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CRITICAL POINTS OF THE LENGTH OF A KILLING VECTOR FIELD

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Introduction

Let M be a complete Riemannian manifold, X a Killing vector field on M, and φ_t its 1-parameter group of isometries of M, and denote by Crit $(|X|^2)$ (resp. Crit (φ_t)) the critical point set of the function $|X|^2$ (resp. $\delta^2_{\varphi t}$, where $\delta_{\varphi_t}(p)$ is the distance from p to $\varphi_t(p)$). In this paper we prove that if M is compact, then there is a number $a > 0$ such that Crit $(|X|^2) =$ Crit (φ_t) for every $|t| < a$. In the proof we make use of a slight generalization of the period bounding lemma of ordinary differential equations; The only version of this lemma which we have seen in the literature (see for example [1]) makes a mild transversality assumption which we eliminate.

1. Period bounding lemma

Let M be a compact $C^r(r \geq 2)$ manifold of dimension n, and X^τ, $\tau \in (-\tau_0, \tau_0)$ and $\tau_0 > 0$, be a parameterized C^r vector field on M. Then $X: (-\tau_0, \tau_0) \times M \to TM$ is a C^r map such that $\pi(X^\tau_p) = p$ for every $(\tau, p) \in (-\tau_0, \tau_0) \times M$, where $\pi: TM \to M$ is the projection of the tangent bundle TM of M. Let ψ^τ_s be the parameterized flow of X^τ, so that, for each fixed $\tau \in (-\tau_0, \tau_0)$, ψ^τ_s is the 1-parameter group of diffeomorphisms of M generated by X^τ.

Lemma. *For each $0 \leq \bar\tau < \tau_0$ there is a number $a(\bar\tau) > 0$ such that for every $|\tau| \leq \bar\tau$ each closed orbit of ψ^τ_s has least period $\geq a(\bar\tau)$.*

Proof. Suppose the lemma is false. Then there are a sequence $p_i \in M$ and sequences $\tau_i \in [-\bar\tau, \bar\tau]$, $\alpha_i \in \boldsymbol{R}$ such that the orbit $\{\psi^{\tau_i}_s(p_i) \mid s \in \boldsymbol{R}\}$ is closed and has least period $\alpha_i > 0$ with $\alpha_i \to 0$ as $i \to \infty$. By choosing subsequences if necessary, we may assume $p_i \to p_* \in M$ and $\tau_i \to \tau_* \in [-\bar\tau, \bar\tau]$. Then $X^{\tau_i}_{p_i} \to X^{\tau_*}_{p_*}$ as $i \to \infty$. Now either $X^{\tau_*}_{p_*} = 0$ or $X^{\tau_*}_{p_*} \neq 0$. If $X^{\tau_*}_{p_*} \neq 0$, then $X^\tau_p \neq 0$ for all (τ, p) near (τ_*, p_*). There is a neighborhood U of p_* such that for each τ near τ_* there is a coordinate system $(x^\tau_1, \cdots, x^\tau_n)$ in U satisfying $X^\tau = \partial/\partial x^\tau_1$. But since the periods of the orbits $\{\psi^{\tau_i}_s(p_i) \mid s \in \boldsymbol{R}\}$ approach 0, these curves eventually lie in arbitrarily small neighborhoods of p_*, contradicting the fact that they are level curves of coordinate systems valid in all of U. Therefore

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we may assume $X_{p_*}^{\tau_*} = 0$. Now choose a fixed coordinate system $(x_1, \cdots, x_n)$ in a neighborhood U of p_*, and assume $p_i \in U$ for all i. Thus we may assume that the parameterized family of vector fields X^τ is defined in a neighborhood V of 0 in $\boldsymbol{R}^n$, and p_i is a sequence of points of V converging to 0 as $i \to \infty$. (Identify $p_* \equiv 0$). Moreover, we may assume the 1-parameter groups ψ_s^τ of the X^τ are defined in V. Let $\gamma_i(s) = \psi_s^{\tau_i}(p_i)$ be the i-th orbit in the sequence. For each i, let P_i be the hyperplane in $\boldsymbol{R}^n$ through p_i and orthogonal to γ_i at p_i, and let $v_i = X_{p_i}^{\tau_i}$ be the tangent to γ_i at p_i. Let $s_i \in (0, \alpha_i)$ be the largest value such that $q_i = \gamma_i(s_i) \in P_i$. Then q_i is the last point of intersection of γ_i with P_i before p_i, and the points $\gamma_i(s), s_i < s < \alpha_i$, lie on the opposite side of P_i from the vector v_i. Let $\tilde{v}_i = (\psi_{s_i}^{\tau_i})_* v_i$, tangent to γ_i at s_i. By the construction, $v_i \perp P_i$ and $\tilde{v}_i$ either lies in P_i or points into the half-space on the other side of P_i from v_i. In any case, the angle between v_i and $\tilde{v}_i$ is always $\geq \pi/2$. (Clearly, $v_i \neq 0$, and $\tilde{v}_i \neq 0$.) By choosing a subsequence if necessary, we may assume that the sequence of unit vectors $v_i/|v_i|$ converges to a unit vector v. Then the sequence of hyperplanes P_i converges to a hyperplane $P \perp v$ through p_*. Since $0 < s_i < \alpha_i$ and $\alpha_i \to 0$, we have $s_i \to 0$ as $i \to \infty$; therefore $(\psi_{s_i}^{\tau_i})_* \to \mathrm{id}\colon T_{p_*}M \to T_{p_*}M$ as $i \to \infty$. Consequently, $\lim_{i\to\infty} (\psi_{s_i}^{\tau_i})_*(v_i/|v_i|) = v = \lim_{i\to\infty} v_i/|v_i|$. But the angles $\sphericalangle\, (v_i/|v_i|, (\psi_{s_i}^{\tau_i})_*(v_i/|v_i|)) \geq \pi/2$ for all i, so $\sphericalangle\, (v, \lim_{i\to\infty} (\psi_{s_i}^{\tau_i})_*(v_i/|v_i|)) \geq \pi/2$, which is a contradiction.

Remark. This result clearly applies to compact neighborhoods of arbitrary (i.e., possibly noncompact) manifolds.

2. Application to Killing vector fields

Suppose M is a complete Riemannian manifold of class C^∞, and $f\colon M \to M$ is an isometry such that for every $p \in M$ there is a unique minimizing geodesic from p to $f(p)$; such an isometry is said to have "small displacement". Let $\delta_f\colon M \to \boldsymbol{R}$ be defined by: $\delta_f(p) =$ distance from p to $f(p)$, and let Crit (f) be the critical point set of δ_f^2. In [3] we showed that for isometries f of small displacement δ_f^2 is C^∞ so that Crit (f) has meaning, and that $p \in$ Crit (f) if and only if f preserves the minimizing geodesic from p to $f(p)$ (in the sense that f is a simple translation along this geodesic). In [2], R. Hermann studied the analogous problem for Killing vector fields, and showed that if X is a Killing vector on M, then the critical point set Crit $(|X|^2)$ of the function $|X|^2$ consists of those points of M whose orbits by the 1-parameter group of isometries φ_t generated by X are geodesics. It is then clear that Crit $(|X|^2) \subset$ Crit (φ_t) for all t such that φ_t has small displacement, and it is not hard to show that Crit $(|X|^2) = \bigcap_{0<t<t_0}$ Crit (φ_t), where t_0 is so small that φ_t has small displacement if $|t| < t_0$. We prove here that if M is compact, then there is a number $a > 0$ such that Crit $(|X|^2) =$ Crit (φ_t) if $0 < |t| < a$.

From now on, we assume M is a compact Riemannian manifold of class C^∞

and X is a Killing vector field on M. Suppose that there is no number $a > 0$ such that Crit $(|X|^2) =$ Crit (φ_t) for all $0 < |t| < a$. Then there are sequences $t_i \in \boldsymbol{R}$ and $p_i \in M$ such that $t_i > 0$, $t_i \to 0$ as $i \to \infty$, and $p_i \in (\text{Crit}\,(\varphi_{t_i}) - \text{Crit}\,(|X|^2))$ for all i. We may take t_i to be strictly decreasing. Since M is compact, we may assume, by taking a subsequence if necessary, that $p_i \to p \in M$ as $i \to \infty$.

Lemma 1. $p \in \text{Crit}\,(|X|^2)$.

Proof. Let γ_i be the minimizing geodesic from p_i to $\varphi_{t_i}(p_i)$. Since the vector fields tangent to the γ_i lie in a compact neighborhood in TM (restrict to the portion of γ_i between p_i and $\varphi_{t_i}(p_i)$) we can assume, by choosing a subsequence if necessary, that the γ_i converge to a geodesic γ through p. Now γ_i intersects the orbit $\{\varphi_t(p_i) \,|\, t \in \boldsymbol{R}\}$ at the points $\varphi_{ti}^m(p_i) = \varphi_{mt_i}(p_i)$, $m \in \boldsymbol{Z}$. We see that since $t_i \to 0$, these points approach a dense set of points on γ at which the orbit $\varphi_t(p)$ meets γ. Therefore $\gamma = \{\varphi_t(p) \,|\, t \in \boldsymbol{R}\}$, and $p \in \text{Crit}\,(|X|^2)$. q.e.d.

Now either $X_p = 0$ or $X_p \neq 0$. If $X_p = 0$, then p is a fixed point of all the φ_t, $t \in \boldsymbol{R}$. Also, since $p_i \notin \text{Crit}\,(|X|^2)$, p_i is not fixed by all φ_t, $t \neq 0$.

Lemma 2. *There is a number $\bar{t} > 0$ such that p_i is not fixed by any φ_t, $0 < |t| < \bar{t}$.*

Proof. Assume to the contrary that there is a sequence $t_k \to 0$ such that $t_k > 0$ and p_i is fixed by φ_{t_k}. Then p_i is fixed by $\varphi_{t_k}^m = \varphi_{mt_k}$ for all $m \in \boldsymbol{Z}$, so p_i is fixed by φ_t for a dense subset of $\boldsymbol{R}$. Consequently p_i is fixed by all φ_t, $t \in \boldsymbol{R}$, which is a contradiction. q.e.d.

Let Zero $(X) = \{p \,|\, X_p = 0\}$.

Lemma 3. *There is $\bar{t} > 0$ such that* Fix $(\varphi_t) =$ Zero (X) *for all $0 < t \leq \bar{t}$.*

Proof. Suppose the lemma is false. Then there are sequences $t_i \to 0$ and $p_i \in (\text{Fix}\,(\varphi_{t_i}) - \text{Zero}\,(X))$. By taking subsequences if necessary, we may assume $p_i \to p \in M$. Since $\varphi_{ti}^m(p_i) = p_i$ for all $m \in \boldsymbol{Z}$, $\varphi_t(p) = p$ for a dense set of $t \in \boldsymbol{R}$. Therefore $p \in$ Zero (X). We may assume $t_i > 0$ is minimal such that $\varphi_{t_i}(p_i) = p_i$, for if no minimal positive t_i exists then $p_i \in$ Zero (X) by Lemma 2. Now the curves $\{\varphi_t(p_i) \,|\, t \in \boldsymbol{R}\}$ are periodic solutions of the differential equation X in a neighborhood of p, and their least periods coverage to 0. This contradicts the period bounding lemma. q.e.d.

Now assuming $X_p = 0$, we have a sequence $p_i \to p$ with $\varphi_{t_i}(p_i) \neq p_i$, such that φ_{t_i} preserves the minimizing geodesic γ_i from p_i to $\varphi_{t_i}(p_i)$. Since φ_{t_i} preserves γ_i and fixes p, the geodesic γ_i never gets farther away from p than $r_i = \max\{\rho(p, \gamma_i(s)) \,|\, 0 \leq s \leq \rho(p_i, \varphi_{t_i}(p_i))\}$, where $\rho(p, q)$ is the distance from p to q. Since $p_i \to p$ and $t_i \to 0$, it is clear that $r_i \to 0$ as $i \to \infty$. Thus we have a sequence of geodesics γ_i which converges to a point; this is impossible. Therefore $X_p \neq 0$. Then $X \neq 0$ in a neighborhood of p, and we may choose a coordinate system $(x_1, \cdots, x_n)$ in a neighborhood U of p such that $x_i(p) = 0$, $1 \leq i \leq n$, and $X = \partial/\partial x_1$ in U. Let $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$ be the coefficients of the Riemannian metric in these coordinates, where $\langle\ ,\ \rangle$ is the Riemannian inner product. Then

$$Xg_{ij} = \langle [\partial/\partial x_1, \partial/\partial x_i], \partial/\partial x_j \rangle + \langle \partial/\partial x_i, [\partial/\partial x_1, \partial/\partial x_j] \rangle = 0 \qquad \text{for all } 1 \le i, j \le n$$

because X is a Killing vector field, so the g_{ij} are independent of x_1. Consequently, all the Christoffel symbols Γ^k_{ij} are also independent of x_1. The orbits $\{\varphi_t(q) \mid t \in \boldsymbol{R}\}$ are integral curves of X and therefore have the form:

$$t \mapsto (x_1(q) + t,\ x_2(q), \cdots, x_n(q)) \qquad \text{for all } q \in U\ .$$

Thus φ_t: $(x_1(q), \cdots, x_n(q)) \mapsto (x_1(q) + t,\ x_2(q), \cdots, x_n(q))$. Now let $\gamma_i(s) = (x^i_1(s), \cdots, x^i_n(s))$ be the minimizing geodesic from p_i to $\varphi_{t_i}(p_i)$ with arc length s. Since φ_{t_i} preserves γ_i, we have $\varphi_{t_i}\gamma_i(s) = \gamma_i(s + \alpha_i)$ for some constant $\alpha_i > 0$ and all $s \in \boldsymbol{R}$. Since $\alpha_i = \rho(p_i, \varphi_{t_i}(p_i))$, we see that $\alpha_i \to 0$ as $i \to \infty$. (Note that since $t_i \to 0$, there is a sequence $m_i \in \boldsymbol{Z}$ such that $m_i \to \infty$ as $i \to \infty$, and $\varphi^k_{t_i}(p_i) \in U$ for all $|k| \le m_i$.) In local coordinates, the equation $\varphi_{t_i\gamma_i}(s) = \gamma_i(s + \alpha_i)$ becomes:

$$(x^i_1(s) + t_i,\ x^i_2(s), \cdots, x^i_n(s)) = (x^i_1(s + \alpha_i),\ x^i_2(s + \alpha_i), \cdots, x^i_n(s + \alpha_i))\ .$$

Thus $x^i_1(s) + t_i = x^i_1(s + \alpha_i)$, and the $x^i_j(s)$, $2 \le j \le n$, are periodic of period α_i. Then the functions $\bar{x}^i_1(s) \equiv x^i_1(s) - (t_i/\alpha_i)s$, $\bar{x}^i_j(s) \equiv x^i_j(s)$, $2 \le j \le n$, are all periodic of period α_i. Since the functions x^i_j, $1 \le j \le n$, satisfy the differential equations for a geodesic:

$$\frac{d^2x^i_k}{ds^2} + \sum_{l,m=1}^{n} \Gamma^k_{lm} \frac{dx^i_l}{ds} \frac{dx^i_m}{ds} = 0\ , \qquad 1 \le k \le n\ ,$$

the functions $\bar{x}^i_k$ satisfy the system:

$$\frac{d^2\bar{x}^i_k}{ds^2} + \sum_{l,m=1}^{n} \Gamma^k_{lm}(x^i_2(s), \cdots, x^i_n(s)) \frac{d\bar{x}^i_l}{ds} \frac{d\bar{x}^i_m}{ds} + 2\left(\frac{t_i}{\alpha_i}\right) \sum_{m=1}^{n} \Gamma^k_{lm}(\cdots) \frac{d\bar{x}^i_m}{ds} + \Gamma^k_{11}(\cdots)\left(\frac{t_i}{\alpha_i}\right)^2 = 0\ .$$

Here Γ^k_{lm} is a function of $\bar{x}^i_2(s), \cdots, \bar{x}^i_n(s)$ alone, since it is independent of x_1. Equivalently, we have the first-order system:

$$(*) \qquad \begin{aligned} & d\bar{x}^i_k/ds = y^i_k\ , \\ & \frac{dy^i_k}{ds} + \sum_{l,m=1}^{n} \Gamma^k_{lm} y^i_l y^i_m + 2\left(\frac{t_i}{\alpha_i}\right) \sum_{m=1}^{n} \Gamma^k_{lm} y^i_m + \Gamma^k_{11}\left(\frac{t_i}{\alpha_i}\right)^2 = 0\ . \end{aligned}$$

The system $(*)$ is autonomous for each i. Assume now that X is normalized so that the parameter t of φ_t is the arc length along the geodesic $\gamma(t) = \varphi_t(p)$, i.e., $|X_{\gamma(t)}| = 1$ for all t.

Lemma 4. $\lim_{i\to\infty} (t_i/\alpha_i) = 1$.

Proof. Let $C_i(t) = \varphi_t(p_i)$ be the orbit of p_i. Since $p_i \to p$, we know that $C_i(t) \to \gamma(t)$ uniformly in some compact neighborhood of p. Since the sequence of geodesics γ_i also has this property, we see that $\lim_{i\to\infty} (L(C_i)/L(\gamma_i)) = 1$, where $L(C_i)$ (resp. $L(\gamma_i)$) is the length of C_i (resp. γ_i). Now $L(\gamma_i) = \alpha_i$, and $L(C_i) = \int_0^{t_i} |X_{C_i(t)}|dt = t_i|X_{C_i(\tilde{t}_i)}|$ for some $0 < \tilde{t}_i < t_i$; so $\dfrac{t_i}{\alpha_i} = \dfrac{1}{|X_{C_i(\tilde{t}_i)}|} \cdot \dfrac{L(C_i)}{L(\gamma_i)}$. Since $C_i(\tilde{t}_i) \to p$ as $i \to \infty$, $|X_{C_i(\tilde{t}_i)}| \to 1$, and the lemma is proved. q.e.d.

Now consider the following autonomous system with parameter τ, defining a parameterized vector field Y^τ in a neighborhood of 0 in $\boldsymbol{R}^{2n}$:

$$
\begin{aligned}
&dx_k/ds = y_k\ , \\
(**)\qquad &\frac{dy_k}{ds} + \sum_{l,m=1}^{n} \Gamma^k_{lm} y_l y_m + 2(1+\tau) \sum_{m=1}^{n} \Gamma^k_{lm} y_m + (1+\tau)^2 \Gamma^k_{11} = 0\ .
\end{aligned}
$$

If $1 + \tau_i = t_i/\alpha_i$, then we see that the sequence of functions $\eta^i = (\bar{x}^i_1, \cdots, \bar{x}^i_n, y^i_1, \cdots, y^i_k)$ which we constructed earlier satisfies $(**)$ with parameter values τ_i. Moreover, $\tau_i \to 0$ as $i \to \infty$ since $t_i/\alpha_i \to 1$, and the solution η^i is periodic of period α_i approaching 0 as $i \to \infty$. This contradicts the period bounding lemma. Therefore our original assumption that the number $a > 0$ does not exist is false. Hence we have proved:

Theorem. *Let M be a compact Riemannian manifold of class C^∞, X a Killing vector field on M, and φ_t the 1-parameter group of isometries generated by X. Then there is a number $a > 0$ such that* Crit $(|X|^2) =$ Crit (φ_t) *for* $|t| < a$.

Example. We construct a simple example of a (noncompact) manifold M and a 1-parameter group of isometries φ_t of M such that Crit $(|X|^2) \neq$ Crit (φ_{t_0}) for some $t_0 > 0$, where X is the Killing vector field associated to φ_t. Let $M = \boldsymbol{R}^5$ with the usual metric, and define

$$
\varphi_t(x_1 \cdots x_5) = \begin{pmatrix} 1 & & & & \\ & \cos t & \sin t & \multicolumn{2}{c}{0} \\ & -\sin t & \cos t & & \\ \multicolumn{3}{c}{0} & \cos 2t & \sin 2t \\ & & & -\sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
$$

φ_t is clearly a 1-parameter group of isometries, and the only geodesic of $\boldsymbol{R}^5$ which is preserved by φ_t for all t is the line $t \mapsto (t, 0, \cdots, 0)$. Crit $(|X|^2)$ therefore equals this line. The set Crit (φ_π) of points lying on geodesics preserved by φ_π is: $\{(x_1, 0, 0, x_4, x_5)\}$, and Crit $(\varphi_{2\pi}) = \boldsymbol{R}^5$.

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