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A CLASS OF VARIATIONALLY COMPLETE REPRESENTATIONS

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Introduction

Let M be a complete Riemannian manifold on which a compact connected Lie group K acts as a group of isometries. If $M = \boldsymbol{R}^n$, then K has a fixed point, hence we lose no generality in assuming for this case that the action of K is a linear orthogonal representation.

Bott and Samelson [5] have defined the concept of variational completeness. Roughly speaking, the action of K on M is variationally complete if it produces enough Jacobi fields along geodesics to determine the multiplicities of focal points to the K-orbits. This notion remains interesting and useful for the case $M = \boldsymbol{R}^n$ (e.g., cf. [4], [5]).

In [6] we formulated the notion of a "K-transversal domain". This is a closed connected flat totally geodesic imbedded submanifold $T \subset M$ which meets all K-orbits and is orthogonal to every K-orbit at each point of intersection. We showed that the existence of a K-transversal domain implies variational completeness, and deduced strong structure theorems for the singular set, the Weyl group, and the Bott-Samelson K-cycles. For $M = \boldsymbol{R}^n$, such a T is evidently a linear subspace.

The theorems of [6], applied to the case $M = \boldsymbol{R}^n$, show that those orthogonal respresentations of K which admit a K-transversal domain bear striking resemblances to the isotropy representations associated to compact symmetric spaces (hereafter referred to as *s-representations*). Indeed, s-representations constitute the principal class of known examples. This suggests that further such analogies should be sought and exploited, the ultimate aim being a complete structure theory and classification.

We will present here three theorems which advance the above program. For this purpose we employ a linear map (due to Kostant)

$$R\colon \Lambda^2(\boldsymbol{R}^n) \to \boldsymbol{k} = \text{Lie algebra of } K\ ,$$

which will be called the curvature tensor of the representation. R is defined for an arbitrary orthogonal representation of K, and in the case of an s-representation it will actually coincide with the Riemann tensor [9]. Usually

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we interpret R as an antisymmetric bilinear map on $\boldsymbol{R}^n$ and write $R(X, Y)$ for $R(X \wedge Y)$. The precise definition and fundamental properties of R will be reviewed carefully in § 1.

Definition. A linear subspace V of $\boldsymbol{R}^n$ is R-flat if and only if $R(V, V) = 0$.

Our first theorem shows that the properly generalized theorem of Cartan-Hunt [7] not only holds, but characterizes our class of representations.

Theorem I. *The orthogonal representation of K on $\boldsymbol{R}^n$ admits a K-transversal domain if and only if any two maximal R-flat subspaces of $\boldsymbol{R}^n$ are conjugate under K. In this case the K-transversal domains are precisely the maximal R-flat subspaces.*

For the statement of the second theorem recall from [6] the notion of the Weyl group W. If $T \subset \boldsymbol{R}^n$ is a K-transversal domain, then W is the group of transformations of T produced by those elements of K which leave T invariant. By [6, Theorem III] this is a Coxeter group. Indeed, the singular varieties $P_1, \cdots, P_r$ in T are linear subspaces of codimension one, and W is generated by the reflections of T in these subspaces.

Theorem II. *Let $T \subset \boldsymbol{R}^n$ be a K-transversal domain, and W the Weyl group. Then K is reducible on $\boldsymbol{R}^n$ if and only if W is reducible on T. Indeed, each orthogonal W-invariant decomposition $T = T_1 \oplus T_2$ corresponds to an orthogonal K-invariant decomposition $\boldsymbol{R}^n = V_1 \oplus V_2$ such that T_i is a K-transversal domain for V_i, $i = 1, 2$.*

The curvature tensor R is a useful tool in the proof of this theorem.

Recall from [6, (3.6)] that, corresponding to the singular varieties $P_1, \cdots, P_r$ in T, there is a direct sum decomposition $\boldsymbol{k} = \boldsymbol{k}_T \oplus \boldsymbol{m}_1 \oplus \cdots \oplus \boldsymbol{m}_r$ as vector spaces, where $\boldsymbol{k}_T$ is the annihilator of T in $\boldsymbol{k}$, $\boldsymbol{m}_i \perp \boldsymbol{k}_T$ for all i, and $\boldsymbol{k}_T \oplus \boldsymbol{m}_i$ is the annihilator of P_i in $\boldsymbol{k}$.

s-representations are well known to have the following important properties which, however, are not enjoyed by every representation having a K-transversal domain:

(a) $\boldsymbol{m}_i \perp \boldsymbol{m}_j$ for all $i \neq j$.

(b) If $\boldsymbol{R}^n = V_1 \oplus V_2$ is a nontrivial orthogonal decomposition into K-invariant subspaces, then there is a decomposition $\boldsymbol{k} = \boldsymbol{k}_1 \oplus \boldsymbol{k}_2$ into nontrivial complementary ideals such that $\boldsymbol{k}_i(V_j) = 0$, $i \neq j$.

For purposes of classification, (b) is a desirable property. If anything, (a) is even more desirable. Indeed, we will prove (again with the help of R)

Theorem III. (a) *implies* (b).

Actually, (a) has a number of pleasant consequences which we hope to discuss in some later paper. For instance, it enables one to use R to define a finite system $\mathfrak{R}$ of linear functions on T whose kernels prove to be the singular varieties P_i (for s-representations $\mathfrak{R}$ is the set of "restricted roots"). We conjecture that these systems $\mathfrak{R}$ will classify the representations in question and that interesting new examples will appear.

In § 6 we discuss two examples (announced in [6]) which clearly violate (b), hence also violate (a). We conjecture, however, that (a) will fail only in a finite number of exceptional cases.

Notations and conventions. Lie groups will be denoted by upper case Roman letters (K, G, K_T, etc.) and their Lie algebras by corresponding lower case boldface latters ($\boldsymbol{k}, \boldsymbol{g}, \boldsymbol{k}_T$, etc.). We remark, however, that the subspaces $\boldsymbol{m}_i$ of $\boldsymbol{k}$ introduced above are *not* subalgebras.

If $X \in \boldsymbol{R}^n$, then $K_X \subset K$ will denote the connected stabilizer of X in K.

The standard negative definite inner product on $\boldsymbol{R}^n$ will be denoted $\langle\ ,\ \rangle$. On the Lie algebra $\boldsymbol{so}(n)$ this same symbol will denote the trace form, i.e., the negative definite form $\langle A, B\rangle = \operatorname{tr}(AB)$, $A, B \in \boldsymbol{so}(n)$.

The standard identification $\Lambda^2(\boldsymbol{R}^n) = \boldsymbol{so}(n)$ will be realized by the formula

$$\langle A, X \wedge Y\rangle = \langle A(X), Y\rangle\ , \qquad A \in \boldsymbol{so}(n), X, Y \in \boldsymbol{R}^n\ .$$

Let $\mathfrak{A}$ be the vector space of bilinear forms $Q\colon \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{so}(n)$. Then $SO(n)$ is represented on $\mathfrak{A}$ by $(xQ)(X, Y) = xQ(x^{-1}X, x^{-1}Y)x^{-1}$ for $x \in SO(n)$, $Q \in \mathfrak{A}, X, Y \in \boldsymbol{R}^n$. The corresponding Lie algebra representation of $\boldsymbol{so}(n)$ on $\mathfrak{A}$ takes the form

$$A(Q)(X, Y) = -Q(A(X), Y) - Q(X, A(Y)) - [Q(X, Y), A]$$

for $A \in \boldsymbol{so}(n), Q \in \mathfrak{A}, X, Y \in R^n$.

1. The curvature tensor

We define the antisymmetric tensor R. The representation of K on $\boldsymbol{R}^n$ is a homomorphism $K \to SO(n)$, hence induces a Lie algebra homomorphism $\boldsymbol{k} \to \boldsymbol{so}(n)$. If $\boldsymbol{k}_0$ is the kernel of this homomorphism, and $\boldsymbol{k}_*$ is the orthogonal complement (under any invariant negative definite inner product) of $\boldsymbol{k}_0$ in $\boldsymbol{k}$, then $\boldsymbol{k} = \boldsymbol{k}_0 \oplus \boldsymbol{k}_*$ is a decomposition into complementary ideals, and we may consider $\boldsymbol{k}_* \subset \boldsymbol{so}(n)$, the inclusion being determined by the given representation. Let $P\colon \boldsymbol{so}(n) \to \boldsymbol{k}_*$ be the orthogonal projection relative to $\langle\ ,\ \rangle$.

In $\boldsymbol{k}_*$ we have the negative semidefinite Killing form $B(\ ,\ )$ and the negative definite form $\langle\ ,\ \rangle$ restricted from that on $\boldsymbol{so}(n)$, hence the negative definite form $(\ ,\ ) = \langle\ ,\ \rangle + B(\ ,\ )$. Let $S\colon \boldsymbol{k}_* \to \boldsymbol{k}_*$ be the nonsingular self adjoint linear transformation such that $(A, B) = \langle A, S(B)\rangle, A, B \in \boldsymbol{k}_*$.

Definition. $R\colon \Lambda^2(\boldsymbol{R}^n) \to \boldsymbol{k}$ is the composition

$$\Lambda^2(\boldsymbol{R}^n) = \boldsymbol{so}(n) \xrightarrow{P} \boldsymbol{k}_* \xrightarrow{S^{-1}} \boldsymbol{k}_* \subset \boldsymbol{k}\ .$$

If $X, Y \in \boldsymbol{R}^n$, we write $R(X, Y)$ for $R(X \wedge Y)$, hence R is interpreted as an antisymmetric tensor.

Remarks. This definition seems to be due to Kostant. The S^{-1} in the definition is not essential for our purposes, but is needed if R is to agree with the

Riemann tensor in the case of an s-representation. Kostant shows (unpublished) that the antisymmetric 4-tensor A^* defined by

$$A^*(X, Y, Z, W) = \langle R(X, Y)Z + R(Y, Z)X + R(Z, X)Y, W\rangle ,$$

for all $X, Y, Z, W \in \boldsymbol{R}^n$, can be used to characterise the isotropy representations associated to homogeneous spaces G/K with G compact. Indeed, viewing $A^* \in \Lambda^4(\boldsymbol{R}^n)$ and remarking that $\Lambda^*(\boldsymbol{R}^n)$ can be viewed as an (ungraded) algebra under Clifford multiplication, we can assert

Theorem (*Kostant*). *An orthogonal effective representation of K on $\boldsymbol{R}^n$ is equivalent to the isotropy representation for G/K, G being some compact Lie group, if and only if there is $B^* \in \Lambda^3(\boldsymbol{R}^n)$ whose Clifford square has A^* as its 4-component. In this case R is the curvature tensor for Nomizu's canonical connection on the bundle $G \to G/K$.*

Since we are interested in both comparing and contrasting a certain class of representations with the class of s-representations, the following result has potential value for our program.

Theorem (*Cartan-Kostant*). *An orthogonal effective representation of K on $\boldsymbol{R}^n$ is equivalent to an s-representation if and only if $A^* = 0$.*

(The author is grateful to the referee for remarking that this theorem is generalized in the work of Nomizu, Amer. J. Math., 1954.)

This second theorem is not too difficult to verify. Indeed, define a negative definite inner product $(\; , \;)$ on $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{R}^n$ by demanding $\boldsymbol{k} \perp \boldsymbol{R}^n$, letting $(\; , \;)$ on $\boldsymbol{k}$ be as already defined, and setting $(\; , \;) = \langle \; , \; \rangle$ on $\boldsymbol{R}^n$. $\boldsymbol{g}$ is made into a Lie algebra by defining

$$[A, B] \text{ as usual} , \qquad A, B \in \boldsymbol{k} ,$$
$$[A, X] = -[X, A] = A(X) , \qquad A \in \boldsymbol{k}, X \in \boldsymbol{R}^n ,$$
$$[X, Y] = R(X, Y) , \qquad X, Y \in \boldsymbol{R}^n .$$

The assumption $A^* = 0$ guarantees the Jacobi identity. The inner product $(\; , \;)$ is invariant relative to this Lie structure and is negative definite, so $\boldsymbol{g}$ becomes a Lie algebra of compact type. Since $[\boldsymbol{R}^n, \boldsymbol{R}^n] \subset \boldsymbol{k}$, the decomposition $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{R}^n$ is the Cartan decomposition corresponding to a compact symmetric space G/K. For the converse, first cf. [9, Theorem 6] for the proof that R is a multiple of the Riemann tensor. Then $A^* = 0$ follows from the Bianchi identity for R.

We now verify the elementary properties of R which will be needed.

(1.1) Lemma. *$(R(X, Y), B) = \langle Y, B(X)\rangle$, for all $X, Y \in \boldsymbol{R}^n$ and all $B \in \boldsymbol{k}_*$.*

Proof. $(R(X, Y), B) = (S^{-1}P(X \wedge Y), B) = \langle P(X \wedge Y), B\rangle = \langle X \wedge Y, B\rangle = \langle Y, B(X)\rangle$. q.e.d.

Denote by $\mathfrak{A}_*$ the subspace of $Q \in \mathfrak{A}$ such that $IM(Q) \subset k_*$. Then the composition of $K \to SO(n)$ with the representation of $SO(n)$ on $\mathfrak{A}$ induces a representation of K on $\mathfrak{A}_*$.

(1.2) Proposition. *$R \in \mathfrak{A}_*$ is invariant under the action of K. Thus also $A(R) = 0$ for all $A \in \boldsymbol{k}$.*

Proof. Remark that the ideal $\boldsymbol{k}_*$ is invariant under $\mathrm{Ad}(K)$. Thus for any $B \in \boldsymbol{k}_*$ and $x \in K$ we have

$$\begin{aligned}(xR(x^{-1}X, x^{-1}Y)x^{-1}, B) &= (R(x^{-1}X, x^{-1}Y), x^{-1}Bx) \\ &= \langle x^{-1}Y, x^{-1}B(X)\rangle = \langle Y, B(X)\rangle = (R(X, Y), B)\end{aligned}$$

for all $X, Y \in \boldsymbol{R}^n$.

(1.3) Proposition. *An orthogonal decomposition $\boldsymbol{R}^n = V_1 \oplus V_2$ is K-invariant if and only if $R(V_1, V_2) = 0$.*

Proof. $(R(V_1, V_2), \boldsymbol{k}_*) = \langle V_2, \boldsymbol{k}_*(V_1)\rangle$. This is identically zero if and only if $\boldsymbol{k}_*(V_1) \subset V_1$, hence if and only if $\boldsymbol{k}(V_1) \subset V_1$.

(1.4) Proposition. *A linear subspace $V \subset \boldsymbol{R}^n$ is an R-flat subspace if and only if V is orthogonal to every K-orbit which it meets.*

Proof. If $X \in V$, then the tangent space to $K \cdot X$ at X is $T_X = \boldsymbol{k}_*(X)$. Thus

$$\langle V, T_X\rangle = \langle V, \boldsymbol{k}_*(X)\rangle = (R(X, V), \boldsymbol{k}_*) .$$

But V is R-flat if and only if $R(X, V) = 0$ for all $X \in V$.

2. Proof of Theorem I

The theorem will be proven in a series of fairly easy propositions and lemmas.

(2.1) Proposition. *If every pair of maximal R-flat subspaces of $\boldsymbol{R}^n$ are conjugate under K, then every such subspace is a K-transversal domain.*

Proof. Let T be a fixed maximal R-flat subspace of $\boldsymbol{R}^n$. Let $Y \in \boldsymbol{R}^n$ and let T' be a maximal R-flat subspace containing Y. T' clearly exists since a one dimensional subspace containing Y is already R-flat. Let $x \in K$ such that $xT' = T$. Then, in particular, $xY \in T$, so T meets every K-orbit. By (1.4), T is orthogonal to each K-orbit at each point of intersection, hence T is a K-transversal domain.

(2.2) Proposition. *If $T \subset \boldsymbol{R}^n$ is a K-transversal domain, then T is a maximal R-flat subspace.*

Proof. T meets every orbit orthogonally, hence by (1.4) T is R-flat. If T is not maximal, let T' be an R-flat subspace properly containing T. Let $X \in T$ be a point whose orbit N is principal. By [6, (1.1)] it follows that $\dim(T') > n - \dim(N)$. But by (1.4), T' meets N orthogonally at X, so that $\dim(T') \leq n - \dim(N)$. This contradiction proves the maximality of T. q.e.d.

Now suppose that $T \subset \boldsymbol{R}^n$ is a K-transversal domain. In view of the above, Theorem I will be proven if we can show that any maximal R-flat subspace $T' \subset \boldsymbol{R}^n$ is conjugate to T under K.

Select $0 \neq Y \in T'$ such that the K-orbit of Y has maximal dimension among all the orbits meeting T'.

(2.3) Lemma. *No generality is lost in assuming $Y \in T \cap T'$.*

This lemma is an immediate consequence of (1.2) and the fact that T meets all K-orbits.

Let S_Y be a slice at Y in the sense of [1, pp. 105–108]. Thus S_Y is a small convex open set in the normal space to the orbit of Y at Y, and $X \in S_Y$ implies $K_X \subset K_Y$. Let $U_Y = \{X \in S_Y : K_X = K_Y\}$.

(2.4) Lemma. $T' \cap S_Y \subset U_Y$.

Proof. If $X \in T' \cap S_Y$, then $K_X \subset K_Y$ and, by the maximality condition on the orbit of Y, $\dim(K_X) \geq \dim(K_Y)$. It follows that $K_X = K_Y$.

(2.5) Lemma. $U_Y \subset T$.

Proof. Let $X \in U_Y$. Since $X \in S_Y$, [6, (1.3)] shows that there is $x \in K_Y$ such that $xX \in T$. But $K_X = K_Y$, so $xX = X$.

(2.6) Proposition. $T' = T$.

Proof. T' is R-flat, hence is orthogonal to the orbit of Y. Thus $T' \cap S_Y$ is an open subset of T'. But, by (2.4) and (2.5), $T' \cap S_Y \subset T$. It follows that $T' \subset T$. But T is R-flat and T' is maximal R-flat, so $T' = T$.

(2.7) Corollary. *If $\boldsymbol{R}^n$ admits a K-transversal domain, then any two maximal R-flat subspaces are conjugate under K.*

This completes the proof of Theorem I.

3. A decomposition of $\boldsymbol{R}^n$

As usual, let $P_1, \cdots, P_r$ be the singular varieties in T and let $\boldsymbol{k} = \boldsymbol{k}_T \oplus \boldsymbol{m}_1 \oplus \cdots \oplus \boldsymbol{m}_r$ be the corresponding direct sum decomposition. If $\boldsymbol{k}^i$ is the annihilator of P_i in $\boldsymbol{k}$, then $\boldsymbol{k}^i = \boldsymbol{k}_T \oplus \boldsymbol{m}_i$ is an orthogonal decomposition.

Definition. $U_i = \boldsymbol{k}^i(T) = \boldsymbol{m}_i(T) \subset \boldsymbol{R}^n$.

(3.1) Lemma. *U_i is a linear subspace of $\boldsymbol{R}^n$ with* $\dim(U_i) = \dim(\boldsymbol{m}_i)$.

Proof. Choose $X \in T - P_i$. Since $\boldsymbol{m}_i(P_i) = 0$, we have $\boldsymbol{m}_i(X) = U_i$. Thus $L: \boldsymbol{m}_i \to \boldsymbol{R}^n$ defined by $L(A) = A(X)$ is a linear map with $U_i = L(\boldsymbol{m}_i)$. Furthermore, if $L(A) = 0$, then $A(T) = 0$ and $A \in \boldsymbol{k}_T \cap \boldsymbol{m}_i = 0$. L is therefore one-one.

(3.2) Lemma. $\langle T, U_i \rangle = 0$.

Proof. Indeed, $U_i = \boldsymbol{m}_i(X) \subset \boldsymbol{k}(X) =$ tangent space to the K-orbit of X at X. Since T is K-transversal, $\langle T, U_i \rangle = 0$.

(3.3) Lemma. *If $A, B \in \boldsymbol{k}$ and $X, Y \in T$, then $\langle X, AB(Y) \rangle = \langle X, BA(Y) \rangle$.*

Proof. Indeed, $[A, B] \in \boldsymbol{k}$, so $\langle X, [A, B](Y) \rangle \in \langle T, [A, B](Y) \rangle \subset \langle T, \boldsymbol{k}(Y) \rangle = 0$. The assertion is immediate.

(3.4) Lemma. *$\langle U_i, U_j \rangle = 0$ if $i \neq j$.*

Proof. Let $X \in T - P_i$ and $Y \in P_i - P_j$. Thus $\boldsymbol{m}_i(X) = U_i$, $\boldsymbol{m}_j(Y) = U_j$, $\boldsymbol{m}_i(Y) = 0$. Let $A \in \boldsymbol{m}_i$, $B \in \boldsymbol{m}_j$. Since $A(Y) = 0$, we then have

$$\langle A(X), B(Y) \rangle = -\langle X, AB(Y) \rangle = -\langle X, BA(Y) \rangle = 0 .$$

(3.5) Proposition. $R^n = T \oplus U_1 \oplus \cdots \oplus U_r$, *an orthogonal decomposition.*

Proof. By the above lemmas the sum is orthogonal. Furthermore,

$$k(T) = \sum_{i=1}^{r} m_i(T) = \sum_{i=1}^{r} U_i .$$

Let $X \in T$ be a point whose K-orbit is principal. Then $k(X)$ is the orthogonal complement of T in R^n by [6, (1.1)]. Since $k(X) \subset k(T)$, we see that the sum of T and all of the U_i must equal all of R^n. q.e.d.

It would be nice to have $R(T, U_i) = m_i$ for all i, but we can only assert this under the hypothesis of (a) (cf. **Introduction**). Indeed,

(3.6) Proposition. (a) *holds if and only if* $R(T, U_i) = m_i$, $i = 1, \cdots, r$.

Proof. First remark that $(R(T, U_i), k_T) = \langle U_i, k_T(T) \rangle = 0$ and, if $i \neq j$,

$$(R(T, U_i), m_j) = \langle U_i, m_j(T) \rangle = \langle U_i, U_j \rangle = 0 .$$

Now assume (a). Then $R(T, U_i) \subset m_i$ is immediate from the above relations. Take $Z \in T - P_i$ and define a linear map $J\colon U_i \to m_i$ by $J(X) = R(Z, X)$. Writing $X = A(Z)$ for some $A \in m_i$, we get

$$(J(X), A) = (R(Z, X), A) = \langle X, A(Z) \rangle = \langle X, X \rangle ,$$

which vanishes if and only if $X = 0$. Thus J is one-one. Then, by (3.1), J must be onto, hence $R(T, U_i) = m_i$.

For the converse, suppose $R(T, U_i) = m_i$. Then, if $i \neq j$,

$$(m_i, m_j) = (R(T, U_i), m_j) = \langle U_i, m_j(T) \rangle = \langle U_i, U_j \rangle = 0 .$$

4. Proof of Theorem II

Assume that W is reducible on T and write $T = T_1 \oplus T_2$, an orthogonal decomposition into nontrivial W-invariant subspaces.

(4.1) Lemma. *If* $W|_{T_1} =$ *identity, then* $K|_{T_1} =$ *identity.*

Proof. By the structure theory of W, we must have $T_1 \subset P_i$, $i = 1, \cdots, r$. Thus $k^i(T_1) = 0$, $i = 1, \cdots, r$. It follows that $k(T_1) = 0$. q.e.d.

In the above case we set $V = T_1^{\perp}$ and pass to $K|_V$ as the only interesting part. We therefore assume from here on that W has no nonzero fixed point in T.

(4.2) Lemma. *For each singular hyperplane* P_i *in* T, *either* $T_1 \subset P_i$ *or* $T_2 \subset P_i$.

Proof. Let $w_i \in W$ be the reflection in P_i. w_i has exactly a one dimensional -1 eigenspace. Since $w_i|_{T_1}$ and $w_i|_{T_2}$ are involutions, the -1 eigenspace must either be in T_1 or in T_2, and w_i is the identity on the other. q.e.d.

We may suppose that $P_1, \cdots, P_q$ is the set of singular hyperplanes in T such that $T_1 \subset P_i$, hence that $P_{q+1}, \cdots, P_r$ is the set such that $T_2 \subset P_i$.

(4.3) Lemma. *$T_1 = P_1 \cap \cdots \cap P_q$ and $T_2 = P_{q+1} \cap \cdots \cap P_r$.*

Proof. $T_1 \subset P_1 \cap \cdots \cap P_q$. If equality does not hold, find nonzero X in this intersection such that $X \perp T_1$. Thus $X \in T_2 \subset P_{q+1} \cap \cdots \cap P_r$. This makes $X \neq 0$ a fixed point of W contrary to our assumption. T_2 is treated similarly. q.e.d.

Let $U_1, \cdots, U_r$ be as in (3.5). Set

$$V_1 = T_1 \oplus U_{q+1} \oplus \cdots \oplus U_r \,, \qquad V_2 = T_2 \oplus U_1 \oplus \cdots \oplus U_r$$

so that $\boldsymbol{R}^n = V_1 \oplus V_2$ is an orthogonal decomposition by (3.5). We will show that V_1 and V_2 are K-invariant.

(4.4) Lemma. *$T_2 \oplus V_1 = \{Y \in \boldsymbol{R}^n : R(Y, T_2)\} = 0$, and similarly $T_1 \oplus V_2 = \{Y \in \boldsymbol{R}^n : R(Y, T_1)\} = 0$.*

Proof. Indeed, $T_2 \oplus V_1 = T \oplus U_{q+1} \oplus \cdots \oplus U_r$ and $R(T, T_2) = 0$. If $q + 1 \leq i \leq r$, then $(R(T_2, U_i), \boldsymbol{k}^i) = \langle U_i, \boldsymbol{k}^i(T_2)\rangle = 0$ since $T_2 \subset P_i$, and, if $j \neq i$,

$$(R(T_2, U_i), \boldsymbol{k}^j) = \langle U_i, \boldsymbol{k}^j(T_2)\rangle \subset \langle U_i, U_j\rangle = 0 \,.$$

Since the $\boldsymbol{k}^j$'s linearly span $\boldsymbol{k}$, we have $R(T_2, U_i) = 0$. Thus $R(T_2, T_2 \oplus V_1) = 0$.

On the other hand, choose $Z \in T_2 - (P_1 \cup \cdots \cup P_q)$. Then $\boldsymbol{k}^j(Z) = U_i$, $1 \leq i \leq q$. If $A_i \in \boldsymbol{k}^i$ and $Y = \sum_{i=1}^{q} A_i(Z) \neq 0$, then some $A_{i_0}(Z) \neq 0$ and

$$\begin{aligned}(R(Z, Y), A_{i_0}) &= \sum_{i=1}^{q} (R(Z, A_i(Z)), A_{i_0}) = \sum_{i=1}^{q} \langle A_i(Z), A_{i_0}(Z)\rangle \\ &= \langle A_{i_0}(Z), A_{i_0}(Z)\rangle \neq 0 \,.\end{aligned}$$

Thus $R(Z, Y) \neq 0$ for every nonzero $Y \in U_1 \oplus \cdots \oplus U_q$. Since $Z \in T_2$, it follows that $R(T_2, Y) = 0$ if and only if $Y \in T_2 \oplus V_1$.

Similarly, $R(T_1, Y) = 0$ if and only if $Y \in T_1 \oplus V_2$.

(4.5) Proposition. *V_1 and V_2 are K-invariant.*

Proof. Let K^j be the connected subgroup of K corresponding to the Lie algebra $\boldsymbol{k}^j$. Then K is generated by $K^1, \cdots, K^r$. Let $g \in K^j$. Either $T_1 \subset P_j$ or $T_2 \subset P_j$. If $T_1 \subset P_j$, then $g|_{T_1} =$ identity. By (4.4) and the K-invariance of R, we must have $g(T_1 \oplus V_2) = T_1 \oplus V_2$, hence $g(V_2) = V_2$. Therefore also $g(V_1) = V_1$. An entirely parallel argument holds if $T_2 \subset P_j$. Thus $K(V_1) = V_1$ and $K(V_2) = V_2$. q.e.d.

By (1.3) we also know

(4.6) Corollary. *$R(V_1, V_2) = 0$.*

The hardest part of Theorem II is given by (4.5). The remainder is given by the following.

(4.7) Proposition. *If $\boldsymbol{R}^n = V_1 \oplus V_2$ is an orthogonal decomposition into two K-invariant subspaces, and $T \subset \boldsymbol{R}^n$ is a K-transversal domain, then for*

$i = 1, 2$, $T_i = T \cap V_i$ is W-invariant with $T = T_1 \oplus T_2$ and is a K-transversal domain in V_i.

Proof. Clearly T_1 and T_2 are W-invariant. Since T meets every K-orbit in $\boldsymbol{R}^n$ orthogonally, it is immediate that T_i has these same properties for V_i, $i = 1, 2$. Thus we have orthogonal decompositions $V_i = T_i \oplus \boldsymbol{k}(T_i)$, $i = 1, 2$, hence an orthogonal decomposition $\boldsymbol{R}^n = T_1 \oplus T_2 \oplus \boldsymbol{k}(T_1) \oplus \boldsymbol{k}(T_2)$. But $\boldsymbol{k}(T_i) \subset \boldsymbol{k}(T) \perp T$, $i = 1, 2$, so necessarily $T = T_1 \oplus T_2$.

5. Proof of Theorem III

Let $\boldsymbol{R}^n = V_1 \oplus V_2$, a nontrivial orthogonal K-invariant decomposition. Assume that (a) holds.

(5.1) Lemma. *$\boldsymbol{k}_i = R(\Lambda^2(V_i))$ is an ideal in $\boldsymbol{k}$, $i = 1, 2$.*

Proof. $R(\Lambda^2(V_i))$ is a linear subspace of $\boldsymbol{k}$. Let $A \in \boldsymbol{k}$, $X, Y \in V_i$. Then by (1.2),

$$0 = A(R)(X, Y) = -R(A(X), Y) - R(X, A(Y)) - [R(X, Y), A] ,$$

hence $[R(\Lambda^2(V_i), \boldsymbol{k}] \subset R(\Lambda^2(V_i))$.

(5.2) Lemma. *$\boldsymbol{k}_i(V_j) = 0$, $i \neq j$.*

Proof. As usual we can write

$$V_1 = T_1 \oplus U_{q+1} \oplus \cdots \oplus U_r , \qquad V_2 = T_2 \oplus U_1 \oplus \cdots \oplus U_q .$$

By (3.6), $R(T_1, V_1) \subset \boldsymbol{m}_{q+1} \oplus \cdots \oplus \boldsymbol{m}_r$, hence $R(T_1, V_1)(T_2) = 0$. Since $T_1 \oplus T_2$ is maximal R-flat, (1.3) and Theorem I imply that $T_1 \oplus yT_2$ is maximal R-flat for any $y \in K$. By Theorem I there is $x \in K$ such that

$$xT_1 \oplus xT_2 = x(T_1 \oplus T_2) = T_1 \oplus yT_2 ,$$

so $xT_1 = T_1$ and $xT_2 = yT_2$. Thus

$$\begin{aligned} R(T_1, V_1)(yT_2) &= R(xT_1, xV_1)(xT_2) = xx^{-1}R(xT_1, xV_1)(xT_2) \\ &= xR(T_1, V_1)(T_2) = 0 \end{aligned}$$

by (1.2). But V_2 is the union of all yT_2 as y ranges over K, so $R(T_1, V_1)(V_2) = 0$. Finally, if $x \in K$ is arbitrary,

$$\begin{aligned} R(xT_1, V_1)(V_2) &= R(xT_1, xV_1)(xV_2) = xx^{-1}R(xT_1, xV_1)(xV_2) \\ &= xR(T_1, V_1)(V_2) = 0 , \end{aligned}$$

and V_1 is the union of all xT_1. Thus $R(V_1, V_2)(V_2) = 0$, hence $\boldsymbol{k}_1(V_2) = 0$. Similarly $\boldsymbol{k}_2(V_1) = 0$.

(5.3) Lemma. *$\boldsymbol{k}_* = \boldsymbol{k}_1 \oplus \boldsymbol{k}_2$, an orthogonal decomposition.*

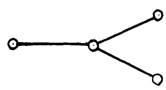
Proof. $(R(V_1, V_1), \boldsymbol{k}_2) = \langle V_1, \boldsymbol{k}_2(V_1) \rangle = 0$, hence $\boldsymbol{k}_1 \perp \boldsymbol{k}_2$. But also $\boldsymbol{k}_* = R(\Lambda^2(\boldsymbol{R}^n)) = R(\Lambda^2(V_1)) + R(\Lambda^2(V_2)) = \boldsymbol{k}_1 + \boldsymbol{k}_2$ since $R(V_1, V_2) = 0$. q.e.d.

These three lemmas give Theorem III as an immediate consequence. Indeed, if $\boldsymbol{k} = \boldsymbol{k}_0 \oplus \boldsymbol{k}_*$ with $\boldsymbol{k}_0 \neq 0$, then replace $\boldsymbol{k}_1$ by $\boldsymbol{k}_0 \oplus \boldsymbol{k}_1$, an ideal in $\boldsymbol{k}$ annihilating V_2, and obtain the desired decomposition of $\boldsymbol{k}$.

6. Examples

In [6] we announced two representations with K-transversal domains of dimension two, neither of which is an s-representation. We give the details here.

Recall that the Dynkin diagram for Spin (8) has the form



and hence has symmetry group isomorphic to the symmetric group $\sum_3$. By [8, p. 46] we can represent $\sum_3$ faithfully as the group of automorphisms of Spin (8) which leaves pointwise fixed a principal three-dimensional subgroup. Let $t, s \in \sum_3$ be elements of order 3 and 2 respectively. t is called a triality automorphism of Spin (8). It is fairly well known that the fixed point set of t is the compact exceptional group $G_2 \subset \mathrm{Spin}\,(8)$ and that of s is $\mathrm{Spin}\,(7) \subset \mathrm{Spin}\,(8)$. Indeed, these groups are seen to be the identity components of the respective fixed point sets by a fairly straightforward application of [8, Chapter II, § 3, and Chapter III, § 1], while the connectedness of these fixed point sets is guaranteed by [2, pp. 224–225]. Let $s_t = tst^{-1}$, the fixed point group of this automorphism being $\mathrm{Spin}_t\,(7) = t(\mathrm{Spin}\,(7))$.

The following two lemmas are well known, at least to experts, but for the sake of completeness we give proofs here.

(6.1) Lemma. $\mathrm{Spin}\,(7) \cap \mathrm{Spin}_t\,(7) = G_2$, *and the natural left action of* $\mathrm{Spin}_t\,(7)$ *on* $\mathrm{Spin}\,(8)/\mathrm{Spin}\,(7) = S^7$ *is transitive.*

Proof. $ss_t = stst^{-1} = t$ by the multiplication table for $\sum_3$, hence G_2 is the fixed point group for ss_t. Thus $\mathrm{Spin}\,(7) \cap \mathrm{Spin}_t\,(7) \subset G_2$. But this intersection is the stabilizer in $\mathrm{Spin}_t\,(7)$ for the base point $o \in \mathrm{Spin}\,(8)/\mathrm{Spin}\,(7)$. The $\mathrm{Spin}_t\,(7)$-orbit of o has dimension ≤ 7 and $\dim\,(\mathrm{Spin}_t\,(7)) = 21$, so

$$\dim\,(\mathrm{Spin}\,(7) \cap \mathrm{Spin}_t\,(7)) \geq 14 = \dim\,(G_2)\ .$$

All assertions follow.

(6.2) Lemma. *Let* $G_2 \subset \mathrm{Spin}\,(7)$ *as above and let* $\mathrm{Spin}\,(6) \subset \mathrm{Spin}\,(7)$ *be the standard imbedding. Then* $G_2 \cap \mathrm{Spin}\,(6) = SU(3)$ *and the natural left action of* G_2 *on* $\mathrm{Spin}\,(7)/\mathrm{Spin}\,(6) = S^6$ *is transitive.*

Proof. $G_2 \cap \mathrm{Spin}\,(6)$ is the isotropy group in G_2 for the basepoint $o \in \mathrm{Spin}\,(7)/\mathrm{Spin}\,(6)$. Let H denote the identity component of $G_2 \cap \mathrm{Spin}\,(6)$. The orbit $G_2 \cdot o$ has dimension ≤ 6 and $\dim\,(G_2) = 14$, so $\dim\,(H) \geq 8$. Thus H cannot be trivial nor of rank one, so that by [3] the only possibilities for H

are (in local notation): $T^2, T^1 \times A_1, A_1 \times A_1, A_2, G_2$. Dimensional considerations rule out all possibilities except A_2 and G_2.

If $H = G_2$, then $G_2 \subset \mathrm{Spin}\,(6)$ and G_2 has codimension one in Spin (6). At the Lie algebra level, there is an orthogonal decomposition $\mathbf{spin}\,(6) = \mathfrak{g}_2 \oplus \boldsymbol{R}$. Then $[\mathfrak{g}_2, \boldsymbol{R}] \subset \boldsymbol{R}$ and $[\boldsymbol{R}, \boldsymbol{R}] = 0$, so $\boldsymbol{R}$ is a proper nontrivial ideal, contradicting the fact that **spin** (6) is simple. Thus $H = A_2$.

Since $\dim(H) = 8$, $\dim(G_2 \cdot o) = 6$ and so $G_2 \cdot o = S^6$. The exact sequences

$$0 = \pi_1(G_2 \cdot o) \to \pi_0(G_2 \cap \mathrm{Spin}\,(6)) \to \pi_0(G_2) = 0\ ,$$
$$0 = \pi_2(G_2 \cdot o) \to \pi_1(G_2 \cap \mathrm{Spin}\,(6)) \to \pi_1(G_2) = 0$$

show that $G_2 \cap \mathrm{Spin}\,(6)$ is connected and simply connected, hence this group is $SU(3)$. q.e.d.

Using the above we will produce two representations satisfying the hypotheses of the following proposition.

(6.3) Proposition. *Let orthogonal representations of K on $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ be given such that K is transitive on the unit sphere S^{n-1} of $\boldsymbol{R}^n$ and such that, if $X \in S^{n-1}$, then K_X is transitive on the unit sphere of S^{m-1} of $\boldsymbol{R}^m$. Then the direct sum of these two representations has a K-transversal domain $T \subset \boldsymbol{R}^{n+m}$ of dimension two.*

Proof. Let $L_1 = \boldsymbol{R} \cdot X$ and let $L_2 \subset \boldsymbol{R}^m$ be any one dimensional subspace. Given $(A, B) \in \boldsymbol{R}^n \oplus \boldsymbol{R}^m = \boldsymbol{R}^{n+m}$, some element of K moves this to a point $(A', B') \in L_1 \oplus \boldsymbol{R}^m$. K_X fixes L_1 pointwise and is transitive on S^{m-1}, hence some element of K_X moves (A', B') to a point on $T = L_1 \oplus L_2$. Thus every K-orbit meets T. $R(T, T) = 0$ is an immediate consequence of (1.3), hence, by (1.4), T is a K-transversal domain. q.e.d.

Let r_1, r_2 be the orthogonal representations of Spin (8) on $\boldsymbol{R}^8$ induced respectively by

$$\mathrm{Spin}\,(8) \xrightarrow{t^{-1}} \mathrm{Spin}\,(8) \xrightarrow{p} SO(8)\ , \qquad \mathrm{Spin}\,(8) \xrightarrow{p} SO(8)$$

where p is the double covering. Let q_1 be the orthogonal representation of Spin (7) on $\boldsymbol{R}^8$ induced by

$$\mathrm{Spin}\,(7) \xrightarrow{t} \mathrm{Spin}_t\,(7) \subset \mathrm{Spin}\,(8) \xrightarrow{p} SO(8)\ ,$$

and let q_2 be the representation on $\boldsymbol{R}^7$ induced by the double covering Spin (7) $\to SO(7)$. By (6.1), (6.2), and (6.3) we conclude

(6.4) Theorem. *$r_1 \oplus r_2$ is an orthogonal representation of* Spin (8) *on $\boldsymbol{R}^{16}$ having a two-dimensional* Spin (8)*-transversal domain, and $q_1 \oplus q_2$ similarly represents* Spin (7) *on $\boldsymbol{R}^{15}$ with a two-dimensional* Spin (7)*-transversal domain.*

Remark that the t^{-1} in the definition of r_1 is essential. $r_2 \oplus r_2$ does not admit a Spin (8)-transversal domain.

In both cases the singular varieties in T are the subspaces L_1 and L_2 in the proof of (6.3). Thus the Weyl group $W = Z_2 \oplus Z_2$.

In the case of $r_1 \oplus r_2$, the group K_T is Spin (7) $\cap$ Spin_t (7) $= G_2$, $K^1 =$ Spin_t (7), $K^2 =$ Spin (7), and so $K^i/K_T = S^7$ as predicted by [6, Theorem IV]. The principal orbits of Spin (8) in $\boldsymbol{R}^{16}$ have the form

$$K/K_T = \mathrm{Spin}\,(8)/G_2 = S^7 \times S^7 .$$

In the case of $q_1 \oplus q_2$, $K_T = G_2 \cap \mathrm{Spin}\,(6) = SU(3)$, $K^1 = G_2$, $K^2 = \mathrm{Spin}\,(6)$ $= SU(4)$, hence

$$K^1/K_T = G_2/SU(3) = S^6 , \qquad K^2/K_T = SU(4)/SU(3) = S^7 ,$$
$$K/K_T = \mathrm{Spin}\,(7)/SU(3) = S^6 \times S^7 .$$

$q_1 \oplus q_2$ can be seen to be equivalent to the isotropy representation for Spin (9)/Spin_t (7) $= S^{15}$.

It is evident that (b) fails for both of these examples, hence so does (a). The fact that (a) fails for $r_1 \oplus r_2$ is also easy to see using the triality automorphism.

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