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TOTAL CURVATURE AND TOTAL ABSOLUTE CURVATURE OF IMMERSED SUBMANIFOLDS OF SPHERES

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1. Introduction

Let M^n be a compact oriented n-dimensional immersed Riemannian submanifold of the $(n + k)$-dimensional Euclidean unit sphere S^{n+k} ($k \geq 1$), and let $p \in S^{n+k}$. Let $\nu(M)$ be the bundle of unit vectors normal to M in S^{n+k}. We define the Gauss map, based at p, $e_p\colon \nu(M) \to S_pS^{n+k}$, where S_pS^{n+k} is the unit sphere in the tangent space T_pS^{n+k} to S^{n+k} at p. We investigate the integral over M of the pullback and the absolute value of the pullback of the normalized volume element of S_pS^{n+k} under e_p. These integrals are called the total curvature and the total absolute curvature of M with respect to the base point p, respectively.

Let $-p$ be the antipode of p in S^{n+k}. If $-p \notin M$, we prove that the total curvature of M with respect to p is the Euler-Poincaré characteristic of M. In addition, if $-p \notin M$, the total absolute curvature of M with respect to p satisfies results similar to those of Chern and Lashof for the total absolute curvature of immersed submanifolds of Euclidean space. If $-p \in M$, and M is even dimensional, then we prove that the total curvature of M with respect to p equals the Euler-Poincaré characteristic less twice the number of times M passes through $-p$. The total absolute curvature with respect to p is also studied when $-p \in M$.

Finally, we consider the average of the total absolute curvatures of M over all base points p in S^{n+k}. Small n-spheres of S^{n+k} for $n = 1, 2$ are characterized by means of this average.

Throughout this paper all manifolds are C^∞, and by a differentiable map we mean a C^∞ differentiable map. A superscript is used to denote the dimension of a manifold, so that M^n is an n-dimensional manifold. We use $\langle\ ,\ \rangle$ for the Riemannian metric on the Euclidean sphere or any submanifold of the sphere with the induced metric.

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2. Definitions

Let S^n be a Euclidean unit sphere, and fix $p \in S^n$. Let $-p$ denote the antipode of p.

Lemma 1. (1) *Let $v \in T_qS^n$ and $q \neq -p$. Then the parallel translate of v to p along any geodesic from q to p is independent of the geodesic.*

(2) *Let $v \in T_{-p}S^n$. Let $v^\perp = \{u \in T_{-p}S^n\colon \langle u, v\rangle = 0\rangle\}$. Then the parallel translate of v to p along any geodesic from $-p$ to p with initial velocity in $v^\perp$ is independent of the geodesic.*

Proof. The proofs of (1) and (2) are straightforward.

Let M^n be an immersed submanifold of S^{n+k}. Define $e_p\colon \nu(M) \to S_pS^{n+k}$ as follows: Let $v \in \nu_q(M)$, that is, let v be a unit vector normal to M at q. If $q \neq p$, let $e_p(v)$ be the parallel translate of v to p along any geodesic from q to p; if $q = -p$, let $e_p(v)$ be the parallel translate of v to p along any geodesic with initial velocity in T_qM. By Lemma 1, the map e_p is well defined.

Lemma 2. *$e_p\colon \nu(M) \to S_pS^{n+k}$ is continuous and differentiable on $\nu(M)\,|\,M\backslash\{-p\}$.*

Proof. The proof is straightforward.

Let $d\alpha^n$ be the volume element of S^n normalized so that

$$\int_{S^n} d\alpha^n = 1 ,$$

for all positive integers n.

According to the preceding paragraphs, if M^n is a compact oriented immersed submanifold of S^{n+k}, we may globally define the Gauss map on M with respect to any base point p. If $-p \in M$ for some $p \in S^{n+k}$, then $e_p\colon \nu(M) \to S_pS^{n+k}$ is continuous but needs only to be differentiable on $\nu(M)\,|\,M\backslash\{-p\}$. Hence $e_p^*(d\alpha^n)$ and $|e_p^*(d\alpha^n)|$ are defined on $\nu(M)\,|\,M\backslash\{-p\}$. Since $\nu(M)\,|\,\{-p\}$ is a set of measure zero we may integrate these forms over $\nu(M)$.

Definition. Set

$$\kappa_p(M) = \int_{\nu(M)} e_p^*(d\alpha^n) , \qquad \tau_p(M) = \int_{\nu(M)} |e_p^*(d\alpha^n)| .$$

We call $\kappa_p(M)$ the total (algebraic) curvature of M with respect to p, and $\tau_p(M)$ the total absolute curvature of M with respect to p.

Clearly $\kappa_p(M)$ equals the algebraic normalized volume covered by e_p. Since e_p is a continuous map from a compact oriented manifold into a compact oriented manifold and both have the same dimension, e_p has a degree and this degree is $\kappa_p(M)$. In particular, note that $\kappa_p(M)$ is integral whether or not $-p \in M$.

Moreover, $\tau_p(M)$ is the normalized volume covered by e_p, and because the volume is normalized $\tau_p(M)$ equals the average number of times any vector in S_pS^{n+1} is taken on by e_p.

Let N^n be an oriented immersed submanifold of E^{n+k}, and $\nu(N)$ the bundle of unit vectors normal to N in E^{n+k}. Then we have the usual Gauss map $e\colon \nu(M) \to S_0^{n+k-1}$, where S_0^{n+k-1} is the unit sphere in E^{n+k} with center 0. The total curvature and total absolute curvature of N in E^{n+k} are defined as above and are denoted $\kappa(N)$ and $\tau(N)$, respectively. The definition for $\tau(N)$ agrees with the one in [3].

3. $\kappa_p(M)$ and $\tau_p(M)$ for $-p \notin M$

Isometrically imbed S^{n+k} in E^{n+k+1}. Let $\sigma_p\colon S^{n+k}\backslash\{-p\} \to E^{n+k}$ be stereographic projection from $-p$ onto the tangent hyperplane E^{n+k} to S^{n+k} at p. For an oriented immersed submanifold M^n of S^{n+k}, set $M(p)$ equal to the image of $M\backslash\{-p\}$ under σ_p. Let $M(p)$ carry the metric induced from E^{n+k}.

We now restate Lemma 5 of [8] for arbitrary positive codimension.

Lemma 3. *Let M^n be an immersed submanifold of S^{n+k}. Then the following diagram is commutative*:

$$\begin{array}{ccc} \nu(M)\,|\,M\backslash\{-p\} & \xrightarrow{\;e_p\;} & S_pS^{n+k} \\ \sigma_p^*\big\downarrow & & \big\downarrow d\sigma_p \\ \nu(M(p)) & \xrightarrow{\;e\;} & S_0^{n+k-1} \end{array}$$

It is clear that σ_p^* and $d\sigma_p\colon S_pS^{n+k} \to S_0^{n+k-1}$ are diffeomorphisms. Thus if $M(p)$ is given the orientation induced from $M\backslash\{-p\}$ by σ_p, the algebraic volumes covered by e and e_p are equal. Hence $\kappa(M(p)) = \kappa_p(M)$. It is equally clear that $\tau(M(p)) = \tau_p(M)$.

Note that for a compact oriented immersed submanifold M of S^{n+k}, $M(p)$ is a compact oriented immersed submanifold of E^{n+k} if $-p \notin M$. If $-p \in M$, then $M(p)$ is a complete open oriented immersed submanifold of E^{n+k}.

Theorem 1. *Let M^n be a compact oriented immersed submanifold of S^{n+k}, and suppose $-p \notin M$. Then $\kappa_p(M) = \chi(M)$ where $\chi(M)$ is the Euler-Poincaré characteristic of M.*

Proof. Since $-p \notin M$, $M\backslash\{-p\} = M$ and hence M and $M(p)$ are diffeomorphic under σ_p. In particular, M and $M(p)$ are topologically equivalent. Hence $\kappa_p(M) = \kappa(M(p)) = \chi(M)$, where the second equality is the Gauss-Bonnet theorem.

Definition. We say that the submanifold $\sum^m$ of S^n is a small m-sphere if for any (and hence every) imbedding of S^n into E^{n+1} we have $\sum^m = S^n \cap L^{m+1}$, where L^{m+1} is an $(m+1)$-dimensional plane in E^{n+1}. For $m = 1$, we say that $\sum^1$ is a small circle. Note that every metric hypersphere of S^n is a small hypersphere of S^n and conversely.

Theorem 2. *Let M^n be a compact oriented immersed submanifold of S^{n+k}. Let $p \in S^{n+k}$ and suppose $-p\notin M$. Then we have the following.*

(1) *$\tau_p(M) \geq \beta(M)$ where $\beta(M)$ is the sum of the Betti numbers of M.*

(2) *$\tau_p(M) < 3$ implies M is homeomorphic to S^n.*

(3) *$\tau_p(M) = 2$ implies M is imbedded as a hypersurface of a small $(n+1)$-sphere $\sum^n$ through $-p$.*

Proof. (1) We know that M and $M(p)$ are topologically equivalent under σ_p. Hence $\tau_p(M) = \tau(M(p)) \geq \beta(M(p)) = \beta(M)$, where the inequality in this chain is due to Chern and Lashof [4].

(2) and (3) are proved in a similar fashion.

4. $\kappa_p(M)$ and $\tau_p(M)$ for $-p \in M$

Throughout this section we suppose M^n is a compact oriented immersed submanifold of S^{n+k}. We want to investigate $\kappa_p(M)$ and $\tau_p(M)$ under the assumption $-p \in M$.

If N^n is an oriented immersed submanifold of E^{n+k}, then $\kappa(N) = 0$ for n odd whether or not N is compact. Hence for M^n with n odd we have $\kappa_p(M) = \kappa(M(p)) = 0 = \chi(M)$ whether or not $-p \notin M$.

If M^{2n} is a compact oriented immersed submanifold of S^{2n+k} and $-p \in M$ for some $p \in S^{2n+k}$, then $\kappa_p(M)$ may not be (in fact, is not) equal to the Euler-Poincaré characteristic of M. For example, let M^{2n} be a small hypersphere through $-p \in S^{2n+1}$. Then the rank of $e_p\colon \nu(M) \to S_pS^{2n+1}$ is zero; see, for example, [8, Theorem 6]. Hence $\kappa_p(M) = 0 \neq 2 = \chi(M)$.

For a compact immersed submanifold M^n of S^{n+k} and $q \in S^{n+k}$, let $\sharp q(M)$ equal the number of times M passes through q. We have the following theorem.

Theorem 3. *Let M^n be a compact oriented immersed submanifold of S^{n+k} and let $p \in S^{n+k}$. Suppose n is even and $-p \in M$. Then*

$$\kappa_p(M) = \chi(M) - 2\sharp_{-p}(M) .$$

Proof. Let $f\colon M^n \to S^{n+k}$ be the immersion of M^n into S^{n+k}. Let $f^{-1}(-p) = \{q_1, \cdots, q_r\}$. Consider $f_t\colon M^n \to S^{n+k}$, $0 \leq t \leq 1$, a continuous deformation of f, i.e., $f_0 = f$ and f_t is an immersion for $0 \leq t \leq 1$. Suppose this deformation has the following properties:

(i) $f_t^{-1}(-p) = f^{-1}(-p)$, for $0 \leq t \leq 1$, and

(ii) $(f_{t*})_{q_i} = (f_*)_{q_i}$, for $0 \leq t \leq 1$, and $i = 1, \cdots, r$.

Denote $f_t(M)$ by M_t, $0 \leq t \leq 1$. Then $\kappa_p(M_t)$ varies continuously with t. However, we observed earlier that $\kappa_p(M)$ is integral for all compact oriented immersed submanifolds M^n of S^{n+k}. Thus $\kappa_p(M_t)$ remains fixed under deformations of the type described. We may therefore assume that f is totally geodesic in a sufficiently small neighborhood about $q_i, i = 1, \cdots, r$, if we are only concerned with computing $\kappa_p(M)$.

For a sufficiently small sphere S_ε sbout $-p$ on S^{n+k}, bounding a ball B_ε^{n+k} on S^{n+k}, the intersection $f(M) \cap B_\varepsilon$ consists of flat discs $f(B_i^n)$, with $q_i \in B_i^n$.

Under stereographic projection σ_{-p} of $f(M\backslash\bigcup_{n=i}^{r} B_i)$ into E^{n+k}, the boundary spheres ∂B_i^n are mapped into the sphere $\sigma_{-p}(S_\varepsilon)$ and each is a great $(n-1)$-dimensional sphere, and σ_{-p} maps $f(B_i^n\backslash q_i)$ into n-planes. We may then find convex n-dimensional surfaces $\sum_i^n$ each with a disc removed in the exterior of $\sigma_{-p}(S_\varepsilon)$ so that $\kappa(\sum_i) = 2$, and so that $(\sigma_{-p}\circ f)(M\backslash\bigcup_{i=1}^{r} B_i) \cup (\bigcup_{i=i}^{r}\sum_q)$ is a smoothly immersed n-manifold in E^{n+k}, homeomorphic to M.

Now $\kappa_p(f(M\backslash\cup B_i)) = \kappa_p(M)$ since $f(B_i)$ is part of a totally geodesic sphere through $-p$. Hence

$$\begin{aligned}\kappa_p(M) &= \kappa_p\Big(f\Big(M\backslash\bigcup_{i=1}^{r} B_i\Big)\Big) = \kappa\Big(\sigma_{-p}\circ f\Big(M\backslash\bigcup_{i=1}^{r} B_i\Big)\Big)\\ &= \kappa\Big[\sigma_{-p}\circ f\Big(M\backslash\bigcup_{i=1}^{r} B_i\Big)\cup\Big(\bigcup_{i=1}^{r}\textstyle\sum_i\Big)\Big] - \kappa\Big(\bigcup_{i=1}^{r}\textstyle\sum_i^n\Big)\\ &= \chi(M) - 2\#_{-p}(M)\ . \quad \text{q.e.d.}\end{aligned}$$

For $A\subset S^n$, let $-A = \{-q\colon q\in A\}$. Let M^n be a compact oriented immersed submanifold of S^{n+k}. It is clear that the function $p\to\tau_p(M)$ is continuous on $S^{n+k}\backslash(-M)$. Equivalently, $\tau_p(M)$ varies continuously as we move M by a continuous 1-parameter family of isometries of S^{n+k} provided at no time $-p\in M$. However, $p\to\tau_p(M)$ is not continuous on S^{n+k}. For example, let M be a small n-sphere in S^{n+1}; if $p\in S^{n+1}\backslash(-M)$, then $\tau_p(M) = 2$, but if $p\in -M$, then $\tau_p(M) = 0$.

The preceding example and Theorem 3 suggest the following.

Conjecture. *Let M^n be a compact oriented immersed submanifold of S^{n+k}. The function*

$$p\to\tau_p(M) + 2\#_{-p}(M)$$

is continuous on S^{n+k}.

We can, however, prove a special case of this conjecture. Let $f\colon M\to S^{n+k}$ be the immersion of M into S^{n+k}. Suppose $f^{-1}(-p) = \{q_1,\cdots,q_r\}$, where $r>0$. Let φ_t, $0\le t\le 1$, be a differentiable 1-parameter family of isometries of S^{n+k} with $\varphi_0 = \text{id}$. Define $\varphi\colon M\times[0,1]\to S^{n+k}$ by $\varphi(q,t) = \varphi_t(f(q))$; φ is differentiable. Set $M_t = \varphi_t(f(M))$.

Theoerm 4. *If $M_t\cap\{-p\} = \emptyset$, $0<t\le 1$, and φ is regular at $(q_i,0)$, $i = 1,\cdots,r$, then*

$$\tau_p(M) + 2\#_{-p}(M) = \lim_{t\to 0}\tau_p(M_t)\ . \tag{1}$$

Sketch of proof. Consider the directed dilitation of S^{n+k} along $-p$, denoted by $S_*^{n+k}(p)$. Now $S_*^{n+k}(p) = S^{n+k}\backslash\{-p\}\cup S_{-p}S^{n+k}$ is a differentiable manifold with boundary $S_{-p}S^{n+k}$, [5]. Also consider the directed dilitation of $M\times[0,1]$ along $\{(q_1,0),\cdots,(q_r,0)\}$, denoted by $(M\times[0,1])_*$. Here $(M\times[0,1])_* = M\times[0,1]\backslash\{(q_1,0),\cdots,(q_r,0)\}\cup\bigcup_{i=1}^{r} G_i$, where $G_i = \{v\in S_{(q_i,0)}M\times[0,1]\colon$

$\langle v, \partial/\partial t_{(q_i,0)}\rangle \geq 0\}$, $\langle\ ,\ \rangle$ being the product metric on $M \times [0, 1]$. φ induces a map Φ: $(M \times [0, 1])_* \to S_*^{n+k}(p)$ since φ is regular at $(q_i, 0), i = 1, \cdots, r$.

There is a natural map ι: $(M \times [0, 1])_* \to M \times [0, 1]$ such that ι is the identity of $\bigcup_{i=1}^r G_i$ and $\iota | G_i = (q_i, 0)$, $i = 1, \cdots, r$. Let $\nu(M_t)$ be the bundle of unit vectors normal to M_t in S^{n+k}. Set $\nu(M \times [0, 1]) = \bigcup_{0 \leq t \leq 1} \nu(M_t)$; this is a bundle over $M \times [0, 1]$. Let $\mu = \iota^* \nu(M \times [0, 1]$. We may define a Gauss map e: $\mu \to S_p S^{n+k}$ so that $e | \nu(M_t)$, $0 < t \leq 1$, and $e | \nu(M \backslash \{q_1, \cdots, q_r\})$ are the usual Gauss maps based at p. For the pair $(v, u) \in \mu | G_i = G_i \times \nu_{q_i}(M)$, $e(v, u)$ is the parallel translate of u to p along the geodesic with initial velocity $\Phi(v)$. Now e: $\mu \to S_p S^{n+k}$ is differentiable.

Define g: $\mu \to R$ such that

(i) $g | \nu(M_t) = |\text{Jacobian } e | \nu(M_t)|$, $0 < t \leq 1$,

(ii) $g | \nu(M \backslash \{q_1, \cdots, q_r\}) = |\text{Jacobian } e | \nu(M \backslash \{q_1, \cdots, q_r\})|$,

(iii) $g | (\mu | G_i) = |\text{Jacobian } e | (\mu | G_i)|$.

Then g is continuous almost everywhere and bounded. Using measure theoretic techniques, one may show

$$\lim_{t \to 0} \int_{\nu(M_t)} g | \nu(M_t) = \int_{\nu(M)} g | \nu(M) + \sum_{i=1}^r \int_{\mu | G_i} g | (\mu | G_i) \ .$$

The integral $\int_{\mu | G_i} g | (\mu | G_i)$ depends only on $T_{q_i} M$ and $\varphi_*(\partial/\partial t_{(q_i,0)})$. Hence one shows by letting M be a small n-sphere in S^{n+k} that $\int_{\mu | G_i} g | (\mu | G_i) = 2$. Hence (1).

For details (in the codimension 1 case) see the author's thesis [9, Chapter IV].

5. Another theorem

Let M^n be a compact oriented immersed submanifold of Euclidean space E^{n+k} $(1 \leq k)$. Suppose there exists an $(n + l)$-plane E^{n+l} $(1 \leq l \leq k)$ in E^{n+k}, which contains M^n. Then it is known that the total absolute curvatures of M^n regarded as a submanifold of E^{n+l} and E^{n+k} are the same. We prove a corresponding result for submanifolds of spheres in this section, and will give an application of this result in the next section.

In the following theorem we consider a compact oriented immersed submanifold M^n of S^{n+k}, which is contained in a small $(n + l)$-sphere $\sum^{n+l}$, $(1 \leq l \leq k)$. For $p \in S^{n+k}$ let $\tau_p(M, S^{n+k})$ be the total curvature of M as a submanifold of S^{n+k} with respect to the base point p. For $p \in \sum^{n+l}$ let $\tau_p(M, \sum^{n+l})$ be the total curvature of M as a submanifold of $\sum^{n+l}$ with respect to the base point p.

Theorem 5. *Let M^n be a compact oriented immersed submanifold of S^{n+k}. Suppose $p \in S^{n+k}$, and M is contained in a small $(n + l)$-sphere $\sum^{n+l}$ $(1 \leq l \leq k)$*

containing $-p$. Let $p' = -(-p)$ in $\sum^{n+l}$, that is, p' is the antipode of $-p$ in $\sum^{n+l}$. Then $\tau_p(M, S^{n+k}) = \tau_{p'}(M, \sum^{n+l})$.

Proof. Isometrically imbed S^{n+k} into E^{n+k+1}. Then we have the stereographic projection $\sigma_p: S^{n+k}\backslash\{-p\} \to E^{n+k}$ from $-p$ onto E^{n+k}, the $(n+k)$-dimensional plane in E^{n+k+1} tangent to S^{n+k} at p. Let L be the $(n+l+1)$-dimensional plane E^{n+k+1} such that $L \cap S^{n+k} = \sum^{n+k}$. Since $-p \in \sum^{n+l}$, under σ_p the small sphere $\sum^{n+l}$ corresponds to the $(n+l)$-dimensional plane $L' = L \cap E^{n+k}$. Hence $\sigma_p(M\backslash\{p\}) = M(p) \subset L'$.

The small sphere $\sum^{n+l}$ is imbedded as a metric sphere in L. Let $\sigma_{p,}$: $\sum^{n+l}\backslash\{-p\} \to L'$ be the stereographic projection in L from $-p$ onto L'. Even though L', in general, is not tangent to $\sum^{n+l}$ at p', Lemma 3 still holds. Hence, if we set $M(p') = \sigma_{p'}(M\backslash\{-p\})$, we have $\tau_{p'}(M, \sum^{n+l}) = \tau(M(p'), L')$, the total curvature of $M(p')$ as a submanifold of L'. Since $\sigma_{p'} = \sigma_p \mid \sum^{n+l}$, we also have $M(p) = M(p')$.

Let $\tau(M(p), E^{n+k})$ be the total curvature of $M(p)$ as a submanifold of E^{n+k}. Then

$$\tau_p(M, S^{n+k}) = \tau(M(p), E^{n+k}) = \tau(M(p'), L') = \tau_{p'}(M, \textstyle\sum^{n+l}) .$$

6. The average total absolute curvature

Let M^n be a compact oriented immersed submanifold of S^{n+k}. Define

$$\bar{\tau}(M) = \int_{S^{n+k}} \tau_p(M) d\alpha^{n+k}(p) ,$$

that is, $\bar{\tau}(M)$ is the average value of $\tau_p(M)$ taken over all possible base points $p \in S^{n+k}$.

Theorem 6. *Let M^n be a compact oriented immersed submanifold of S^{k+k}. Then*

(1) $\bar{\tau}(M) \geq \beta(M) \geq 2$,

(2) $\bar{\tau}(M) = 2$ *if M is imbedded as a small n-sphere.*

Proof. (1) We know by Theorem 2 that for all $p \in S^{n+k}$ with $-p \notin M$, $\tau_p(M) \geq \beta(M)$. Since $\{p \in S^{n+k}: -p \in M\}$ is a set of measure zero, we have $\bar{\tau}(M) = \int_{S^{n+k}} \tau_p(M) d\alpha^{n+k} \geq \int_{S^{n+k}} \beta(M) d\alpha^{n+k} \geq \beta(M)$.

(2) It is easy to show for $p \in S^{n+k}$ with $-p \notin M$ that the image of a small n-sphere under σ_p is a metric sphere in an $(n+1)$-dimensional plane of E^{n+k}. Hence, if M is a small n-sphere and $-p \notin M$, then $\tau_p(M) = \tau(M(p)) = 2$. Thus $\bar{\tau}(M) = 2$. q.e.d.

It is natural to ask to what extend the converse of part (2) of Theorem 6 is true. If $\bar{\tau}(M) = 2$, then $\tau_p(M) = 2$ for all $p \in S^{n+k}$ such that $-p \notin M$. This is true since the function $p \to \tau_p(M)$ is continuous and ≥ 2 on $\{p \in S^{n+k}: -p \notin M\}$. In particular, there is at least one $p \in S^{n+k}$ with $-p \notin M$ such that $\tau_p(M) = 2$. By Theorem 2 there exists a small $(n+1)$-sphere $\sum^{n+1}$ containing $-p$ in which M is imbedded and M is homeomorphic to S^n. By Theorem 5 it follows immediately that $\bar{\tau}(M, \sum^{n+1}) = 2$, where $\bar{\tau}(M, \sum^{n+1})$ is the average total absolute curvature of M as a submanifold of $\sum^{n+1}$. So, to find out to what extent the converse of part (2) of Theorem 6 is true, we need only to study manifolds M^n homeomorphic to S^n, which are imbedded in S^{n+1} with $\bar{\tau}(M) = 2$. In particular, when these M^n are imbedded as small spheres.

If L^n is a hyperplane of E^{n+1}, its complement $E^{n+1}\backslash L^n$ is the disjoint union of two sets D_1 and D_2 with closures $\bar{D}_i = D_i \cup L^n$, $i = 1, 2$. A set A in E^{n+1} has the *two-piece property* (TPP) if $A \cap \bar{D}_i$ is path connected, for either complementary component D_i, $i = 1, 2$, of any hyperplane L^n of E^{n+1}.

If $\sum^n$ is a metric hypersphere of S^{n+1}, its complement $S^{n+1}\backslash \sum^n$ is the disjoint union of two open sets D_1 and D_2 with closures $\bar{D}_i = D_i \cup \sum^n$, $i = 1, 2$. A set A in S^{n+1} has the *spherical-two-piece-property* (STPP) if $A \cap \bar{D}_i$ is path connected, for either complementary component D_i, $i = 1, 2$, of any metric hypersphere $\sum^n$ in S^{n+1}. For example, it follows from Proposition 3.1 of [1] that every metric hypersphere of S^{n+1} has the STPP.

Let L^n be a hyperplane of E^{n+1}, and let $L(\varepsilon)$ equal the set of all points whose distance from L^n is less than ε. We say a set A contained in E^{n+1} is asymptotic to L^n if given $\varepsilon > 0$ there exists an $R > 0$ such that for all $r > R$, $N\backslash B_0(r) \neq \emptyset$ and $N\backslash B_0(r) \subset L(\varepsilon)$, where $B_0(r)$ is the open ball of radius r centered at the origin of E^{n+1}.

Lemma 4. *Let N^n be a complete imbedded hypersurface of E^{n+1} asymptotic to a hyperplane L^n of E^{n+1}. If N^n has the TPP, then $N^n = L^n$.*

Proof. Suppose $N^n \neq L^n$. Let d be the metric on E^{n+1}. Let $p \in N$ so that $d(p, L) = \rho$ is a maximum. Such a point exists since N is asymptotic to L. Let P equal the connected component of $\{q \in N\colon d(q, L) = \rho\}$, which contains p. Let K^n be the hyperplane through p at a distance ρ from L. Clearly $P \subset K$.

Let the origin 0 of E^{n+1} be the base point of the perpendicular from p to L. Since N is asymptotic to L, there is a sequence of points q_i, $i = 1, 2, \cdots$, in N such that $\lim_{i\to\infty}\|q_i\| = +\infty$. Consider the sequence $q_i/\|q_i\|$, $i = 1, 2, \cdots$, in the unit sphere of E^{n+1} about 0. We may assume by taking a subsequence if necessary that $\lim_{i\to\infty} q_i/\|q_i\| = u$. Clearly $u \in L$. Note that P is bounded since N is asymptotic to L. Hence let $p' \in p$ so that $(p' - p)\cdot u = c$ is a maximum. Let J^{n-1} be the $(n - 1)$-plane in K through p' orthogonal to u. Rotate K about J so that the unit normal to K pointing away from L rotates toward u. Let D_1 be the complementary component of K, which does not contain L. For a small enough rotation of K, the path component of p' in $N \cap \bar{D}_1$ is at least a distance $\frac{1}{2}\rho$ from L. Since $N \cap \bar{D}_1$ must also contain a point q_i which is closer to L than $\frac{1}{2}\rho$ for i sufficiently large, L does not satisfy the TPP. This is a contradiction. Hence $L = N$.

Lemma 5. *Let M^n be a compact imbedded hypersurface of S^{n+1}. If M has the STPP with respect to all metric hyperspheres through an umbilic point of M, then M is a metric sphere.*

Proof. Let q be the umbilic point of M. Consider the stereographic projection σ from q. Then $N^n = \sigma(M\backslash\{q\})$ with metric induced from E^{n+1} is a complete imbedded hypersurface of E^{n+1}. Since M is umbilic at q, there exists a metric hypersphere $\sum^n$ through q, which makes second order contact with M. Then $L^n = \sigma\{\sum\backslash\{q\})$ is a hyperplane of E^{n+1}.

Let L_1 and L_2 be two hyperplanes parallel to L with one on each side of L^n. Under the stereographic projection σ, L_1 and L_2 correspond to metric spheres through q, Σ_1, and Σ_2, with one on each side of Σ. Since Σ makes second order contact with M at q, in a small enough neighborhood about q, M lies between Σ_1 and Σ_2. Hence outside a large enough ball about 0 in E^{n+1}, N lies between L_1 and L_2. It is now clear that N is asymptotic to L.

Since M has the STPP with respect to all metric spheres through q, N has the TPP. Thus the hypotheses of Lemma 4 are satisfied so that $N = L$. Hence $M = \Sigma$, that is, M is a metric sphere.

Lemma 6. *Let M^n be a manifold homeomorphic to S^n imbedded in S^{n+1} with $\bar{\tau}(M) = 2$. If* (1) *$n \leq 2$ or* (2) *$n \geq 3$ and M has an umbilic point, then M is imbedded as a small sphere.*

Proof. Let $p \in S^{n+1}$ such that $-p \notin M$. Since $\bar{\tau}(M) = 2$, we have $\tau_p(M) = 2$. Thus $\tau(M(p)) = 2$, which implies that $M(p)$ is imbedded as a convex hypersurface of E^{n+1}. In particular, $M(p)$ has the TPP so that M has the STPP with respect to all metric spheres through $-p$. Hence for all $q \notin M$, M has the STPP with respect to all metric spheres through q. Since every metric sphere passes through some point not in M unless M already is a metric sphere, M has the STPP. If $n = 1$, then every point of M is umbilic. If $n = 2$, we have $\chi(M) = \chi(S^2) = 2 \neq 0$. If M did not have an umbilic point, then the second fundamental form of M in S^{n+1} determines a field of tangent line elements corresponding to, say, the larger eigenvalue of the second fundamental form. Hence, according to the comments following Theorem 40.13 in [7], $\chi(M) = 0$. For $n \geq 3$, we have assumed the existence of an nmbilic point. Now apply Lemma 5 to get the result since metric hyperspheres of S^{n+1} are small hyperspheres.
q.e.d.

We now present an alternate proof of Lemma 6 for the cases $n = 1$ and $n = 2$.

Proof ($n = 1$). It is clear that $\bar{\tau}(M^1)$ equals the total central curvature of M^1 as defined in [2] where it is shown that the total central cuvature of a closed curve $M^1 \subset S^2$ equals the total absolute curvature of the curve as a curve in E^3. Consequently if $\bar{\tau}(M) = 2$, then $M^1 \subset S^2$ is imbedded as a convex curve in a hyperplane in E^3. Thus M^1 is a small circle.

Proof ($n = 2$). Since $\tau_p(M) = 2$ for all p such that $-p \notin M$, by Theorem 4 we have $\tau_p(M) = 0$ for p with $-p \in M$. Hence if $-p \in M$, then $\tau(M(p)) = \int_{M(p)} |K| = 0$, which implies $K \equiv 0$ on $M(p)$. Since $M(p)$ is complete, $M(p)$ is a generalized cylinder [6]. Also M has an umbilic point, for $\chi(M) \neq 0$ since

M is a topological sphere. Hence if we choose p so that $-p$ is the umbilic point, we also have $M(p)$ asymptotic to a hyperplane L^n of E^{n+1}.

Clearly, an imbedded generalized cylinder asymptotic to a hyperplane must be that hyperplane, so $M(p) = L$. Thus M is a small sphere.

Using Lemma 6 and the comments at the beginning of this section, we have the following.

Theorem 7. *Let M^n be a compact oriented immersed submanifold of S^{n+k}, where $n \leq 2$. If $\bar{\tau}(M) = 2$, then M is imbedded as a small n-sphere.*

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