J. DIFFERENTIAL GEOMETRY
10 (1975) 181-200

THE FRENET FRAME OF AN IMMERSION

M. ROCHOWSKI

Introduction

It is a known theorem of Jacobi that the indicatrix of the principal normal of a curve in a Euclidean three-space E^3 divides the unit sphere S^2 into two pieces of equal area. In this paper a generalization of this theorem is given in the sense that the curve is replaced by a two-sphere S^2 imbedded in a Euclidean 4-space E^4.

To define a principal normal of an immersion $x\colon M^n \to E^{n+N}$ of a manifold M^n into a Euclidean space E^{n+N} we proceed as follows. If we take the boundary of a small tubular neighborhood of a curve in the three-space and examine the maximal value of the Gaussian curvature along a fiber over a fixed point of the curve, then the point of the boundary of the tubular neighborhood, at which the Gaussian curvature attains its maximal value, together with the fixed point of the curve defines the principal normal of the curve. This construction can be generalized to a manifold M^n immersed in E^{n+N} by replacing the tubular neighborhood by the normal bundle B_ν of the immersion and the Gauss curvature by the Killing-Lipschitz curvature as defined in [2], and the invariant local cross sections in B_ν thus obtained are called the Frenet frame of the immersion x. These cross sections enable us to define in an obvious way also local invariant cross sections in the tangent bundle B_τ. However we shall not need them in this paper, and therefore their construction will be omitted. For $n = 2$, $N = 2$ the construction of a Frenet frame in our sense was given by T. Ōtsuki in [5].

The construction of a Frenet frame leads to the definition of new invariants of the immersed manifold $x(M^n)$ called mixed curvatures, by means of which we can generalize to closed even-dimensional manifolds the K. Borsuk's theorem [1] concerning the total curvature of a closed curve in a Euclidean n-space, $n \geq 3$.

Furthermore, we give another proof of a result of D. Ferus [4] concerning the total curvature of a knotted sphere of codimension two imbedded in a Euclidean space.

In this paper all manifods and mappings are supposed to be of class C^∞.

Received June 2, 1972, and, in revised form, September 16, 1973.

1. Preliminaries

Let

$$x: M^n \to E^{n+N} \tag{1.1}$$

be an immersion of a differentiable manifold M^n in E^{n+N}. By $F(M^n)$ we denote the family of orthonormal frames $x(p)e_1 \cdots e_n e_{n+1} \cdots e_{n+N}$, such that e_i, $1 \leq i \leq n$, are tangent to the manifold $x(M^n)$ at $x(p) \in x(M^n)$, $p \in M^n$. In the sequel we use the following convention concerning indices

$$1 \leq i, j, k \leq n\,, \quad n+1 \leq r, s, t \leq n+N\,, \quad 1 \leq A, B, C \leq n+N\,.$$

In $F(M^n)$ the connection forms ω_A, ω_{AB} are defined such that

$$\omega_r = 0\,, \tag{1.2}$$

$$\omega_{ir} = \sum_j A_{rij}\omega_j\,, \qquad A_{rij} = A_{rji}\,, \tag{1.3}$$

$$\begin{aligned} d\omega_i &= \sum_k \omega_k \wedge \omega_{ki}\,, \\ d\omega_{ik} &= \sum_j \omega_{ij} \wedge \omega_{jk} + \sum_t \omega_{it} \wedge \omega_{tk}\,. \end{aligned} \tag{1.4}$$

For details see [2].

2. The Killing-Lipschitz curvature

Let e denote a unit vector of the Euclidean space E^{n+N}, which in the following is regarded also as a point on the unit sphere $S^{n+N-1} \subset E^{n+N}$. By $B_\nu \to M^n$ we denote the normal bundle of the immersion (1.1) with the base space M^n and the bundle space

$$B_\nu = \{(p, e) \,|\, e \cdot dx(p) = 0, p \in M^n, e \in S^{n+N-1}\}\,.$$

The manifold B_ν can be endowed with the Riemannian metric

$$ds^2 = \omega_1^2 + \cdots + \omega_n^2 + \omega_{n+1,n+N}^2 + \cdots + \omega_{n+N-1,n+N}^2\,,$$

so that

$$dV_{n+N-1} = dV_n \wedge d\sigma_{N-1} \tag{2.1}$$

is the measure density of this metric, where

$$dV_n = \omega_1 \wedge \cdots \wedge \omega_n\,, \tag{2.2}$$

$$d\sigma_{N-1} = (-1)^{N-1}\omega_{n+1,n+N} \wedge \cdots \wedge \omega_{n+N-1,n+N}\,. \tag{2.3}$$

The form (2.2) is the measure density of M^n induced by the immersion (1.1), and the form (2.3) is the measure density of the fiber $S^{N-1}(p)$ described by e_{n+N}. Let $\nu: B_\nu \to S^{n+N-1}$ be the mapping $(p, e) \to e$, $(p, e) \in B_\nu$. The measure density $d\sigma_{n+N-1}$ of S^{n+N-1} induced by this mapping has the form

$$\nu^* d\sigma_{n+N-1} = (-1)^{n+N-1}\omega_{1,n+N} \wedge \cdots \wedge \omega_{n+N-1,n+N} ,$$

where the forms $\omega_{i,n+N}$ are defined by (1.3), and S^{n+N-1} is described by e_{n+N}. Thus using (2.1) we have

$$\nu^* d\sigma_{n+N-1} = (-1)^n \det (A_{n+N,ij}) dV_{n+N-1} . \tag{2.4}$$

The function $L(p, e_{n+N}) = (-1)^n \det (A_{n+N,ij})$ is the Killing-Lipschitz curvature of B_ν at $(p, e_{n+N}) \in B_\nu$; for this see Chern-Lashof [2].

With the aid of the concept of the Killing-Lipschitz curvature we shall construct a Frenet frame of an immersion, namely, by $\tilde{e}_{n+N}(p)$ we denote such a vector e_{n+N} for which and a fixed $p \in M^n$, $L(p, e_{n+N})$ takes the maximal value. If the vectors $\tilde{e}_{n+N}, \tilde{e}_{n+N-1}, \cdots, \tilde{e}_r(p), n + 2 \leq r \leq n + N$, are defined, then $\tilde{e}_{r-1}(p)$ denotes such a vector e_{n+N} for which $L(p, e_{n+N})$ attains its maximal value, where $e_{n+N}(p)$ varies on the sphere $S^{r-n-1}(p) \subset S^{N-1}(p)$ and is orthogonal to the vectors $\tilde{e}_{n+N}, \cdots, \tilde{e}_r$. The uniqueness of this construction depends on the immersion (1.1) and will be assumed throughout this paper.

3. The Frenet frame of an immersion

Suppose $n \geq 2$. By

$$\tilde{e}_{n+1}(p), \tilde{e}_{n+2}(p), \cdots, \tilde{e}_{n+N}(p) , \qquad p \in U \subset M^n , \tag{3.1}$$

we denote mutually orthonormal local cross sections in $B_\nu \to M^n$, where U denotes a neighborhood of p in M^n. Then

$$e_r = \sum_s a_{rs} \tilde{e}_s , \tag{3.2}$$

where $\| a_{rs} \|$ is an orthogonal matrix. Thus

$$\omega_{ir} = \sum_s a_{rs} \tilde{\omega}_{is} , \tag{3.3}$$

where $\tilde{\omega}_{ir} = de_i \cdot \tilde{e}_r = -e_i \cdot d\tilde{e}_r$. Substituting (3.3) in (2.4) we get

$$\begin{aligned} &\omega_{1,n+N} \wedge \omega_{2,n+N} \wedge \cdots \wedge \omega_{n,n+N} \\ &= \sum_{k_1 + \cdots + k_N = n} a^{k_1}_{n+N,n+1} \cdots a^{k_N}_{n+N,n+N} \sum_{(i_1, \cdots, i_n)} \frac{1}{k_1 !} \cdots \frac{1}{k_N !} \operatorname{sign} (i_1, \cdots, i_n) \end{aligned} \tag{3.4}$$

$$\cdot \tilde{\omega}_{i_1,n+1} \wedge \cdots \wedge \tilde{\omega}_{i_{k_1},n+1} \wedge \tilde{\omega}_{i_{k_1+1},n+2} \wedge \cdots \wedge \tilde{\omega}_{i_{k_1+k_2},n+2} \wedge \cdots \wedge \tilde{\omega}_{i_n,n+N} ,$$

where $1 \leq i_k \leq n$, $n+1 \leq s_i \leq n+N$, $k_\rho \geq 0$, $1 \leq \rho \leq N$, and $(i_1, \cdots, i_n)$ denotes a permutation of $(1, \cdots, n)$. We suppose that $k_\rho = 0$ implies $a^{k_\rho}_{n+N,n+\rho} = 1$ and $a_{n+N,n+\rho} = 0$.

With the aid of (1.3) the expression (3.4) can be written in the form

$$L(p, e_{n+N})dV_n = (-1)^n \sum_{k_1+\cdots+k_N=n} a^{k_1}_{n+N,n+1} \cdots a^{k_N}_{n+N,n+N}$$
$$\cdot \sum_{(i_1,\cdots,i_n)} \frac{1}{k_1!} \cdots \frac{1}{k_N!} \operatorname{sign}(i_1, \cdots, i_n) \det(\tilde{A}_{n+1,i_1 j}, \cdots, \tilde{A}_{n+1,i_{k_1} j},$$
$$\tilde{A}_{n+2,i_{k_1+1} j}, \cdots, \tilde{A}_{n+2,i_{k_1+k_2} j}, \cdots, \tilde{A}_{n+N,i_{k_1+\cdots+k_{N-1}+1} j}, \cdots, \tilde{A}_{n+N,i_n j})dV_n ,$$

where $L(p, e_{n+N})$ denotes the Killing-Lipschitz curvature, and dV_n is defined by (2.2).

Remark 1. The functions $\tilde{A}_{rij}$ depends on $p \in M^n$ and on parameters α_{ij} defined by $e_i = \sum_j \alpha_{ij} \tilde{e}_j$, where $\tilde{e}_1, \cdots, \tilde{e}_n$ denote fixed orthonormal cross sections in the tangent bundle B_τ of M^n.

Definition 1. The function

$$C_{k_1\cdots k_N} = (-1)^n \sum_{(i_n,\cdots,i_n)} \frac{1}{k_1!} \cdots \frac{1}{k_N!} \operatorname{sign}(i_1, \cdots, i_n) \det(A_{n+1,i_1 j}, \cdots, \tag{3.5}$$
$$A_{n+1,i_{k_1} j}, A_{n+2,i_{k_1+1} j}, \cdots, A_{n+2,i_{k_1+k_2} j}, \cdots, A_{n+N,i_{k_1+\cdots+k_{N-1}+1} j}, \cdots, A_{n+N,i_n j})$$

is called the mixed curvature of the type $(k_1, \cdots, k_N)$. The mixed curvature of the type $(0, \cdots, 0, k_\rho, 0, \cdots, 0)$, $k_\rho = n$, $1 \leq \rho \leq N$, is the Killing-Lipschitz curvature of $B_\nu \rightarrow M^n$.

Remark 2. The mixed curvature is a function defined in the principal bundle B^*_ν of B_ν, i.e., the bundle over M^n, whose fiber over $p \in M^n$ consists of all orthonormal frames $x(p)e_{n+1} \cdots e_{n+N}$ of the space $E^N(p)$ normal to $x(M^n)$ at $x(p)$ determined by the fiber $S^{N-1}(p)$ of B_ν.

Let us denote

$$\oplus B_\nu = \underbrace{B_\nu \oplus \cdots \oplus B_\nu}_{N} , \tag{3.6}$$

where $\oplus$ on the right of (3.6) is the Whitney sum of bundles. The fiber of $\oplus B_\nu$ over $p \in M^n$ is denoted by $\oplus S^{N-1}(p)$. We have the inclusion map

$$B^*_\nu \rightarrow \oplus B_\nu .$$

The function $C_{k_1\cdots k_N}(p, e_{n+1}, \cdots, e_{n+N})$ has a prolongation on $\oplus B_\nu$ defined as follows: for fixed $\tilde{e}_{n+1}, \cdots, \tilde{e}_{r-1}, \tilde{e}_{r+1}, \cdots, \tilde{e}_{n+N}$, $n+1 \leq r \leq n+N$, $C_{k_1\cdots k_N}(p,$

$\tilde{e}_{n+1}, \cdots, \tilde{e}_{r-1}, e_r, \tilde{e}_{r+1}, \cdots, \tilde{e}_{n+N})$ is given by (3.5) for every $e_r \in S^{N-1}(p)$. This prolongation is denoted by the same symbol $C_{k_1\cdots k_N}(p, e_{n+1}, \cdots, e_{n+N})$.

We define a function on M^n by

$$\int_{\oplus S^{N-1}(p)} C_{k_1\cdots k_N}(p, e_{n+1}, \cdots, e_{n+N})(d\sigma_{N-1})^N , \tag{3.7}$$

where $(d\sigma_{N-1})^N = d\sigma_{N-1}^{n+1} \wedge \cdots \wedge d\sigma_{N-1}^{n+N}$, and

$$d\sigma_{N-1}^r = (-1)^{N-1}\omega_{n+1,n+r} \wedge \cdots \wedge \omega_{n+r-1,n+r} \wedge \omega_{n+r+1,n+r} \wedge \cdots \wedge \omega_{n+N,n+r} .$$

From (2.3) it follows $d\sigma_{N-1}^{n+N} = d\sigma_{N-1}$.

The function (3.7) is called the mixed curvature of M^n of the type $(k_1, \cdots, k_N)$ induced by the immersion (1.1).

Let us take the polynomial

$$P = \sum_{k_1+\cdots+k_N=n} \tilde{C}_{k_1\cdots k_N} a_{n+N,n+1}^{k_1} \cdots a_{n+N,n+N}^{k_N} \tag{3.8}$$

with coefficients $\tilde{C}_{k_1\cdots k_N} = C_{k_1\cdots k_N}(p, \tilde{e}_{n+1}, \cdots, \tilde{e}_{n+N})$ evaluated for the cross sections (3.1).

To find necessary conditions for the polynomial (3.8) to attain its maximal value at the point

$$a_{n+N,n+1} = \cdots = a_{n+N,n+N-1} = 0 , \qquad a_{n+N,n+N} = 1 \tag{3.9}$$

under the additional assumption

$$Q = a_{n+N,n+1}^2 + \cdots + a_{n+N,n+N}^2 - 1 = 0 ,$$

by equating to zero the partial derivatives of $P + \lambda Q$, where λ is a real number, with respect to $a_{n+N,n+\rho}$, $1 \le \rho \le N$, at the point (3.9) we obtain

$$\tilde{C}_{0\cdots 0k_\rho 0\cdots 0n-1} = 0 , \quad k_\rho = 1 , \quad 1 \le \rho \le N-1 . \tag{3.10_N}$$

Denote

$$G_{k_1\cdots k_N} = C_{k_1\cdots k_N} dV_n , \tag{3.11}$$

where dV_n is the form (2.2). Then with the use of (3.4) and (3.5), (3.10_N) can be written in the following form:

$$\begin{aligned} \tilde{G}_{0\cdots 0k_\rho 0\cdots 0n-1} &= \tilde{C}_{0\cdots 0k_\rho 0\cdots 0n-1} dV_n \\ &= (-1)^n \sum_{(i_1,\cdots,i_n)} \frac{1}{(n-1)!} \operatorname{sign}(i_1, \cdots, i_n) \tilde{\omega}_{i_1,n+\rho} \\ &\qquad \wedge \tilde{\omega}_{i_2,n+N} \wedge \cdots \wedge \tilde{\omega}_{i_n,n+N} \end{aligned} \tag{$3.10'_N$}$$

$$= (-1)^n \sum_{k=1}^{n} \tilde{\omega}_{1,n+N} \wedge \cdots \wedge \tilde{\omega}_{k,n+\rho} \wedge \cdots \wedge \tilde{\omega}_{n,n+N} = 0 ,$$

$$1 \leq \rho \leq N - 1 .$$

If in (3.10_N) or $(3.10'_N)$ we delete the wave line, which is significant for the fixed cross sections (3.1), we get the sought for equations, which are satisfied if the Killing-Lipschitz curvature $L(p, e_{n+N})$ attains its maximal value at $e_{n+N} = \tilde{e}_{n+N}$ for fixed $p \in M^n$.

Applying succesively the above process, which leads to the vector $\tilde{e}_{n+N}$ defined by $(3.10'_N)$, and the definition of the Frenet frame formulated at the end end of § 2 we get the system of equations

$$(3.10'_{N-\sigma}) \quad \sum_{k=1}^{n} \omega_{1,n+N-\sigma} \wedge \cdots \wedge \omega_{k,n+\rho} \wedge \cdots \wedge \omega_{n,n+N-\sigma} = 0 ,$$

$0 \leq \sigma \leq N - 2$, $1 \leq \rho \leq N - \sigma - 1$, for determining the vector $\tilde{e}_{n+N-\sigma}$ if $\tilde{e}_{n+N}, \cdots, \tilde{e}_{n+N-\sigma+1}$, $1 \leq \sigma \leq N - 2$, have already been chosen. The vector $\tilde{e}_{n+1}$ is defined as a unit vector orthogonal to the vectors

$$e_1, \cdots, e_n, \tilde{e}_{n+2}, \cdots, \tilde{e}_{n+N}$$

such that $F_{x(p)} = x(p)e_1 \cdots e_n\tilde{e}_{n+1} \cdots \tilde{e}_{n+N}$ is coherently oriented with E^{n+N}.

4. The sphere-image of an imbedded manifold

Let

$$(4.1) \qquad x : M^n \to E^{n+2}$$

be an imbedding of a closed manifold, and suppose that there exists a manifold $(M^{n+1}, \partial M^{n+1})$ with boundary $\partial M^{n+1} = M^n$, and the imbedding (4.1) is a restriction of an immersion $x : M^{n+1} \to E^{n+2}$. Then the principal bundle, associated with the normal bundle B_ν of the imbedding (4.1), and the normal bundle itself are trivial. Therefore the Frenet frame, i.e., the vectors $\tilde{e}_{n+1}$ and $\tilde{e}_{n+2}$ can be defined on the whole of M^n. In the following we consider only such imbeddings (4.1) for which the vector $\tilde{e}_{n+2}(p)$, $p \in M^n$ of the Frenet frame is uniquely determined on the whole of M^n.

Definition 2. The mapping

$$(4.2) \qquad \tilde{e}_{n+2} : M^n \to S^{n+1}$$

is called a sphere-mapping of the imbedding (4.1), and the set $\tilde{e}_{n+2}(M^n)$ the sphere-image of M^n.

In the following we suppose that (4.2) is also an imbedding. Thus S^{n+1} is divided into two regions D^{n+1} and D'^{n+1} with the common boundary $\tilde{e}_{n+2}(M^n)$.

The system $(3.10'_{N-\sigma})$ for $N = 2$ and $\sigma = 1$ takes the form

$$(4.3)\quad (-1)^n G_{1,n-1} = \sum_{k=1}^{n} \omega_{1,n+2} \wedge \cdots \wedge \omega_{k,n+1} \wedge \cdots \wedge \omega_{n,n+2} = 0$$

and is satisfied for

$$\tilde{\omega}_{i,n+2} = de_i \cdot \tilde{e}_{n+2}\ , \qquad \tilde{\omega}_{i,n+1} = de_i \cdot \tilde{e}_{n+1}\ ,$$

where $\tilde{e}_{n+1}(p)$, $\tilde{e}_{n+2}(p)$ are the Frenet cross sections. Thus we have

$$\begin{aligned}
&(-1)^n dG_{1,n-1} \\
&\quad = \sum_{i\neq k} (-1)^i \omega_{1,n+2} \wedge \cdots \wedge \omega_{k,n+2} \wedge \omega_{ik} \wedge \cdots \wedge \omega_{k,n+1} \\
&\qquad\qquad \wedge \cdots \wedge \omega_{n,n+2} \\
&\quad + \sum_{i\neq k} (-1)^i \omega_{1,n+2} \wedge \cdots \wedge \omega_{n+1,n+2} \wedge \omega_{i,n+1} \wedge \cdots \wedge \omega_{k,n+1} \\
&\qquad\qquad \wedge \cdots \wedge \omega_{n,n+2} \\
&\quad + \sum_{j,k} (-1)^k \omega_{1,n+2} \wedge \cdots \wedge \omega_{j,n+1} \wedge \omega_{kj} \wedge \cdots \wedge \omega_{n,n+2} \\
&\quad + \sum (-1)^k \omega_{1,n+2} \wedge \cdots \wedge \omega_{n+2,n+1} \wedge \omega_{k,n+2} \wedge \cdots \wedge \omega_{n,n+2} \\
&\quad = (-1)^n n \omega_{1,n+2} \wedge \omega_{2,n+2} \wedge \cdots \wedge \omega_{n,n+2} \wedge \omega_{n+1,n+2} \\
&\qquad - 2 \sum_{i<k} \omega_{n+1,n+2} \wedge \omega_{1,n+2} \wedge \cdots \wedge \omega_{i,n+1} \wedge \cdots \wedge \omega_{k,n+1} \\
&\qquad\qquad \wedge \cdots \wedge \omega_{n,n+2}\ .
\end{aligned}$$

With the use of the definition of the measure density of S^{n+1} and Definition 1 of the mixed curvatures the above formula can be rewritten in the form

$$(4.4)\qquad (-1)^{n+1} dG_{1,n-1} = n\nu^* d\sigma_{n+1} + 2G_{2,n-2} \wedge d\sigma_1\ ,$$

where $d\sigma_1 = \omega_{n+1,n+2}$.

Let the manifold M^n be the sphere S^n. Then the integral formula

$$(i_n)\qquad \int_{D^{n+1}} n d\sigma_{n+1} + 2\int_{D^{n+1}} G_{2,n-2} \wedge d\sigma_1 = 0\ , \qquad n \geq 2\ ,$$

is valid, where $D^{n+1} \subset S^{n+1}$ is bounded by $\tilde{e}_{n+2}(M^n)$.

Proof of (i_n). Let $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n+1}$ denote mutually orthogonal cross sections in the tangent bundle of the sphere S^{n+1} over D^{n+1}. Then from Stokes' theorem and (4.4) it follows

$$(4.5)\quad (-1)^{n+1} \int_{\tilde{e}_{n+2}(M^n)} \bar{G}_{1,n-1} = \int_{D^{n+1}} n d\sigma_{n+1} + 2\int_{D^{n+1}} \bar{G}_{2,n-2} \wedge d\bar{\sigma}_1\ ,$$

where the bar over the forms in (4.5) means that they are evaluated for the fixed cross sections. The cross sections can be defined, e.g., by a stereographic projection of the orthogonal net of a Euclidean space E^{n+1} onto the sphere S^{n+1} from a point $q' \in D'^{n+1} \subset S^{n+1}$ such that its antipodal q belongs to $D^{n+1} \backslash \tilde{e}_{n+2}(M^n)$.

In E^{n+1} we introduce the spherical coordinates with the pole at the image of q. By means of the stereographic projection this coordinate system defines an orthogonal coordinate system in $S^{n+1} \backslash \{q'\}$ with the exeptional point q. We denote the unit vectors tangent to the new coordinat curves in $S^{n+1} \backslash \{q'\}$ again by

$$\bar{e}_\lambda , \qquad 1 \leq \lambda \leq n+1 , \tag{4.6}$$

an orthonormal base of the tangent space T_p^n, $p \in M^n$, of the surface $\tilde{e}_{n+2}(M^n) \subset S^{n+1}$ by

$$\hat{e}_1, \cdots, \hat{e}_n , \tag{4.7}$$

and the unit normal at $p \in M^n$ to the surface $\tilde{e}_{n+2}(M^n) \subset S^{n+1}$ by $\hat{e}_{n+1}$.

We prove that the left-hand member of (4.5) vanishes if M^n is the sphere S^n ($n \geq 2$). Since (4.2) is supposed to be an inbedding, we have

$$\det (\tilde{A}_{n+2,ij}) \neq 0$$

for every $p \in M^n$, or equivalently

$$\tilde{\omega}_{n+2,1} \wedge \cdots \wedge \tilde{\omega}_{n+2,n} \neq 0 . \tag{4.8}$$

From

$$d\tilde{e}_{n+2} = \tilde{\omega}_{n+2,i} e_i + \tilde{\omega}_{n+2,n+1} \tilde{e}_{n+1}$$

and (4.8) it follows that T_p^n spanned by the vectors (4.7) is transversal to $\tilde{e}_{n+1}$ for every $p \in M^n$. Thus the vectors $\hat{e}_{n+1}$, $\tilde{e}_{n+1}$ are in the same half-space of the tangent space T_p^{n+1} of S^{n+1} at $\tilde{e}_{n+2}(p)$ defined by T_p^n, so that we can suppose that the plane spanned by $\hat{e}_n, \hat{e}_{n+1}$ coincides with that spanned by $\hat{e}_n$, $\tilde{e}_{n+1}$. Then the formula

$$e_{n+1}^t = \hat{e}_n \sin \alpha t + \hat{e}_{n+1} \cos \alpha t , \qquad 0 \leq t \leq 1 , \tag{4.9}$$

where $\alpha = \arccos (\hat{e}_{n+1} \cdot \tilde{e}_{n+1})$, defines a homotopy which joins $\hat{e}_{n+1}$ with $\tilde{e}_{n+1}$ such that e_{n+1}^t remains transversal to T_p^n for every t, $0 \leq t \leq 1$.

Since

$$\hat{e}_\lambda = a_{\lambda i} e_i + a_{\lambda n+1} e_{n+1} , \qquad 1 \leq \lambda, \gamma \leq n+1 ,$$

it follows

$$\hat{\omega}_{i,n+1} = -a_{i\lambda} da_{n+1,\lambda} + a_{i\lambda} a_{n+1,\gamma} \bar{\omega}_{\lambda\gamma} , \tag{4.10}$$

$$\hat{\omega}_{i,n+2} = a_{i\lambda}\tilde{\omega}_{\lambda,n+2} \; . \tag{4.11}$$

By the use of (4.10) and (4.11) we get

$$\begin{aligned}(-1)^n\hat{G}_{1,n-1} &= \sum_{k=1}^{n} \hat{\omega}_{1,n+2} \wedge \cdots \wedge \hat{\omega}_{k,n+1} \wedge \cdots \wedge \hat{\omega}_{n,n+2} \\ &= \sum_{k=1}^{n} \hat{\omega}_{1,n+2} \wedge \cdots \wedge a_{k\lambda}da_{n+1,\lambda} \wedge \cdots \wedge \hat{\omega}_{n,n+2} \\ &\quad + \sum_{k=1}^{n} \tilde{\omega}_{1,n+2} \wedge \cdots \wedge \tilde{\omega}_{k,n+1} \wedge \cdots \wedge \tilde{\omega}_{n,n+2} \; .\end{aligned} \tag{4.12}$$

Let k, $1 \leq k \leq n$, be a fixed integer, and suppose that the mutually orthogonal cross sections (4.7) define a spherical coordinate net on $M^n = S^n$ such that the equations

$$\hat{\omega}_{1,n+2} = \cdots = \hat{\omega}_{k-1,k+2} = \hat{\omega}_{k+1,n+2} = \cdots = \hat{\omega}_{n,n+2} = 0 \tag{4.13}$$

defines a family of circles. Let us take a fixed circle S^1 defined by (4.13). On S^1 we consider the linear form $a_{k\lambda}(s)da_{n+1,\lambda}(s)$, where s denotes the parameter on S^1. In the coordinate system $e_1, \cdots, e_n, \tilde{e}_{n+1}$ we have

$$\begin{aligned}&\hat{e}_k(s) = (a_{k1}(s), \cdots, a_{k,n+1}(s)) \; , \\ &\hat{e}_{n+1} = (a_{n+1,1}(s), \cdot\cdot, a_{n+1,n+1}(s)) \; , \qquad \tilde{e}_{n+1}(s) = (0, \cdots, 0, 1) \; ,\end{aligned} \tag{4.14}$$

so that

$$a_{k\lambda}(s)da_{n+1,\lambda}(s) = -da_{k\lambda}(s)a_{n+1,\lambda}(s) \; .$$

Since by (4.9) and the transversality of $\tilde{e}_{n+1}$ to T^n_p the mappings

$$\hat{e}_{n+1}: S^1 \to S^1 \; , \qquad \tilde{e}_{n+1}: S^1 \to S^1$$

have the same degree, we obtain, with the use of (4.14),

$$\begin{aligned}\int_{\tilde{e}_{n+2}(S^1)} a_{k\lambda}(s)da_{n+1,\lambda}(s) &= -\int_{\tilde{e}_{n+2}(S^1)} da_{k\lambda}(s)a_{n+1,\lambda}(s) \\ &= -\int_{\tilde{e}_{n+2}(S^1)} da_{k,n+1}(s) = 0 \; ,\end{aligned} \tag{4.15}$$

and therefore

$$\begin{aligned}&\int_{\tilde{e}_{n+2}(M^n)} \hat{\omega}_{1,n+2} \wedge \cdots \wedge a_{k\lambda}da_{n+1,\lambda} \wedge \cdots \wedge \hat{\omega}_{n,n+2} \\ &\quad = \pm\int \hat{\omega}_{1,n+2} \wedge \cdots \wedge \hat{\omega}_{k-1,n+2} \wedge \hat{\omega}_{k+1,n+2} \wedge \cdots \wedge \hat{\omega}_{n,n+2} \\ &\qquad \cdot \int_{\tilde{e}_{n+2}(S^1)} a_{k\lambda}da_{n+1,\lambda} = 0 \; .\end{aligned} \tag{4.16}$$

By repeating the cconstruction for $k = 1, 2, \cdots, n$, from (4.3), (4.12) and (4.16) we have

$$\int_{\tilde{e}_{n+2}(M^n)} \hat{G}_{1,n-1} = 0 \qquad \text{for } M^n = S^n . \tag{4.17}$$

Let us set

$$\hat{e}_\lambda = b_{\lambda\gamma}\bar{e}_\gamma , \qquad 1 \leq \lambda, \gamma \leq n+1 ,$$

where $\bar{e}_\lambda$ are the vectors (4.6). As in the previous case (see (4.10), (4.11) and (4.12)) we get

$$\begin{aligned}(-1)^n&\hat{G}_{1,n-1}\\ &= \sum_{i=1}^{n} \hat{\omega}_{1,n+2} \wedge \cdots \wedge b_{i\lambda}db_{n+1,\lambda} \wedge \cdots \wedge \hat{\omega}_{n,n+2} + (-1)^n\bar{G}_{1,n-1} .\end{aligned} \tag{4.18}$$

The vector field $\hat{e}_{n+1}$ defined on $\tilde{e}_{n+2}(M^n)$ and the vector field $\bar{e}_{n+1}$ restricted to $\tilde{e}_{n+2}(M^n)$ are homotopic. Indeed, if we project S^{n+1} from $q' \in D'^{n+1}$ by a stereographic projection s on E^{n+1}, then $s \circ \tilde{e}_{n+2}(M^n) \subset E^{n+1}$ is a topological sphere such that $s(D^{n+1})$ contains the origin $s(q)$ of E^{n+1}, $s_*\hat{e}_{n+1}$ is a normal vector field of the surface $s \circ \tilde{e}_{n+2}(M^n)$, and $s_*\bar{e}_{n+1}$ is a vector field, whose vectors lie on straight lines passing through the origin of E^{n+1}. Thus in both cases the degrees of the mappings

$$s_*\hat{e}_{n+1} \colon M^n \to S^n , \quad s_*\bar{e}_{n+1} \colon M^n \to S^n , \quad M^n = S^n ,$$

are equal to ± 1. This implies that $s_*\hat{e}_{n+1}$ and $s_*\bar{e}_{n+1}$ are homotopic, and therefore $\hat{e}_{n+1}$, $\bar{e}_{n+1}$ are homotopic. In particular, $\hat{e}_{n+1}$, $\bar{e}_{n+1}$ are homotopic on every circle $S^1 \subset M^n = S^n$, and therefore as in the case (4.15) we get

$$\int_{\tilde{e}_{n+2}(S^1)} b_{k\lambda}db_{n+1,\lambda} = 0 ,$$

which implies (see (4.16))

$$\sum_{i=1}^{n} \int_{\tilde{e}_{n+2}(M^n)} \hat{\omega}_{1,n+2} \wedge \cdots \wedge b_{i\lambda}db_{n+1,\lambda} \wedge \cdots \wedge \hat{\omega}_{n,n+2} = 0 . \tag{4.19}$$

From (4.17), (4.18) and (4.19) we have

$$\int_{\tilde{e}_{n+2}(M^n)} \bar{G}_{1,n-1} = 0 \qquad \text{for } M^n = S^n . \tag{4.20}$$

Let $S^n_\varepsilon \subset s(D^{n+1})$ denote a sphere with the center $s(q) = 0$ and small radius ε. The spherical coordinate system introduced in $D^{n+1} \backslash K^{n+1}_\varepsilon$, where K^{n+1}_ε is the inverse image of the ball contained in E^{n+1} and bounded by S^n_ε, is regular.

Since q is an exceptional point of the spherical coordinate net in formula (4.5), the left-hand member must be replaced by

$$\int_{\bar{e}_{n+2}(M^n)} \bar{G}_{1,n-1} - \int_{\partial K_\varepsilon^{n+1}} \bar{G}_{1,n-1} . \tag{4.21}$$

By (4.20) the proof of (i_n) is finished if we show that the second member of (4.21) tends to zero with ε tending to zero. For this purpose, in S^{n+1} we define spherical coordinates by the formulas

$$\begin{aligned} x_1 &= \cos \varphi_1 , \\ x_\lambda &= \sin \varphi_1 \cdots \sin \varphi_{\lambda-1} \cos \varphi_\lambda , \qquad 2 \leq \lambda \leq n+1 , \\ x_{n+2} &= \sin \varphi_1 \cdots \sin \varphi_n \sin \varphi_{n+1} . \end{aligned}$$

Suppose that 0 as a point of E^{n+2} has coordinates $(1, 0, \cdots, 0)$. Then $\varphi_1 =$ const is the equation of $\partial K_\varepsilon^{n+1} = S^n(\varphi_1)$. Since $e_{n+2} = (x_1, \cdots, x_{n+2})$, where x_λ are defined by the spherical coordinates, and

$$e_\lambda = \frac{\partial e_{n+2}}{\partial \varphi_{n-\lambda+2}} \cdot \frac{1}{d} ,$$

where

$$d = \left| \frac{\partial e_{n+2}}{\partial \varphi_{n-\lambda+2}} \right| , \qquad 2 \leq \lambda \leq n+1 ,$$

we have

$$\begin{aligned} \bar{e}_1 &= (0, \cdots, -\sin \varphi_{n+1}, \cos \varphi_{n+1}) , \\ \bar{e}_\lambda &= (0, \cdots, -\sin \varphi_{n-\lambda+2}, \cos \varphi_{n-\lambda+2} \cos \varphi_{n-\lambda+3}, \\ &\qquad \cdots, \cos \varphi_{n-\lambda+2} \sin \varphi_{n-\lambda+3} \cdots \sin \varphi_{n+1}) . \end{aligned}$$

From

$$\bar{\omega}_{k,n+1} = d\bar{e}_k \cdot \bar{e}_{n+1} , \qquad \bar{\omega}_{i,n+2} = d\bar{e}_i \cdot e_{n+2} ,$$

it follows that

$$\begin{aligned} &\sum_{k=1}^{n} \bar{\omega}_{1,n+2} \wedge \cdots \wedge \bar{\omega}_{k,n+1} \wedge \cdots \wedge \bar{\omega}_{n,n+2} \\ &\qquad = (-1)^{\frac{1}{2}n(n+1)} n \cos \varphi_1 \sin^{n-1} \varphi_2 \cdots \sin \varphi_n d\varphi_2 \wedge \cdots \wedge d\varphi_{n+1} , \end{aligned}$$

and therefore that

$$\int_{\partial K_\varepsilon^{n+1}} \bar{G}_{1,n-1}$$
$$= \pm n \cos\varphi_1 \sin^{n-1}\varphi_1 \int_{S^n(\varphi_1)} \sin^{n-1}\varphi_2 \cdots \sin\varphi_n d\varphi_2 \wedge \cdots \wedge d\varphi_{n+1} .$$

If $\varepsilon \to 0$, then $\varphi_1 \to 0$, and we get

$$\lim_{\varphi_1\to 0} \int_{S^n(\varphi_1)} \bar{G}_{1,n-1} = 0 ,$$

which completes the proof of (i_n).

Remark 3. In the case $n = 1$, i.e., for closed curves $x(C)$ in E^3, (4.3) is reduced to

$$\omega_{13} = 0 .$$

Thus we have $d\omega_{13} = \omega_{12} \wedge \omega_{23}$, and the formula (4.5) for $n = 1$ takes the form

$$\text{(4.22)} \qquad \int_{\tilde{e}_2(C)} \bar{\omega}_{13} = \int_D \bar{\omega}_{12} \wedge \bar{\omega}_{23} = |D| ,$$

where $\tilde{e}_2$ is the principal normal, $|D|$ is the area of the region $D \subset S^2$ bounded by $\tilde{e}_2(C) \subset S^2$, and the differential forms are evaluated in the spherical coordinate system (ϑ, φ) defined by

$$x_1 = \cos\vartheta , \qquad x_2 = \sin\vartheta\cos\varphi , \qquad x_3 = \sin\vartheta\sin\varphi$$

with the pole belonging to the interior of D. As in the case $n \geq 2$ the left-hand member of (4.22) vanishes. On the circle $\vartheta = \text{const}$ we have $\bar{\omega}_{13} = d\bar{e}_1 \cdot \bar{e}_3 = -\cos\vartheta d\varphi$, where

$$\bar{e}_1 = (0, -\sin\varphi, \cos\varphi) , \qquad e_3 = (-\sin\vartheta, \cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi) .$$

This together with (4.22) implies the known theorem of Jacobi: $|D| = 2\pi$.

As in Remark 3 a similar geometric interpretation of the formula (i_2) is possible, namely, for $n = 2$ we have

$$(i_2) \qquad \int_{D^3} \omega_{14} \wedge \omega_{24} \wedge \omega_{34} + \int_{D^3} \omega_{13} \wedge \omega_{23} \wedge \omega_{43} = 0 .$$

The form $\omega_{14} \wedge \omega_{24} \wedge \omega_{34}$ is the measure density of S^3, while $\bar{\omega}_{13} \wedge \bar{\omega}_{23} \wedge \bar{\omega}_{43}$ gives an interpretation if e_4 varies in D^3, and the bar over the forms indicates that they are evaluated in the spherical coordinate system of S^3 with the pole q. Let

$$\bar{e}_1 \to \bar{e}_1 , \quad \bar{e}_2 \to \bar{e}_2 , \quad \bar{e}_3 \to e_4 , \quad e_4 \to -\bar{e}_3$$

be the change of coordinates in the second member of (i_2). If e_4 varies in D^3, then $\bar{e}_3$ varies in $H^3 \backslash G^3$, where H^3 is the half-sphere of S^3 with the line of symmetry qq', and G^3 is the domain of S^3 bounded by $\tilde{e}_3(M^2)$, where $M^2 = S^2$. Hence

$$\int_{D^3} \bar{\omega}_{13} \wedge \bar{\omega}_{23} \wedge \bar{\omega}_{43} = \int_{H^3 \backslash G^3} \bar{\omega}_{14} \wedge \bar{\omega}_{24} \wedge \bar{\omega}_{34}$$

is the measure of $H^3 \backslash G^3$, yielding our promised generalized theorem of Jacobi.

Theorem 1. *If $x: S^2 \to E^4$ is an imbedding such that $\tilde{e}_4: S^2 \to S^3$ is also an imbedding, then the algebraic sum of the volumes of the domains on S^3 bounded by $\tilde{e}_3(S^2)$ and $\tilde{e}_4(S^2)$ is equal to the measure of the half-sphere:*

$$\pm |D^3| \pm |G^3| = \pi^2 ,$$

the signes being chosen independently for each member.

5. A theorem of D. Ferus

In this section a generalization of Borsuk's theorem [1] concerning the total curvature of a closed curve in the Euclidean n-space, $n \geq 3$, is proved. Furthermore we give another proof of a result of D. Ferus [4] concerning the total curvature of knotted spheres.

It is known that

$$\text{degree of} \quad \nu = \chi(M^n) , \tag{5.1}$$

where $\chi(M^n)$ denotes the Euler number of M^n, and ν is defined in § 2; for the proof of (5.1) see for instance [4]. The formula (5.1) together with the interpretation of the Killing-Lipschitz curvature $L(p, e)$, $p \in M^n$, $e \in S^{n+N-1}$, as the Jacobien determinant of the mapping ν yields

$$\int_{B_\nu} L(p, e) dV_{n+N-1} = c_{n+N-1} \chi(M^n) , \tag{5.2}$$

and (see [3])

$$\int_{B_\nu} |L(p, e)| \, dV_{n+N-1} \geq c_{n+N-1} \sum_{k=0}^{n} b_k ,$$

where c_{n+N-1} denotes the volume of the unit sphere S^{n+N-1}, and b_k is the k-th Betti number of M^n.

The invariant $L(p, \tilde{e}_r)$ of the surface $x(M^n)$, where $\tilde{e}_r$ is the r-th vector of the Frenet frame, is called the r-th curvature of $x(M^n)$. Our next purpose is the calculation of the integral

$$\int_{M^n} L(p, \tilde{e}_{n+N}) dV_n \; .$$

Let us set $e_{n+N} = t_r \tilde{e}_r$. Then using (2.4) we get

$$\nu^* d\sigma_{n+N-1} = (-1)^n \det (t_r \tilde{A}_{rij}) dV_n \wedge \sum_{\alpha=1}^{N} (-1)^{\alpha-1} t_{n+\alpha} dt_{n+1} \wedge \cdots \wedge dt_{n+\alpha-1} \wedge dt_{n+\alpha+1} \wedge \cdots \wedge dt_{n+N} \; .$$

From the definition of the mixed curvatures it follows

$$\begin{aligned} L(p, e_{n+N}) &= (-1)^n \det (t_r \tilde{A}_{rij}) \\ &= \sum_{k_1 + \cdots + k_N = n} t_{n+1}^{k_1} \cdots t_{n+N}^{k_N} \tilde{C}_{k_1 \cdots k_N} \; , \end{aligned} \tag{5.3}$$

where $0 \leq k_\rho \leq n$, $1 \leq \rho \leq N$. In S^{N-1} we introduce the spherical coordinates

$$\begin{aligned} t_{n+1} &= \cos \theta_{n+1} \; , \\ t_{n+\rho} &= \sin \theta_{n+1} \cdots \sin \theta_{n+\rho-1} \cos \theta_{n+\rho} \; , \qquad 2 \leq \rho \leq N-1 \; , \\ t_{n+N} &= \sin \theta_{n+1} \cdots \sin \theta_{n+N-1} \; , \end{aligned}$$

where $0 \leq \theta_{n+\rho} \leq \pi$ for $1 \leq \rho \leq N-2$, and $0 \leq \theta_{n+N-1} < 2\pi$. We have

$$\begin{aligned} &\sum_{k_1 + \cdots + k_N = n} t_{n+1}^{k_1} \cdots t_{n+N}^{k_N} \sum_{\alpha=1}^{N} (-1)^{\alpha-1} t_{n+\alpha} dt_{n+1} \wedge \cdots \wedge dt_{n+\alpha-1} \\ &\qquad\qquad \wedge dt_{n+\alpha+1} \wedge \cdots \wedge dt_{n+N} \\ &= \sum_{k_1 + \cdots + k_N = n} \cos^{k_1} \theta_{n+1} \sin^{k_2 + \cdots + k_N} \theta_{n+1} \sin^{N-2} \theta_{n+1} \cos^{k_2} \theta_{n+2} \\ &\qquad \cdot \sin^{k_3 + \cdots + k_N} \theta_{n+2} \sin^{N-3} \theta_{n+2} \cdots \cos^{k_{N-1}} \theta_{n+N-1} \sin^{k_N} \theta_{n+N-1} \\ &\qquad\qquad \cdot d\theta_{n+1} \wedge \cdots \wedge d\theta_{n+N-1} \; . \end{aligned} \tag{5.4}$$

The integration of the function $L(p, e_{n+N})$ with respect to $\theta_{n+\rho}$, $1 \leq \rho \leq N-1$, is reduced by (5.3) and (5.4) to the integrals

$$\int_0^\pi \cos^{k_\rho} \theta_{n+\rho} \sin^{k_{\rho+1} + \cdots + k_N + N - \rho - 1} \theta_{n+\rho} d\theta_{n+\rho} \; , \qquad 1 \leq \rho \leq N-2 \; , \tag{5.5}$$

$$\int_0^{2\pi} \cos^{k_{N-1}} \theta_{n+N-1} \sin^{k_N} \theta_{n+N-1} d\theta_{n+N-1} \; . \tag{5.6}$$

Thus we get the formula

$$\int_{B_\nu} L(p, e_{n+N}) dV_n \wedge d\sigma_{N-1} = \begin{cases} (2\pi)^{N/2} K \; , & \text{for } N \text{ even} \; , \\ 2^{(N+1)/2} \pi^{(N-1)/2} K, & \text{for } N \text{ odd} \; , \end{cases} \tag{5.7}$$

where

$$(5.8) \qquad K = \sum_{k_1+\cdots+k_N=n} \frac{(k_1-1)!!\cdots(k_N-1)!!}{(n+N-2)!!}\int_{M^n}\tilde{C}_{k_1\cdots k_N}dV_n ,$$

$k_1, \cdots, k_N$ are even integers or zero, $k!! = 1\cdot 3\cdot\cdots\cdot k$ for k odd, $k!! = 2\cdot 4\cdot\cdots\cdot k$ for k even, and $(-1)!! = 1$.

If in the sequence $k_1, \cdots, k_N$ at least one number is an odd integer, then at least one of the integrals (5.5), (5.6) vanishes. Since $k_1 + \cdots + k_N = n$, the integral (5.7) can be different from zero only for n even.

From (5.2) and (5.7) with the use of (5.8) we hence obtain a generalization of Borsuk's theorem.

Theorem 2. *If the immersion $x: M^n \to E^{n+N}$ of an orientable even-dimensional closed manifold is such that $\tilde{C}_{k_1\cdots k_N}(p) \leq 0$ for every $p \in M^n$ and $k_N \neq n$, then*

$$\int_{M^n} L(p, \tilde{e}_{n+N})dV_n \geq c_n\chi(M^n) .$$

Remark 4. If $M^n \subset E^{n+1} \subset E^{n+N}$, then $\tilde{C}_{k_1\cdots k_N}(p) = 0$ for every $p \in M^n$ and $k_N \neq n$. In the next section we prove the converse statement.

Let $M^n = S^n$, and let $x: S^n \to E^{n+2}$ be an imbedding such that $x(S^n) \subset E^{n+2}$ is a knotted sphere. Then we have

Theorem 3 (*D. Ferus*). *If $x(S^n) \subset E^{n+2}$ is a knotted sphere, then*

$$(5.9) \qquad \int_{B_\nu} |L(p, e)|\, dV_n \wedge d\sigma_1 \geq 4c_{n+1} .$$

From (5.9) we see that the degree of the mapping $\nu: B_\nu \to S^{n+1}$ is at least four for almost every $p \in S^n$. Hence Theorem 3 follows from the following lemma.

Lemma. *Let $x: S^n \to E^{n+2}$, $n \geq 2$, be an imbedding with the property: there exists a neighborhood $U \subset S^{n+1}$ such that for every $e \in U$ the function $e\cdot x(p)$, $p \in S^n$, has exactly two nondegenerate critical points. Then the imbedding x is topologicaly equivalent to the standard inclusion $S^n \subset E^{n+1} \subset E^{n+2}$.*

By a critical point of $e\cdot x(p)$ we mean a point $p \in S^n$ for which $e\cdot dx(p) = 0$, and hence $(p, e) \in B_\nu$. A critical point is nondegenerate if $e\cdot d^2x(p)$ is a nondegenerate quadratic form or, equivalently,

$$L(p, e) \neq 0 .$$

The formula (5.9) follows from the lemma by the remark that the number of nondegenerate critical points for a sphere can change only by an even number.

Proof of the lemma. For the proof it suffices to construct an isotopy

$$\varphi_\tau: S^n \to E^{n+2} , \qquad 0 \leq \tau \leq 1 ,$$

such that $\varphi_0(S^n) = x(S^n)$, $\varphi_1(S^n) \subset E^{n+1}$. Indeed, for an isotopy φ_τ there exists a diffeotopy (see [4])

$$\phi_\tau : E^{n+2} \to E^{n+2}$$

such that

$$\phi_\tau \circ \varphi_0 = \varphi_\tau \ , \quad 0 \leq \tau \leq 1 \ , \quad \phi_0 = \text{identity on } E^{n+2} \ .$$

The construction of φ_τ. Let $p_1, p_2 \in S^n$ denote the only nondegenerate critical points of $e \cdot x(p)$ for a fixed $e \in U$. We assume that $e = e_{n+2}$, $x(p_2) = (0, \cdots, 0, 1)$ and therefore $e \cdot x(p_2) = 1$. Moreover we can assume that $x(p_1) = (0, \cdots, 0, -1) = -e_{n+2}$. Indeed, if $x(p_1) \neq -e_{n+2}$, then the vector

$$-(e_{n+2} + x(p_1))e^{1-c^{-2}} \ , \qquad 0 \leq c \leq 1 \ ,$$

defines a displacement T_c of the hyperplane $x_{n+2} = -c$ such that T_1 leads $x(p_1)$ into $-e_{n+2}$, and all the T_c, $0 \leq c \leq 1$, define in an obvious way a diffeotopy of the imbedding x such that x remains unchanged for $x_{n+2} \geq 0$.

On S^n the vector field grad $e \cdot x(p)$ is a nonzero field except at the points p_1, p_2. Indeed, assume that for some $p_3 \in S^n$ different from p_1 and p_2 we have grad $e \cdot x(p_3) = 0$. Then p_3 would be a critical point of $e \cdot x(p)$. But by Sard's theorem we can first suppose that $e \cdot x(p)$ has only nondegenerate critical points. Then $e \cdot x(p)$ would have at least three nondegenerate critical points, contrary to the assumption of the lemma.

It follows that the height function $e \cdot x(p) = e_{n+2} \cdot x(p)$ is monotonic on every integral line of grad $e \cdot x(p)$, and therefore every hyperplane $x_{n+2} = c$, $-1 < c < 1$, intersects $x(S^n)$ in a set which is diffeomorphic to the sphere S^{n-1}. Suppose that the tangent space of $x(S^n)$ at $x(p_2)$ is defined by the equations $x_{n+1} = 0$, $x_{n+2} = 1$. Then there exists a neighborhood $V \subset S^n$ of p_2 such that the imbedding $x(p) = (x_1(p), \cdots, x_{n+1}(p), x_{n+2}(p))$, $p \in S^n$, can be represented by the functions

$$x_{n+1} = g(x_1, \cdots, x_n) \ , \qquad x_{n+2} = f(x_1, \cdots, x_n) \ ,$$

where $(x_1, \cdots, x_n) \in V_1$, and V_1 denotes an open subset of the image of the mapping $p \to (x_1(p), \cdots, x_n(p))$ for $p \in V$, which contains $(x_1(p_2), \cdots, x_n(p_2))$. In particular, the projection of $x(V)$ into the subspace defined by $x_{n+1} = 0$ is a diffeomorphism at least for an open subset contained in V and containing p_2, which we suppose to coincide with V. We deform the imbedding x by a diffeotopy which acts only on the coordinate $x_{n+1}(p)$ for $p \in V$ in the following manner. Let us assume that

$$a \leq x_{n+2}(p) \leq 1 \qquad \text{for } p \in V \ , 0 < a < 1 \ ,$$

and define $\eta = \frac{1}{4}(1 - a)$. Let $\varepsilon(z)$ be a real C^∞ function such that

$$\varepsilon(z) = \begin{cases} 1 & \text{for } -\infty < z \leq 1 \,, \\ 0 & \text{for } 1 + \eta \leq z < \infty \,, \end{cases}$$

and is decreasing for $1 \leq z \leq 1 + \eta$. Define

$$\varepsilon_\tau(z) = \varepsilon((1 - \tau)z + \tau(z + 2\eta)) \,, \qquad 0 \leq \tau \leq 1 \,.$$

The formulas

$$p \to (x_1(p), \cdots, \varepsilon_\tau(x_{n+2}(p))x_{n+1}(p), x_{n+2}(p)) \qquad \text{for } p \in V$$

and $p \to x(p)$ for $p \in S^n \backslash V$ define a diffeotopy with the properties:

a) $x_0(p) = x(p)$ for every $p \in S^n$,

b) $x_1(p)$ for $1 - \eta < x_{n+2} \leq 1$ can be represented in the form

$$(5.10) \quad x_{n+2} = f(x_1, \cdots, x_n) \,, \quad x_{n+1} = 0 \,, \quad f(0) = 1 \,, \quad f_i(0) = 0 \,,$$

where $f_i(0)$ denotes the i-th derivative of f evaluated at the origin. We suppose also that the second derivatives f_{ij} vanish at the origin for $i \neq j, 1 \leq i, j \leq n$. Thus we can assume that the imbedding $x\colon S^n \to E^{n+2}$ already allows a representation (5.10) in a neighborhood of p_2.

Since an integral curve

$$(x_1(t), \cdots, x_{n+1}(t), x_{n+2}(t))$$

of the gradient field is such that $x_{n+2}(t)$ is a monotone function, we can represent the curve in the form

$$(\bar{x}_1(x_{n+2}), \cdots, \bar{x}_{n+1}(x_{n+2}), x_{n+2}) \,.$$

For a fixed point $p \in S^n$ of the integral curve regarded as a curve on S^n we define the isotopy φ_τ by the formulas

$$(5.11) \quad \begin{aligned} x_\alpha(p, \tau) &= \sqrt{\frac{1 - x_{n+2}^2(p)}{1 - [\tilde{x}_{n+2}(p, \tau)]^2}}\,\tilde{x}_\alpha(p, \tau) \,, \\ &\qquad 1 \leq \alpha \leq n + 1 \,, 0 \leq \tau < 1 \,, \end{aligned}$$

$$(5.12) \quad x_{n+2}(p, \tau) = x_{n+2}(p) \,, \qquad 0 \leq \tau \leq 1 \,,$$

where

$$\begin{aligned} \tilde{x}_{n+2}(p, \tau) &= (1 - \tau)x_{n+2}(p) + \tau \,, \qquad -1 < x_{n+2}(p) < 1 \,, \\ \tilde{x}_\alpha(p, \tau) &= \bar{x}_\alpha(\tilde{x}_{n+2}(p, \tau)) \,, \quad \bar{x}_\alpha(x_{n+2}(p)) = x_\alpha(p) \,, \quad 1 \leq \alpha \leq n + 1 \,. \end{aligned}$$

For $x_{n+2} = \pm 1$ we set respectively

$$(5.13)\qquad \begin{aligned} x(p_2, \tau) &= (0, \cdots, 0, 1)\,, & 0 \le \tau < 1\,, \\ x(p_1, \tau) &= (0, \cdots, 0, -1)\,, & 0 \le \tau \le 1\,. \end{aligned}$$

If $1 - \eta < \tilde{x}_{n+2} \le 1$, then the isotopy (5.11), (5.12) takes the form

$$(5.14)\qquad \begin{aligned} f\Big(\sqrt{\frac{1-[(1-\tau)x_{n+2}+\tau]^2}{1-x_{n+2}^2}}x_1, \cdots, \sqrt{\frac{1-[(1-\tau)x_{n+2}+\tau]^2}{1-x_{n+2}^2}}x_n\Big) \\ = (1-\tau)x_{n+2} + \tau\,, \end{aligned}$$

where f denotes the function (5.10), and $(x_1, \cdots, x_n)$ belongs to the image of the mapping $p \to (x_1(p), \cdots, x_n(p))$, $p \in V_1$. We complete the formula (5.14) for $\tau = 1$ by

$$(5.15)\qquad -\frac{1}{1-x_{n+2}^2}(f_{11}x_1^2 + \cdots + f_{nn}x_n^2) = 1\,,$$

where f_{ii} denotes the second derivative of f with respect to x_i evaluated at the origin. Then the formula (5.13) is also valid for $\tau = 1$.

Since (5.15) is the equation of an ellipsoid, it follows that $\varphi_1(p) = x(p, 1)$ is an ellipsoid of E^{n+1} spanned by the vectors $e_1, \cdots, e_n, e_{n+2}$. This proves the lemma.

6. A property of the mixed curvatures

Let us suppose that

$$(6.1)\qquad \tilde{C}_{0\cdots 0k_\rho 0\cdots 0(n-k_\rho)}(p) = 0 \qquad \text{for } p \in U \subset M^n\,,$$

$1 \le k \le n$, $1 \le \rho \le \sigma$, $1 \le \sigma \le N-1$, and that for some r, $n+\rho+1 \le r \le n+N$ the quadratic form

$$(6.2)\qquad \tilde{A}_{rij}(p)t_it_j \qquad \text{for } p \in U$$

is positive (or negative) definite. Then every matrix $\|\tilde{A}_{n+\rho,ij}\|$, $1 \le \rho \le \sigma$, is a zero matrix.

If moreover

$$(6.3)\qquad \tilde{\omega}_{rs}(p) = d\tilde{e}_r(p)\cdot\tilde{e}_s(p) = 0\,, \qquad p \in U\,,$$

where $n+1 \le r \le n+\sigma$, $n+1 \le s \le n+N$, then

$$(6.4)\qquad x(U) \subset E^{n+N-\sigma} \subset E^{n+N}\,.$$

We prove that $\|\tilde{A}_{n+1,ij}\|$ is a zero matrix. The proof for the remaining matrices is the same up to notation. We assume that in (6.2) $r = n + N$.

From (6.1) it follows

$$\tilde{C}_{n0\cdots 0}(p) = L(p, \tilde{e}_{n+1}) = 0 \qquad \text{for } p \in U , \tag{6.5}$$

so that we can assume that the row

$$\tilde{A}_{n+1,11}, \cdots, \tilde{A}_{n+1,1n} \tag{6.6}$$

of $L(p, \tilde{e}_{n+1})$ depends on the remaining rows. We can suppose that (6.6) represents a zero vector. Indeed, if not, then by a suitable change of coordinates in the tangent space of $x(U)$ at $p \in U$, defined by

$$e_i = \alpha_{ik} f_k ,$$

where $\|\alpha_{ik}\|$ is an orthogonal matrix, we get, in consequence of (1.3),

$$\Omega_r = \alpha^T A_r \alpha \theta ,$$

where Ω_r denotes the one-column matrix $\|f_k \cdot d\tilde{e}_r\|$, $A_r = \|\tilde{A}_{rij}\|$, and θ is the one-column matrix $\|f_k \cdot dx\|$. Since α is arbitrary and A_r is symmetric, we can achieve that $\alpha^T A_{n+1}\alpha$ will be a diagonal matrix, and therefore by (6.5) we can suppose that the numbers (6.6) are all zeroes.

From (6.1) it follows

$$\tilde{C}_{n-10\cdots 01}(p) = 0 \qquad \text{for } p \in U . \tag{6.7}$$

The left-hand side of (6.7) is the sum of n determinants; and the k-th determinant of the sum arises from $L(p, \tilde{e}_{n+1})$ if we replace its k-th row by the same row of $L(p, \tilde{e}_{n+N})$. Thus from the assumption that (6.6) is a zero vector it follows that (6.7) is the determinant $L(p, \tilde{e}_{n+1})$ whose first row (6.6) is replaced by

$$\tilde{A}_{n+N,11}, \cdots, \tilde{A}_{n+N,1n} . \tag{6.8}$$

Since the assumption that (6.2) is positive definite implies that

$$\tilde{A}_{n+N,ii} \neq 0 , \tag{6.9}$$

the row (6.8) cannot depend on the remaining $n - 1$ rows of $L(p, \tilde{e}_{n+1})$. Thus by (6.7) we can assume that in the determinant $L(p, \tilde{e}_{n+1})$ the first two rows represent zero vectors.

The left-hand side of the equation

$$\tilde{C}_{n-20\cdots 02}(p) = 0 \qquad \text{for } p \in U \tag{6.10}$$

is the sum of $\binom{n}{2}$ determinants such that the determinant with the indices (j, k)

arises from $L(p, \tilde{e}_{n+1})$ if we replace its j-th and k-th rows by the same rows of $L(p, \tilde{e}_{n+N})$ with the same indices. Since the first two rows of $L(p, \tilde{e}_{n+1})$ are zero vectors, except the determinant with the indices $(1, 2)$ each determinant of (6.10) contains a row which represents a zero vector. As above, from (6.9) and (6.10) it then follows that $L(p, \tilde{e}_{n+1})$ contains three rows which represent zero vectors. This process terminates if from

$$\tilde{C}_{10\cdots 0n-1}(p) = 0 \qquad \text{for } p \in U \tag{6.11}$$

we obtain inductively that the last row of $L(p, \tilde{e}_{n+1})$ represents a zero vector. This shows that A_{n+1} is a zero matrix. Hence we have proved (see (1.3))

$$\tilde{\omega}_{ir} = 0 \qquad \text{for } n + 1 \leq r \leq n + \sigma . \tag{6.12}$$

From (6.12) together with (6.3) we get $\tilde{e}_r(p) = \text{const}$ for $p \in U$, $n + 1 \leq r \leq n + \sigma$, and hence (6.4) follows.

References

[1] K. Borsuk, *Sur la courbure totale des courbes férmees,* Ann. Soc. Polon. Math. **20** (1947) 251–265.
[2] S. S. Chern & R. K. Lashof, *On the total curvature of immersed manifolds,* Amer. J. Math. **79** (1957) 306–318.
[3] S. S. Chern & R. K. Lashof, *On the total curvature of immersed manifolds.* **II,** Michigan Math. J. **5** (1958) 5–12.
[4] D. Ferus, *Totale Absolutkrümmung in Differentialgeometrie und-topologie,* Lecture Notes in Math. Vol. 66, Springer, Berlin, 1968.
[5] T. Ōtsuki, *On the total curvature of surfaces in Euclidean spaces,* Japan. J. Math. **35** (1966) 61–71.

SILESEAN UNIVERSITY, KATOWICE, POLAND