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COMPACT COMPLEX SUBMANIFOLDS IMMERSED IN COMPLEX PROJECTIVE SPACES

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0. Introduction

J. Simons [17], H. B. Lawson [9], and S. S. Chern-M. do Carmo-S. Kobayashi [6], etc. studied minimal submanifolds of spheres. One of the beautiful results is as follows: Let M be an n-dimensional compact submanifold minimally immersed in a unit sphere S^{n+p} of dimension $n+p$, and let S denote the square of the length of the second fundamental form. Then

$$\int_M \left[\left(2 - \frac{1}{p}\right)S - n\right] S*1 \geq 0 \tag{0.1}$$

holds, where $*1$ denotes the volume element of M. Since the scalar curvature R of M is given by $R = n(n-1) - S$, (0.1) can be rewritten as an integral inequality concerning the scalar curvature. The classification of M with $S = n(2 - 1/p)$ was given in [6], [9].

With respect to the complex version of (0.1), K. Ogiue [12] obtained an inequality, which was applied to scalar curvature and holomorphic pinchings in [14]. In the present paper, we generalize these results.

Let CP^{m+q} be a complex projective space of complex dimension $m+q$ with the Fubini-Study metric of constant holomorphic sectional curvature 1.

Theorem A. *Let M be a compact complex submanifold of complex dimension m immersed in CP^{m+q}, and assume that the scalar curvature R of M with respect to the induced Kählerian metric satisfies*

$$R \geq m(m+1) - \tfrac{1}{3}(m+2)\ . \tag{0.2}$$

(1) *If the inequality in* (0.2) *holds at some point of M, then M is imbedded as a projective subspace CP^m in CP^{m+q}.*

(2) *If the equality in* (0.2) *holds on M, then $m = 1$ and M is imbedded as a complex quadric CQ^1 in some CP^2 in CP^{1+q}.*

Applying Theorem A to holomorphic or Riemannian pinchings, we have

Theorem B. *Let M be a compact complex submanifold of complex dimen-*

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sion m immersed in CP^{m+q}, *and assume that the holomorphic sectional curvature* $K(X, JX)$ *of* M *with respect to the induced Kählerian structure satisfies*

$$K(X, JX) \geq 1 - \frac{m+2}{6m^2} \qquad \text{for } q \geq 2\,, \tag{0.3}$$

$$K(X, JX) \geq 1 - \frac{m+2}{6m} \qquad \text{for } q = 1 \tag{0.3$'$}$$

for any tangent vector X.

(1) *If the inequality in* (0.3) *and* (0.3)$'$ *holds for some* X *at some point of* M, *then* M *is imbedded as a projective subspace* CP^m *in* CP^{m+q}.

(2) *If the equality in* (0.3) *and* (0.3)$'$ *holds on* M, *then* $m = 1$ *and* M *is imbedded as a complex quadric* $CQ^1 \subset CP^2 \subset CP^{1+q}$.

Theorem C. *Let* M *be a compact complex hypersurface immersed in* CP^{m+1}, $m \geq 2$. *If the sectional curvature* $K(X, Y)$ *of* M *with respect to the induced Kählerian metric satisfies*

$$K(X, Y) \geq \frac{1}{4}\left(1 - \frac{m+2}{3m}\right)\,, \tag{0.4}$$

then M *is imbedded as a projective hypersurface* CP^m *in* CP^{m+1}.

If a compact complex submanifold M is imbedded in CP^{m+q}, then by Chow's theorem M is algebraic. K. Nomizu and B. Smyth [11], K. Nomizu [10], and K. Ogiue [16] studied imbedded (or nonsingular) submanifolds and, as a special case, compact nonsingular complex curves in CP^{m+q}. In § 6, we generalize some of their theorems to the case of immersed complex curves in CP^{1+q}.

In § 7 we give some remarks. Throughout this paper all manifolds are assumed to be connected.

1. Preliminaries

To obtain the Laplacian of the second fundamental form for immersion of Kählerian manifolds, we first consider a submanifold M of real dimension n minimally immersed in an $(n + p)$-dimensional locally symmetric Riemannian manifold N', and use the same notations as those in [6] by S. S. Chern-M. do Carmo-S. Kobayashi. Let $e_1, \cdots, e_{n+p}$ be a local field of orthonormal frames in N' such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and $e_{n+1}, \cdots, e_{n+p}$ are normal to M. It is known that

$$\sum_{\alpha,i,j} h^\alpha{}_{ij} \Delta h^\alpha{}_{ij} = \sum_{\alpha,\beta,i,j,k} (4K^\alpha{}_{\beta ki} h^\beta{}_{jk} h^\alpha{}_{ij} - K^\alpha{}_{k\beta k} h^\alpha{}_{ij} h^\beta{}_{ij})$$

$$
\begin{aligned}
(1.1) \qquad & + \sum_{\alpha,i,j,k,l} (2K^l{}_{kik}h^\alpha{}_{lj}h^\alpha{}_{ij} + 2K^l{}_{ijk}h^\alpha{}_{lk}h^\alpha{}_{ij}) \\
& - \sum_{\alpha,\beta,i,j,k,l} (h^\alpha{}_{ik}h^\beta{}_{jk} - h^\alpha{}_{jk}h^\beta{}_{ik})(h^\alpha{}_{il}h^\beta{}_{jl} - h^\alpha{}_{jl}h^\beta{}_{il}) \\
& - \sum_{\alpha,\beta,i,j,k,l} h^\alpha{}_{ij}h^\alpha{}_{kl}h^\beta{}_{ij}h^\beta{}_{kl} ,
\end{aligned}
$$

where $1 \leq i, j, k, l \leq n$, $n + 1 \leq \alpha$, $\beta \leq n + p$, $h^\alpha{}_{ij}$'s denote the second fundamental forms, $\Delta h^\alpha{}_{ij}$'s denote their Laplacians, and $K^\alpha{}_{\beta ki}$'s denote the components of the curvature tensor of N' with respect to the above frames (cf. [6, (2.23)]).

Now let CP^{m+q} be a complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and M be a compact complex submanifold of complex dimension m immersed in CP^{m+q}. As is well known, M is minimal in CP^{m+q}. We denote the complex structure tensor by J and the Kählerian metric of CP^{m+q} by g. M has the induced Kählerian structure tensor (J, g) denoted by the same letters. On CP^{m+q}, we have

$$
(1.2) \qquad K^A{}_{BCD} = \tfrac{1}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}) ,
$$

where $J_{AB} = \sum g_{AC}J^C{}_B$, and $1 \leq A, B, C, D \leq n + p = 2(m + q)$ for $n = 2m$, $p = 2q$.

We can assume that our local field of orthonormal frames is of J-basis such that, restricted to M, $(e_A) = (e_r, e_{m+r} = Je_r, e_a, e_{q+a} = Je_a)$, where we use the following convention on the ranges of indices:

$$
\begin{gathered}
1 \leq A, B, C, D \leq n + p = 2(m + q) ; \\
1 \leq r, s, t \leq m ; \qquad 1 \leq i, j, k, l \leq n = 2m ; \\
n + 1 \leq a, b \leq n + q ; \qquad n + 1 \leq \alpha, \beta, \gamma \leq n + p = 2(m + q) ;
\end{gathered}
$$

and $r^* = m + r$, $a^* = q + a$. Such a local field of orthonormal frames is said to be adapted.

Substituting (1.2) into (1.1), we have (cf. K. Ogiue [12])

$$
\begin{aligned}
(1.3) \qquad \sum_{\alpha,i,j} h^\alpha{}_{ij}\Delta h^\alpha{}_{ij} = & - \sum_{\alpha,\beta,i,j} (\sum_k h^\alpha{}_{ik}h^\beta{}_{kj} - \sum_k h^\beta{}_{ik}h^\alpha{}_{kj})^2 \\
& - \sum_{\alpha,\beta,i,j,k,l} h^\alpha{}_{ij}h^\alpha{}_{kl}h^\beta{}_{ij}h^\beta{}_{kl} + \tfrac{1}{2}(m + 2) \sum_{\alpha,i,j} (h^\alpha{}_{ij})^2 .
\end{aligned}
$$

By noticing that $\sum J_{ij}h^a{}_{jk} = h^{a^*}{}_{ik}$ and $\sum J_{ij}h^a{}_{jk} = -\sum h^a{}_{ij}J_{jk}$, a direct calculation gives (cf. K. Ogiue [16])

$$
(1.4) \qquad - \sum_{\alpha,\beta,i,k} (\sum_k h^\alpha{}_{ik}h^\beta{}_{kj} - \sum_k h^\beta{}_{ik}h^\alpha{}_{kj})^2 = -8 \sum_{a,b,i,j,k,l} h^a{}_{ij}h^a{}_{jk}h^b{}_{kl}h^b{}_{li} .
$$

By w^A and $w^A{}_B$ we denote the dual of e_A and the connection forms on CP^{m+q}. Since J is parallel ($J^A{}_{B,C} = 0$), we have

$$\sum J^A{}_{B,C}w^C = dJ^A{}_B + \sum w^A{}_C J^C{}_B - \sum w^C{}_B J^A{}_C = 0 .$$

By putting $A = i$ and $B = \beta$, the above equation becomes $\sum w^i{}_\alpha J^\alpha{}_\beta - \sum w^j{}_\beta J^i{}_j = 0$. Because $w^\alpha{}_i = \sum h^\alpha{}_{ij} w^j$ and $w^\alpha{}_i = -w^i{}_\alpha$, we get

$$\sum_\alpha h^\alpha{}_{ik} J^\alpha{}_\beta = \sum_j h^\beta{}_{jk} J^i{}_j . \tag{1.5}$$

Now we put $S_{\alpha\beta} = \sum h^\alpha{}_{ij} h^\beta{}_{ij}$. Then by (1.5) we have

$$\begin{aligned}\sum_{\alpha,\beta} J^\alpha{}_\gamma S_{\alpha\beta} J^\beta{}_\delta &= \sum_{\alpha,\beta,i,j} (J^\alpha{}_\gamma h^\alpha{}_{ij})(h^\beta{}_{ij} J^\beta{}_\delta) = \sum_{i,j,k,l} (h^\gamma{}_{lj} J^i{}_l)(h^\delta{}_{kj} J^i{}_k) \\ &= \sum_{i,j,k,l} h^\gamma{}_{lj} h^\delta{}_{kj}(-J^\gamma{}_i J^i{}_k) = S_{\gamma\delta} ,\end{aligned}$$

which means that $S_{\alpha\beta}$ is diagonalized to the form

$$('S_{\alpha\beta})_x = \left(\begin{array}{ccc:ccc} S_{n+1} & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & S_{n+q} & & & \\ \hdashline & & & S_{n+1} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & S_{n+q} \end{array}\right)$$

at a (fixed) point x of M, by operating an orthogonal transformation (or real representation of a unitary transformation) to e_α-part of adapted frames; $(e_\alpha) \to ('e_\alpha = \sum U^\beta{}_\alpha e_\beta)$, where $U^\beta{}_\alpha$ are constant and $('e_A) = (e_i, 'e_\alpha)$ is defined on the domain where (e_A) is defined. The eigenvalues S_a are all real and nonnegative.

Let S denote the square of the length of the second fundamental form. Then

$$\sum_{\alpha,\beta,i,j} 'h^\alpha{}_{ij}'h^\beta{}_{ij} = \sum_{\alpha,i,j} 'h^\alpha{}_{ij}'h^\alpha{}_{ij} = S = \sum_\alpha S_\alpha = 2\sum_a S_a$$

at x, where $'h^\alpha{}_{ij}$'s denote the components with respect to the new frame field $('e_A)$. By (1.3) and (1.4), we get

$$-\sum_{\alpha,i,j} 'h^\alpha{}_{ij}\Delta' h^\alpha{}_{ij} = 8 \sum_{a,b,i,j,k,l} 'h^a{}_{ij}'h^a{}_{jk}'h^b{}_{kl}'h^b{}_{li} + 2\sum_a S^2{}_a - \tfrac{1}{2}(m+2)S \tag{1.6}$$

at x. Now we show that

$$8 \sum_{i,j,k,l} 'h^a{}_{ij}'h^a{}_{jk}'h^b{}_{kl}'h^b{}_{li} \leq 4S_a S_b \tag{1.7}$$

holds at x. Since $'h^b{}_{kl}$ is symmetric in k and l, as is well known, by operating an orthogonal transformation (or real representation of a unitary transformation) to e_i-part of adapted frames: $(e_i) \to (*e_i = \sum U^j{}_i e_j)$, where $U^j{}_i$ are constant, $('h^b{}_{kl})$ is diagonalized to the following form

$$(1.8)\qquad ({}^*h^b{}_{kl})_x = \left[\begin{array}{ccc:ccc} \lambda_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & \lambda_m & & & \\ \hdashline & & & -\lambda_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -\lambda_m \end{array}\right], \qquad 0 \le \lambda_1 \le \cdots \le \lambda_m$$

at the point, where ${}^*h^b{}_{kl}$'s denote the components with respect to $({}^*e_A) = ({}^*e_i, {}'e_\alpha)$. Then

$$\begin{aligned} 8 \sum_{i,j,k,l} {}^*h^a{}_{ij}{}^*h^a{}_{jk}{}^*h^b{}_{kl}{}^*h^b{}_{li} &= 8 \sum_{i,j} ({}^*h^a{}_{ij})^2({}^*h^b{}_{ii})^2 \\ &\le 4 \sum_{i,j} ({}^*h^a{}_{ij})^2(\lambda_m{}^2 + \lambda_m{}^2) \\ &= 4S_a(2\lambda_m{}^2) \le 4S_aS_b \end{aligned}$$

at x, where we have used

$$(1.9)\qquad 2\lambda_m{}^2 \le 2 \sum_r \lambda_r{}^2 = S_b \ .$$

Consequently, (1.6) and (1.7) imply

$$\begin{aligned} (1.10)\qquad - \sum_{\alpha,i,j} {}^*h^\alpha{}_{ij}\Delta{}^*h^\alpha{}_{ij} &\le 4 \sum_{a,b} S_aS_b + 2 \sum_a S_a{}^2 - \tfrac{1}{2}(m+2)S \\ &= 4(\sum_a S_a)^2 + [2(\sum_a S_a)^2 - 4 \sum_{a<b} S_aS_b] - \tfrac{1}{2}(m+2)S \\ (1.11)\qquad &\le 6(\sum_a S_a)^2 - \tfrac{1}{2}(m+2)S = \tfrac{3}{2}S^2 - \tfrac{1}{2}(m+2)S \end{aligned}$$

at x. Since S is independent of the choice of adapted frames, and $\sum h^\alpha{}_{ij}\Delta h^\alpha{}_{ij}$ is also invariant under orthogonal transformations of the adapted frames, we have

$$- \sum_{\alpha,i,j} h^\alpha{}_{ij}\Delta h^\alpha{}_{ij} \le \tfrac{3}{2}S^2 - \tfrac{1}{2}(m+2)S$$

on the domain where (e_A) is defined. On the other hand,

$$(1.12)\qquad - \sum_{\alpha,i,j} h^\alpha{}_{ij}\Delta h^\alpha{}_{ij} = \sum_{\alpha,i,j,k} (h^\alpha{}_{ijk})^2 - \tfrac{1}{2}\Delta S \ ,$$

where $h^\alpha{}_{ijk}$'s are defined by the first equation of (2.1)(cf. [6]). Integration of (1.12) and relations above yield the following integral inequalities:

$$(1.13)\qquad 0 \le \int_M \sum_{\alpha,i,j,k} (h^\alpha{}_{ijk})^2 *1 \le \int_M \frac{1}{2}[3S^2 - (m+2)S] *1 \ .$$

Theorem 1. *Let M be a compact complex submanifold of complex dimension m immersed in CP^{m+q}. Then the square S of the length of the second fundamental form satisfies*

$$\int_M [3S - (m+2)]S{*}1 \geq 0 . \tag{1.14}$$

Consequently, we have

Theorem 2. *Let M be a compact complex submanifold of complex dimension m immersed in CP^{m+q}, and assume that $S \leq \frac{1}{3}(m+2)$ holds on M.*

(1) *If inequality holds at some point of M, then $S = 0$.*

(2) *Otherwise, $S = \frac{1}{3}(m+2)$.*

Proof. If $S < \frac{1}{3}(m+2)$ on M, (1.14) implies $S = 0$ on M since S is nonnegative.

If $S < \frac{1}{3}(m+2)$ on a nonempty open set W and $S = \frac{1}{3}(m+2)$ on the nonempty closed set $M - W$, then we have $S = 0$ on W. This is a contradiction since S is continuous.

2. Complex submanifolds with $S = \frac{1}{3}(m+2)$

Let M be a compact complex submanifold of complex dimension m immersed in CP^{m+q} with $S = \frac{1}{3}(m+2)$. Then we have equality in (1.9), (1.11) and (1.13). By (1.13) and (1.11), we have

$$\sum_k h^\alpha{}_{ijk} w^k = dh^\alpha{}_{ij} - \sum_k h^\alpha{}_{kj} w^k{}_i - \sum_k h^\alpha{}_{ik} w^k{}_j + \sum_\beta h^\beta{}_{ij} w^\alpha{}_\beta = 0 , \tag{2.1}$$

$$\sum_{a<b} S_a S_b = 0 . \tag{2.2}$$

We consider these at an arbitrarily fixed point x as in § 1. By (2.2) at most one S_a is nonvanishing. Since $S = 2\sum S_a = \frac{1}{3}(m+2)$, changing the order if necessary we have $S_{n+1} = \frac{1}{6}(m+2)$, $S_a = 0$ for $a \geq n+2$. Denote by $[S]$ the field of operators to normal vectors such that $[S]X = \sum S^\alpha{}_\beta X^\beta e_\alpha$, where $S^\alpha{}_\beta = \sum g^{\alpha\gamma} S_{\gamma\beta}$ and X^β's denote the components of a vector field X normal to M. Then we see that $[S]J = J[S]$. Let Y, Z_a $(a \geq n+2), JY, JZ_a$ be fields (on a domain D in M) of normal vectors such that they are orthonormal at x and satisfy

$$([S]Y)_x = \tfrac{1}{6}(m+2)Y_x , \qquad ([S]Z_a)_x = 0 .$$

Define E_{n+1} and E_a $(a \geq n+2)$ by $E_{n+1} = [S]Y$ and $E_a = ([S] - \frac{1}{6}(m+2))Z_a$ for $a \geq n+2$. Then E_a, JE_a $(a = n+1, \cdots, n+q)$ are differentiable. E_{n+1} satisfies $[S]E_{n+1} = \frac{1}{6}(m+2)E_{n+1}$ on D, since $([S] - \frac{1}{6}(m+2))[S]Y = 0$ which follows from the fact that $(t - \frac{1}{6}(m+2))t$ is the minimal polynomial of $[S]$. Similarly, we have $[S]E_a = 0$ for $a \geq n+2$. Therefore, if we take a sufficiently small domain D_0 in D, we have e_{n+1} and Je_{n+1} (normalizing E_{n+1}

and JE_{n+1}) and e_a, Je_a for $a \geq n+2$ (orthonormalizing within E_a, JE_a for $a \geq n+2$) such that

$$[S] = \left[\begin{array}{cccc:cccc} \frac{1}{6}(m+2) & & & & & & & \\ & 0 & & 0 & & & & \\ & & \ddots & & & & 0 & \\ & 0 & & 0 & & & & \\ \hdashline & & & & \frac{1}{6}(m+2) & & & \\ & & 0 & & & 0 & & 0 \\ & & & & & & \ddots & \\ & & & & & 0 & & 0 \end{array}\right]$$

holds on D_0 with respect to the new frame field (e_α) which is assumed to be an extended frame field on a domain in CP^{m+q} containing D_0.

Next, putting $\lambda_m = \lambda$, by equality in (1.9) we have (for $b = n+1$)

$$(2.3) \qquad ({}^*h^{n+1}{}_{ij}) = \left[\begin{array}{cccc:cccc} 0 & & & & & & & \\ & \ddots & 0 & & & & 0 & \\ 0 & & 0 & & & & & \\ & & & \lambda & & & & \\ \hdashline & & & & 0 & & & \\ & 0 & & & & \ddots & 0 & \\ & & & & 0 & & 0 & \\ & & & & & & & -\lambda \end{array}\right],$$

$$(2.4) \qquad ({}^*h^{n+q+1}{}_{ij}) = (\sum_k J_{ik}{}^*h^{n+1}{}_{kj}) = \left[\begin{array}{cccc:cccc} & & & & 0 & & & \\ & 0 & & & & \ddots & 0 & \\ & & & & 0 & & 0 & \\ & & & & & & & \lambda \\ \hdashline 0 & & & & & & & \\ & \ddots & 0 & & & & 0 & \\ 0 & & 0 & & & & & \\ & & & \lambda & & & & \end{array}\right]$$

at x. We show that there is a local field on D_1 in D_0 of adapted frames such that (2.3) and (2.4) hold on D_1. Denote by $[h]$ the field of linear operator such that $[h]X = (\sum h^{n+1i}{}_j X^j e_i)$ where $h^{n+1i}{}_j = \sum g^{ik} h^{n+1}{}_{kj}$ and X^j's denote components of a vector field X on M. Then $[h]$ satisfies $[h]J = -J[h]$ and $[h][h]J = J[h][h]$. From (2.3) it follows that $[h][h]$ has exactly two eigenvalues 0 and λ^2, where $\lambda^2 = (m+2)/12$ by $S = 2\sum S_a = 4\lambda^2$. Hence, similar to $[S]$ we have a local field (on D_1 in D_0) of orthonormal frames $e_1, \cdots, e_m, Je_1, \cdots, Je_m$ such that

$$[h][h]e_m = \lambda^2 e_m , \qquad [h][h]Je_m = \lambda^2 Je_m ,$$
$$[h][h]e_i = 0 \qquad \text{for } i = 1, \cdots, m-1 .$$

Since $(e_1, \cdots, e_{m-1}, Je_1, \cdots, Je_{m-1})$ defines a $(2m-2)$-dimensional distribution on D_1, its distribution is the same as the distribution $\{X; [h]X = 0\}$. If we restrict $[h]$ to the field of 2-planes spanned by (e_m, Je_m), $[h]$ has two eigenvalues λ and $-\lambda$. Therefore we have a local field of frames e_m, Je_m (denoted by the same letters) such that $[h]e_m = \lambda e_m$ and $[h]Je_m = -\lambda Je_m$. We extend (e_i) on a domain in CP^{m+q} containing D_1. Summerizing, we have a local field of adapted frames (e_A) such that $S_{\alpha\beta}$ is diagonal with nonvanishing S_{n+1}, and $h^{n+1}{}_{ij}$, $h^{n+q+1}{}_{ij}$ are diagonal as in (2.3), (2.4), holding on D_1. From now on in this section, we use this (e_A).

In (2.1) we put $(\alpha = n+1; i = m; j \neq m, j \neq n)$ and $(\alpha = n+1; i = m+m; j \neq m, j \neq n)$. Then

$$\text{(2.5)} \qquad w^m{}_j = w^{m+m}{}_j = 0 \qquad \text{for } j \neq m,\ j \neq m+m = n .$$

Since

$$\begin{aligned} dw^m{}_j &= -\sum_k w^m{}_k \wedge w^k{}_j + \Omega^m{}_j \\ &= -\sum_k w^m{}_k \wedge w^k{}_j + \tfrac{1}{4}\sum_{k,l} [K^m{}_{jkl} + \sum_\alpha (h^\alpha{}_{mk}h^\alpha{}_{jl} - h^\alpha{}_{ml}h^\alpha{}_{jk})]w^k \wedge w^l , \end{aligned}$$

by (1.2) and (2.5), we have

$$0 = dw^m{}_r = \tfrac{1}{4}(w^m \wedge w^r + w^{m+m} \wedge w^{m+r})$$

for $r \neq m$ on D_1. Since w^m and w^{m+m} are nonvanishing, $m \neq 1$ gives a contradiction, so that $m = 1$, and $S = 1$ and $\lambda^2 = \frac{1}{4}$ follow. Thus the curvature form of M is given by

$$\Omega^1{}_2 = w^1 \wedge w^2 + w^3{}_1 \wedge w^3{}_2 + w^{3+q}{}_1 \wedge w^{3+q}{}_2 = (1 - 2\lambda^2)w^1 \wedge w^2 = \tfrac{1}{2}w^1 \wedge w^2 .$$

which implies that the Kählerian manifold M is of constant curvature $\frac{1}{2}$, and is therefore simply connected. Hence M is complex analytically isometric to a 1-dimensional complex quadric CQ^1 in CP^2. Applying E. Calabi's rigidity theorem [4, Theorems 9, 10], we thus have

Theorem 3. *Let M be a compact complex submanifold of complex dimension m immersed in CP^{m+q}. If $S = \frac{1}{3}(m+2)$ holds on M, then $m = 1$ and M is imbedded as a complex quadric CQ^1 in some CP^2 in CP^{1+q}.*

3. Scalar curvature

The scalar curvature R of a complex submanifold of complex dimension m immersed in CP^{m+q} is given by (cf. K. Ogiue [14], etc.)

$$R = m(m+1) - S \; . \tag{3.1}$$

By Theorems 1, 2, and 3, we have

Theorem 4. *For a compact complex submanifold M of complex dimension m immersed in CP^{m+q}, the scalar curvature R of M with respect to the induced Kählerian structure satisfies*

$$\int_M (3m^2 + 2m - 2 - 3R)(m^2 + m - R) *1 \geq 0 \; . \tag{3.2}$$

Assume that on M, R satisfies

$$R \geq m(m+1) - \tfrac{1}{3}(m+2) \; . \tag{3.3}$$

(1) *If the inequality in* (3.2) *holds at some point of M, then $R = m(m+1)$ holds on M and M is imbedded as a projective subspace CP^m in CP^{m+q}.*

(2) *If the equality in* (3.2) *holds on M, then $m = 1$ and $R = 1$, and M is imbedded as a complex quadric $CQ^1 \subset CP^2 \subset CP^{1+q}$.*

It may be remarked that in (3.2), etc. the codimension q is not involved.

4. Holomorphic pinchings

Denote by $K(e_i, e_j) = K_{ij}$ the sectional curvature for a 2-plane (e_i, e_j) (with respect to the induced Kählerian structure on M). Then

$$R = 2 \sum_r \sum_{s \neq r} (K_{rs} + K_{rs*}) + 2 \sum_r K_{rr*} \tag{4.1}$$

If the holomorphic sectional curvature is δ-pinched; i.e., if $\delta \leq K(X, JK) \leq 1$, then we have (cf. M. Berger [2])

$$K_{rs} + K_{rs*} \geq \delta - \tfrac{1}{2} \qquad \text{for } r \neq s \; . \tag{4.2}$$

By noticing that the holomorphic sectional curvature of M is actually ≤ 1 (cf. (4.7) below) and considering (4.1) and (4.2), we thus get

$$R \geq m(2\delta m - m + 1) \; . \tag{4.3}$$

Theorem 5. *Let M be a compact complex submanifold of complex dimension m immersed in CP^{m+q}, and assume that on M the holomorphic sectional curvature with respect to the induced Kählerian structure satisfies*

$$K(X, JX) \geq 1 - \frac{m+2}{6m^2} \; . \tag{4.4}$$

(1) *If the inequality in* (4.4) *holds for some X at some point of M, then M is imbedded as a projective subspace CP^m in CP^{m+q}.*

(2) *If the equality in* (4.4) *holds on* M, *then* $m = 1$, $K(X, JX) = \frac{1}{2}$, *and* M *is imbedded as a complex quadric* $CQ^1 \subset CP^2 \subset CP^{1+q}$.

Proof. By (4.3) and (4.4) we have $S \leq \frac{1}{3}(m+2)$. Thus we have either $M = CP^m$ or $M = CQ^1$. The inequality in (4.4) for some X implies $K(X, JX) > \frac{1}{2}$ and $M \neq CQ^1$, and hence $M = CP^m$. The equality in (4.4) on M implies $K(X, JX) \neq 1$ and $M \neq CP^m$, and hence $M = CQ^1$.

If $q = 1$, then (4.4) is improved.

Theorem 6. *Let* M *be a compact complex hypersurface immersed in* CP^{m+1}. *If the holomorphic sectional curvature of* M *with respect to the induced Kählerian structure satisfies*

$$K(X, JX) \geq 1 - \frac{m+2}{6m}\,, \tag{4.5}$$

then we have the conclusions (1), (2) *of Theorem* 5.

Proof. From the expression of the sectional curvature $K(X, Y)$:

$$K(X, Y) = \tfrac{1}{4}[1 + 3(g(X, JX))^2] + \sum_{\alpha} [h^\alpha(X, X)h^\alpha(Y, Y) - (h^\alpha(X, Y))^2]\,, \tag{4.6}$$

it follows that

$$K(X, JX) = 1 - 2 \sum_{\alpha} [h^\alpha(X, X)]^2\,. \tag{4.7}$$

Since $q = 1$, we can diagonalize $(h^{n+1}{}_{ij})$ to the form (1.8), so that $K_{rr*} = 1 - 2\lambda_r{}^2$. Putting $K_{rr*} \geq \delta$, we have $1 - \delta \geq 2\lambda_r{}^2$, which, together with $S = 2S_{n+1} = 4 \sum \lambda_r{}^2$, yields

$$2m(1 - \delta) \geq S\,. \tag{4.8}$$

Thus $\frac{1}{3}(m+2) \geq 2m(1-\delta)$ implies $\frac{1}{3}(m+2) \geq S$ for $\delta = 1 + \frac{1}{6}(m+2)/m$. Then the rest of the proof is the same as that of Theorem 5.

Corollary. *Let* M *be a compact complex hypersurface immersed in* CP^3. *If the holomorphic sectional curvature of* M *satisfies*

$$K(X, JX) \geq 2/3\,, \tag{4.9}$$

then M *is imbedded as a projective hypersurface* CP^2 *in* CP^3.

Remark. For an imbedded hypersurface "$K(X, JX) > \frac{1}{2}$" is the best result (cf. K. Ogiue [16, Theorem 3.2]).

5. Positive curvature

By a similar technique as in the proof of Theorem 3.3 in [16], we have

Theorem 7. *Let* M *be a compact complex hypersurface immersed in* CP^{m+1} *where* $m \geq 2$. *If the sectional curvature of* M *with respect to the induced Kählerian structure satisfies*

$$K(X,Y) \geq \tfrac{1}{4}\left(1 - \frac{m+2}{3m}\right), \tag{5.1}$$

then M is imbedded as a projective hypersurface CP^m in CP^{m+1}.

Proof. We first diagonalize $(h^{n+1}{}_{ij})$ as in (1.8), and then use (4.6) to obtain

$$K(e_r + e_s, Je_r - Je_s) = \tfrac{1}{4} - \tfrac{1}{2}(\lambda_r^2 + \lambda_s^2) \tag{5.2}$$

for $r \neq s$. By putting $K(X,Y) \geq \delta$ we thus have $\frac{1}{2} - 2\delta \geq \lambda_r^2 + \lambda_s^2$. According as the dimension m is even or odd, let $m = 2w$ or $m = 2w + 1$. By noticing that $\lambda_1^2 = \min\{\lambda_i^2\} \leq \frac{1}{4} - \delta$, we get

$$S = 4\sum_r \lambda_r^2 = 4[(\lambda_1^2 + \lambda_2^2) + \cdots + (\lambda_{2w-1}^2 + \lambda_{2w}^2)] \leq m(1 - 4\delta),$$

$$S = 4[\lambda_1^2 + (\lambda_2^2 + \lambda_3^2) + \cdots + (\lambda_{2w}^2 + \lambda_{2w+1}^2)] \leq m(1 - 4\delta),$$

respectively. Thus $m(1 - 4\delta) \leq \frac{1}{3}(m+2)$ implies $S \leq \frac{1}{3}(m+2)$ for $\delta = \frac{1}{4}[1 - \frac{1}{3}(m+2)/m]$. Since $m \geq 2$, Theorems 3 and 4 complete the proof.

Remarks. (i) For $m = 1$, Theorem 6 is valid.

(ii) (5.1) means that M is δ'-pinched, $\delta' \geq \frac{1}{4}[1 - \frac{1}{3}(m+2)/m]$. In fact, we have $K(X, JX) \leq 1$ by (4.7), and $K(X,Y) \leq 1$ by Theorem 8.2 of R. L. Bishop and S. I. Goldberg [3].

(iii) Theorem 7 is a generalization of the results of K. Nomizu [10, Theorem 2], and K. Abe [1, Corollary 4.2.1].

6. Singular or nonsingular complex curves

Theorem 8. *Let M be a compact complex curve immersed in CP^{1+q}. If the sectional curvature of M with respect to the induced Kählerian structure is $\geq \frac{1}{2}$ and the inequality holds at some point, then M is a projective line.*

Proof. This follows from Theorem 5 with $m = 1$.

Remark. For a compact nonsingular complex curve, Theorem 8 was obtained by K. Nomizu and B. Smyth [11, Theorem 9] for $q = 1$, and by K. Ogiue [16, Theorem 4.1].

Theorem 9. *Let M be a compact complex curve immersed in CP^{1+q}. If the sectional curvature of M with respect to the induced Kählerian structure satisfies $\frac{1}{2} \leq K(X,Y) < 1$, then M is imbedded as a complex quadric $CQ^1 \subset CP^2 \subset CP^{1+q}$.*

Proof. If $K(X,Y) \geq \frac{1}{2}$, we have $M = CP^1$ or $M = CQ^1$. $K(X,Y) \neq 1$ implies $M = CQ^1$.

Remark. For a compact nonsingular complex curve, see [11], [16].

7. Remarks

(i) It is known that an odd-dimensional unit sphere $S^{2r+1}(1)$ (of constant sectional curvature 1) is a circle bundle over a complex projective space $CP^r(4)$

(of constant holomorphic sectional curvature 4) (i.e., Hopf fibration $\pi: S^{2r+1} \to CP^r$). Corresponding to the Kählerian structure on $CP^r(4)$ we have a Sasakian structure on $S^{2r+1}(1)$.

For a compact complex submanifold M of complex dimension m immersed in $CP^r(4)$ ($r = m + q$) we have an invariant Sasakian submanifold $\pi^{-1}M$ in $S^{2r+1}(1)$ of real dimension $u = 2m + 1$. Since invariant submanifolds are minimal (cf. for example, [20]), J. Simons' result (0.1) is applied to $\pi^{-1}M$ and hence also to M. In the latter case, (3.3) becomes

$$R > m(m+1) - (m + \tfrac{1}{2})/(4 - 1/p) . \tag{7.1}$$

(ii) By using (3.10) in [6] K. Ogiue [14] generalized (7.1) to

$$R > m(m+1) - (m+2)/(4 - 1/p) . \tag{7.2}$$

(iii) (3.3) is a generalization of (7.2). Consequently (3.3) can be extended to a proposition for an invariant Sasakian submanifold of $S^{2r+1}(1)$, which is better than Theorem 4.2 in [20]. Since the scalar curvature R' of $\pi^{-1}M$ in $S^{2r+1}(1)$ and the scalar curvature R^* of M in $CP^r(4)$ are related by $R' = R^* - (\dim \pi^{-1}M - 1)$ (cf. (5.12) in [19]), we have $R' = 4R - 2m$, where R denotes the scalar curvature of M as a submanifold of $CP^r = CP^r(1)$. Therefore we obtain the following result:

Let N be an invariant submanifold of $S^{2r+1}(1)$ as a Sasakian manifold, let $\dim N = u = 2m + 1$, *and assume that the scalar curvature R' of N satisfies*

$$R' \geq u(u-1) - \tfrac{2}{3}(u+3) . \tag{7.3}$$

If the inequality holds at some point of N, then $R' = u(u-1)$ and $N = S^u(1)$ in $S^{2r+1}(1)$; if the equality holds on N, then $u = 3$.

An example of Sasakian submanifold N of dimension 3 with equality in (7.3) is as follows: $N = \pi^{-1}CQ^1$ for $CQ^1 \subset CP^2 \subset CP^{1+q}$.

(iv) If a compact complex submanifold M is imbedded in CP^{m+q}, then M is algebraic. Hence stronger results are expected. In fact, for hypersurface M, $R > m^2$ implies that M is a projective hypersurface in CP^{m+1} (K. Ogiue [15], [16]).

(v) If the scalar curvature is constant, the best results for imbedded hypersurfaces are known (cf. S. S. Chern [5], S. Kobayashi [7]).

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