle) = g$,

(3.2) (ii) $f^{-1}(D_{\langle,\rangle}) = \nabla\,|\,U$ where $D_{\langle,\rangle}$ is the unique Riemannian connection on $(R^q, \langle\,,\ \rangle)$.

Note that a Riemannian foliation defines an $R\Gamma_q$-structure, and we will henceforth use synonomously the expression $R\Gamma_q$-foliation and Riemannian foliation.

With the above understood, if $\mathscr{H}$ is an $\overline{R\Gamma}_q$-structure on M, and $\mathscr{S}$ is a trivialization of $\nu(\mathscr{H})$, then as in Definition 1.13 we can define a map of graded differential complexes

$$\delta_{\mathscr{H},\mathscr{S}}\colon RW_q \to A^*(M)\ . \tag{3.3}$$

Furthermore, if $\mathscr{H}$ associated to the $\overline{R\Gamma}_q$-foliation $\mathscr{F}_g$ then $\delta_{\mathscr{H},\mathscr{S}} = \delta_{\mathscr{F}_g,\mathscr{S}}$.

Finally recall that $R\Gamma_q$-structures pull back with respect to smooth maps, [4]. If $\mathscr{H}$ is a smooth $R\Gamma_q$-structure on M, and $\varphi\colon N \to M$ is a smooth map of manifold, then $\varphi^{-1}(\mathscr{H})$ denotes the pull back of M to N.

Theorem 3.1 (*Naturality*). *Let $\mathscr{H}$ be an $\overline{R\Gamma}_q$-structure on M, and $\mathscr{S}$ an orthonormal trivialization of $\nu(\mathscr{H})$. If $\varphi\colon N \to M$ is a smooth map of manifolds, then*

$$\delta_{\varphi^{-1}(\mathscr{H}),\varphi^{-1}(\mathscr{S})} = \varphi^* \circ \delta_{\mathscr{H},\mathscr{S}}\ .$$

Proof. The normal bundle $\nu(\varphi^{-1}(\mathscr{H})) = \varphi^{-1}(\nu(\mathscr{H}))$ and the unique Riemannian connection on $\nu(\varphi^{-1}(\mathscr{H}))$ is $\varphi^{-1}(\nabla)$ where ∇ is the unique Riemannian connection on $\nu(\mathscr{H})$. With respect to the framings $\mathscr{S}$ and $\varphi^{-1}(\mathscr{S})$ we have

$$K(\varphi^{-1}(\nabla) = \varphi^*(K(\nabla))\ . \tag{3.4}$$

Furthermore, for $P \in I^*(SO_q)$

$$\Delta_P(\varphi^{-1}(\nabla), D_{\varphi^{-1}(\mathscr{S})}) = \varphi^*(\Delta_P(\nabla, D_{\mathscr{S}}))\ . \tag{3.5}$$

Comparing (3.4), (3.5) with Definition 1.13 completes the proof. q.e.d.

Suppose $(\mathscr{H}_0, \mathscr{S}_0)$ and $(\mathscr{H}_1, \mathscr{S}_1)$ are smooth $\overline{R\Gamma}_q$-structures on M with $\mathscr{S}_i$ a framing of $\nu(\mathscr{H}_i)$ for $i = 0, 1$. These two $\overline{R\Gamma}_q$-structures are said to be smoothly homotopic if there exist a smooth $(\mathscr{H}, \mathscr{S})$ on $M \times [0, 1]$ which satisfies

$$i^*(\mathscr{H}, \mathscr{S}) = (\mathscr{H}_t, \mathscr{S}_t) \qquad \text{for } t = 0, 1\ , \tag{3.6}$$

where $i_t\colon M \to M \times [0, 1]$ is given by $i_t(m) = (m, t)$.

Corollary 3.2 (*Homotopy invariance*). *If $(\mathscr{H}_0, \mathscr{S}_0)$ and $(\mathscr{H}_1, \mathscr{S}_1)$ are* *smoothly homotopic $\overline{R\Gamma}_q$-structures on M, then*

$$\delta^*_{\mathscr{H}_0,\mathscr{S}_0} = \delta^*_{\mathscr{H}_1,\mathscr{S}_1} .$$

Proof. Letting $\mathscr{H}$ be as in (3.6). Then by Theorem 3.1

$$\delta^*_{\mathscr{H}_t,\mathscr{S}_t} = i_t^* \circ \delta_{\mathscr{H},\mathscr{S}} \qquad \text{for } t = 0, 1 .$$

In cohomology $i_0^* = i_1^*$.

Corollary 3.3. *Suppose $(B, \langle, \rangle)$ is a q-dimensional parallelizible Riemannian manifold with $\mathscr{S}$ a framing for the tangent bundle of B. Let $\mathscr{F}_{\langle,\rangle}$ be the $\overline{R\Gamma}_q$-foliation of B by points. If $\varphi\colon M \to B$ is a smooth map, then $\delta_{\varphi^{-1}(\mathscr{F}_{\langle,\rangle}),\varphi^{-1}(\mathscr{S})}\colon H^{(r)}(RW_q) \to H^{(r)}(M)$ is the zero map for $r > q$.*

Remark 3.4. Examples 2.1 and 2.3 show that if B is not parallelizible, then $\delta_{\varphi^{-1}(\mathscr{F}_{\langle,\rangle}),\varphi^{-1}(\varphi)}$ may be nonzero for a framing of $\nu(\varphi^{-1}(\mathscr{F}_{\langle,\rangle}))$.

If φ is a submersion Corollary 3.3 is a special case of the following theorem which gives an idea of what is being measured by the secondary characteristic classes.

Theorem 3.5. *Suppose $\mathscr{F}_g$ is an $\overline{R\Gamma}_q$-foliation, and $\mathscr{S} = \{s_1, \cdots, s_q\}$ is an orthonormal framing of $\nu(\mathscr{F}_g)$. If for every X tangent to the leaves of*

$$\nabla_X s_i = 0 \qquad \textit{for } i = 1, \cdots, q ,$$

then

$$\delta^{(r)}_{\mathscr{F}_g,\mathscr{S}} = 0 \qquad \textit{for } r > q .$$

Proof. We will show that if $\alpha \in RW_q$ and $\dim(\alpha) > q$, then $\delta_{\mathscr{F}_g,\mathscr{S}}(\alpha) \equiv 0$ on M. If U is an open subset of M and $f\colon U \to R^q$ is a submersion whose fibres are the local leaves of $\mathscr{F}$, then the hypothesis implies that there exists a framing S of R^q so that $f^{-1}(S) = \mathscr{S} \mid U$ and it follows that the flat connection D as in § 1.2 satisfies $D_{\mathscr{S}} \mid U = f^{-1}(D_S)$ where D_S is the flat connection on R^q associated to S. From § 1.1 the unique Riemannian connection ∇_g is, over U, pulled back from R^q, and thus $\mathscr{D}$ of (1.2), defining $\Delta_P(\nabla_g, D)$, is, over $U \times [0, 1]$, pulled back from R^q. In purticular, $K(\mathscr{D}) \mid U \times [0, 1]$ and $K(\nabla_g) \mid U$ are pulled back from R^q, and it follows from Definition 1.13 that $\delta_{\mathscr{F}_g,\mathscr{S}}(\alpha) \mid U \equiv 0$. Since M is covered by open sets U of the above type, the theorem is proved.

3.2. The classifying space $\overline{BR\Gamma}_q$

There exist a classifying space $BR\Gamma_q$ for $R\Gamma_q$-structures and a map $v_q\colon BR\Gamma_q \to BO_q$ classifying the normal bundle of the universal $R\Gamma_q$-structure (see [4], [12]). Let $\overline{BR\Gamma}_q$ be the homotopy theoretic fibre of the map v_q. The space $\overline{BR\Gamma}_q$ classifies $\overline{R\Gamma}_q$-structure with a framing of the normal bundle. As in [2] we can use the naturality given by Theorem 3.1 to define a canonical homomorphism

$$\delta_q^* : H^*(RW_q) \to H^*(\overline{BR\Gamma_q}; R) .$$

Examples 2.1, 2.2, 2.3 show that δ_q^* is not zero on certain elements of $H^*(RW_q)$.

Conjecture. *δ_q^* is an injection for all q.*

4. Dependence on the trivialization of the normal bundle

In order to describe the dependence of the map $\delta^*_{\mathscr{H},\mathscr{S}}$ on the framing $\mathscr{S}$ we need to recall the transgression map τ,

$$\tau : H^r(BSO_q; \Lambda) \to H^{r-1}(SO_q; \Lambda) \tag{4.1}$$

where $r \geq 1$, and Λ will be a coefficient ring which for our purpose will be either the integers or the reals. The map τ is a homomorphism of the additive structure but *not* a ring homomorphism; τ maps primitive elements to primitive elements and maps products to zero. For a definition and basic properties of τ see [6] or [1].

A polynomial $P \in I^r(SO_q)$ can be viewed by the Weil homomorphism ([9] or [2]) as an element of $H^{2r}(BSO_q; R)$. From [6] we have an explicit formula for a differential form on SO_q, denote $\check{\tau}P$, which represents τP in the de Rham cohomology:

$$\check{\tau}P = \left(-\frac{1}{2}\right)^{r-1} \frac{r!(r-1)!}{(2r-1)!} P(w, \underbrace{[w, w], \cdots, [w, w]}_{r-1}) , \tag{4.2}$$

where w is the Maurer-Cartan form on SO_q.

Before giving the main theorem of this section we state without proof a proposition which follows from the work of J. Vey (cf. [6]).

Proposition 4.1. *The cohomology algebra $H^*(RW_q)$ is generated by the set of elements γ of one of the following forms:*

(i) $\gamma = \{h_j\}$, *where j is even, and $\frac{1}{2}q < j < q$.*

(ii) $\gamma = \{c_{i_1} \cdots c_{i_p} h_{j_1} \cdots h_{j_l}\}$ *where*

(a) *the i's are either even integers $\leq \frac{1}{2}q$ or possibly χ in case q is even and* $I = \dim (c_{i_1} \cdots c_{i_p}) \leq q$,

(b) *the j's are distinct even integers $\leq \frac{1}{2}q$ or possibly χ in case q is even,*

(c) *letting* $j_0 = \min \{\dim (h_{j_k}) \mid l = 1, \cdots, l\}$, $I + 2j_0 > q$.

(If $I > q$, then γ is prima facie zero; $I + 2j_0 > q$ is the cocycle condition.)

Theorem 4.2. *Let $\mathscr{H}$ be an $\overline{R\Gamma_q}$-structure on a manifold M, and let $\mathscr{S} = \{s_1, \cdots, s_q\}$ and $\mathscr{S}' = \{s'_1, \cdots, s'_q\}$ be coherently oriented orthonormal trivializations of $\nu(\mathscr{H})$. Define $\varphi: M \to SO_q$ by*

$$s'_i = \sum_{j=1}^{q} \varphi_{ij} s_j \qquad \textit{for } i = 1, \cdots, q \,.$$

Then

(4.3) (i) $\delta^*_{\mathscr{H},\mathscr{S}}(\{h_j\}) - \delta^*_{\mathscr{H},\mathscr{S}'}(\{h_j\}) = \varphi^*(\tau c_j)$ *for j even and* $\frac{1}{2}q < j < q$,
(ii) *for γ of form* 2 *in Proposition* 4.1

$$\delta^*_{\mathscr{H},\mathscr{S}}(\gamma) - \delta^*_{\mathscr{H},\mathscr{S}'}(\gamma) = \sum_{k=1}^{l} (-1)^{k-1} \varphi^*(\tau c_{j_k}) \tag{4.4}$$

$$\cdot \left\{ \delta_{\mathscr{H},\mathscr{S}'}(c_{i_1} \cdots c_{i_p}) \prod_{\substack{1 \le k' \le l \\ k' \ne k}} (\delta_{\mathscr{H},\mathscr{S}'}(h_{j_{k'}}) + \varphi^*(\tau c_{j_{k'}})) \right\} .$$

(In case $q \equiv 2 \pmod 4$ replace τc_χ by $-\tau c_\chi$ in the above.)

Note. In understanding (4.4) it is important to note that $\delta_{\mathscr{H}_k\mathscr{S}'}(h_{j_{k'}})$ is *not* in general closed ($j_{k'} \le \frac{1}{2}q$). However $\delta_{\mathscr{H},\mathscr{S}'}(c_{i_1} \cdots c_{i_p})$ multiplied by any number of terms $\delta_{\mathscr{H},\mathscr{S}'}(h_{j_{k'}})$ *is* in general closed and the right hand side of (4.4) can be expanded and shown to depend only on the map $\delta^*_{\mathscr{H},\mathscr{S}'}$ and φ^*.

Corollary 4.3. *If γ of the above forms contains only a single h_j ($j \le \frac{1}{2}q$), then $\delta_{\mathscr{H},\mathscr{S}}(\gamma)$ is independent of the framing $\mathscr{S}$.*

Proof. Apply (4.4) and observe that $\delta_{\mathscr{H},\mathscr{S}'}(c_{i_1} \cdots c_{i_p})$ is zero in cohomology.

Remark. In case γ contains two h_j's, then formula (4.4) becomes

$$\begin{aligned} \delta^*_{\mathscr{H},\mathscr{S}}(\gamma) - \delta^*_{\mathscr{H},\mathscr{S}'}(\gamma) &= \varphi^*(\tau c_{j_1}) \delta^*_{\mathscr{H},\mathscr{S}'}(\{c_{i_1} \cdots c_{i_p} h_{j_2}\}) \\ &\quad - \varphi^*(\tau c_{j_2}) \delta^*_{\mathscr{H},\mathscr{S}'}(\{c_{i_1} \cdots c_{i_p} h_{j_1}\}) \,. \end{aligned} \tag{4.5}$$

One can easily check that Corollary 4.3 is consistent with reversing the roles of $\mathscr{S}$ and $\mathscr{S}'$ above since φ is then replaced by φ^{-1} and $(\varphi^{-1})^* = -\varphi^*$.

Corollary 4.4. *If $q = 2$, then $\delta^*_{\mathscr{H},\mathscr{S}}$ is independent of the choice of framing $\mathscr{S}$.*

Proof. Observe that RW_2 contains only h_χ and apply Corollary 4.3.

Corollary 4.5. *$(2\pi)^{-j}\delta^*_{\mathscr{H},\mathscr{S}}(h_j)$ is a well defined R/Z class independent of the framing $\mathscr{S}$ for $j > \frac{1}{2}q$.*

Proof. From [8] the polynomial $(2\pi)^{-j}c_j$ represents a class in $H^{2j}(BSO_q; Z)$ and, from [1], τ $((2\pi)^{-j}c_j)$ is in $H^{2j-1}(SO_q; Z)$. The corollary then follow directly from (4.3).

We begin the proof of Theorem 4.2 with a lemma which is due to H. Blaine Lawson and James Heitsch. We are happy to thank H. Blaine Lawson for communicating the essential ideas to us.

Observe that $\delta_{\mathscr{H},\mathscr{S}}(h_j) - \delta_{\mathscr{H},\mathscr{S}'}(h_j)$ is a closed form.

Lemma 4.6. $\{\delta_{\mathscr{H},\mathscr{S}}(h_j) - \delta_{\mathscr{H},\mathscr{S}'}(h_j)\} = (-1)^j \varphi^*(\check{\tau c_j})$ *for j even, $j < q$, or possibly $j = \chi$ in case q is even where $(-1)^j = (-1)^{\frac{1}{2}q}$ in case $j = \chi$.*

Proof of Lemma 4.6. Let ∇ be the unique Riemannian connection on $\nu(\mathscr{H})$ and let D and D' be the flat connections on $\nu(\mathscr{H})$ associated to the trivializations $\mathscr{S}$ and $\mathscr{S}'$ respectively.

For any polynomial $P \in I^*(SO(q))$ we have [5] (a consequence of Theorem 1, p. 382) that modulo exact forms

$$\Delta_P(\nabla, D) + \Delta_P(D, D') + \Delta_P(D', \nabla) = 0 \ . \tag{4.6}$$

It follows directly from the definition that

$$\Delta_P(D', \nabla) = -\Delta_P(\nabla, D') \ , \tag{4.7}$$

and thus comparing (1.13) with (4.6) we have modulo exact forms

$$\delta_{\mathscr{H},\mathscr{S}}(h_j) - \delta_{\mathscr{H},\mathscr{S}'}(h_j) = \Delta_{c_j}(D', D) \ . \tag{4.8}$$

To compute $\Delta_{c_j}(D', D)$ consider $M \times [0, 1]$ as in (1.2) and let $\mathscr{D}$ be the connection on $\pi^{-1}(\nu(\mathscr{H}))$ given by

$$\mathscr{D} = tD' + (1 - t)D \ . \tag{4.9}$$

We now find the connection matrix of D' with respect to the framing $\mathscr{S}$:

$$\begin{aligned} D's_i &= D'\left(\sum_{j=1}^{q} \varphi_{ij}s_j\right) , \\ 0 &= \sum_{j=1}^{q} (d\varphi_{ij} \otimes s_j + \varphi_{ij}D's_j) , \\ D's_k &= -\sum_{j=1}^{q} \left(\sum_{i=1}^{q} (\varphi^{-1})_{ki} d\varphi_{ij}\right) \otimes s_j \ . \end{aligned} \tag{4.10}$$

Thus the connection matrix of D' with respect to $\mathscr{S}$ is $-\varphi^{-1}d\varphi$, the connection matrix of $\mathscr{D}$ with respect to $\pi^{-1}(\mathscr{S})$ is

$$-t\varphi^{-1}d\varphi \ , \tag{4.11}$$

and the curvature of $\mathscr{D}$ is

$$K(\mathscr{D}) = -dt\varphi^{-1}d\varphi + (t - t^2)\varphi^{-1}d\varphi\varphi^{-1}d\varphi \ . \tag{4.12}$$

By (1.3) and the symmetry and linearity properties of c_j

$$\begin{aligned}\Delta_{c_j}(D', D) &= \pi_*(c_j(\underbrace{K(\mathscr{D}), \cdots, K(\mathscr{D})}_{j})) \\ &= j\pi_*(c_j(-dt\varphi^{-1}d\varphi, \underbrace{(t - t^2)\varphi^{-1}d\varphi\varphi^{-1}d\varphi, \cdots}_{j-1})) \\ &= -j\left(\int_0^1 (t - t^2)^{j-1}dt\right)c_j(\varphi^{-1}d\varphi, \underbrace{\varphi^{-1}d\varphi\varphi^{-1}d\varphi, \cdots}_{j-1}) \\ &= -\frac{(j)!(j-1)!}{(2j-1)!}c_j(\varphi^{-1}d\varphi, \underbrace{\varphi^{-1}d\varphi\varphi^{-1}d\varphi, \cdots}_{j-1}) .\end{aligned} \tag{4.13}$$

If w is the Maurer-Cartan form on SO_q, then

$$\varphi^*(w) = \varphi^{-1}d\varphi . \tag{4.14}$$

Since $[\varphi^*(w), \varphi^*(w)] = 2\varphi^{-1}d\varphi\varphi^{-1}d\varphi$ it follows that

$$\Delta_{c_j}(D', D) = -\left(\frac{1}{2}\right)^{j-1}\frac{j!(j-1)!}{(2j-1)!}\varphi^*(c_i(w, \underbrace{[w, w], \cdots, [w, w]}_{j-1}) . \tag{4.15}$$

Comparing (4.2) completes the proof of the lemma.

Proof of Theorem 4.2. Part (i) of the theorem follows directly from Lemma 4.3. We prove part (ii). By the lemma,

$$\delta_{\mathscr{H},\mathscr{S}}(h_j) = \delta_{\mathscr{H},\mathscr{S}'}(h_j) + \varphi^*(\tau\check{c}_j) + d\xi_j , \tag{4.16}$$

where $\xi_j \in A^{2j-2}(M)$, and, in case $q = 2 \pmod 4$ and $j = \chi$, replace $\tau\check{c}_\chi$ by $-\tau\check{c}_\chi$.

Since $\delta_{\mathscr{H},\mathscr{S}}$ and $\delta_{\mathscr{H},\mathscr{S}'}$ are homomorphisms of RW_q into $A^*(M)$ and $\delta_{\mathscr{H},\mathscr{S}}(c_i) = \delta_{\mathscr{H},\mathscr{S}'}(c_i)$, it follows that

$$\begin{aligned}&\delta_{\mathscr{H},\mathscr{S}}(c_{i_1} \cdots c_{i_p}h_{j_1} \cdots h_{j_l}) \\ &\quad = \delta_{\mathscr{H},\mathscr{S}'}(c_{i_1} \cdots c_{i_p}) \prod_{1\le k\le l} (\delta_{\mathscr{H},\mathscr{S}'}(h_{j_k}) + \varphi^*(\tau\check{c}_{j_k}) + d\xi_{j_k}) .\end{aligned}$$

The proof of (4.4) is completed by passing to cohomology and partially expanding the product.

5. Continuous deformations

In this section the behavior of the secondary characteristic classes on a differentiable family of $\overline{R\Gamma_q}$-structures is discussed. It is shown that a nonzero variation in these classes can occur for those classes which lie in cohomology dimension q or $q + 1$, and that analogous to [5], rigid classes are generated in dimension greater than $q + 1$. Furthermore, it is proved that for a fixed $\overline{R\Gamma_q}$-

foliation the classes in cohomology dimension greater than q are foliation-invariants independent of the choice of preserved Riemannian metric on the normal bundle.

We begin with three examples.

Example 5.1. In this example we will have a fixed foliation $\mathscr{F}$, and for each real value u, $u > 0$, we have a "preserved" metric g_u on $\nu(\mathscr{F})$. The manifold is $SU(2)$, and $\mathscr{F}$ is the foliation of Example 2.1 defined by the right action of S^1 given by (2.4). Let V, W_1, W_2 be the right invariant vector fields on $SU(2)$ represented in the Lie algebra by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively. For $u > 0$, let $\langle , \rangle_u$ be the unique right invariant metric on $SU(2)$ for which $\{uV, W_1, W_2\}$ is an orthonormal framing. For each $u > 0$, $\langle , \rangle_u$ induces, as in Example 2.1, a preserved metric g_u on $\nu(\mathscr{F})$. From Proposition 2.1 and Theorem 5.6 below we will see that

$$\delta^*_{\mathscr{F}_{g_u}}\{\chi h_\chi\} = 8\beta_2\beta_1\alpha \ , \tag{5.1}$$

where we have suppressed the framing of $\nu(\mathscr{F}_{gu})$ since by Corollary 4.4 in codimension 2, the secondary classes are independent of framing.

Example 5.2. This example was suggested to us by Raoul Bott. We are happy to thank him for his interest and help. As is Example 5.1, this is a differentiable variation of Example 2.1. Let $\langle , \rangle$ be as in Example 2.1 and for each real u define an isometric action of R on $(SU(2), \langle , \rangle)$ by the formula

$$t\cdot\sigma = \begin{pmatrix} e^{iut} & 0 \\ 0 & e^{-iut} \end{pmatrix}\cdot\sigma\cdot\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \tag{5.2}$$

for $t \in R$ and $\sigma \in SU(2)$. The actions are isometric since $\langle , \rangle$ is bi-invariant. For each u the orbits of the associated action are the integral curves of $X + uV$ where V is as in Example 5.1, and these orbits give a foliation of $SU(2)$ for $u \neq \pm 1$. For each u, $-1 < u < 1$, let $\mathscr{F}_u$ be the induced Riemannian foliation of $SU(2)$.

Proposition 5.1. *The function* $\delta^*_{\mathscr{F}_u}(\{\chi h_\chi\})$ *is a nonconstant function of* u.

For a proof of this proposition see Appendix I.

Example 5.3. This example is due to Chern-Simons [3], and is a continuation of Example 2.2. For $u > 0$, let $\langle , \rangle_u$ be the left invariant metric on $SU(2)$ for which $\mathscr{S}_u = \{uX, Y_1, Y_2\}$ is an orthonormal basis. Let $\mathscr{F}_{\langle , \rangle_u}$ be a 3-codimensional $\overline{R\Gamma}_3$-foliation of $SU(2)$ as in Example 2.2. A direct computation yields a formula for $\delta^*_{\mathscr{F}_{\langle , \rangle_u}, \mathscr{S}_u}(\{h_2\}$ and

Proposition 5.2. *The function* $\delta^*_{\mathscr{F}_{\langle , \rangle}, \mathscr{S}_u}(\{h_2\})$ *is a nonconstant function of* u.

Remark 5.4. As in [2] Propositions 5.1 and 5.2 yield the following

(i) $\pi_3(\overline{BR\Gamma}_2)$ is uncountable,

(ii) $\pi_3(\overline{BR\Gamma}_3)$ is uncountable.

In Examples 5.2 and 5.3 above we have seen that nonzero variation of the secondary class can occur in the codim and codim plus one. In contract, following Heitsch [5] we have

Theorem 5.5. *If* $(\mathscr{H}_u, \mathscr{S}_u)$ *for* $u \in [0, 1]$ *is a differentiable family of* $\overline{R\Gamma}_q$-*structures on a manifold* M, *then*

$$\delta^*_{\mathscr{H}_0,\mathscr{S}_0} = \delta^*_{\mathscr{H}_u,\mathscr{S}_u}$$

for $u \in [0, 1]$ *on classes generated in cohomology dimension greater than* $q + 1$.

Proof. Let ∇_u be the unique Riemannian connection on $\nu(\mathscr{H}_u)$, and let θ_u be the matrix of ∇_u with respect to $\mathscr{S}_u$. By [5, p. 382] modulo exact forms

$$\frac{\partial}{\partial u}\delta_{\mathscr{H}_u,\mathscr{S}_u}(h_j) = jc_j\Big(\frac{\partial}{\partial u}\theta_u, \underbrace{K(\nabla_u), \cdots, K(\nabla_u)}_{j-1}\Big) . \tag{5.3}$$

By an argument which uses the fact that $K(\nabla_u)$ is locally pulled back from R_q and is entirely similar to [5, p. 384] we conclude that for γ as in Proposition 4.1 and of dim $> q + 1$ and with one h

$$\frac{\partial}{\partial u}(\delta_{\mathscr{H}_u,\mathscr{S}_u}(\gamma)) = 0 , \tag{5.4}$$

modulo exact forms. q.e.d.

Theorem 5.5 can be strengthened for variations of the type given by Example 5.1.

Theorem 5.6. *Suppose that* $\mathscr{F}$ *is a fixed q-codimensional foliation of a manifold* M, *and* $g_u, u \in [0, 1]$, *is a differentiable family of preserved Riemannian metric on* $\nu(\mathscr{F})$. *Let* $\mathscr{S}_u$ *be a differentiable family of* g_u-*orthonormal framings on* $\nu(\mathscr{F})$. *Then for* $u \in [0, 1]$ *and lasses generated in dimension greater than* q

$$\delta^*_{\mathscr{F}_{g_0},\mathscr{S}_0} = \delta^*_{\mathscr{F}_{g_u},\mathscr{S}_u} .$$

Corollary 5.7. *The secondary characteristic classes of an* $\overline{R\Gamma}_q$-*foliation* $\mathscr{F}_g$ *are independent of the preserved metric* g *on* $\nu(\mathscr{F}_g)$. *That is, if* g_0 *and* g_1 *are both preserved metrices on* $\nu(\mathscr{F})$, *and* $\mathscr{S}_0$ *and* $\mathscr{S}_1$ *are homotopic orthonormal trivializations of* $\nu(\mathscr{F}_{g_0})$ *and* $\nu(\mathscr{F}_{g_1})$ *respectively, then for cohomology dimension* $r > q$

$$\delta^*_{\mathscr{F}_{g_0},\mathscr{S}_0} = \delta^*_{\mathscr{F}_{g_1},\mathscr{S}_1} .$$

Proof of Corollary 5.7. Let $g_u = ug_0 + (1 - u)g_1$ for $u \in [0, 1]$. Observe that this gives a differentiable family of preserved metrices on $\nu(\mathscr{F})$. Since $\mathscr{S}_0$ is homotopic to $\mathscr{S}_1$, we can find a continuous family $\mathscr{S}_u, u \in [0, 1]$, of trivializations of $\nu(\mathscr{F})$ with $S_0 = \mathscr{S}_0$, $S_1 = \mathscr{S}_1$ and by Gram-Schmidt a continuous family $\mathscr{S}_u$ where for each $u \in [0, 1]$, $\mathscr{S}_u$ is g_u-orthonormal. Now apply Theorem 5.6 to the family $(\mathscr{F}_{g_u}, \mathscr{S}_u)$.

Proof of Theorem 5.6. First observe that comparing with Theorem 5.5 for q odd, there is nothing to prove since $q + 1$ is even and $H^*(RW_q)$ is zero in even dimensions.

For q even, we consider two cases $q > 2$ and $q = 2$. For q even and greater than 2, observe that if $\gamma \in H^{q+1}(RW_q)$, then γ does not involve c_χ or h_χ. For such γ by Remark 1.15, $\delta_{\mathscr{F}_g, \mathscr{S}}(\gamma)$ can be defined with $\mathscr{S}$ not necessarily g-orthonormal and it is straightforward to show that Theorem 4.2 carries over with τ replaced by the transgression $\tau\colon H^*(BGL_q) \to H^*(GL_q)$.

For $\gamma \in H^{q+1}(RW_q)$, $q > 2$, it follows that

$$\delta^*_{\mathscr{F}_{g_u}, \mathscr{S}_0}(\gamma) = \delta^*_{\mathscr{F}_{g_u}, \mathscr{S}_u}(\gamma) \,, \tag{5.5}$$

since the change of coordinates map φ_u between $\mathscr{S}_0$ and $\mathscr{S}_u$ is homotopically trivial. Furthermore, if $\gamma \in H^{q+1}(RW_q)$ is of type (i) or (ii) in Proposition 4.1, then γ contains only a single h_j, and combining Remark 1.15 with (5.5) yields

$$\delta^*_{\mathscr{F}_{g_u}, \mathscr{S}_u}(\gamma) = \{\Delta_P(\nabla_u, D_{\mathscr{S}_0})\} \,, \tag{5.6}$$

where ∇_u is the unique Riemannian connection on $\nu(\mathscr{F}_{g_u})$, and P is a polynormal of degree $\frac{1}{2}q + 1$, $P \in I^*(GL_q)$.

In case $q = 2$, then $\{\chi h_\chi\}$ is a basis for $H^3(RW_2)$, and it is not difficult to check (compare [3]) that

$$\delta_{\mathscr{F}_g, \mathscr{S}}(\{\chi h_\chi\}) = \{\Delta_{c_2}(\nabla_g, D_{\mathscr{S}})\} \,, \tag{5.7}$$

where c_2 is the determinant polynomial. (In comparing with [3] note that $c_2 = \chi^2$, and thus $Tc_2(\nabla) = \chi(K(\nabla)) \wedge T\chi(\nabla) +$ exact.) Since $c_2 \in I^*(GL_2)$, we may compute $\delta_{\mathscr{F}_g, \mathscr{S}}(\{\chi h_\chi\})$ with respect to a framing which is not necessarily g-orthonormal, and (5.6) holds for $\gamma\{\chi h_\chi\}$.

We will need the following lemma.

Lemma 5.8. *Suppose U is an open subset of M, and $f\colon U \to R^q$ is a submersion with fibres the local leaves of $\mathscr{F}$. Then for any polynomial $P \in I^*(GL_q)$, $\dfrac{\partial}{\partial u}(\Delta_P(\nabla_u, D_{\mathscr{S}_0}) \,|\, U$ is a section of $f^-(\Lambda^*(T^*(R^q))$, that is, a linear combination of differential forms pulled back from R^q.*

Proof of Lemma 5.8. Let θ_u be the connection matrix of ∇_u with respect to $\mathscr{S}_0$, and let $\psi_u = \dfrac{\partial}{\partial u}\theta_u$. Suppose P is homogeneous of degree r. By [5]

$$\frac{\partial}{\partial u}\nabla_P(\nabla_u, D_{\mathscr{S}_0}) = rP(\psi_u, \underbrace{K(\nabla_u), \cdots, K(\nabla_u)}_{r-1}) \,.$$

Recall that we have metrics $\langle\ ,\ \rangle_u$ on R^q, and $\nabla_u | U$ is pulled back from the unique Riemannian connection on $(R^q, \langle\ ,\ \rangle_u)$. It is standard (cf. [12]) that with respect to any framing of $\nu(\mathscr{F}) | U$ the entries of $K(\nabla_u) | U$ are linear combinations of differential forms pulled back from R^q. The proof will be completed by showing that $\psi_u | U$ is similarly pulled back from R^q.

Observe that ψ_u, like $K(\nabla_u)$, is a tensorial object even though θ_u is not tensorial. Specifically, suppose $\mathscr{S}' = \{s_1', \cdots, s_q'\}$ is a framing for $\nu(\mathscr{F}) | U$, θ_u' is the connection matrix of ∇_u with respect to $\mathscr{S}'$, $\psi_u' = \dfrac{\partial}{\partial u}\Theta_u'$, and $\lambda \colon U \to GL_q$ satisfies $s_i' = \sum_{j=1}^q \lambda_{ij} s_j$. Then

$$\psi_u = \lambda^{-1} \circ \psi_u' \circ \lambda \,. \tag{5.8}$$

Let S be a fixed framing for $T(R^q)$, and let $\mathscr{S}' = f^{-1}(S)$. Then the connection matrix θ_u' is pulled back from R^q, and therefore ψ_u' from R^q. By (5.8), ψ_u is a linear combination of differential forms pulled back from R^q. q.e.d.

Comparing Lemma 5.8 with (5.6) yields

$$\frac{\partial}{\partial u}\delta^*_{\mathscr{F}_{g_u}, \mathscr{S}_u}(\gamma) = 0 \,,$$

since for each open set U as in the lemma

$$\frac{\partial}{\partial u}\Delta_p(\nabla_u, D_{\mathscr{S}_0}) \equiv 0 \qquad \text{on } U$$

by a dimensionality argument, and M is covered by such open sets. Hence the proof of Theorem 5.6 is complete.

Appendix

Here we present a proof of Example 5.2 in a computational manner.

To do this computation we view S^3 as the set of quaternions $q = q_0 + q_1 i + q_2 j + q_2 k$ of unit length. The foliation $\mathscr{F}_u$ is generated by the vector field $Y = L_*(i) + uR_*(i)$. Then $Y_q = L_*(i + u \operatorname{Ad}(q^{-1})i)_q$. We let X_1, X_2, X_3 be the left invariant vector fields $L_*(i), L_*(j), L_*(k)$ respectively. Let x_1, x_2, x_3 be the dual basis of left invariant forms. $\operatorname{Ad}(q^{-1})i = a_{11}X_1 + a_{12}X_2 + a_{13}X_3$ where $a_{11} = q_0^2 + q_1^2 - q_2^2 - q_3^2$, $a_{12} = 2(q_1q_2 - q_0q_3)$, $a_{13} = 2(q_0q_2 + q_1q_3)$. For later purposes we will need the following dota.

$$\begin{aligned} &X_1(a_{11}) = 0 \,, && X_2(a_{11}) = -2a_{13} \,, && X_3(a_{11}) = 2a_{12} \,, \\ &X_1(a_{12}) = 2a_{13} \,, && X_2(a_{12}) = 0 \,, && X_3(a_{12}) = -2a_{11} \,, \\ &X_1(a_{13}) = -2a_{12} \,, && X_2(a_{13}) = 2a_{11} \,, && X_3(a_{13}) = 0 \,, \end{aligned}$$

$$da_{11} = -2a_{13}x_2 + 2a_{12}x_3 \,,$$

$$da_{12} = 2a_{13}x_1 - 2a_{11}x_3 \,,$$
$$da_{13} = -2a_{12}x_1 + 2a_{11}x_2 \,.$$

Recall that $\langle , \rangle$ is the bi-invariant metric on S^3, and we construct a global orthonormal framing of the orthogonal complement to $\mathscr{F}_u$, namely,

$$Z_2 = X_2 - \frac{\langle X_2, Y\rangle}{\langle Y, Y\rangle} Y \left(\langle X_2, X_2\rangle - \frac{\langle X, Y\rangle^2}{\langle Y, Y\rangle}\right)^{-\frac{1}{2}} ,$$
$$Z_3 = X_3 - \frac{\langle X_3, Y\rangle}{\langle Y, Y\rangle} Y - \langle X_3, Z_2\rangle Z_2 \left(\langle X_3, X_3\rangle - \frac{\langle X_3, Y\rangle^2}{\langle Y, Y\rangle} - \langle X_3, Z_2\rangle^2\right)^{-\frac{1}{2}} .$$

Then $s = \{Z_2, Z_3\}$ is an orthonormal framing of $\mathscr{F}_u^{\perp}$. Let ∇ be the unique Riemannian torsion-free connection on this normal bundle, and D the connection which is globally flat relative to s. Then $\nabla s = s\theta$ where

$$\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} ,$$

and a simple computation shows that $\delta_{\mathscr{F}_u}(\{\chi h_\chi\}) = \{\theta d\theta\}$.

Now to show that the cohomology class of $\delta_{\mathscr{F}_u}(\{\chi h_\chi\})$ varies continuously with u, we will expand everything in powers of u and *drop all terms involving powers of u greater than* u^2. Thus

$$Y = (1 + ua_{11})X_1 + a_{12}uX_2 + a_{13}uX_3 \,,$$
$$Z_2 = (a_{11}a_{12}u^2 - a_{12}u)X_1 + (1 - \tfrac{1}{2}a_{12}^2u^2)X_2 - u^2a_{12}a_{13}X_3 \,,$$
$$Z_3 = (a_{11}a_{13}u^2 - a_{13}u)X_1 + (1 - \tfrac{1}{2}a_{13}^2u^2)X_3 \,.$$

Now write $\theta = \theta(X_1)x_1 + \theta(X_2)x_2 + \theta(X_3)x_3$, and so

$$-\theta(X_1) = \langle \nabla_{X_1} Z_2, Z_3\rangle = \tfrac{1}{2}Z_2\langle X_1, Z_3\rangle - \tfrac{1}{2}Z_3\langle X_1, Z_2\rangle + \tfrac{1}{2}\{2\langle [X_1, Z_2], Z_3\rangle + \langle [Z_3, Z_2], \pi X_1\rangle\} \,,$$
$$-\theta(X_2) = \langle \nabla_{X_2} Z_2, Z_3\rangle = \tfrac{1}{2}Z_2\langle X_2, Z_3\rangle - \tfrac{1}{2}Z_3\langle X_2, Z_2\rangle + \tfrac{1}{2}\{2\langle [X_2, Z_2], Z_3\rangle + \langle [Z_3, Z_2], \pi X_2\rangle\} \,,$$
$$-\theta(X_3) = \langle \nabla_{X_3} Z_2, Z_3\rangle = \tfrac{1}{2}Z_2\langle X_3, Z_3\rangle - \tfrac{1}{2}Z_3\langle X_3, Z_2\rangle + \tfrac{1}{2}\{2\langle [X_3, Z_2], Z_3\rangle + \tfrac{1}{2}\langle [Z_3, Z_2], \pi X_3\} \,.$$

Here π is the projection on the orthogonal complement, and we have used (2.1) and the standard formula for the torsion-free Riemannian connection in terms of brackets, inner products and derivations. We have also used the fact that X_1, X_2, X_3 are Killing vector fields. Then

$$\begin{aligned}[Z_2, Z_3] &= (9a_{11}^2u^2 - 5u^2 - 4a_{11}u + 2)X_1 + (-4a_{11}a_{12}u^2 + 2a_{12}u)X_2 \\ &\quad + (-6a_{11}a_{13}u^2 + 2a_{13}u)X_3 , \\ Z_2\langle X_1, Z_3\rangle &= -Z_3\langle X_1, Z_2\rangle = (4a_{11}^2 - 2)u^2 - 2a_{11}u , \\ Z_2\langle X_2, Z_3\rangle &= 0 , \qquad Z_3\langle X_2, Z_2\rangle = 2a_{11}a_{12}u^2 , \\ Z_2\langle X_3, Z_3\rangle &= -Z_3\langle X_3, Z_2\rangle = -2a_{11}a_{13}u^2 , \\ \langle[X_1, Z_2], Z_3\rangle &= 2 + u^2(a_{12}^2 - a_{13}^2) , \\ \langle[X_2, Z_2], Z_3\rangle &= -4a_{11}a_{12}u^2 + 2a_{12}u , \\ \langle[X_3, Z_2], Z_3\rangle &= -2a_{11}a_{13}u^2 + 2a_{13}u .\end{aligned}$$

Thus

$$\begin{aligned}\pi X_1 &= (a_{12}^2 + a_{13}^2)u^2X_1 + (a_{11}a_{12}u^2 - a_{12}u)X_2 + (a_{11}a_{13}u^2 - a_{13}u)X_3 , \\ \pi X_2 &= (a_{11}a_{12}u^2 - a_{12}u)X_1 + (1 - a_{12}^2u^2)X_2 - a_{12}a_{13}u^2X_3 , \\ \pi X_3 &= (a_{11}a_{13}u^2 - a_{13}u)X_1 - a_{12}a_{13}u^2X_2 + (1 - a_{13}^2u^2)X_3 ,\end{aligned}$$

and therefore

$$\langle[Z_2, Z_3], \pi X_1\rangle = 0 , \quad \langle[Z_2, Z_3], \pi X_2\rangle = 2a_{11}a_{12}u^2 , \quad \langle[Z_2, Z_3], \pi X_3\rangle = 0 .$$

Hence

$$\begin{aligned}\theta(X_1) &= -2 + 2a_{11}u - (2a_{11}^2 - a_{12}^2 - 3a_{13}^2)u^2 , \\ \theta(X_2) &= 6a_{11}a_{12}u^2 - 2a_{12}u , \\ \theta(X_3) &= 4a_{11}a_{13}u^2 - 2a_{13}u ,\end{aligned}$$

so that $\theta = \theta(X_1)x_1 + \theta(X_2)x_2 + \theta(X_3)x_3$.

Let us write $\theta = -2x_1 + u\omega_1 + u^2\omega_2$ where

$$\begin{aligned}\omega_1 &= 2a_{11}x_1 - 2a_{12}x_2 - 2a_{13}x_3 , \\ \omega_2 &= (-2a_{11}^2 + a_{12}^2 + 3a_{13}^2)x_1 + 6a_{11}a_{12}x_2 + 4a_{11}a_{13}x_3 .\end{aligned}$$

Then

$$\begin{aligned}\theta d\theta &= -8x_1x_2x_3 + u(4x_2x_3\omega_1 - 2x_1d\omega_1) \\ &\quad + u^2(-2x_1d\omega_2 + 4x_2x_3\omega_2 + \omega_1d\omega_1) .\end{aligned}$$

Now it is easily seen that the coefficient of u integrates to zero over S^3. Let $C = -2x_1d\omega_2 + 4x_2x_3\omega_3 + \omega_1d\omega_1$, which is the coefficient of u^2. Since $-2x_1d\omega_2 = 4x_2x_3\omega_2 + d\gamma$ for some γ, we have

$$C = 8x_2x_3\omega_2 + \omega_1d\omega_1 + d\gamma ,$$

$$\omega_1 d\omega_1 = -8(2a_{11}^2 + 1)x_1x_2x_3 ,$$
$$8x_2x_3\omega_2 = 8(-2a_{11}^2 + a_{12}^2 + 3a_{13}^2)x_1x_2x_3 .$$

Thus $C = 8(-5a_{11}^2 + 2a_{13}^2)x_1x_2x_3 + d\gamma$,

Since $\int_{S^3} x_1x_2x_3 = 2\pi^2$ and $\int_{S^3} a_{11}^2x_1x_2x_3 = \int_{S^3} a_{12}^2x_1x_2x_3 = \int_{S^3} a_{13}^2x_1x_2x_3 = \frac{2}{3}\pi^2$,

$\int_{S^3} C = -16\pi^2$. Finally, up to the second order

$$\delta^*_{\mathscr{F}_u}(\{\chi h_\chi\})[S^3] = \int_{S^3} \theta d\theta = -16\pi^2(1 + u^2) .$$

Hence this class varies continuously with u.

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HERBERT H. LEHMAN COLLEGE,
CITY UNIVERSITY OF NEW YORK
UNIVERSITY OF ROCHESTER