FORMAL LINEARIZATION OF VECTOR FIELDS AND RELATED COHOMOLOGY. II

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1. Introduction

In Part I, we have discussed some algebraic features of the problem of "linearization" of a Lie algebra of vector fields near an invariant submanifold. There, we encountered certain cohomology groups, known as "obstructions", to the linearization. It is our aim now to relate those cohomology groups to certain geometric invariants.

To explain what these invariants are, let us consider the following situation: K is a Lie group acting on a manifold M, and N is an orbit of K. Let $N^\perp$ be the normal vector bundle to N. Then, of course, K acts linearly on $N^\perp$ and also on any tensor bundle E constructed from $N^\perp$. Let $\Gamma(E)$ be the space of cross-sections of E. Then K acts geometrically as a linear transformation group on $\Gamma(E)$, as does $\boldsymbol{K}$, the Lie algebra of K. Thus the cohomology groups of $\boldsymbol{K}$, defined relative to this representation of $\boldsymbol{K}$ in $\Gamma(E)$, are natural invariants of the orbit N.

We shall show that the cohomology obstructions to the linearization of $\boldsymbol{K}$ near the orbit N are essentially determined by the cohomology groups of $\boldsymbol{K}$ with coefficients in $\Gamma(E)$, for a certain class of tensor bundles constructed from $N^\perp$. One may also remark that these cohomology groups also play a basic role in the problem of "deformation" of infinite dimensional linear representations of Lie groups [2].

2. Construction of the vector bundles

All data will be of differentiability class C^∞. We refer to [3] for the notations of differential geometry.

Let N be a manifold, and E a vector bundle over N. $\Gamma(E)$ will denote its space of cross sections. $\Gamma(E)$ is a module over $F(N)$, the ring of real-valued C^∞ functions on N. It is a general principle of differential geometry that most ideas can usually be described in an optimally elegant fashion in terms of these modules.

If E, E' are vector bundles over N, a *differential operator* from E to E' is a

real-linear mapping; $\Gamma(E) \to \Gamma(E')$ which, in terms of local coordinate systems for N and local product structures for E and E', takes the form of differential operator in the classical sense. The *order* of such an operator is its "classical" order on terms of local coordinates. In particular, let us denote by $D_h^1(E, E')$ the space of differential operators $\Delta: \Gamma(E) \to \Gamma(E')$ which, in terms of local coordinates, are given by first-order homogeneous differential operators in the classical sense. (Notice that "homogeneity" is a property which is independent of local coordinates for first-order differential operators, but not for higher order operators.)

Let K be a group of transformations of N which also acts linearly on the vector bundles E and E'. Thus K acts as a transformation group on $\Gamma(E)$ and $\Gamma(E')$ as follows:

If $\psi: N \to E$ is an element of $\Gamma(E)$, and $k \in K$, then

$$k(\psi)(p) = k\psi(k^{-1}p) \qquad \text{for } p \in N .$$

Suppose that $t \to k(t)$, $\infty < t < \infty$, is a one-parameter subgroup of K. Its infinitesimal generator, which we will denote by X, may be considered as an element of $\underline{K}$, the Lie algebra of K. Given $\psi \in \Gamma(E)$, one may define the *Lie derivative* of ψ by X, denoted by $X(\psi)$, as follows:

$$X(\psi) = \frac{d}{dt}k(t)(\psi)|_{t=0} .$$

This action defines a linear representation of $\underline{K}$ by operators on $\Gamma(E)$.

K and K' also act on $D_h^1(E, E')$: For $\Delta \in D_h^1(E, E')$, $k \in K$, and $\psi \in \Gamma(E)$ we have

$$k(\Delta)(\psi) = k(\Delta(k^{-1}\psi)) .$$

The action of $\underline{K}$ on Δ is

$$(2.1) \qquad\qquad X(\Delta)(\psi) = X(\Delta(\psi)) - \Delta(X(\psi)) .$$

$D_h^1(E, E')$ may also be identified with the space of cross sections of the vector bundle $\Gamma(T(N) \otimes E \otimes E'^*)$. The Lie derivative action (2.1) agrees with the action obtained by letting $\underline{K}$ act in a natural way on the bundle $E \otimes E'^*$.

($\otimes$ denotes the tensor product bundle. E'^* denotes the dual bundle of E', i.e., the fibre over each point is the dual space of the fibre of the original bundle. $T(N)$ denotes the tangent bundle to N. In making these identifications one should keep in mind the following fact from linear algebra; If V, V' are vector spaces, the space of linear maps: $V \to V'$ can be identified with $V \otimes V'^*$.)

Let us apply these general remarks to the following situation: K acts as a

transformation group in a manifold M, while N is a submanifold of M which is left invariant by the action of K.

Let $N^\perp$ be the normal vector bundle to N. Then the fibre of $N^\perp$ over a point $p \in N$ is the quotient M_p/N_p of the tangent spaces to M and N at p. $\Gamma(N^\perp)$ can be identified with the quotient $F(N)$-module $\Gamma(T(M))/\Gamma(T(N))$. Of course, since $T(M)$ and $T(N)$ are the tangent bundles of M and N, $\Gamma(T(M))$ and $\Gamma(T(N))$ can be identified with $V(M)$ and $V(N)$, the spaces of vector fields on M and N.

Let $V(M, N)$ denote the Lie algebra vector fields on M which are tangent to N. Our geometric assumptions then imply that K can be identified with a subalgebra of $V(M, N)$.

We will now define a mapping: $V(M, N) \to D^1_h(N^\perp, N^\perp)$. Let $X \in V(M, N)$. Then $\operatorname{Ad} X: Y \to [X, Y]$ maps $V(M)$ into itself, and acts there as a first-order differential operator. Let $\psi \in \Gamma(N^\perp)$. Choose a vector field $Y \in V(M)$ whose restriction to N, followed by the projection on the normal bundle, is the cross-section ψ. (For simplicity of terminology, we will refer to this operator as "projection of Y on $\Gamma(N^\perp)$".) Now set

(2.2) $\Delta_X(\psi) = $ projection of $[Y, X]$ on $\Gamma(N^\perp)$.

Since the kernel of the projection map: $V(M) \to \Gamma(N^\perp)$ is $V(M, N)$, this definition is independent of the ψ chosen (algebraically, of course, this is nothing but the action of $\operatorname{Ad}(V(M, N))$ in $V(M)/V(M, N)$.)

Let $V^2(M, N)$ be the set of $X \in V(M, N)$ such that $\Delta_X = 0$. We will define Δ^2_X as a bilinear map: $\Gamma(N^\perp) \times \Gamma(N^\perp) \to \Gamma(N^\perp)$ as follows:

For $\psi_1, \psi_2 \in \Gamma(N^\perp)$, choose $Y_1, Y_2 \in V(M)$ whose projections on $\Gamma(N^\perp)$ are ψ_1, ψ_2. Set

(2.3) $\Delta_X(\psi_1, \psi_2) = $ projection of $[Y_1, [Y_2, X]]$ on $\Gamma(N^\perp)$.

Lemma 2.1.

(2.4) $\Delta_X(\psi_1, \psi_2) = \Delta_X(\psi_2, \psi_1)$ for $\psi_1, \psi_2 \in \Gamma(N^\perp), X \in V^2(M, N)$.

Proof.   Using the Jacobi identity, we have

$$[Y_1, [Y_2, X]] = [[Y_1, Y_2], X] + [Y_2, [Y_1, X]] .$$

Since $X \in V^2(M, N)$, the projection of $[[Y_1, Y_2], X]$ on $\Gamma(N^1)$ vanishes, which proves (2.4).

Set

(2.5) $E = N^\perp, E^2 = E \circ E^3, E = E \circ E \circ E$, etc.

($\circ$ denotes the "symmetric tensor product of vector bundles".) Thus we can regard (2.3) as defining a map:

(2.6) $V^2(M, N) \to D_h^1(E^2, E)$.

Now let $V^3(M, N)$ denote the kernel of the map (2.6). One can now define a map: $V^3(M, N) \to D_h^q(E^3, E)$ in a similar way.

$$\varDelta_X(\psi_1, \psi_2, \psi_3) = \text{projection of } [Y_1, [Y_2, [Y_3, X]]] \text{ on } \Gamma(N^1) .$$

In a way similar to that used in Lemma 2.1, $\varDelta_X$ depends symmetrically on ψ_1, ψ_2, ψ_3.

This construction may be continued indefinitely. The result is a sequence $\{V^r(M, N)\}$ of subspaces of $V(M, N)$, $r = 1, 2$, such that

(2.7) $V(M, N) = V^1(M, N) \supset V^2(M, N) \supset V^3(M, N) \supset \cdots$.

A map:

(2.8) $V^2(M, N) \to D_h^1(E^r, E)$

is defined naturally.

(2.9) $[V^r(M, N), V^s(M, N)] \subset V^{r+s-1}(M, N)$ for $r, s \geq 1$.

(Lemma 3.1 of Part I applies to prove (2.9)). Notice that (2.7) and (2.9) imply that

(2.10) $[V(M, N), V^r(M, N)] \subset V^r(M, N)$.

In particular, if K is a subalgebra of $V(M, N)$, then Ad K passes to the quotient to define a representation of K in $V^r(M, N)/V^{r+1}(M, N)$. The cohomology groups of K defined by this representation are obviously geometric invariants of the action of K. Our goal is to see their relation to the cohomology groups defined in Part I, which governs the "linearization" of K.

To this end, let us determine the $V^r(M, N)$ in local coordinates. Choose the following range of indices and summation conventions:

$$1 \leq i, j \leq m = \dim M ; \; 1 \leq a, b \leq n = \dim N ; \; n + 1 \leq u, v \leq m .$$

Suppose (X_1) are coordinates for M, such that $X_u = 0$ determines N. A vector fields $Y \in V(M)$ of the form $Y = b_j \partial/\partial x_j$ belongs to $V(M, N)$ if and only if $b_u(N) = 0$.

Let us first determine $V^2(M, N)$. Suppose $X \in V(M, N) = V^1(M, N)$. Then X is of the form

(2.11) $X = A_a \partial/\partial X_a + b_{uv} x_v \partial/\partial x_u$.

First, if $X \in V^2(M, N)$ we must have

(2.12) $[\partial/\partial x_u, X] \in V(M, N)$.

But, in view of (2.11), the condition for (2.12) is $b_{uv}(N) = 0$. Thus X satisfies (2.12) if and only if it is of the following form:

(2.13) $X = A_a \partial/\partial x_a + b_{uvv'} x_v x_{v'} \partial/\partial x_u$.

With (2.3) satisfied, let us now determine the condition that

(2.14) $[b_u \partial/\partial x_u, X] \in V(M, N)$

for all $b_u \in F(M)$. For this, we must have

$X(b_u)(N) = 0$, i.e. , $X(F(M)) = 0$ and N ,   or $A_a(N) = 0$.

Finally then X is of the form:

(2.15) $X = A_{au} x_u \partial/\partial x_a + A_{uvv'} x_v x_{v'} \partial/\partial x_u$.

We have proved the following:

Theorem 2.2.   *A vector field X on M belongs to $V^2(M, N)$ if and only if, in terms of local coordinates, it admits a description of form (2.15) with coefficients $A_{au}, A_{uvv'} \in F(M)$.*

We can continue in this fashion to determine $V^r(M, N)$. For example, for $r = 2$, it requires that $[\partial/\partial x_u, [\partial/\partial x_{u'}, X]]$ belongs to $V(M, N)$, then requires that $[b_u \partial/\partial x_u, [b_{u'} \partial/\partial x_{u'}, X]]$ belongs to $V(M, N)$. The obvious result can be summed up as follows:

Theorem 2.3.   *A vector field X on M of the form:*

$$X = A_j \partial/\partial x_i$$

belongs to $V^r(M, N)$ if and only if: a) *the functions A_a vanish to the $(r-1)$-st order on N, and* b) *the functions A_u vanish to the r-th order on N.*

We can relate the results to those of Part I concerned with "linearization". Define

$$V^r(M, N)/V^{r+1}(M, N) = V_h^r(M, N)$$

(the subscript "h" indicates "homogeneous"). Then, in those local coordinates, $V_h^r(M, N)$ is identified as a vector space with the space of vector fields X of the form:

(2.16)
$$X = A_{au_1 \dots u_{r-1}}(x_1, \cdots, x_n) x_{u_1} \cdots x_{u_{r-1}} \partial/\partial x_a$$
$$+ A_{uu_1 \dots u_r}(x_1, \cdots, x_n) x_{u_1} \cdots x_{u_r} \partial/\partial x_u$$.

Let $Z = x_u \partial/\partial x_u$. Then Z is geometrically the vector field representing "dilitations" in a direction nomal to N. Further, for X of form (2.16),

(2.17) $[Z, X] = rX$.

Conversely, any X satisfying (2.17) can be written in form (2.16). This determines the spaces labelled "V^r" in §7 of Part I with the spaces now labelled "$V_h^r(M, N)$", and hence identifies the cohomology groups of K which represent the obstructions to "linearizing" K, as described in Part I, with the cohomology group of K determined by the action of space $D_h^q(E^r, E)$. Now, in turn, $D_h^q(E', E)$ may be identified with the space of cross-sections of a vector bundle over N. Hence we may sum up by saying that the cohomology obstructions are determined by the representations of K in the space of cross-sections of vector bundles over N which are associated with the normal vector bundle to N.

3. Lie algebra cohomology associated with homogeneous vector bundles

As we have seen, computing the "obstruction" to formal linearization of Lie algebras of vector fields in the neighborhood of an invariant submanifold reduces to computing the cohomology of the Lie algebra, as determined by the representation of the Lie algebra on the space of cross-sections of various vector bundles over the submanifold. We now turn independently of the linearization question to the following problem.

Suppose that G is a Lie group, and that a maximal rank, onto map $\pi: E \to N$ determines E as a vector bundle over N. Let us suppose that G acts as a transformation group on both E and N, and that the following conditions are satisfied:

a) π is an intertwining map for the action of G,

b) G acts transitively on N, i.e., if the isotropy subgroup of G at one point is a subgroup L, then N is the coset space G/L,

c) G acts linearly on E, i.e., if $p \in N$, with the fibre $\pi^{-1}(p) = E(p)$ a vector space, then g maps $E(p)$ linearly onto $E(gp)$.

To abbreviate the terminology, we will call such an object a *homogeneous vector bundle*. For the rest of the section one such object will be fixed. $\Gamma(E)$ will denote the space of cross-sections of the vector bundle. G and $\mathbf{G}$ (the Lie algebra of G) act linearly on $\Gamma(E)$. For $X \in \mathbf{G}$, and $\psi \in \Gamma(E)$, $X(\psi)$ will denote the transformation of ψ by x. This "Lie derivative" satisfies the following rule:

$$X(f\psi) = X(f)\psi + fX(\psi) \qquad \text{for } \psi \in \Gamma(E),\ f \in F(N) .$$

Let p be the point of N at which the isotropy subgroup of G is L. Let V denote the vector space $E(p) = \pi^{-1}(p)$. The action of G on E determines a linear action of L on V, and hence a representation of L by linear transformations on V. As is well-known, this linear representation of L determines the vector bundle [1]. In fact, if $\psi \in \Gamma(E)$, and $X \in L$, then this "linear isotropy represen-

tation" assigns to $\psi(p)$ the vector $X(\psi)(p)$.

Suppose that $X \to \omega(X)$ is a one-cocycle of G with coefficients in $\Gamma(E)$, i.e., ω is a linear map: $G \to \Gamma(E)$. (For simplicity, we will only consider one-dimensional cohomology since this determines the linear "deformations" with which we are concerned.) The "cocycle" condition is just

$$(3.1) \qquad X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y]) \qquad \text{for } X, Y \in G .$$

Our basic problem is to decide when such an ω cobounds, i.e., to determine when there is an element $\psi \in \Gamma(E)$ such that

$$(3.2) \qquad\qquad X(\psi) = \omega(X) \qquad \text{for all } X \in G .$$

As the first step to decide this question, we can define an operation of "restriction to the point p" for 1-cocycles, namely, given $\omega G \to \Gamma(E)$ satisfying (3.1) we can define $\omega(p): L \to \pi^{-1}(p)$ as follows:

$$(3.3) \qquad\qquad \omega(p)(X) = \omega(X)(p) \qquad \text{for } X \in L .$$

Obviously, $\omega(p)$ is a 1-cocycle of L, with coefficients defined by the linear isotropy representation.

We now ask the following question: Suppose $\omega(p)$ cobounds. Does ω itself cobound? We will examine this point locally. Choose the following range of indices and the summation convention:

$$1 \le i, j \le n = \dim N ,$$
$$1 \le a, b \le m = \dim G ,$$
$$1 \le u, v \le r = \dim \pi^{-1}(p) .$$

Suppose (x_i) is a coordinate system for N, (X_a) is a basis of G, and that (ψ_u) is a basis for cross-sections of E. Then

$$(3.4) \qquad\qquad [X_a, X_b] = C_{abc} X_c .$$

Suppose that

$$(3.5) \qquad\qquad X_a(\psi_u) = \alpha_{avu} \psi_v ,$$

$$(3.6) \qquad\qquad X_a = \beta_{abi} \partial/\partial x_i ,$$

$$(3.7) \qquad\qquad \omega(X_a) = \gamma_{au} \psi_u .$$

The coefficients in (3.4) to (3.7) are functions on N, and C_{abc} are constants, the structure constants of the Lie algebra G.

Let us now express (3.1) in terms of this local data.

$$X_b(\gamma_{au}\psi_u) - X_a(\gamma_{bu}\psi_u) = X_b(\gamma_u)\psi_u + \gamma_{au}\alpha_{bvu}\psi_v$$
$$- X_a(\gamma_{bu})\psi_u - \gamma_{bu}\alpha_{avu}\psi_v ,$$
$$\omega([X_a, X_b]) = C_{abc}\omega(X_c) = C_{abc}\gamma_{cu}\psi_u .$$

(3.1) then takes the form

(3.8) $$X_b(\gamma_{av}) + \gamma_{au}\alpha_{bvu} - X_a(\gamma_{bv}) - \gamma_{bu}\alpha_{avu} = C_{abc}\gamma_{cv} .$$

Let us now try to solve (3.3). Suppose that we look for ψ of the form $\psi = f_u\psi_u$. Then (3.3) takes the form

$$X_a(f_u)\psi_u + f_u\alpha_{avu}\psi_v = \gamma_{au}\psi_u ,$$

or

(3.9) $$X_a(f_v) + f_u\alpha_{avu} + \gamma_{au} .$$

Note that conditions (3.8) are the compatibility conditions resulting from first applying X_b to both sides of (3.9) and then permuting a and b and substracting. One can thus prove, using the classical methods, the existence of solutions of (3.9) in a neighborhood of a point p. Further, this solution is unique if its value at p is prescribed. Let us then summarize as follows:

Theorem 3.1. *Suppose ω is a 1-cocycle of G with coefficients in $\Gamma(E)$. Let p be a point of N, such that $\omega(p)$ of L is the coboundary of an element $\psi(p) \in \pi^{-1}(p)$. Then p has a neighborhood U such that ω, restricted to U, is a coboundary.*

Now let us attempt to make the argument global. First, suppose that N is connected, and that $\omega(p)$ cobounds for one point of N. Then notice that the set of all points p such that $\omega(p)$ cobounds is both open and closed, and hence is all of N.

Thus we can cover N with contractable open sets $\{U\}$ such that in each U, ω is the coboundary of a 0-cochain, i.e., of an element ψ_U of $\Gamma(E_U)$, where E_U denotes the vector bundle E restricted to U. In the intersection $U \cap U'$ of two such open sets, set $\psi_{UU'} = \psi_U - \psi_{U'}$. Then $\psi_{UU'}$ is a 0-cocycle, i.e.,

(3.10) $$G(\psi_{UU'}) = 0 .$$

Consider now the sheaf of germs of cross-sections of E which satisfy (3.10). By the existence and uniquness theorem, the stalks of this sheaf are finite dimensional. Notice that $(U, U') \to \psi_{UU'}$ defines a 1-(Cech)-cocycle of N with coefficients in this sheaf. Now, we have:

Theorem 3.2. *Suppose that the first (Cech) cohomology group of N with real coefficients vanishes. Assume also that N is connected, and that ω is a 1-cocycle of G with coefficients in $\Gamma(E)$ such that $\omega(p)$ cobounds for one point $p \in N$. Then ω itself cobounds.*

Proof. Our assumption about the first cohomology group implies that the 1-Cech cocycle $\{\psi_{UU'}\}$ cobounds, i.e., to each U in the covering one can assign an element $\psi_{U'}$ such that

$$\text{a)}\; G(\psi'_U) = 0 \,, \qquad \text{b)}\; \psi_{UU'} = \psi'_U - \psi'_{U'} = \psi_U - \psi_{U'} \,,$$

in the intersection $U \cap U'$ of of two such open sets. Then $\psi_U - \psi'_U$ agrees with $\psi_{U'} - \psi'_{U'}$ in the intersection $U \cap U'$, and hence defines a globally defined cross-section of E, i.e., a 0-cochain of G with coefficients in $\Gamma(E)$. By condition a), the coboundary of this 0-cochain is ω cobounds.

Remark. The general features of this argument are very reminiscent of A. Weil's proof of the de Rham theorem [4] connecting differential form cohomology and Cech cohomology. Presumably, it is a special case of a theorem relating Lie algebra cohomology with coefficients determined by sheaf cohomology and the action of the Lie algebra on cross-sections of vector bundles. (See the comments by the referee at the end of this paper.)

We can now present another point of interest.

Theorem 3.3. *Suppose that both the first cohomology group of L with coefficients in the linear isotropy representation and the first Cech cohomology group of N with real coefficients are finite dimensional (as real vector spaces). Then $H^1(G, \Gamma(E))$ is finite dimensional.*

For the proof, notice that we have defined a "restriction" linear map: $H^1(G, \Gamma(E)) \to H^1(L, \pi^{-1}(p))$. We have also defined a homomorphism of the kernel of this map into $H^1(N, R)$. The argument of Theorem 3.2 shows that this map is one-one, whence the conclusion of the theorem.

This result is of interest for group representation theory. Recall [2] that the first cohomology group may be considered as the "tangent space" to the equivalence classes of representation. Theorem 3.2 then reinforces the intuitive belief that the equivalence classes of representation of a Lie group form a "finite dimensional" family.

Bibliography

[1]   R. Hermann, *Lie groups for physicists,* Benjamin, New York, 1966.
[2]   ——, *Analytic continuation of group representations.* I–III, Comm. Math. Phys. **2** (1966) 251–270, **3** (1966) 53–74, **3** (1966) 75–97.
[3]   ——, *Differential geometry and the calculus of variations,* Academic Press, New York, 1968.
[4]   A. Weil, *Sur les théorèmes de de Rham,* Comment. Math. Helv. **26** (1952) 119–145.

REFEREE'S COMMENTS ON THE PRECEDING TWO PAPERS

The second paper and part of the first deal with the probrem of linearizing a Lie group or algebra near an invariant manifold. Some comments are in order with regards to this problem. Let G be a Lie group acting on a manifold M and acting transitively on an invariant submanifold N. The problem is whether the action of G in a tubular neighborhood of N is equivalent to the induced linear action on the normal bundle of N. Let H be the isotropy group of a point of N. We have the following:

Lemma.   *The action of G is equivalent to its normal bundle action if and only if there exists a submanifold through p transversal to N, which is invariant under H and on which the action of H is equivalent to a linear action.*

Proof.   If G can be linearized, then the fibre of the linear action of G provides the desired submarifold through p. Conversely, suppose such a submanifold exists. We are thus given a linear representation of H. Construct the associated vector bundle over N. We are given a map of a neighborhood of the origin in the fibre over p of this bundle into M which is equivariant with respect to the action of H. The action of G then induces a map of a neighborhood of the zero section into M which is equivariant with respect to G. This then provides the desired linearization. Notice that in this argument we could replace Lie group by local Lie group and thus by Lie algebra as well. The important point is that what really counts is the behaviour of H. We mention two corollaries:

If H is semi-simple, the action of G can always be linearized.

In fact, according to [1] of the first paper, we can linearize H in a whole neighborhood of p. The tangent space to N is invariant, and has an invariant complement since H is semi-simple. Thus the hypotheses of the lemma are satisfied. Another consequence is:

If H has an action near a fix point which cannot be linearized, then we can construct an M and an N for G which cannot be linearized.

In fact, starting with G and N and the given action of H, just construct the associated bundle over N.

Some more comments about the second paper: The relationship between the cohomology of the Lie algebra of G with values in the sections of the homogeneous vector bundle and the cohomology of the Lie algebra of H with values in the fiber is well known. This is, for example, the content of equation (2a) in Proposition 4.2 of H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, 1956, p. 275.